# Modulo $p$ representations of $GL(2, Q_p)$ and $(\varphi, \Gamma)$-modules

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Two deep theories were independently developped this last 50 years in number theory, the $p$-adic theory of Fontaine and the theory of automorphic representations by Langlands. These two theories emerge together these last 10 years giving rise to the local $p$-adic and modulo $p$ Langlands correspondence for $GL(2, Q_p)$.

We fix a finite field $k$ of characteristic $p$ with $q$ elements.

**Lectures**

1. *Etale $(\varphi, \Gamma)$-modules over $k((T))$ and the mirabolic monoid $P^+$ of $GL(2, Q_p)$.*
2. *Irreducible $k$-representations of $GL(2, F_p)$ and of $GL(2, Q_p)$.*
3. *From smooth $k$-representations of the mirabolic $P$ of $GL(2, Q_p)$ to etale $(\varphi, \Gamma)$-modules over $k((T))$.*
4. *Coefficient system on the tree and finite dimensional etale $(\varphi, \Gamma)$-modules over $k((T))$.*

**References**

1. Laurent Berger On some modular representations of the modular subgroup of $GL_2(Q_p)$.
2. Pierre Colmez $(\varphi, \Gamma)$-modules et représentations du mirabolique de $GL_2(Q_p)$
   Reprsentations de $GL_2(Q_p)$ et $(\varphi, \Gamma)$-modules
3. Yongquan Hu Diagrammes canoniques et représentations modulo $p$ de $GL_2(F)$

We consider the groups $Gal(Q_p^{ac}/Q_p)$, where $Q_p^{ac}$ is an algebraic closure of $Q_p$ and $GL(n, Q_p)$.

We put on these groups the topology such that the finite index subgroups of $Gal(Q_p^{ac}/Q_p)$ and of $GL(n, Z_p)$ are all open.

This is the usual topology ***.

**Definition 0.1.** *Let $G$ be a topological group. A $k$-representation of $G$ is a group morphism $G \to GL_k(V)$ where $V$ is a $k$-vector topogical space; the representation is continuous if the map*

$$G \times V \to V \to V \quad , \quad (g, v) \mapsto g.v$$

*is continuous. The $k$-dimension of $V$ is the dimension of the representation, and can be infinite.*

When the topology of $V$ is discrete, the $k$-representation of $G$ on $V$ is continuous if and only if for any $v \in V$ the subgroup of $g \in G$ such that $v_r(g)v = v$ is open. One says that the representation is smooth.

When the $k$-dimension is finite we suppose that $V$ is discrete. When $V = k$, the $k$-representation is called a character.

**Lemma 0.2.** *Any finite dimensional $k$-representation of $Gal(Q_p^{ac}/Q_p)$ and of $GL(n, Z_p)$ is smooth.*

*Proof.* Any finite index subgroup is open. □

A profinite group is a projective limit of finite groups with the projective limit topology. A topological group is finitely generated when it contains finitely many elements generating a dense subgroup. Any finite dimensional $k$-representation of finitely generated profinite group is smooth because any finite index subgroup is open (Segal Nikolov, annals of math 165 (2007)).

The aim of the local Langlands correspondence over $Q_p$ is to compare in a meaningful way the $n$-dimensional $k$-representations of $Gal(Q_p^{ac}/Q_p)$ and the smooth $k$-representations of $GL(n, Q_p)$.

Fontaine showed that the category of $n$-dimensional $k$-representations of $Gal(Q_p^{ac}/Q_p)$ is equivalent to the category of $n$-dimensional etale $(\varphi, \Gamma)$-modules over the field $k((T)) = \{\sum_{n \geq r} a_n T^n \quad , \quad a_n \in k, \ r \in Z\}$ of Laurent series in one indeterminate $T$ with coefficients in $k$.

We denote $k[[T]] = \{\sum_{n \geq 0} a_n T^n \quad , \quad a_n \in k, \ r \in Z\}$ the ring of Taylor series in one indeterminate $T$ with coefficients in $k$.

**Definition 0.3.** *A $n$-dimensional etale $(\varphi, \Gamma)$-module $D$ over $k((T))$ is :*

*1) a $k((T))$-vector space $D$.*

*2) A $k$-linear endomorphism $\varphi$ of $D$ which is semi-linear in the sense that*

$$\varphi(P(T)x) = P(T^p)\varphi(x)$$

*for all $x \in D, P(T) \in k((T))$ and etale in the sense that the image of a $k((T))$-basis $e_1, \ldots, e_n$ of $D$ by $\varphi$ is a $k((T))$-basis $\varphi(e_1), \ldots, \varphi(e_n)$ of $D$.*

*2) A continous representation of $\Gamma = Gal(Q_p^{cyc}/Q_p)$ where $Q_p^{cyc}$ is the $p$-adic cyclotomic field, on the $k$-vector space $D$ with the topology with neigbourhood basis of $0$ equal to $(T^n L)_{n \in L}$, where $L$ is the $k[[T]]$-module generated by a $k((T))$-basis of $D$, which is semi-linear in the sense that*

$$\gamma(P(T)x) = P((1+T)^{\chi(\gamma)} - 1)\gamma(x)$$

*for all $x \in D, P(T) \in k((T))$, where $\chi : \Gamma \to Z_p^*$ is the isomorphism given by the cyclotomic character, and commutes with $\varphi$*

$$\gamma \circ \varphi = \varphi \circ \gamma.$$

As an exercise, show that $\varphi$ is etale if and only if $\varphi$ is injective and

$$D = \oplus_{i=0}^{p-1} (1+T)^i \varphi(D) \quad .$$

As an exercise, show that for $x = a_0 + pa_1 + \ldots + p^n a_n + \ldots$ with $a_i \in \{0, \ldots, p-1\}$, the sequence $(1+T)^{a_0 + pa_1 + \ldots + p^n a_n}$ converges in $k[[T]]$. By definition, the limit is $(1+T)^x$.

**Definition 0.4.** *Let $D, D'$ be two finite dimensional etale $(\varphi, \Gamma)$-modules over $k((T))$. A morphism $f : D \to D'$ is a $k$-linear morphism $f : D \to D'$ which is $\varphi$ and $\Gamma$-equivariant: $f \circ \varphi_D = \varphi_{D'} \circ f \quad , \quad f \circ \gamma_D = \gamma_{D'} \circ f$.*

As an exercise, show that the category of finite dimensional etale $(\varphi, \Gamma)$-modules over $k((T))$ is abelian.

# 1 The case $n = 1$

The characters $GL(1, Q_p) = Q_p^* \to k^*$ are easy to describe. We have $Q_p^* = p^{\mathbb{Z}} Z_p^*$ and $k^*$ is a cyclic group of order prime to $p$ and divisible by $p-1$. A character of $Z_p^*$ factorizes by the reduction map $Z_p^* \to F_p^*$. A character $\eta : Q_p^* \to k^*$ is given by $\eta(p) \in k^*$ and a $k$-character of $F_p^*$. There are $(q-1)(p-1)$ character $Q_p^* \to k^*$.

**Proposition 1.1.** *The isomorphism classes of etale 1-dimensional $(\varphi, \Gamma)$-modules over $k((T))$ are in bijection with the $(q-1)(p-1)$ characters $Q_p^* \to k^*$.*

*Proof.* We associate to a character $\eta : Q_p^* \to k^*$ the etale $(\varphi, \Gamma)$-module $D_\eta = k((T))e$ of basis $e$ such that

$$\varphi(e) = \eta(p))e \quad , \quad \gamma.e = \eta(\chi(\gamma))e \quad .$$

We must check two things:

1) When $\eta \neq \eta'$ then $D_\eta$ and $D_{\eta'}$ are not isomorphic.

2) Any 1-dimensional etale $(\varphi, \Gamma)$-module over $k((T))$ is isomorphic to some $D_\eta$.

*** Let $D$ be a 1-dimensional etale $(\varphi, \Gamma)$-module over $k((T))$. We choose a non zero element $e \in D$. Then $D = k((T))e$. The semilinear endomorphism $\varphi$ and the semilinear action of $\Gamma$ on $D$ commuting with $\varphi$ are given by

$$\varphi(e) = a(T)e \quad , \quad \gamma.e = b_\gamma(T)e$$

for some $a(T) \in k((T))$ and a 1-dimensional continuous $k((T))$-representation $\gamma \mapsto b_\gamma : \Gamma \to k((T))^*$, satisfying

$$a((1+T)^{\chi(\gamma)} - 1)b_\gamma(T) = b_\gamma(T^p)a(T) \ .$$

The endomorphism $\varphi$ is etale if and only if $a(T) \neq 0$. ***

□

## 2 $k$-representations of $Z_p$

The compact group $Z_p$ is the projective limit of the finite groups $Z/p^nZ = Z_p/p^nZ_p$ with the projective limit topology,

$$Z_p = projlim Z/p^nZ$$

The subgroup $Z$ is dense in $Z_p$. The group $Z_p$ is topologically cyclic generated by 1.

