
BANACH ℓ -ADIC REPRESENTATIONS OF p -ADIC GROUPS

by

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Abstract. — Let $p \neq \ell$ be two distinct prime numbers, let F be a p -adic field and let E be an ℓ -adic field. We prove that the smooth part and the completion are inverse equivalences of categories between the category of admissible Banach unitary E -representations of $GL(n, F)$ and the category of admissible smooth E -representations of $GL(n, F)$ equipped with a commensurability class of lattices. We formulate the ℓ -adic local Langlands correspondence as a canonical bijection between the n -dimensional ℓ -adic representations of the absolute Galois group Gal_F and the topologically irreducible admissible Banach unitary ℓ -adic representations of $GL(n, F)$.

Résumé (Représentations ℓ -adiques de groupes p -adiques). — Soient $p \neq \ell$ deux nombres premiers distincts, soit F un corps p -adique et soit E un corps ℓ -adique. Nous démontrons que la partie lisse et la complétion définissent des équivalences de catégories inverses l'une de l'autre entre la catégorie des représentations admissibles de Banach unitaires de $GL(n, F)$ sur E et la catégorie des représentations lisses admissibles de $GL(n, F)$ sur E munies d'une classe de commensurabilité de réseaux. Nous formulons la correspondance de Langlands locale ℓ -adique comme une bijection canonique entre les représentations ℓ -adiques de dimension n du groupe de Galois absolu Gal_F et les représentations topologiquement irréductibles admissibles de Banach unitaires ℓ -adiques de $GL(n, F)$.

1. Introduction

Let p be a prime number, let F be a finite extension of \mathbf{Q}_p or a field of Laurent series $k((T))$ over a finite field k of characteristic p , let \bar{F} be an algebraic closure of F and let n be an integer ≥ 1 .

For any topological field C , the continuous representations of $GL(n, F)$ on topological vector spaces over C are interesting for their applications in arithmetic, geometry or physics, via the theory of L -functions associated to automorphic representations.

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When C varies, the theories of C -representations of $GL(n, F)$ present simultaneously strong similarities and strong different features but the Langlands insight, when C is the complex field, to use the smooth complex representations of $\text{Gal}_F = \text{Gal}(\overline{F}/F)$ as a classifying scheme, seems to be extend to other fields.

Why moving the coefficient field C ? There are many reasons.

1) The representations of Gal_F appearing naturally are not smooth complex. In the étale cohomology of proper smooth algebraic varieties, they are continuous ℓ -adic representations on finite dimensional vector spaces V over finite extensions E/\mathbf{Q}_ℓ , for a prime number ℓ . By a reduction of a stable O_E -lattice of V , they give smooth mod ℓ -representations over the residual field of E .

2) The local Langlands correspondence for $GL(n, F)$, over any algebraically closed field R of characteristic different from p , is a bijection

$$\pi \leftrightarrow (\rho, N)$$

between the equivalence classes of the smooth irreducible R -representations π of $GL(n, F)$ and of the pairs (ρ, N) where ρ is a n -dimensional smooth semi-simple R -representation of the Weil group W_F and N a nilpotent endomorphism of the space of ρ such that $\rho(w)N = N|w|\rho(w)$ where $|?|$ is the unramified R -character of W_F sending a geometric Frobenius to q , the order of the residual field of F .

Our purpose is to obtain a local Langlands correspondence for continuous ℓ -adic representations.

Theorem 1 *Let ℓ be a prime number different from p . The ℓ -adic local Langlands correspondence for $GL(n, F)$ is a canonical bijection between the equivalence classes of*

- a) *n dimensional continuous ℓ -adic representations of Gal_F with a semi-simple action of the Frobenius,*
- b) *Topologically irreducible admissible Banach unitary ℓ -adic representations of $GL(n, F)$.*

This theorem⁽¹⁾ is motivated by the fascinating work and conjectures of Christophe Breuil on the p -adic local Langlands correspondence, where topologically irreducible admissible Banach unitary p -adic representations of $GL(2, \mathbf{Q}_p)$ appear naturally.

With the existing literature, one translates the local Langlands complex correspondence for $GL(n, F)$ into a canonical bijection between the isomorphism classes of a) and of

- c) *Irreducible smooth $\overline{\mathbf{Q}}_\ell$ -representations of $GL(n, F)$ with a stable lattice.*

Indeed, as is well known,

1. proved in a letter to Breuil in september 2003, and announced in the Emmy Noether lectures 2005 of Goettingen.

