Münster J. of Math. **7** (2014), 363–379
urn:nbn:de:hbz:6-22359592466
**Münster Journal of Mathematics**


# The pro-$p$-Iwahori–Hecke algebra of a reductive $p$-adic group, II

Marie-France Vignéras

(Communicated by Christopher Deninger)

*This paper is dedicated to Peter Schneider on the occasion of its 60-th birthday, in appreciation for his insights, precision and generosity in mathematics.*

**Abstract.** For any commutative ring $R$ and any reductive $p$-adic group $G$, we describe the center of the pro-$p$-Iwahori–Hecke $R$-algebra of $G$. We show that the pro-$p$-Iwahori–Hecke algebra is a finitely generated module over its center and is a finitely generated $R$-algebra. When the ring $R$ is noetherian, the center is a finitely generated $R$-algebra and the pro-$p$-Iwahori–Hecke $R$-algebra is noetherian. This generalizes results known only for split groups.

## 1. Results

Let $R$ be a commutative ring, let $F$ be a local nonarchimedean field, of finite residue field $k$ with $q$ elements and of characteristic $p$, let $G, T, Z, N$ be the groups of $F$-points of a connected reductive $F$-group $\mathbf{G}$ of maximal $F$-split torus $\mathbf{T}$ of $\mathbf{G}$-centralizer $\mathbf{Z}$ and $\mathbf{G}$-normalizer $\mathbf{N}$. The group $Z$ admits a unique parahoric subgroup $Z_0$, and $Z_0$ admits a unique pro-$p$-Sylow $Z_0(1)$. The unicity of these groups imply that they are normalized by $N$. The group $Z_k = Z_0/Z_0(1)$ is the group of $k$-points of a $k$-torus.

Let $W_0 = N/Z$, $W = N/Z_0$ and $W(1) = N/Z_0(1)$. Then $W_0$ is the relative finite Weyl group of $G$, the quotient map $W \to W_0$ of kernel $\Lambda = Z/Z_0$ splits, the group $W(1)$ is an extension of $W$ by $Z_k$.

*We will denote by $X(1)$ the inverse image in $W(1)$ of a subset $X$ of $W$, and we write $\tilde{w}$ for an arbitrary element of $W(1)$ of image $w \in W$.*

The group $\Lambda$ is commutative but the group $\Lambda(1) = Z/Z_0(1)$ may be not commutative. A conjugacy class $C$ of $W$ is contained in $\Lambda$ or disjoint from $\Lambda$. The same is true for $W(1)$ and $\Lambda(1)$.

The Iwahori–Hecke $R$-algebra $\mathcal{H}_R$ of $G$ is a deformation of the group $R$-algebra $R[W]$ and the pro-$p$-Iwahori–Hecke $R$-algebra $\mathcal{H}_R(1)$ of $G$ is a deformation of the group $R$-algebra $R[W(1)]$, cp. [7].

The center of the $R$-algebra of the group $W$ or of $W(1)$ can be easily determined using the following following property:

**Lemma 1.1.** *A conjugacy class $C$ of $W$ is finite if and only if $C$ is contained in $\Lambda$. The same is true for $W(1)$ and $\Lambda(1)$.*

As a corollary, the center of the $R$-algebra $R[W]$ is equal to $R[\Lambda]^{W_0}$ and is $R$-free of basis

$$z_C = \sum_{\lambda \in C} \lambda,$$

for all finite conjugacy classes $C$ of $W$. The group algebra $R[W]$ is a finitely generated module over its center. If the ring $R$ is noetherian, the center is a finitely generated algebra and $R[W]$ is a noetherian algebra. The same is true for $R[W(1)]$. We will prove that these properties remain true for the Iwahori and pro-$p$-Iwahori $R$-algebras.

For each (spherical) orientation $o$, we denote by $(E_o(w))_{w\in W}$, resp. $(E_o(w))_{w\in W(1)}$, the (alcove walk or Bernstein) basis of $\mathcal{H}_R$, resp. $\mathcal{H}_R(1)$ (cp. [7, Cor. 5.26, Cor. 5.28]).