We denote by $k[G]$ the group $k$-algebra of a group $G$. We denote by $[g]$ the element $g \in G$ in $k[G]$.

**Definition 2.1.** *The completed $k$-group algebra of $Z_p$ is*

$$k[[Z_p]] = projlim k[Z_p/p^nZ_p]$$

*with the projective limit topology.*

Clearly $k[Z_p]$ embeds as a dense subalgebra of $k[[Z_p]]$.

**Theorem 2.2.** *The completed $k$-group algebra $k[[Z_p]]$ is topologically isomorphic to $k[[T]]$ by the map sending $u$ to $1+T$.*

*Proof.* Alain Robert *** □

In particular a $k[[T]]$-module is a $k$-representation of $Z_p$. Conversely, is a $k$-representation $V$ of $Z_p$ always a $k[[T]]$-module ?

A topological $k$-vector space $V$ which is a projective limite $V = \operatorname{proj\,lim} V_n$ of finite $k$-vector spaces $V_n$ with the profinite topology, is called profinite.

A finite $k$-vector space or a finitely generated $k[[T]]$-module with the topology induced by $k[[T]]$ is a profinite $k$-vector space.

**Proposition 2.3.** *Let $M$ be a profinite $k$-vector space. A continuous $k$-representation of $Z_p$ on $M$ is the same than a structure of topological $k[[Z_p]]$-module on $M$.*

*Proof.* **** Wilson Profinite groups (1998) 7.2.4 □

The monoid $Z_p - \{0\} = Z_p^o$ acts continuously by multiplication on the group $Z_p$ (in the additive notation) and acts continuously the $k$-algebra $k[[Z_p]]$. In the multiplicative notation $x \in Z_p^o$ sends $[1]$ to $[x]$. By the theorem we get a continous action of $Z_p^o$ on the $k$-algebra $k[[T]]$ such that $x.(1+T) := (1+T)^x$ for $x \in Z_p^0$. The group $Z_p^o$ acts on the field $k((T))$. ****

# 3 Etale $k$-representations of $P^+$

The produit semidirect $Z_p \rtimes Z_p^o$ where $Z_p^o = Z_p - \{0\} = p^{\mathbb{N}} Z_p^*$, is isomorphic to the mirabolic submonoid $P^+$ of $GL(2, Q_p)$ defined by

$$P^+ := \begin{pmatrix} Z_p^o & Z_p \\ 0 & 1 \end{pmatrix} = P_0 t^N = P_0 t^N P_0$$

where

$$N_0 := \begin{pmatrix} 1 & Z_p \\ 0 & 1 \end{pmatrix} \ , \ P_0 := \begin{pmatrix} Z_p^* & Z_p \\ 0 & 1 \end{pmatrix}, \ t := \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \quad .$$

**Definition 3.1.** *Let $D$ be a $k$-representation of $P^+$. The action $\varphi$ of $t$ is called etale, if $\varphi$ is injective and*

$$D = \oplus_{i=1}^{p-1} \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix} \circ \varphi(D) \quad .$$

*When the action of $t$ is etale, one says that the $k$-representation $D$ of $P^+$ is etale.*

As an exercise, show that the action of any element of $P^+$ is etale when the action of $t$ is etale.

Show as an exercise, that $D = \oplus_{i=1}^{p-1} \theta_i \circ \varphi(D)$ is equivalent to $D = \oplus_{\theta \in \Theta_1} \theta \circ \varphi(D)$ for any system $\Theta_1$ of representatives of $N_0 / t N_0 t^{-1}$.

A finite dimensional $(\varphi, \Gamma)$-module $D$ over $k((T))$ is a continuous $k$-representation of $P^+$, where

$$\begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix} x = (1+T)^i x \quad , \quad tx = \varphi(x) \quad , \quad \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} = \chi^{-1}(a)(x) \quad ,$$

where $x \in D, a \in Z_p^*$. We leave as an exercise to show that $D$ is etale as a finite dimensional $(\varphi, \Gamma)$-module over $k((T))$ if and only if $D$ is etale as a $k$-representation of $P^+$.

As an exercise show that the category of etale $k$-representations of $P^+$ is abelian.

The interpretation of a finite dimensional $(\varphi, \Gamma)$-module over $k((T))$ as an etale continuous $k$-representation of the monoid $P^+$ is due to Colmez and is the first step towards the local modulo $p$ Langlands correspondence for $GL(2, Q_p)$.

**Definition 3.2.** *Let $D$ be an etale $k$-representation of $P^+$. The injective endomorphism $\varphi$ has a canonical left inverse $\psi$ with kernel*

$$D^{\psi=0} = (N_0 - t N_0 t^{-1}) \varphi(D) = \oplus_{i=1}^{p-1} \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix} \varphi(D) \quad .$$

Note that $D = \varphi(D) \oplus D^{\psi=0}$. The $k$-endomorphism $e := \varphi \circ \psi$ satisfies $e \circ e = e$ because $\psi \circ \varphi = id_D$. Hence $e$ is a projector from $D$ onto $\varphi(D)$. The $k$-endomorphism $e_g := g \circ \varphi \circ \psi \circ g^{-1}$ for any $g \in P^+$ is also a projector.

**Proposition 3.3.** *The projectors $e_{\begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix}}$ are orthogonal for $0 \leq i \leq p-1$ of sum $id_D$.*

*Proof.* Let $\Theta_1$ be any system of representatives of $N_0/tN_0t^{-1}$. $x = \varphi\varphi(x) + \sum_{\theta\in\Theta_1} \theta\varphi(x_\theta)$ with $x_\theta \in D$. Left multiply by $\theta^{-1}$ and use that $\theta^{-1}(\Theta_1 - \theta)$ is a system of representatives of $(N_0 - N_1)/N_1$. We leave the orthogonality as an exercise. $\square$

As an exercise show that the projectors $e_\theta$ for $\theta$ in a system $\Theta_k$ of representatives of $N_0/N_k$ and $N_k = t^k N_0 t^{-k}$ are orthogonal of sum $id_D$ for any integer $k \geq 1$.

The $k$-endomorphism $\psi$ does not respect the product but we have:

**Lemma 3.4.** *Let $a \in k[[T]], x \in D$. We have*

$$\psi(a\varphi(x)) = \psi(a)x \quad , \quad \psi(\varphi(x)a) = x\psi(a)$$

*Proof.* We leave the proof as an exercise. The second formula is easier. $\square$

# 4 Lattices

Let $D$ be a finite dimensional etale continuous $(\varphi, \Gamma)$-module over $k((T))$.

We want to find a canonical $k[[T]]$-lattice $D^\sharp$ which is $P^-$-stable and on which the action of $\psi$ is surjective.

A lattice in $D$ is a $k[[T]]$-submodule generated by a $k((T))$-basis of $D$, or equivalently a compact $k[[T]]$-submodule generating the $k((T))$-vector space $D$. A $k[[T]]$ module containing a lattice and contained in a lattice is a lattice.

The image by $\Gamma$ of a lattice $L$ generates a $\Gamma$-stable lattice because $\Gamma$ is compact.

Why does it exist a $\psi$-stable lattice in $D$ ?

When $D = k((T))$, then $k[[T]]$ is $\varphi$ and $\psi$-stable and $\psi$ is surjective is $k[[T]]$. Show as an exercise that $T^{-1}k[[T]]$ is the maximal lattice of $k((T))$ where $\psi$ is surjective and that $k[[T]]$ is the minimal lattice where $\psi$ is surjective.

In general one cannot find a lattice in $D$ which is stable by $\varphi$ and by $\varphi$.