(i) The smooth complex local Langlands correspondence $LL(\rho, N)$ twisted by a suitable unramified character,

$$(\rho, N) \leftrightarrow LL(\rho, N) \otimes |\det?|^{-(n-1)/2},$$

called the smooth complex local Hecke correspondence, is $\text{Aut } \mathbf{C}$ -equivariant [H prop.6].

(ii) Transporting the correspondence (i) with an algebraic isomorphism $j : \mathbf{C} \simeq \overline{\mathbf{Q}}_\ell$, we obtain the smooth local Hecke $\overline{\mathbf{Q}}_\ell$ -correspondence, which does not depend on the choice of the isomorphism j .

(iii) N disappears when one considers continuous $\overline{\mathbf{Q}}_\ell$ -representations of W_F instead of smooth $\overline{\mathbf{Q}}_\ell$ -representations. The pairs (ρ, N) are in bijection

$$(\rho, N) \leftrightarrow \sigma$$

with the n -dimensional ℓ -adic representations σ of W_F with a semi-simple action of the Frobenius. The reason is that the kernel of the natural morphism $t : I_F \rightarrow \mathbf{Z}_\ell$ is a profinite group prime to ℓ . There is a nilpotent endomorphism N of the space of σ such that $\sigma(?) = \exp(t(?)N)$ on a subgroup of finite index of I_F [8].

(iv) The n -dimensional ℓ -adic representation σ of W_F in (iii) extends by continuity to an ℓ -adic representation of Gal_F if and only if ρ has a bounded image (i.e. the values of determinants of the irreducible components of ρ are units) [8].

(v) ρ has a bounded image if and only if $\pi = LL(\rho, N)$ is integral [11] §1.4; moreover all stable lattices in π are commensurable [10] theorem 1.

Our task is to show that the completion with respect to a stable lattice gives a bijection between the isomorphism classes of b) and of c).

The beginning of the proof is valid for any locally profinite group G , with a countable fundamental system of neighbourhoods of the unit, consisting of open profinite groups of pro-order not divisible by ℓ (section 2). We prove (theorem 2.12) that the completion and the smooth part induce equivalences of categories between the category $\mathcal{M}_\ell(G)^{adm}$ of admissible smooth ℓ -adic representations of G equipped with a commensurability class of lattices, and the category $\mathcal{B}_\ell(G)^{adm}$ of admissible Banach unitary ℓ -adic representations of G .

Then we consider the group of rational points G_F of any reductive connected group over a local non archimedean field F of residual characteristic $p \neq \ell$ (section 3). We prove (theorem 3.6) that the completion and the smooth part induce equivalences of categories between the category $\text{Mod}_{\overline{\mathbf{Q}}_\ell}^{int, fl}(G_F)$ of integral smooth $\overline{\mathbf{Q}}_\ell$ -representations of G_F of finite length and the category $\mathcal{B}_\ell(G_F)^{adm, fl}$ of admissible Banach unitary ℓ -adic representations of topological finite length of G_F . We deduce the wanted bijection between the isomorphism classes of b) and c) by restricting to irreducible representations and choosing $G_F = GL(n, F)$.

A natural question was raised by the referee: Is a topologically irreducible Banach unitary ℓ -adic representation of G_F always admissible? L. Clozel noticed that the examples of B. Diarra [4] th. 4 (van Rooj), give examples of topologically irreducible representations $V \in \mathcal{B}_E(GL(1, F))$ where any non zero intertwining operator is bijective, which are *not admissible*.

2.

2.1. The two categories. — Let $\ell \neq p$ be two distinct prime numbers, let E/\mathbf{Q}_ℓ be a finite extension of ring of integers O_E , of uniformizer p_E , and of residual field k_E , and let G be a topological group admitting a *countable* fundamental system of neighbourhoods of the unit consisting of open *pro- ℓ'* -subgroups (profinite subgroups of pro-order *prime to ℓ*).

After having recalled some definitions and properties concerning the representations of the group G on E -vector spaces, we will introduce the two categories of representations $\mathcal{M}_E(G)$ and $\mathcal{B}_E(G)$ which will be compared in this paper.