**Lemma 4.1.** *There are two lattices $L_0$ and $L_1$ in $D$ such that*

$$\varphi(L_0) \subset T^{-1}L_0 \subset L_1 \subset k[[T]]\varphi(L_1)$$

*Proof.* Start with any $k((T))$-basis $e_1, \ldots, e_d$ of $D$. Because $D$ is etale, $\varphi(e_1), \ldots, \varphi(e_d)$ is also a $k((T))$-basis of $D$. There are $a'_{ij} \in k((T))$ and $b'_{ij} \in k((T))$ such that

$$\varphi(e_j) = \sum_i a'_{ij} e_i \quad , \quad e_j = \sum_i b'_{ij} \varphi(e_i) \quad .$$

Choose $T^n$ such that $T^n a'_{ij} \in k[[T]]$ and $T^n b'_{ij} \in k[[T]]$. Take for $L_0$ the lattice of $k[[T]]$-basis $(T^n e_i)$ and for $L_1$ the lattice of $k[[T]]$-basis $(T^{-n} e_i)$. They satisfy the lemma because

$$\varphi(T^n e_i) = T^{np}\varphi(e_i) = T^{n(p-1)} \sum_i a_{ij} T^n e_i$$

and

$$T^{-n}e_j = T^{-n}\sum_i b'_{ij}\varphi(e_i) = T^{-n}\sum_i b'_{ij}T^{np}\varphi(T^{-n}e_i) = T^{n(p-2)}\sum_i b_{ij}\varphi(T^{-n}e_i) \quad .$$

□

**Lemma 4.2.** *The lattice $L_0$ is $\varphi$-stable. The lattice $L_1$ of $D$ is $\psi$-stable.*

*Proof.* $\varphi(L_0) \subset T^{-1}L_0 \subset L_0$ and $\psi(L_1) \subset \psi(k[[T]])L_1 = L_1$. □

For $n \in N$ let $M_n = \psi^n(N_0)$. Then $M_n$ is a lattice contained in $L_1$ and the sequence $M_n$ is increasing. As $k[[T]]$ is noetherian, it has a limit $M_\infty$ which is a lattice such that $\psi(M_\infty) = M_\infty$. The sequence $\psi^n(T^{-1}M_\infty)$ is a decreasing sequence of lattices containing $M_\infty$. It has a limit $D^\sharp$ such that $\psi(D^\sharp) = D^\sharp$.

**Proposition 4.3.** *$D$ contains a maximal lattice $D^\sharp$ satisfying $\psi(D^\sharp) = D^\sharp$.*
*$D$ contains a minimal lattice $D^\natural$ satisfying $\psi(D^\natural) = D^\natural$.*
*$D$ contains a maximal lattice $D^+$ stable by $\varphi$.*
*These three lattices are $\Gamma$-stable and $D^+ \subset D^\natural \subset D^\sharp$.*

# 5 $k$-representations of the mirabolic group $P$

The subgroup $P$ of $GL(2, Q_p)$ generated by $P^+$ is the mirabolic subgroup

$$P = \begin{pmatrix} Q_p^* & Q_p \\ 0 & 1 \end{pmatrix} \simeq Q_p \rtimes Q_p^*$$

We denote $N = \begin{pmatrix} 1 & Q_p \\ 0 & 1 \end{pmatrix}$.

The second step is to associate to an etale $k$-representation $D$ of the mirabolic monoid $P^+$ a $k$-representation of the mirabolic group $P$.

There is a classical method, called induction and denoted by $ind_H^G$ which associates a $k$-representation of a group $G$ to a $k$-representation of a submonoid $H$.

**Definition 5.1.** *Let $H$ be a submonoid of a group $G$. Let $V$ be $k$-representation of $H$. The group $G$ acts on the space*

$$ind_H^G D := \{f : G \to V \quad , \quad f(hg) = hf(g) \quad \text{for } g \in GP, h \in H \quad \}$$

*by right translations.*

The induction from $H$ to $G$ is the right adjoint of the restriction from $G$ to $H$, and is a left exact functor. The induction from $H$ to $G$ behaves better when the elements of $H$ acts surjectively on $V$.

Let

$$P^- := \{g^{-1} \mid g \in P^+ \} = t^{-N}P_0$$

be the inverse monoid. An etale $k$-representation $D$ of $P^+$ has a canonical structure of $k$-representation of $P^-$, which coincide on $P_0$ with the original action and such that the action of $t^{-1}$ is the canonical left inverse $\psi$ of $\varphi$ defined as follows. The canonical lattices $D^\natural \subset D^\sharp$ of $D$ are $P^-$-stable and the action of $P^-$ is surjective on these lattices. We consider the $k$-representations of $P$

$$ind_{P^-}^P D^\natural \subset ind_{P^-}^P D^\sharp \quad .$$

**Proposition 5.2.** *(i) If $D$ is irreducible, $dim_{k((T))} D \geq 2$, then $D^\natural = D^\sharp$.*

*(ii) the functor $D \to ind_{P^-}^P D^\sharp$ is exact (this is not true for $D^\natural$),*

*(iii) If $D, D'$ are two finite dimensional etale $(\varphi, \Gamma)$-modules over $k((T))$ such that $ind_{P^-}^P D^\sharp \simeq ind_{P^-}^P D^\sharp$ then $D \simeq D'$.*

We do not prove the proposition but we prove the following corollary of (i).

**Corollary 5.3.** *If $D$ is irreducible, $dim_{k((T))} D \geq 2$, then the representation of $P$ on $\psi^{-\infty}(D^\sharp)$ is topologically irreducible (a closed $P$-stable subspace of is trivial).*

*Proof.* If $M$ is a non zero $P$-stable subspace of $\psi^{-\infty}(D^\sharp)$ the $n$-th projection of $M$ is a $\psi$-stable non zero $k[[T]]$-submodule of $D^\sharp$ hence is equal to $D^\sharp$ by (i) in the last proposition. This implies that $M$ is dense in $\psi^{-\infty}(D^\sharp)$. □

***** Not done in the lecture, until the end of this section. The representation $ind_{P^-}^P D$ has two other useful models.

**Lemma 5.4.** *$P = \cup_{n \in N} P^- t^n$ (disjoint union).*

*Proof.* We have $t^{-n} P_0 t^n = \begin{pmatrix} Z_p^* & p^{-n} Z_p \\ 0 & 1 \end{pmatrix}$ and $t^{-n} P_0 t^m = \begin{pmatrix} p^{m-n} Z_p^* & p^{-n} Z_p \\ 0 & 1 \end{pmatrix}$. Let $p = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \in P$. Write $a = p^r a'$ with $a' \in Z_p^*$ and $r \in Z$. Choose $r' \in N$ such that $b \in p^{-r'} Z_p$. For $n \geq r'$ we have $p^{-r'} Z_p \subset p^{-n} Z_p$. Choose $n$ such that $n \leq r$ and take $m = r - n$. □

**Proposition 5.5.** *Let $D$ be an etale $k$-representation of $P^+$. The map $f \mapsto (f(t^n))_{n \in N}$ is a bijection from $ind_{P^-}^P D$ to the space*

$$\psi^{-\infty}(D) := \{(x_n)_{n \in N} \mid x_n = \psi(x_{n+1}) \text{ for all } n \in N\}$$

*and the restriction to $N$ is a $N$-equivariant bijection from $ind_{P^-}^P D$ to $ind_{N_0}^N D$.*

*Proof.* The disjoint union

$$P = \cup_{n \in N} P^- t^n$$

show that $f$ is determined by its restriction to $t^N$. We have $\psi f(t^{n+1}) = t^{-1} f(t^{n+1}) = f(t^n)$ for $n \in N$. and conversely $f(p) = p^-(x_n)$ if $p = p^- t^n$ for $p \in P$ equal to $p^- t^n$ with $p^- \in P^-, n \in N$.

The second assertion is deduced from the first assertion and the formula

$$f(t^k) = \sum_{\theta \in N_0/N_k} \theta \varphi^k \psi^k \theta^{-1}(f(t^k)) = \sum_{\theta \in N_0/N_k} \theta \varphi^k t^{-k} \theta^{-1}(f(t^k)) = \sum_{\theta \in N_0/N_k} \theta \varphi^k (f(t^{-k} \theta^{-1} t^k)) \quad ,$$

$$f(t^{-k}\theta t^k) = \psi^k\theta^{-1}(f(t^k)) \quad .$$

for any $k \in N$. Note that the group $N = \cup_{k\in N} t^{-k}N_0 t^k$. We leave the rest of the proof as an exercise. $\square$

As an exercise, give the action of $P$ in the two models $\psi^{-\infty}(D)$ and $ind_{N_0}^N D$ of the representation of $P$ on $ind_{P^-}^P D$, obtained by restriction to the submonoid $t^N$ and to the subgroup $N := \begin{pmatrix} 1 & Q_p \\ 0 & 1 \end{pmatrix}$.