Let Mod_E be the category of E -vector spaces and let $M \in \text{Mod}_E$ non zero. A *line* in M is a subspace of dimension 1. A *lattice* L in M is a O_E -submodule of M which contains *no line* and contains a basis of M over E . Note that a quotient of a lattice may contain a line. When the dimension of M over E is *countable*, a lattice L in M is a *free* O_E -submodule of M generated by a basis of M over E [9] I Appendice C.5. Two lattices L, L' in M are *commensurable* when there exists an element $a \in O_E$ such that $aL \subset L'$, $aL' \subset L$. We denote by $[L]$ the commensurability class of L .

Remark 2.1. — *An O_E -submodule L of $M \in \text{Mod}_E$ is a lattice in M if and only if any non zero element $m \in M$ satisfies the two conditions:*

- a) *there exists an integer $n \in \mathbf{N}$ such that $\ell^n m$ belongs to L ,*
- b) *there exists an integer $n \in \mathbf{N}$ such that $\ell^{-n} m$ does not belong to L .*

Two lattices L, L' in M are commensurable if and only if there exists an integer $n \in \mathbf{N}$ such that $\ell^n L \subset L'$, $\ell^n L' \subset L$.

A representation (= a linear action) of G on M is called *admissible* when $\dim_E M^H < \infty$, for any open *pro- ℓ'* -subgroup H of G , where $M^H \in \text{Mod}_E$ is the subspace of H -invariant vectors of M . The representation M is called *irreducible* when $M \neq 0$ and 0 and M are the only G -stable subspaces of M , *finitely generated* when M is a finitely generated EG -module, *of finite length* when there exists a finite G -stable filtration $0 \subset M_1 \subset \dots \subset M_n = M$ with *irreducible quotients*. The length of the filtration and the isomorphism classes of the quotients, up to the order, do not depend on the choice of the filtration.

A lattice L in the representation of G on M will always be a G -stable lattice in M ; the lattice will be called *finitely generated* when it is a finitely generated $O_E G$ -module. A representation of G on M containing a lattice is called *integral* (we do not suppose that the lattice is O_E -free as in [9]). There exist finitely generated lattices in a finitely generated integral representation; they form a commensurability class, and any lattice contains a finitely generated lattice.

A *continuous* E -representation of G is a topological Hausdorff E -vector space M equipped with a continuous action of G , i.e. such that the map $(g, v) \rightarrow gv : G \times M \rightarrow M$ is continuous. It is called *topologically irreducible* when $M \neq 0$ and 0 and M are the only closed G -stable subspaces of M . It is called of *finite topological length* when there exists a finite filtration by G -stable closed subspaces $0 \subset M_1 \subset \dots \subset M_n = M$ with *topologically irreducible quotients*.

The category $\mathcal{C}_E(G)$ of continuous representations of G on topological Hausdorff complete E -vector spaces with continuous G -equivariant E -linear morphisms, called *intertwining operators*, contains the subcategory $\mathcal{B}_E(G)$ of Banach unitary representations, defined below. We indicate by an upper index *adm* or *fl* or *adm, fl* or *int* or *int, fl* the full subcategories representations which are admissible or of finite topological length or admissible and of finite topological length or integral or integral and of finite topological length. Example: $\mathcal{C}_E(G)^{adm}, \mathcal{B}_E(G)^{adm}$ for admissible representations.

A representation of G on an E -vector space W is *smooth* when the stabilizer in G of any vector of W is open. Let $\text{Mod}_E(G)$ be the category of smooth E -representations of G , with morphisms the G -equivariant E -linear maps, called intertwining operators. The upper indices *adm* or *int* or *fl* have the same signification than above (forgetting the topology). Example: $\text{Mod}_E(G)^{adm, fl}$ for admissible finite length representations.

A *Banach unitary* E -representation V of G is a Hausdorff complete topological E -vector space with a topology given by a norm, equipped with a continuous action of G which respects the norm. A unit ball of V is $L = \{v \in V : \|v\| \leq 1\}$ for some norm $v \mapsto \|v\|$ on V defining the topology [Sch I.3, III]; it is a lattice in V . The unit balls of two norms on V giving the same topology are commensurable.

An E -linear map $f : V_1 \rightarrow V_2$ between two Banach E -vector spaces V_1, V_2 is continuous if and only if there exists some non zero $a \in E$ such that $f(L_1) \subset af(L_2)$ for some unit balls L_1, L_2 of V_1, V_2 [Sch I.3.1]. The topology quotient on the image of f is the topology induced by V_2 if and only if $f(L_1)$ and $L_2 \cap f(V_1)$ are commensurable (this does not depend on the choice of the unit balls L_1, L_2). When f is continuous and bijective, the inverse of f is continuous [Sch I.8.7].