In the model $\psi^{-\infty}(D)$, it is convenient to identity $(x_n)_{n\in N}$ o $(x_n)_{n\in Z}$ where $x_n = \psi^{-n}(x_0)$ when $n \leq -1$.

For $a \in Z_p^*, b \in Q_p$

$$t(x_n)_{n\in N} = (x_{n+1})_{n\in N} \quad , \quad \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} (x_n)_{n\in N} = (ax_n)_{n\in N} \quad , \quad \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} (x_n)_{n\in N} = (y_n)_{n\in N}$$

where $y_n = \begin{pmatrix} 1 & p^n b \\ 0 & 1 \end{pmatrix} x_n$ if $p^n b \in Z_p$ and $y_n = \psi^{-v_p(b)-n}(y_{-v_p(b)})$ if $n \leq -v_p(b)$.

We see easily that $\psi^{-\infty}(D) = D$ when $\varphi$ is invertible.

**Proposition 5.6.** *If $0 \to D_1 \to D \to D_2 \to 0$ is an exact sequence of of representations of $P^-$ such that the action $\psi$ of $t^{-1}$ on $D_1$ is surjective. Then $0 \to \psi^{-\infty}(D_1) \to \psi^{-\infty}(D) \to \psi^{-\infty}(D_2) \to 0$ is an exact sequence of representations of $P$.*

*Proof.* To prove the surjectivity of the map $\psi^{-\infty}(D) \to \psi^{-\infty}(D)$ we have to show that for $x \in D_2$ and $y \in D$ of image in $D_2$ equal to $\psi(x)$ there exists $z \in D$ of image $x \in D_2$ with $\psi(z) = y$. Choose $z' \in D$ with image $x$ and set $y' := \psi(z')$. Consider $D_1$ embedded in $D$. Then $y' - y \in D_1$. As $\psi$ is surjective on $D_1$ choose $t \in D_1$ with $\psi(t) = y' - y$. Take $z := z' + t$. $\square$

**Definition 5.7.** *Let $Res_{N_0}$ the $k$-endomorphism of $ind_{N_0}^N D$ sending $f \in ind_{N_0}^N D$, the function $Res_{N_0}(f) \in ind_{N_0}^N D$ vanishes outside $N_0$ and equal to $f$ on $N_0$.*

Clearly $Res_{N_0}$ is a projector of $ind_{N_0}^N D$.

In the $\psi^{-\infty}(D)$-model the projector $Res_{N_0}$ admits the following description. The map

$$\iota : D \to \psi^{-\infty}(D) \quad , \quad x \mapsto (\varphi^n(x))_{n\in N} \quad ,$$

corresponds to the map $D \to ind_{N_0}^N D$ sending $x \in D$ to the function vanishing outside $N_0$ and value $x$ at 1. It is injective and $P^+$-equivariant. The map

$$\pi : \psi^{-\infty}(D) \to D \quad , \quad (x)_{n\in N} \mapsto x_0 \quad ,$$

corresponds to the map $ind_{N_0}^N D \to D$ sending $f$ to $f(1)$. It is surjective and $P^-$-equivariant. We have $\pi \circ \iota = id_D$.

**Lemma 5.8.** *The projector $\iota \circ \pi$ in the $\psi^{-\infty}(D)$-model correspponds to the projector $Res_{N_0}$ in the $ind_{N_0}^N D$ model.*

We write an element $g \in P$ as $g = nt^k a$ with $n \in N, k \in Z$ and $a \in Z_p^*$. We have $hN_0h^{-1} = N_k$.

**Lemma 5.9.** *For $g \in P$ as above, the projector $g \circ \iota \circ \pi \circ g^{-1}$ depends only on the set $nN_k$ in $N$.*

*Proof.* It is true to prove $g \circ \iota \circ \pi \circ g^{-1} = h \circ \iota \circ \pi \circ h^{-1}$ for $h = n't^k$ with $n' \in N$ such that $n'N_k = nN_k$. We have $g^{-1}h = a^{-1}t^{-k}n^{-1}n't^k$ and $n^{-1}n' \in t^kN_0t^{-k}$. Hence $g^{-1}h \in P_0$. Clearly $Res_{N_0} = \iota \circ \pi$ commutes with $P_0$. $\square$

We denote $Res_{nN_k} := g \circ \iota \circ \pi \circ g^{-1}$ when $g = nt^k a$ as above. A open compact subset $U$ of $N$ is a finite disjoint union of $\cup_{n \in N/N_k} nN_k$ (the group $N$ is commutative) for some $k \in Z$. We define $Res_U = \sum_{n \in N/N_k} Res_{nN_k}$.

As an exercise, show that $Res_U$ does not depend of the choice of $k \in Z$ and for $g \in P$ we have $g \circ Res_{gU} = Res_U \circ g$ for all $g \in P$.

As an exercise, show that the projector $1 - \varphi \circ \psi : D \to D^{\psi=0}$ corresponds to the restriction to $D$ embedded canonically in $ind_{P^-}^P D$ of the projector $Res_{N_0^*}$.

**Proposition 5.10.** *??? The map*

$$Res : C_c^\infty(N; k) \to End_k ind_{P^-}^P D$$

*defined by*

$$1_U \mapsto Res_U$$

*fro all open compact subsets $U$ of $N$ characteristic function $1_U$, is well defined $k$-linear.*

The group $P$ acts naturally on $End_k ind_{P^-}^P D$. For $f : D \to D$ and $p \in P$ we have $(p.f)(x) = p.f(p^{-1}.x)$.

**Proposition 5.11.** *??? The map*

$$Res : C_c^\infty(N; k) \to End_k ind_{P^-}^P D$$

*is $P$-equivariant .*

## 6 Irreducible smooth $k$-representations of the mirabolic $P$

**Proposition 6.1.** *Let $V$ be a topological $k$-vector space. If $V$ is discrete (resp. profinite) then $V^* = Hom_{cont}(V, k)$ is profinite (resp. discrete) and $V^{**} = V$. If $V$ is a smooth $k$-representation of $Z_p$, then $V^{Z_p}$ is finite if and only if $V^*$ is a fnitely generated $k[[Z_p]]$-module.*

*Proof.* 1) Topological Nakayama lemma (Howson).

Claim: If $M$ is a profinite $k$-vector space which is a topological $k[[T]]$-module such that $M = TM$ then $M = 0$.

Proof. Assume that $M \neq 0$ and let $U$ be an open neighborhood of 0 in $M$ with $U \neq M$. Let $m \in M$. There exists a neighborhood $U_m$ of $m$ in $M$ such that $T^n M \subset U$ for $n \in N$ large enough. We cover the compact space $M$ by finitely many $U_m$. For $n$ large enough we have $T^n M \subset U$. But $T^n M = M$. Hence we get a contradiction.

Claim: If $M/TM$ is a finite dimensional $k$-vector space then $M$ is a finitely generated $k[[T]]$-module.

Proof of claim. Let $N = k[[T]]e_1 + \ldots + k[[T]]e_n$ such that $M = N + TM$. The quotient $M/N$ is compact and Hausdorff. We have $T(M/N) = (TM + N)/N = 0$. Hence $M/N = 0$ and $M = N$.

□

The dual $\Omega(D^\natural)$ of c-ind$_{P^-}^P(D^\natural)$ is a quotient of the dual $\Omega(D^\sharp)$ of c-ind$_{P^-}^P(D^\sharp)$. They are smooth $k$-representations of $P$ for the contragredient action

$$< bv^*, bv >=< v^*, v >$$

By duality we obtain :

**Corollary 6.2.** *(i) If $D$ is irreducible, $dim_{k((T))} D \geq 2$. Then $\Omega(D)$ is an irreducible smooth $k$-representation of $P$.*

*(ii) the functor $D \to ind_{P^-}^P D^\sharp$ is exact and contravariant.*

*(iii) If $D, D'$ are two finite dimensional etale continuous $(\varphi, \Gamma)$-modules over $k((T))$ such that $\Omega(D) = \Omega(D')$ then $D \simeq D'$.*

We consider the upper triangular subgroup

$$B := \begin{pmatrix} Q_p^* & Q_p \\ 0 & Q_p^* \end{pmatrix} = P \times Z \quad , \quad ,$$

where

$$Z := \{ \begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix} \quad , \quad d \in Q_p^* \}$$

is the center of $GL(2, Q_p)$.

**Proposition 6.3.** *We have the Bruhat decomposition $GL(2, Q_p) = B \cup BsB$ where $s := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and*

$$BsB = BsN = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, Q_p) \quad , \quad c \neq 0 \}$$

*is open in $GL(2, Q_p)$.*

*Proof.* □

To extend a representation of $P$ to a $k$-representation of $GL(2, Q_p)$, the action of $Z$ is given by a character $Z \simeq Q_p^* \to k^*$, and we look for a compatible action of $s$ .

**Theorem 6.4.** *When $D$ is irreducible and $dim_{k((T))} D = 2$, the representation $\Omega(D)$ of $P$ extends to a smooth irreducible $k$-representation of $GL(2, Q_p)$.*

*Proof.* (Berger) The proof uses the classification of the smooth irreducible $k$-representations of $GL(2, Q_p)$ with a central character and results in characteristic 0. It would be nice to have a direct proof. □

This is no more the case when $D$ is irreducible and $dim_{k((T))}D \geq 3$ The representation $\Omega(D)$ is not seen by $GL(2,Q_p)$. Is it seen by $GL(d,Q_p)$ ? In which way ?

When $D = D_\eta$ is the 1-dimensional $(\varphi,\Gamma)$-module associated to a character $\eta : Q_p^* \to k$, we take $D^\natural = k[[T]]e$ for the $e \in D$ non zero with $\varphi(e) = \eta(p), \gamma(e) = \eta(\chi^{-1}(\gamma))$ and not $D^\sharp = T^{-1}k[[T]]e$ to define $\Omega(D)$.

# 7 The special representation

Let $Z$ be the center of $GL(2,Q_p)$. The group $B = PZ$ is the subgroup of upper triangular matrices in $GL(2,Q_p)$. Let $s :=$. By the Bruhat decomposition $G = B \cap BsN$ and $B\backslash BsN \simeq N$ by $Bsn \mapsto n$. The space $C^\infty(B\backslash GL(2,Q_p;k)$ of locally constant $k$-valued functions $f : B\backslash GL(2,Q_p \to k$ is a smooth representation of $GL(2,Q_p)$ such that $g.f(x) = f(xg)$ for $g \in GL(2,Q_p)$. The center $Z$ acts trivially. The representation is not irreducible because the subspace of constant functions is stable by $GL(2,Q_p)$. The quottient of $C^\infty(B\backslash GL(2,Q_p;k)$ by the constant functions is a smooth representation of $GL(2,Q_p)$ called the special $k$-representation $Sp$ of $GL(2,Q_p)$.

The restriction of $C_c(B\backslash G;k)$ to $P$ contains the functions with support in $BsN$ which form a $P$-stable subspace isomorphic to $sp$.

Formula

$$s\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix} = \begin{pmatrix}0 & 1\\ -1 & x\end{pmatrix} \quad ,$$

$$\begin{pmatrix}0 & 1\\ -1 & x\end{pmatrix}\begin{pmatrix}a & b\\ 0 & d\end{pmatrix} = \begin{pmatrix}d & 0\\ 0 & a\end{pmatrix}\begin{pmatrix}0 & 1\\ -1 & \frac{dx-b}{a}\end{pmatrix} \quad .$$

We have $ZPsP = ZPsN$ and the map $s\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}$ is a representative system of the cosets $ZP\backslash ZPsN$. The space $C_c^\infty(ZP\backslash ZPsP;k)$ has a natural action of $B$ trivial on $Z$ and isomorphic to $C_c^\infty(Q_p;k)$ by the map $f \to r(x) = f(s\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix})$. The induced action of $B$ trivial on $Z$ on $C_c^\infty(Q_p;k)$ is

$$\begin{pmatrix}a & b\\ 0 & d\end{pmatrix}.r(x) = (\begin{pmatrix}a & b\\ 0 & d\end{pmatrix}.f)(\begin{pmatrix}0 & 1\\ -1 & x\end{pmatrix}) = f(\begin{pmatrix}0 & 1\\ -1 & x\end{pmatrix}\begin{pmatrix}a & b\\ 0 & d\end{pmatrix} = r(\frac{dx-b}{a})$$

We give now a very useful proposition.

**Proposition 7.1.** *1) Let $H$ be a finite $p$-group acting on a non zero $k$-vector space $V$. Then $V^H \neq 0$.*

*2) Let $G$ be a finite group containing $H$ acting on a $k$-vector space $V$. When $\dim_k V^H = 1$ and $V$ is the $k$-space generated by the $G$-orbits of $V^H$, the $k$-representation of $G$ on $V$ is irreducible.*

*3) 1) and 2) remain true when $H$ is a pro-$p$-subgroup of a profinite group acting smoothly on $V$*

*Proof.* 1) Let $v \in V$ non zero and let $W$ be the non zero $k$-vector space generated by the $H$-orbit of $v$. The number of elements of $W$ is finite and is a power of $p$. For $w \in W$ the order of the $H$-orbit of $w$ is a power of $p$, equal to 1 if and only if $w \in W^H$. Hence $W^H$ which is not empty because it contains 0 is divisible by $p$.

2) Let $V'$ be a non zero subspace of $V$ which is stable by $G$. Then $(V')^H$ is a non zero $k$-subspace of $V^H$. As $\dim_k V^H = 1$ we have $(V')^H = V^H$. Hence $V \subset V' \subset V$.

3) The orbits of $G$ in $V$ are finite in the profinite case, and this is all what the proof is using.

□

**Proposition 7.2.** *The $k$-vector space $C_c^\infty(Q_p; k)$ of locally constant compactly supported $k$-valued functions $r : Q_p \simeq N$ with the action of $(a, b) \in Q_p^* \rtimes Q_p \simeq P$ given by*

$$(a, b).r(x) = r(\frac{x-b}{a})$$

*is an irreducible $k$-representation of $P$.*

*It is isomorphic to the restriction to $P$ of the special representation*

$$Sp := C_c(B\backslash G; k)/\text{constant functions}$$

*of $G$.*

*Proof.* Proof of the irreducibility. Let $f \in C_c^\infty(Q_p; k)$ non zero generating a subrepresentation $W$. There exists $n \in N$ such that the support of $f$ is contained in $p^{-n}Z_p$. Hence $f \in C_c^\infty(p^{-n}Z_p; k)$. A function in $C_c^\infty(p^{-n}Z_p; k)$ fixed by $p^{-n}Z_p$ is constant. The subrepresentation $W_n$ of $p^{-n}Z_p$ generated by $f$ is contained in $C_c^\infty(p^{-n}Z_p; k)$ and has a vector fixed by $p^{-n}Z_p$, hence $W_n$ contains the characteristic function of $p^{-n}Z_p$. We deduce that $W$ contains the characteristic functions of $b + p^{-m}Z_p$ for any $b \in Q_p, m \in N$. Hence $W = C_c^\infty(Q_p; k)$.

□

**Proposition 7.3.** *$sp \otimes (\eta^{-1} \circ \det) = \Omega(D_\eta)$.*

*Proof.* *** □

**Proposition 7.4.** *The special $k$-representation $Sp$ of $GL(2, Q_p)$ is irreducible.*

*Proof.* The restriction to the mirabolic group $P$ of $Sp$ is isomorphic to the special representation of $P$ which is irreducible. □

# 8 Bruhat-Cartan-Iwasawa decompositions

Let $B := \begin{pmatrix} Q_p^* & Q_p \\ 0 & Q_p^* \end{pmatrix}$ be the upper triangular group. We have $B = PZ$. Let

$$s = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad t = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \quad , \quad st = \begin{pmatrix} 0 & 1 \\ p_F & 0 \end{pmatrix}$$

**Theorem 8.1.** *Bruhat decomposition $G = B \cup BsB$ disjoint union and $BsB$ is open in $G$*

*Cartan decomposition $G = \cup_{n \in N} KZt^{nN}K$*

*Iwasawa decomposition $G = BK$*

*Proof.* With the tree *** (Rachel) □

# 9 Irreducible $k$-representations of $GL(2, F_p)$

For $r \in \{0, \ldots, p-1\}$, let $k[X,Y]_r$ be the space homogenous of polynomials of degree $r$ with two indeterminates $X, Y$ and coeffcients in $k$. This is a $k$-vector space of dimension $r+1$ of basis $X^iY^j$ for $i, j \in N$ such that $i + j = r$. The group $GL(2, F_p)$ acts on $k[X,Y]_r$ by

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} X^iY^j = (aX + cY)^i(bX + dY)^j \quad .$$

This representation is denoted $Sym^r$ and is of dimension $r+1$. When $r = 0$ we have $k[X,Y]_0 = k$ and $GL(2, F_p)$ acts trivially. When $r = p-1$ the representation $Sym^{p-1}$ of dimension $p$ is isomorphic to the special $k$-representation of $GL(2, F_p)$ (exercise).