We will compare $\mathcal{B}_E(G)$ with the category $\mathcal{M}_E(G)$ of smooth E -representations W of G equipped with a commensurability class $[L]$ of lattices; a morphism $(W, [L]) \rightarrow (W', [L'])$ is a morphism $f : W \rightarrow W'$ in $\text{Mod}_E(G)$ such that $f(L) \subset aL'$ for some

$a \in E$. The pair $(W, [L])$ is called admissible or of finite length when W is admissible or of finite length, and $\mathcal{M}_E(G)^{adm}$ or $\mathcal{M}_E(G)^{fl}$ is the full subcategory of admissible or of finite length pairs in $\mathcal{M}_E(G)$.

2.2. The two functors. — We introduce two natural functors in opposite directions between the categories $\mathcal{M}_E(G)$ and $\mathcal{B}_E(G)$.

There is the natural functor $\mathcal{C}_E(G) \rightarrow \text{Mod}_E(G)$ sending $M \in \mathcal{C}_E(G)$ to its *smooth part*

$$M^\infty := \cup_H M^H,$$

for all open pro- ℓ' -subgroups H of G . When $V \in \mathcal{B}_E(G)$ is a Banach unitary representation of G , the smooth part $L^\infty = V^\infty \cap L$ of a unit ball L of V is a lattice of V^∞ . Two unit balls of V are commensurable and their smooth parts are commensurable, hence $(V^\infty, [L^\infty]) \in \mathcal{M}_E(G)$ is well defined. A continuous morphism $f : V_1 \rightarrow V_2$ of Banach unitary E -representations of G with unit balls L_1, L_2 , restricts to a morphism $f^\infty : (V_1^\infty, [L_1^\infty]) \rightarrow (V_2^\infty, [L_2^\infty])$. We get a functor

$$\mathcal{B}_E(G) \rightarrow \mathcal{M}_E(G).$$

In the opposite direction there is the natural functor

$$\mathcal{M}_E(G) \rightarrow \mathcal{B}_E(G)$$

sending $(W, [L])$ to the *completion* of W for the L -adic topology [Sch 7.5]:

$$\hat{W}_L := \varprojlim_n W/\ell^n L \simeq E \otimes_{O_E} \hat{L}, \quad \hat{L} := \varprojlim_n L/\ell^n L.$$

Any element $v \in \hat{W}_L$ is written

$$(1) \quad v = (w_n + \ell^n L)_n, \quad w_n \in W, \quad w_{n+1} \in w_n + \ell^n L,$$

for all $n \in \mathbf{N}$. The lattice \hat{L} is a unit ball of \hat{W}_L for the gauge norm $\|v\| = \inf_{a \in E, v \in a\hat{L}} |a|$. The completions of W defined by two commensurable lattices of W are the same. The group G acts naturally on \hat{W}_L , for $g \in G$ and v as above,

$$gv = (gw_n + \ell^n L)_{n \in \mathbf{N}},$$

and \hat{W}_L is a Banach unitary E -representation of G of unit ball \hat{L} , well defined by $(W, [L])$. A morphism $f : (W, [L]) \rightarrow (W', [L'])$ in $\mathcal{M}_E(G)$ extends by continuity to an intertwining operator $\hat{f} : \hat{W}_L \rightarrow \hat{W}'_L$.

Remark 2.2. — *The map $W \mapsto \hat{W}_L$ sending w to $(w + \ell^n L)_{n \in \mathbf{N}}$ is injective, because L contains no line. We will identify W with its image in \hat{W}_L .*

2.3. To study the two functors, smooth part and completion, between $\mathcal{M}_E(G)$ and $\mathcal{B}_E(G)$, the key point is the exactness of the H -invariants functor.

Proposition 2.3. — *Let H be any open pro- ℓ' -subgroup of G . The H -invariants functor*

$$M \mapsto M^H : \mathcal{C}_E(G) \rightarrow \text{Mod}_E$$

is exact.