**Lemma 9.1.** *$kX^r$ is the subspace of elements of $k[X,Y]_r$ fixed by $\begin{pmatrix} 1 & F_p \\ 0 & 1 \end{pmatrix}$).*

*$kY^r$ is the subspace of elements of $k[X,Y]_r$ fixed by $\begin{pmatrix} 1 & 0 \\ F_p & 1 \end{pmatrix}$).*

*The $k$-subspace generated by the $GL(2, F_p)$-orbit of $X^r$ (or $Y^r$) is equal to $k[X,Y]_r$.*

*Proof.* Exercise. □

**Theorem 9.2.** *The irreducible $k$-representations of $GL(2, F_p)$ are $Sym^r \otimes (\eta \otimes det)$ for $0 \leq r \leq p-1$ and a morphism $\eta : F_p^* \to k^*$.*

*Proof.* These $k$-representations are irreducible and they are not isomorphic (exercise). Their number is $p(p-1)$. This is true for any $k \subset F_p^{ac}$. Hence they remain irreducible and not isomorphic when one extends the scalar to $F_p^{ac}$

By the theory of Brauer, the number of isomorphism classes of irreducible $F_p^{ac}$-representations of $GL(2, F_p)$ is equal to the number of conjugacy classes of elements pf order prime to $p$ (Serre Linear representations of finite groups). The number of conjugacy classes of elements of order prime to $p$ is $p(p-1)$ (Exercise). □

Let $K := GL(2, Z_p)$. The reduction is a surjective morphism $K \to GL(2, F_p)$. We inflate $Sym^r$ to a $k$-representation of $K$.

Let $Z := \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ , $a \in Q_p^*$ the center of $GL(2, F_p)$. We inflate $Sym^r$ to a representation of $KZ$ where $p$ acts trivially.

**Theorem 9.3.** *Modulo isomorphism, the irreducible smooth $k$-representations of $KZ$ are $Sym^r \otimes (\eta \otimes det)$ for $0 \leq r \leq p-1$ and a continous morphism $\eta : Z_p^* \to k^*$. Their number is $p(p-1)$.*

*Proof.* It remains only to prove that $\begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix}$ acts by a scalar in an irreducible smooth $k$-representation of $KZ$. *** □

Let $\text{c-ind}_{KZ}^G Sym^r$ be the space of functions $f : G := GL(2, Q_p) \to k[X,Y]_r$ with compact support modulo $Z$ such that $f(kg) = Sym^r(k)f(g)$ for $k \in KZ, g \in GL(2, Q_p)$. The group $G = GL(2, Q_p)$ acts by right translation. We have $gf(x) = f(xg)$. This representation is called the compact induction of $Sym^r$ to $G$.

Fix $v_r \in \text{c-ind}_{KZ}^G\, Sym^r$ for the function with support $KZ$ and $v_r(1) = X^r$. Then $v_r$ generates $\text{c-ind}_{KZ}^G\, Sym^r$, i.e. the $k$-vector space generated by the $G$-orbit of $v_r$ is $\text{c-ind}_{KZ}^G\, Sym^r$.

It is known that $End_{kG}\, \text{c-ind}_{KZ}^G\, Sym^r \simeq k[\mathcal{T}]$ for some $\mathcal{T}$. To define $\mathcal{T}$ is suffices to defined $\mathcal{T}v_r$. Let (Paskunas, restriction to the Borel)

When $r = 0$

$$\mathcal{T}v_r = \begin{pmatrix} 1 & 0 \\ p & 0 \end{pmatrix} v_r + \sum_{i=0}^{p} \begin{pmatrix} p & i \\ 0 & 1 \end{pmatrix} v_r$$

When $r = 1, \ldots, p-1$

$$\mathcal{T}v_r = \sum_{i=0}^{p} \begin{pmatrix} p & i \\ 0 & 1 \end{pmatrix} v_r \quad .$$

For any $\lambda \in k$, the image $\mathcal{T} - \lambda \in End_{kG}\, \text{c-ind}_{KZ}^G\, Sym^r$ is a subrepresentation $k$-representation of $G$. Let $\eta : Q_p^* \to k^*$ be a smooth character and let

$$\pi(r, \lambda, \eta) := \frac{\text{c-ind}_{KZ}^G\, Sym^r}{(\mathcal{T} - \lambda)\, \text{c-ind}_{KZ}^G\, Sym^r} \otimes (\eta \circ det) \quad .$$

The representation $\pi(r, \lambda, \eta)$ is not irreducible if and only if $\lambda = \pm 1$ and $r \in \{0, p-1\}$ (theorem of Barthel-Livne-Breuil). When $r = 0$, $\pi(0, \pm 1, \eta)$ has a unique irreducible subrepresentation $Sp \otimes (\eta\mu_{\pm 1} \circ \det)$ and the quotient is $\eta\mu_\lambda \circ det$. When $r = p-1$, $\pi(p-1, \pm 1, \eta)$ has a unique irreducible subrepresentation $\eta\mu_{\pm 1} \circ \det$ and the quotient is $Sp \otimes (\eta\mu_{\pm 1} \circ \det)$, where $\mu_\lambda$ is the character of $Q_p^*$ trivial on $Z_p^*$ sending $p$ on $\lambda$ and $Sp$ is the special representation of $G$, equal to the quotient of the space $C^\infty(PZ \backslash G; k)$ of locally constant functions by the constant functions, with the natural action of $G$ by translation. I will prove later that it is irreducible.

The representations with $\lambda = 0$ are all irreducible; they called supersingular by the number theorists and supercuspidal by the group theorists. Their number when the action of $p$ is fixed is $(p^2 - p)/2$. It is also the number of irreducible $k$-representations of $Gal(Q_p^{ac}/Q_p)$ of dimension 2 where the determinant of a Frobenius is fixed.

The number of irreducible $k$-representations of $Gal(Q_p^{ac}/Q_p)$ of dimension $n$ where the determinant of a Frobenius is fixed, is the number of unitary irreducible polynomials in $F_p[X]$ of degree $n$

$$n^{-1} \sum_{d|n} \mu(n/d) q^d$$

**Theorem 9.4.** *The smooth irreducible $k$-representations of $GL(2, Q_p)$ with $p$ acting by a scalar are*

*(1) the characters $\eta \circ \det$ where $\eta : Q_p^* \to k^*$ is a smooth character*

*(2) the special representations $Sp \otimes (\eta \circ \det)$ where $Sp$ is the "special" representation.*

*(3) the supercuspidal (=supersingular) representations $\pi(r, 0, \eta)$*

*(4) the irreducible representations $\pi(r, \lambda, \eta)$ with $\lambda \neq 0$.*

*The only isomorphisms occur when $\lambda = 0$ and are*

$$\pi(r, 0, \eta) \simeq \pi(r, 0, \eta\mu_{-1}) \simeq \pi(p-1-r, 0, \eta\omega^r) \simeq \pi(p-1-r, 0, \eta\omega^r\mu_{-1})$$

*where $\mu_{-1}$ is the unramified quadratic character of $Q_p^*$, trivial on $Z_p^*$ sending $p$ to $-1$ and $\omega$ is the character of $Q_p^*$ trivial on $p^Z$ given by the reduction $Z_p^* \to F_p^*$.*

**Corollary 9.5.** *An irreducible smooth $k$-representation of $G = GL(2, Q_p)$ where $p$ acts by a scalar is a quotient $\pi : \text{c-ind}_{KZ}^G V_0 \to V$ where $V_0$ is an irreducible $k$-representation of $KZ$ and the kernel $R$ of $\pi$ is a finitely generated smooth $k$-representation of $G$.*

*Proof.* The kernel is $W + (T-\lambda)\,\text{c-ind}_{KZ}^G V_0$ where $W = 0$ or $\eta \circ \det$ or $Sp \otimes (\eta \circ \det)$. The $k$-vector spaces $\text{c-ind}_{KZ}^G V_0$ and $(T-\lambda)\,\text{c-ind}_{KZ}^G V_0$ are generated by a single $G$-orbit, the same is true for any irreducible $k$-representation. □

The irreducible $k$-representations $\pi(r, \lambda, \eta)$ of $GL(2, F)$ are called principal series. They are the irreducible $k$-representations $\text{c-ind}_B^G \eta_1 \otimes \eta_2$ induced from a $k$-character $\eta_1 \otimes \eta_2$ of $B$

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \to \eta_1(a) \otimes \eta_2(b)$$

The principal series are $\text{c-ind}_B^G \eta_1 \otimes \eta_2$ with $\eta_1 \neq \eta_2$ and there are no isomorphic.