Proof. — This is well known for the category $\text{Mod}_E(G)$ of smooth representations. The exactness results from the existence of a Haar O_E -measure dg on G such that the volume $\text{vol}(H, dg)$ of H is a unit in O_E . The function e_H equal to $\text{vol}(H, dg)^{-1}$ on H and 0 on $G - H$, is an idempotent in the convolution algebra $C_c^\infty(G; O_E)$ of locally constant compactly supported functions $G \rightarrow O_E$, for the Haar measure dg . The idempotent e_H acts on $M \in \mathcal{C}_E(G)$, as follows. One chooses a decreasing sequence of normal subgroups H_n of H of finite index such that $\bigcap_{n \in \mathbf{N}} H_n$ is trivial, and a system of representatives X_n in H of H/H_n . The continuity of the action of G on M implies that the sequence

$$v_n = [H : H_n]^{-1} \sum_{g \in X_n} gv$$

converges to a unique element $e_H * v$ in the Hausdorff complete space M . This element $e_H * v$ does not depend on the choice of $(H_n, X_n)_{n \in \mathbf{N}}$ and clearly $v \mapsto e_H * v$ is a linear projector $M \rightarrow M^H$ of its H -invariants. \square

Corollary 2.4. — *The smooth part functor $M \mapsto M^\infty : \mathcal{C}_E(G) \rightarrow \text{Mod}_E(G)$ is exact.*

Proposition 2.5. — *A Banach unitary E -representation V of G is equal to the closure of its smooth part V^∞ .*

Proof. — Let v be an arbitrary element of V and let L be a unit ball of V . For any integer $n \geq 1$, there is an open pro- ℓ' -subgroup H_n of G such that $H_n v \subset v + \ell^n L$, by the continuity of the action of G on V . The element $e_{H_n} * v$ is fixed by H_n and belongs to $v + \ell^n L$. The element $(e_{H_n} * v + \ell^n L)_{n \in \mathbf{N}}$ belongs to the closure of V^∞ and is equal to v . \square

Corollary 2.6. — *The smooth part functor $\mathcal{B}_E(G) \rightarrow \mathcal{M}_E(G)$ is fully faithful.*

Proof. — For $i = 1, 2$, let $V_i \in \mathcal{B}_E(G)$ with unit ball L_i . The embedding $V_i^\infty \rightarrow (V_i^\infty)_{L_i^\infty}$ extends by continuity to an isomorphism $\tau_i : V_i \rightarrow (V_i^\infty)_{L_i^\infty}$ in $\mathcal{B}_E(G)$ by the proposition 2.5 and its proof. We deduce that arbitrary intertwining operators $\phi : (V_1^\infty, [L_1^\infty]) \rightarrow (V_2^\infty, [L_2^\infty])$ and $f : V_1 \rightarrow V_2$ satisfy

$$\phi = (\tau_2^{-1} \hat{\phi} \tau_1)^\infty \quad , \quad f = \tau_2^{-1} (f^\infty) \tau_1 .$$

\square

We show that the completion commutes with the H -invariants.

Proposition 2.7. — *Let V be the completion of an integral smooth E -representation W of G with respect to a lattice L , and let H be an open pro- ℓ' -subgroup of G . The H -invariants V^H of V is equal to the closure of W^H in V ,*

$$V^H = \overline{W^H}.$$

Proof. — For $X = W, L$ or V , we have $e_H * X = X^H$. Let $v = (w_n + \ell^n L)_{n \in \mathbf{N}}$ be an element of V as in (1). Then $e_H * w_{n+1} \in e_H * w_n + \ell^n L$, and $e_H * v = (e_H * w_n + \ell^n L)_{n \in \mathbf{N}}$. □

Corollary 2.8. — *An admissible smooth E -representation of G with a commensurability class of lattices is equal to the smooth part of its completion.*

Proof. — When the representation W is admissible, the E -vector space W^H is finite dimensional and already complete, hence $V^H = W^H$ in the proposition 2.7. □

It is clear that the functor smooth part respect admissible representations, the corollary shows that the completion respects also admissible representations.

Theorem 2.9. — *The smooth part and completion are inverse equivalences of categories between $\mathcal{M}_E^{adm}(G)$ and $\mathcal{B}_E^{adm}(G)$.*

Proof. — Proposition 2.5, Corollaries 2.6, 2.8. □

In particular, the smooth part and the completion induce inverse equivalences of categories between admissible and of finite topological length representations $\mathcal{M}_E^{adm, fl}(G)$ and $\mathcal{B}_E^{adm, fl}(G)$.

We consider now ℓ -adic representations of G . For any finite extensions $E'/E/\mathbf{Q}_\ell$ contained in a fixed algebraic closure $\overline{\mathbf{Q}}_\ell$, the scalar extension $s_{E'/E}$ from E to E'

$$\mathcal{C}_E(G) \rightarrow \mathcal{C}_{E'}(G)$$

sends $M \in \mathcal{C}_E(G)$ to $M_{E'} := E' \otimes_E M = \oplus(e_i \otimes M)$, for a finite basis (e_i) of the E -vector space E' , with the topology induced by M (independent of the choice of the basis) and a morphism $f : M \rightarrow M'$ in $\mathcal{C}_E(G)$ to $\text{id}_{E'} \otimes f$. The inductive limit

$$\mathcal{C}_\ell(G) := \lim_{s_{E'/E}} \mathcal{C}_E(G)$$

is the category of ℓ -adic representations of G . The scalar extension respects smooth representations, and the inductive limit

$$\text{Mod}_\ell(G) := \lim_{s_{E'/E}} \text{Mod}_E(G)$$

is the category of smooth ℓ -adic representations of G , which is a (not full) subcategory of the classical category $\text{Mod}_{\overline{\mathbf{Q}}_\ell}(G)$ of smooth $\overline{\mathbf{Q}}_\ell$ -representations of G .

Let R/E be any extension contained in $\overline{\mathbf{Q}}_\ell$ and let O_R be the ring of integers in R . As the dimension of R/E is countable, O_R is an O_E -free module [9] Appendice C, C.4. We denote by $L_{O_R} := O_R \otimes_{O_E} L$, the scalar extension from O_E to O_R of an O_E -module L .

Lemma 2.10. — *Let $M \in \mathcal{C}_E(G)$ or $M \in \text{Mod}_E(G)$ equipped with a lattice L .*

(i) *The H -invariants commute with the scalar extension, $e_H M_R = (e_H M)_R$, $e_H L_{O_R} = (e_H L)_{O_R}$ for any pro- ℓ' -subgroup H of G (this is true for any extension R/E of fields of characteristic different from p). In particular, M is admissible if and only if M_R is admissible.*

(ii) *The intersection $L = M \cap L'$ of a lattice L' of M_R with M is a lattice in M . In particular, if M_R is integral then M is integral.*

(iii) *The scalar extension L_{O_R} of a lattice L in M is a lattice in M_R . In particular, if M is integral then M_R is integral.*

(iv) *Two lattices L, L' of M are commensurable if and only if their scalar extensions L_{O_R}, L'_{O_R} are commensurable.*

Proof. — (i) is clear. The other properties are clear using the remark 2.1 and $L_{O_R} = \bigoplus_i (e_i \otimes L)$ for a basis (e_i) of the free O_E -module O_R . \square

Lemma 2.11. — *The scalar extension $s_{E'/E}$ from E to a finite extension E' commutes with the smooth part functor $\mathcal{C}_E(G) \rightarrow \text{Mod}_E(G)$ and with the smooth part and completion functors between $\mathcal{B}_E(G)$ and $\mathcal{M}_E(G)$.*

Proof. — We choose a basis (e_i) of the free O_E -module $O_{E'}$. The scalar extension $V_{E'} = \bigoplus_i (e_i \otimes V)$ of the completion V of $(W, [L]) \in \mathcal{M}_E(G)$ is clearly the completion of the scalar extension $(W_{E'} = \bigoplus_i (e_i \otimes W), [L_{O_{E'}} = \bigoplus_i (e_i \otimes L)])$ of $(W, [L])$. The scalar extension of the H -invariants of $V \in \mathcal{B}_E(G)$ is the H -invariants of the scalar extension $V_{E'}$ (lemma 2.10). \square

As the scalar extension $s_{E'/E}$ from E to a finite extension E' respects admissibility, lattices, commensurability of lattices, Banach spaces (lemma 2.10), the inductive limit over $s_{E'/E}$ for all finite extensions $E'/E/\mathbf{Q}_\ell$ contained in $\overline{\mathbf{Q}}_\ell$, defines the categories

- a) $\mathcal{C}_\ell(G)^{adm}$ of admissible ℓ -adic representations of G ,
- b) $\text{Mod}_\ell(G)^{int}$ of integral smooth ℓ -adic representations,
- b) $\mathcal{M}_\ell(G)$ of smooth ℓ -adic representations of G equipped with a commensurability class of lattices,
- c) $\mathcal{B}_\ell(G)$ of Banach unitary ℓ -adic representations of G .

We define the completion and smooth part functors between $\mathcal{M}_\ell(G)$ and $\mathcal{B}_\ell(G)$ using the lemma 2.11.