The Bruhat decomposition $G = B \cup BsB$ implies that the restriction to $P$ of a principal series $\text{c-ind}_B^G \eta_1 \otimes \eta_2$ contains $sp \otimes (\eta_2\eta_1) \circ \det$ and that the quotient is $(\eta_1\eta_2) \circ \det$.

**Theorem 9.6.** *The restrictions to $P$ of the irreducible smooth $k$-representations of $GL(2, Q_p)$ where $p$ acts by a scalar, which are NOT principal series, are irreducible and not isomorphic.*

This is quite surprising if one thinks of the modulo $\ell$ irreducible representations of $GL(2, Q_p)$ where the restrictions to $P$ of generic representations all isomorphic.

# 10 A functor from the smooth $k$-representations of $P$ to $(\varphi, \Gamma)$-modules

Let $V$ be a smooth $k$-representation of $P$. When $V$ is finitely presented, one associates canonically to $V$ an etale $(\varphi, \Gamma)$-module over $k((T))$.

A finitely generated smooth $k$-representation $V$ of $P$ is $k[P]$-generated by finitely many vectors $v_1, \ldots, v_r$. The group $P_0 = \begin{pmatrix} Z_p^* & Z_p \\ 0 & 1 \end{pmatrix}$ is open and compact in $P$. As the representation of $P$ on $V$ is smooth, the $k[P_0]$-submodule $V_0$ of $V$ generated by $v_1, \ldots, v_r$ is finite.

Let $\text{c-ind}_{P0}^P(V_0)$ be the compact induced representation. This is the space of functions $f : P \to V_0$ with compact support such that $f(xg) = xf(g)$ for $x \in P^0, g \in G$ by right translation $(g.f)(x) = f(xg)$ for $x, g \in P$. The group $P$ acts by right translations: $g.f(x) = f(xg)$ for $x, g \in P$. We denote by $[1, v_0]$ the function with value $v_0 \in V_0$ at 1 and vanishing outside $P_0$.

Show as an exercise, that the compact induced representation $ind_{P0}^P(V_0)$ is a smooth $k$-representation of $P$, that the functions $g[1, v_0]$ for $v_0$ in a $k$-basis of $V_0$ and $g \in P/P_0$ form a $k$-basis of $ind_{P0}^P(V_0)$.

The $k$-linear map

$$\pi : ind_{P_0}^P(V_0) \to V \quad , \quad \pi(g[1, v_0]) = g.v_0$$

is $P$-equivariant and surjective. The kernel $R$ of $\pi$ is a subrepresentation of $ind_{P_0}^P(V_0)$.

When the kernel of $\pi$ is a finitely generated representation of $P$, one says that $\pi$ is a finite presentation of $V$. A smooth $k$-representation $V$ of $P$ is finitely presented if it admits a finite presentation.

Let $V$ be a finitely presented smooth $k$-representation of $P$, and let

$$\pi : ind_{P_0}^P(V_0) \to V$$

be a finite presentation of kernel $R$.

Let $P^+V_0$ be the $k$-vector space generated by $g.v$ for $g \in P^+, v \in V^0$. The dual $(P^+V_0)^*$ is a profinite $k$-vector space with an action of $P^-$; in particular it is a $k[[Z_p]]$-module, i.e. a $k[[T]]$-module. Let

$$D := k((T)) \otimes_{k[[T]]} (P^+V_0)^* \quad .$$

**Proposition 10.1.** *The $k((T))$-vector space $D$ does not depend on the choice of $V_0$.*

*Proof.* Claim. Let $V_0'$ be a $P_0$-stable finite $k$-subspace of $V$ generating $V$. Then $P^+V_0'$ is contained in $P^+V_0 + X$ for some $N_0$-stable finite $k$-vector subspace $X$ of $V$.

The claim implies the proposition. By duality $(P^+V_0')^*$ is a quotient of $(P^+V_0)^*$ as a $k[[T]]$-module. The kernel $W$ is a finite $k[[T]]$-module. It is killed by a power of $T$, hence vanishes when one inverts $T$. We have $k((T)) \otimes_{k[[T]]} W = 0$. This proves the proposition.

Proof of the claim. If $p_1, \ldots, p_r \in P$, there exists an integer $n > 0$ such that $t^n p_1, \ldots, t^n p_r \in P^+$. This implies that there is an integer $n > 0$ such that $t^n V_0 \subset P^+V_0'$ for some $n \in N$ hence $P^+t^nV_0 \subset P^+V_0'$.

$P^+ = P^+t^nP_0 + N_0YP_0$ for a finite subset $Y$ of $P^+$. Hence $P^+V_0 = P^+t^nV_0 + X$ for some $N_0$-stable finite $k$-vector subspace $X$ of $V$, because the action is smooth and $N_0, P_0$ are compact. □

The group $P^-$ acts smoothly on $D$ and not $P^+$.

When the kernel of the presentation $\pi : ind_{P_0}^P(V_0) \to V$ is finitely generated, we show that the representation of $P^-$ on $D$ comes from an etale representation of $P^+$ on $D$, i.e. a structure of etale $(\varphi, \Gamma)$-module over $k((T))$.

**Lemma 10.2.** *The set $P - P^+$ is stable by multiplication by $P^-$.*

*Proof.* Let $p \in P$. We write uniquely $p = uat^r$ with $u \in N, a \in Z_p^*, r \in Z$. We have $t^{-1}p = t^{-1}utat^{r-1}$. If $u \notin N_0$ then $t^{-1}ut \notin N_0$. If $r < 0$ then $r - 1 < 0$.

The element $p$ does not belong to $P^+$ if and only if $u \notin N_0$ or $u \in N_0, r < 0$. Hence $p \in P - P^+$ implies $t^{-1}p \in P - P^+$. The monoid $P^-$ is generated by $t^{-1}$ and $N_0$. For $u \in N_0$ it is clear that $p \in P - P^+$ if and only if $up \in P - P^+$

□

The lemma implies that the group $P^-$ acts naturally on

$$\frac{ind_{P_0}^P(V_0)}{R + (P - P^+)[1, V_0]}$$

where $(P - P^+)[1, V_0]$ be the $k$-vector space generated by $g.[1, v_0]$ for $g \in P - P^+, v_0 \in V^0$.

**Lemma 10.3.**

$$\frac{ind_{P_0}^{P}(V_0)}{R+(P-P^+)[1,V_0]} \simeq \frac{P^+V_0}{\Delta} \quad \text{where} \quad \Delta = P^+V_0 \cap (P-P^+)V_0$$

*Proof.* We leave the proof as an exercise. □

**Lemma 10.4.** *When the kernel of $\pi : \pi : ind_{P_0}^{P}(V_0) \to V$ is finite, $\Delta$ is finite.*

The dual $(\frac{P^+V_0}{\Delta})^*$ is a $k$-representation of $P^+$ and

$$(\frac{P^+V_0}{\Delta})^* \subset (P^+V_0)^*$$

We deduce from the lemma that

$$k((T)) \otimes_{k[[T]]} (P^+V_0)^* \simeq k((T)) \otimes_{k[[T]]} (\frac{P^+V_0}{\Delta})^*$$

Let $D_{V_0}^{+} \subset D_{V_0}^{\natural}$ be the natural images of $(\frac{P^+V_0}{\Delta})^* \subset (P^+V_0)^*$ in $D$. Then $D_{V_0}^{+}$ is a $\varphi$-stable lattice and $D_{V_0}^{\natural}$ is a $\psi$-stable lattice. Compare with Schneider Vigneras to find $P^+V_0$ minimal. Check if the minimal case corresponds to a finite presentation and that the lattices which are canonical correspond to $D^+$ and $D^\natural$.

**Theorem 10.5.** *$V \mapsto D(V)$ is a contravariant exact functor from the category of smooth $k$-representations of $P$ to the category of etale $(\varphi,\Gamma)$ $k((T))$-modules.*

What are the finitely presented smooth $k$-representations $V$ of $P$ such that the associated etale $(\varphi,\Gamma)$-module $D$ over $k((T))$ is finite dimensional ?