Theorem 2.12. — *The completion and smooth part functors induce equivalence of categories between the categories $\mathcal{M}_\ell(G)^{adm}$ and $\mathcal{B}_\ell(G)^{adm}$.*

Proof. — Theorem 2.9. □

3.

Let G_F be the group of rational points of a connected reductive group over a local non archimedean field F of residual characteristic p . The group G_F is a locally pro- p -group. As before E/\mathbf{Q}_ℓ is a finite extension contained in $\overline{\mathbf{Q}_\ell}$ and $\ell \neq p$.

Proposition 3.1. — *Let R/R_o be any extension of fields of characteristic different from p . Then $W \in \text{Mod}_{R_o}(G_F)$ has finite length if and only if $W_R \in \text{Mod}_R(G_F)$ has finite length.*

Proof. — [9] II.4.3.c. □

Proposition 3.2. — *Any finite length smooth representation W of G_F over a field of characteristic different from p is admissible.*

Proof. — This is proved in [9] II.2.8 when the field is algebraically closed. The scalar extension is not sensitive to admissibility and finite length (lemma 2.10, proposition 3.1) for any extension of fields of characteristic different from p . □

Proposition 3.3. — *The lattices in an integral finite length representation $W \in \text{Mod}_E(G_F)$ are commensurable (hence finitely generated).*

Proof. — This is proved [10] th.1 when the field is $\overline{\mathbf{Q}_\ell}$. The scalar extension is not sensitive to integrality, commensurability of lattices, and finite length, (proposition 2.10, proposition 3.1). □

Remark 3.4. — *One cannot replace “finite length” by “admissible” in the proposition 3.3.*

One can construct admissible cuspidal representations in $\text{Mod}_E(G_F)$ with infinitely many classes of commensurable lattices.

Lemma 3.5. — *The category of smooth $\overline{\mathbf{Q}_\ell}$ -representation of G_F of finite length is equal to the category of smooth ℓ -adic representations of G_F of finite length,*

$$\text{Mod}_{\overline{\mathbf{Q}_\ell}}(G_F)^{fl} \simeq \text{Mod}_\ell(G_F)^{fl}.$$

Proof. — Let $W \in \text{Mod}_{\overline{\mathbf{Q}_\ell}}(G_F)^{fl}$. There exists a finite extension E/\mathbf{Q}_ℓ and $W_E \in \text{Mod}_E(G_F)^{fl}$ such that W is the scalar extension of W_E . When W is irreducible, this is proved in [9] II.4.7. In general, let H be an open pro- p -subgroup of G_F such that the length of W is equal to the length of the module $e_H W$ over the Hecke algebra $\text{End}_{\overline{\mathbf{Q}_\ell}G_F} \overline{\mathbf{Q}_\ell}[G_F/H]$. Let (w_i) be a finite $\overline{\mathbf{Q}_\ell}$ -basis of $e_H W$. The convolution algebra $\text{End}_{\mathbf{Z}_\ell G_F} \mathbf{Z}_\ell[G_F/H]$ is finitely generated [9] II.2.13, and the dimension of $e_H W$ over $\overline{\mathbf{Q}_\ell}$ is finite. Hence there exists a finite extension E/\mathbf{Q}_ℓ such that the E -vector space $\oplus_i Ew_i$ in $e_H W$ is stable by the Hecke algebra $\text{End}_{EG_F} E[G_F/H]$. The

E -representation U of G_F generated by (w_i) in W satisfies $e_H U = \oplus_i E w_i$. The scalar extension $\overline{\mathbf{Q}}_\ell \otimes_E U$ is equal to W because it is a subrepresentation of W with the same H -invariants. By the proposition 3.1, W_E has finite length.

Let E, E' be two finite extensions. Let $W_E \in \text{Mod}_E(G_F)^{fl}, W_{E'} \in \text{Mod}_{E'}(G_F)^{fl}$, let

$$f : \overline{\mathbf{Q}}_\ell \otimes_E W_E \rightarrow \overline{\mathbf{Q}}_\ell \otimes_{E'} W_{E'}$$

be a $\overline{\mathbf{Q}}_\ell G_F$ -morphism between their scalar extensions to $\overline{\mathbf{Q}}_\ell$. There exists a finite extension E'' containing E, E' such that f is defined on E'' , i.e. induces a $E'' G_F$ -morphism $f_{E''} : E'' \otimes_E W_E \rightarrow E'' \otimes_{E'} W_{E'}$ between their scalar extensions to E'' [9] proof of II.4.7. \square

The scalar extension $s_{E'/E}$ for smooth representations of G_F respects finite length (proposition 3.1) and the category $\text{Mod}_\ell(G_F)^{fl}$ of smooth ℓ -adic representations of G_F of finite length is well defined, contained in the category $\text{Mod}_\ell(G_F)^{adm}$ of admissible smooth ℓ -adic representations of G_F (proposition 3.2). The category $\text{Mod}_\ell(G_F)^{int, fl}$ of integral smooth ℓ -adic representations of G_F of finite length is equivalent by the forgetful functor composite with the completion and the smooth part to the category $\mathcal{B}_\ell(G_F)^{adm, fl}$ of Banach unitary ℓ -adic representations which are admissible and of finite length.

Theorem 3.6. — *The completion and the smooth part define equivalence of categories between $\text{Mod}_{\overline{\mathbf{Q}}_\ell}^{int, fl}(G_F)$ and $\mathcal{B}_\ell(G_F)^{adm, fl}$.*

In particular, they give bijections between the irreducible integral smooth $\overline{\mathbf{Q}}_\ell$ -representations of G_F and the topologically irreducible admissible Banach unitary ℓ -adic representations of G_F .

For $G_F = GL(n, F)$, we deduce the ℓ -adic local Langlands correspondence for $GL(n, F)$, given in the introduction.

A very natural question (asked by the referee) for a Banach unitary ℓ -adic representation V of G_F (notations of the sections 2 and 3) is: does V topologically irreducible imply V admissible? The answer is no.

A necessary condition for a positive answer is that any non zero intertwining operator f of V is surjective (it is clearly injective).

Proposition 3.7. — *When $V \in \mathcal{B}_E(G_F)$ is admissible and topologically irreducible, any non zero intertwining operator f of V is bijective.*

Proof. — The restriction f^∞ of f to the smooth part V^∞ is non zero (theorem 2.9), bijective because V^∞ is irreducible, hence f is bijective. \square

L. Clozel noticed that the examples of B. Diarra [4] th. 4 (van Rooij), give examples of topologically irreducible representations $V \in \mathcal{B}_E(GL(1, F))$ where any non zero intertwining operator is bijective, which are *not admissible*.

When F is replaced by \mathbf{R} , Fokko Du Cloux has proved that a topologically irreducible Banach unitary *complex* representation V of $G_{\mathbf{R}}$ is admissible if and only if any intertwining operator of V is a scalar [5] 3.2.9, 3.2.10. Soergel has proved that there exists topologically irreducible Banach unitary complex representations of $SL(2, \mathbf{R})$ which are not admissible [7].

References

- [1] Bernstein J.-N. “Le centre de Bernstein”. In Bernstein, Deligne, Kazhdan, Vignéras, Représentations des groupes réductifs sur un corps local. Travaux en cours. Hermann, Paris 1984.
- [2] Bourbaki N. “Algèbre commutative Chapitres 1 à 4”. Masson 1985.
- [3] Dat J.-F. “Finitude pour les représentations lisses des groupes p-adiques”. J. Inst. Math. Jussieu , 8(1); 261333, 2009.
- [4] Diarra B. “Sur quelques représentations p-adiques de \mathbf{Z}_p ”. Indagationes Math. 41, 481-493 (1979).
- [5] Du Cloux F. “Sur les représentations différentiables des groupes de Lie algébriques”. Ann. Sci. de l’ENS 24 (1991) 257-318.
- [6] Schneider P. “Nonarchimedean functional analysis.” SMM Springer (2001).
- [7] Soergel W. “An irreducible not admissible Banach representation of $SL(2, \mathbf{R})$.” Proc. of the AMS, vol.104 (1988) 1322-1324.
- [8] Tate John. Number theoretic background, Proc. Symp. Pure Math., 33, part 2 (1979), 3-26.
- [9] Vignéras M.-F. “Représentations l -modulaires d’un groupes réductif p-adique avec $l \neq p$.” Birkhauser Progress in Math.137 (1996).
- [10] Vignéras M.-F. “On highest Whittaker models and integral structures.” Contributions to Automorphic Forms, Geometry and Number Theory: ShalikaFest 2002—a supplemental volume to the American Journal of Mathematics.
- [11] Vignéras M.-F. “Correspondance de Langlands semi-simple pour $GL(n, F)$ modulo $\ell \neq p$.” Invent.math. 144, 177-223 (2001).