**Proposition 10.6.** *If* $\text{c-ind}_{P_0}^{P} V_0 \to V$ *is a finite presentation of $V$ such that $(P^+V_0)^{N_0}$ is finite, then the $k((T))$-vector space $D$ is finite dimensional.*

*Proof.* See the proposition on duality. □

By the topological Nakayama lemma, $\dim_k(P^+V_0)^{N_0}$ is the minimal number of generators of the $k[[T]]$-module $P^+V_0^*$. The ring $k[[T]]$ is principal. By the classification of the finitely generated $k[[T]]$-modules, the $k((T))$-dimension of $D$ is the rank of the $k[[T]]$-free module $P^+V_0^*$ divided by its torsion submodule. Hence the $k((T))$-dimension of $D$ of bounded by the $k$-dimension of $P^+V_0^*$, with equality if the $k[[T]]$-module $P^+V_0^*$ is free.

**Theorem 10.7.** *Let $V$ be an irrreducible smooth $k$-representation of $GL(2,Q_p)$ where $p$ acts by a scalar. Then $V|_P$ is finitely presented and the associated etale $(\varphi,\Gamma)$ $k((T))$-module is a $k((T))$-vector space of dimension*
0 *if $V$ is a character,*
1 *if $V$ is special or a principal series,*
2 *if $V$ is supercuspidal.*

Before explaining the proof of the theorem, we give a corollary based on the following result on extensions

**Proposition 10.8.** *If $V_1 \to V \to V_2 \to 0$ is an exact sequence of smooth $k$-representations of $P$, and if $V_1$ and $V_2$ are finitely presented, then $V$ is finitely presented.*

*Proof.* To be written. □

**Corollary 10.9.** *$V \to D(V)$ is a well defined functor from the category of finite length smooth $k$-representations of $GL(2,Q_p)$ where $p$ acts by a scalar on any irreducible subquotient. It is an exact contravariant functor to the category of finite dimensional etale $(\varphi,\Gamma)$ $k((T))$-modules over $k((T))$.*

This functor is the Colmez functor, and is not faithful because it vanishes on the smooth $k$-characters of $GL(2,Q_p)$.

We give now the proof of the theorem.

**Proposition 10.10.** *Iwasawa decomposition. We have $G = PKZ$ with $P_0 = P \cap KZ$.*

*Proof.* Rachel's course on the tree. □

**Proposition 10.11.** *When $V$ is the restriction to $P$ of an irreducible smooth $k$-representation of $GL(2,Q_p)$ then $V$ is finitely presented.*

*Proof.* By Iwasawa decomposition, for any smooth $k$-representation $V_0$ of $KZ$, the restriction to $P$ of $\text{c-ind}_{KZ}^G V_0$ is $\text{c-ind}_{P_0}^P V_0$ and the restriction to $P$ of a a finitely generated smooth $k$-representation of $G$ is a finitely generated smooth $k$-representation of $P$. We know that $V$ is the unique quotient of some $V(r,\lambda,\eta)$. The presentation of $V|_P$ associated to the image $V_0$ of $Sym^r \otimes (\eta \otimes \det)$ is finite.

□

When $dim_k V$ is finite, $D = 0$ because $k((T)) \otimes_{k[[T]]} (P^+V_0)^* = 0$ when $(P^+V_0)^*$ is finite.

The special representation $sp$ of $P$ is the restriction of the special representation $Sp$ of $GL(2,Q_p)$. The subspace $C^\infty(Z_p;k)$ (functions with support in $Z_p$) in the model $C_c^\infty(Q_p;k)$ is $P^+$-stable and generated by the characteristic function of $Z_p$ as a representation of $P^+$. This implies that

$$D(Sp) = k((T)) \otimes_{k((T))} (C^\infty(Z_p;k))^*$$

because two finitely generated $P^+$-stable submodules generating a smooth $k$-representation of $P$ are equal modulo a finite $N_0$-stable set. The dual of $C_c^\infty(Z_p;k) = \lim C_c^\infty(Z_p;k)^{p^n Z_p}$ is $k[[Z_p]] = \text{proj}\lim (C_c^\infty(Z_p;k)^{p^n Z_p})^* \simeq k[[T]]$. We deduce that $D(Sp) \simeq k((T))$.

$D(Sp \otimes \eta \otimes) \simeq D_{\eta'}$ for two characters $\eta, \eta' \to k^*$. Do we have $\eta' = \eta$ ? or $\eta' = \eta^{-1}$ ?

A smooth $k$-representation $V$ of $GL(2,Q_p)$ is called admissible when $V^C$ is finite for any open compact subgroup $C$ of $GL(2,Q_p)$. Let $I_1$ be the pro-$p$-Iwahori subgroup inverse image of $N(F_p)$ in $K$.

**Proposition 10.12.** *A smooth $k$-representation $V$ of $GL(2,Q_p)$ is admissible if and only $V^C$ is finite for some open compact subgroup $C$ of $GL(2,Q_p)$.*

*The dimension of the $k$-vector space $V^{I_1}$ is 1 when $V$ is a character or a special representation and is 2 for the other irreducible smooth $k$-representations of $GL(2,Q_p)$ with $p$ acting by a scalar.*

We consider now a supersingular irreducible smooth $k$-representation $V(r,\eta)$ of $GL(2,Q_p)$ where $p$ acts by a scalar. Let $V_0$ be the image of $Sym^r \otimes (\eta \circ \det)$ embedding in $V$ by the quotient map $\text{c-ind}_{KZ}^G Sym^r \otimes (\eta \circ \det)$ of kernel $T\, \text{c-ind}_{KZ}^G Sym^r \otimes (\eta \circ \det)$. Let $\Delta = P^+V_0 \cap (P - P^+)V_0$.

**Proposition 10.13.** $\Delta$ *contains* $(P^+V_0)^{N_0}$

As $\Delta$ is finite, we deduce that $(P^+V_0)^{N_0}$ is finite hence the $k((T))$-vector space $D(V)$ is finite dimensional. This is enough for the corollary of the theorem.

*Proof.* The following proof is due to Y. Hu.

a) If $v \in (P^+V_0)^{N_0}$ then $v$ is fixed by some $\begin{pmatrix} 1 & 0 \\ p^{n_v}Z_p & 1 \end{pmatrix}$. The subgroup $C_v$ of $GL(2,Q_p)$ generated by $N_0$ and $\begin{pmatrix} 1 & 0 \\ p^{n_v}Z_p & 1 \end{pmatrix}$ is open and compact, The representation $V$ of $GL(2,Q_p)$ is admissible. Hence $V^{C_v}$ is finite.

b) Let $T_1 = \sum_{i=0}^{p-1} \begin{pmatrix} p & i \\ 0 & 1 \end{pmatrix}$ seen as an operator of $V$. Set $T_n = T_1 \circ \ldots \circ T_1$ $n$-times. Then one can show *** that $T_n v \in V^{C_v}$ when $n \geq n_v$.

c) Let $\Delta_0 = \Delta$ and by induction on $n$ let $\Delta_n = P^+V_0 \cap Kst\Delta_{n-1}$ where $st = \begin{pmatrix} 0 & 1 \\ p & 1 \end{pmatrix}$. Then one can show *** that the sequence of $k$-vector spaces $(\Delta_n)_{n\in N}$ is increasing of union $P^+V_0$. For $v \in P^+V_0$ let $\ell(v)$ be the minimal integer $n \in N$ such that $v \in \Delta_n$. Then one shows that $\ell(T_1 v) = \ell(v) + 1$ if $v \notin \Delta$.

d) By a) and b), the elements $T_n v$ for $n \geq n_v$ are linearly dependent. This implies that $v$ belongs to $\Delta$ because $\ell(T_n v) = \ell(v) + n$ if $v \notin \Delta$.

$\square$

In fact, Hu proved more:

**Proposition 10.14.** $\Delta = V^{I_1}$

*Proof.* *** $\square$

An element of $\Delta$ invariant by $I_1$ belongs to $P^+V_0$ which contains $\Delta$ and is invariant by $N_0$ which is contained in $I_1$. Hence we obtain

**Corollary 10.15.** $(P^+V_0)^{N_0} = V^{I_1}$

We deduce that the dimension of $D(V)$ over $k((T))$ is $\leq 2$ with equality if $(P^+V_0)^*$ is a free $k((T))$-module.

To finish ***

The canonical diagram $\Delta \subset K\Delta$ To finish ***