**Theorem 1.2.** *The center $\mathcal{Z}_R$ of the Iwahori–Hecke $R$-algebra $\mathcal{H}_R$ is $R$-free of basis*

$$E(C) = \sum_{\lambda \in C} E_o(\lambda)$$

*for all finite conjugacy classes $C$ of $W$. The basis elements $E(C)$ do not depend on the choice of the orientation $o$.*

*The Iwahori–Hecke $R$-algebra is finitely generated and is a finitely generated module over its center. If the ring $R$ is noetherian, the center is a finitely generated algebra, and the Iwahori–Hecke $R$-algebra is noetherian.*

*The center $\mathcal{Z}_R(1)$ of the pro-$p$ Iwahori–Hecke $R$-algebra $\mathcal{H}_R(1)$ satisfies the same properties.*

This theorem was proved by Bernstein for affine Hecke complex algebras [3], or when the group $G$ is $F$-split [5], [6]. In his unpublished and nice diplomarbeit [4], Schmidt gave this theorem for certain pro-$p$-Iwahori–Hecke algebras, although his proof contains some gaps but the pattern of his proof is correct and we follow it. The Iwahori–Hecke or pro-$p$-Iwahori–Hecke algebras attached to reductive $p$-adic groups are more general than the affine Hecke algebras of Lusztig and than the pro-$p$-Iwahori–Hecke algebras of Schmidt.

We will prove the theorem for the $R$-algebras $\mathcal{H}_R(q_s, c_s)$ defined in the part I of our work [7] generalizing the pro-$p$-Iwahori–Hecke algebra. This allows to reduce some proofs to the simpler case $q_s = 1$.

We recall the definition of $\mathcal{H}_R(q_s, c_s)$, cp. [7]. There exists an affine Weyl Coxeter system $(W^{\mathrm{aff}}, S^{\mathrm{aff}})$ and a finitely generated commutative subgroup $\Omega$ normalizing $S^{\mathrm{aff}}$, such that

$$W = W^{\mathrm{aff}} \rtimes \Omega.$$

We write $s \sim s'$ for two elements $s, s' \in S^{\mathrm{aff}}$ which are $W$-conjugate. Let $(q_s)_{s\in S^{\mathrm{aff}}/\sim}$ in $R$, and $(c_s)_{s\in S^{\mathrm{aff}}(1)}$ in $R[Z_k]$ satisfying

$$c_{\tilde{s}t} = c_s t,\ c_{s'} = wc_s w^{-1},\ \ (t \in Z_k,\ s' = wsw^{-1},\ s, s' \in S^{\mathrm{aff}}(1),\ w \in W(1)).$$

We set $q_{\tilde{s}} = q_s$. The $R$-algebra $\mathcal{H}_R(q_s, c_s)$ is the free $R$-module of basis $(T_w)_{w\in W(1)}$ endowed with the product satisfying

- The braid relations:

(1) $$T_w T_{w'} = T_{ww'} \ \ (w, w' \in W(1), \ell(w) + \ell(w') = \ell(ww')),$$

  where $\ell(w) = \ell(w^{\mathrm{aff}})$ if the image of $w$ in $W$ is $w^{\mathrm{aff}} u,\ w^{\mathrm{aff}} \in W^{\mathrm{aff}}, u \in \Omega$ and $\ell$ the length of $(W^{\mathrm{aff}}, S^{\mathrm{aff}})$. Therefore the linear map sending $t$ to $T_t$ identifies $R[Z_k]$ to a subalgebra of $\mathcal{H}_R(q_s, c_s)$, and $s^2 \in Z_k$ and $c_s \in R[Z_k]$ identify to elements of $\mathcal{H}_R(q_s, c_s)$.
- The quadratic relations:

(2) $$T_s^2 = q_s s^2 + c_s T_s \ (s \in S^{\mathrm{aff}}(1)).$$

In our basic example, the pro-$p$-Iwahori–Hecke algebra, $q_s = [IsI : I] = [I(1)sI(1) : I(1)]$ where $I$ is an Iwahori group of pro-$p$-Sylow $I(1)$; when $s$ belongs to a certain coset $sZ_{k,s} = Z_{k,s}s$ in $W(1)$ of a subgroup $Z_{k,s}$ of $Z_k$ defined in [7],

$$c_s = (q_s - 1)|Z_{k,s}|^{-1} \sum_{t\in Z_{k,s}} t.$$

The $R$-algebra $\mathcal{H}_R(q_s)$ generalizing the Iwahori–Hecke algebra is simpler. It admits the same definition with $c_s = q_s - 1$ and $W$ instead of $W(1)$. We have

$$\mathcal{H}_R(q_s) = R \otimes_{R[Z_k]} \mathcal{H}_R(q_s, q_s - 1)$$

for the homomorphism sending $t \in Z_k$ to 1.

The affine Weyl group $W^{\mathrm{aff}}$ is generated by the orthogonal reflections with respect to a set of affine hyperplanes in an Euclidean real vector space $V$. The finite Weyl group $W_0$ identifies with the subgroup of $W^{\mathrm{aff}}$ generated by the the reflections with respect to the hyperplanes containing 0. The group $W$ acts by conjugation on $\Lambda$ and

$$W = \Lambda \rtimes W_0.$$

The simply transitive action of the group $W_0$ on the Weyl chambers inflates to an action of $W$ trivial on $\Lambda$ and to an action of $W(1)$ trivial on $\Lambda(1)$. The orientations are in bijection with the Weyl chambers, and inherit this action. For an orientation $o$ and for $w$ in $W_0$ or $W$ or $W(1)$, we denote by $o \bullet w$ the image of $o$ by $w^{-1}$. We denote by $S_o$ the set of reflections with respect to the walls of the Weyl chamber defining $o$. The $R$-basis $(E_o(w))_{w\in W(1)}$ of $\mathcal{H}_R(q_s, c_s)$ associated to $o$ satisfies:

- the product formula [7, Thm. 5.25]:

(3) $$E_o(w)E_{o\bullet w}(w') = q_{w,w'} E_o(ww') \ \ (w, w' \in W(1)),$$

  where $q_{w,w'} = (q_w q_{w'} q_{ww'}^{-1})^{1/2}$ and $q_w = q_{s_1} \dots q_{s_r}$ if $w = \tilde{s}_1 \dots \tilde{s}_r \tilde{u}$ with $s_i \in S^{\mathrm{aff}}, u \in \Omega$ is a reduced decomposition ($r$ minimal).

- the Bernstein relations [7, Thm. 5.45]:

$$E_o(s)E_o(\lambda) - E_o(s\lambda s^{-1})E_o(s) = \sum_{\mu\in\Lambda(1)} a_{o,s,\lambda}(\mu)E_o(\mu) \tag{4}$$

  for $\lambda \in \Lambda(1)$, $s \in S_o(1)$, with explicit coefficients $a_{o,s,\lambda}(\mu) \in R$.

We have the subalgebra $\mathcal{A}_o(1)$ of basis $(E_o(\lambda))_{\lambda\in\Lambda(1)}$ with the natural action of $W(1)$ coming from the conjugation on $\Lambda(1)$, (cp. [7, Cor. 5.26, Cor. 5.28]). Theorem 1.2 is a corollary of:

**Theorem 1.3.** *The center $\mathcal{Z}_R(q_s, c_s)$ of $\mathcal{H}_R(q_s, c_s)$ is the ring $\mathcal{A}_o(1)^{W(1)}$ of $W(1)$-invariant elements of $\mathcal{A}_o(1)$. It is a free $R$-module of basis*

$$E(C) = \sum_{\lambda\in C} E_o(\lambda)$$

*for all finite conjugacy classes $C$ of $W(1)$. The basis elements $E(C)$ do not depend on the choice of the orientation $o$.*

*The $R$-algebra $\mathcal{H}_R(q_s, c_s)$ is finitely generated and is a finitely generated module over its center $\mathcal{Z}_R(q_s, c_s)$. If the ring $R$ is noetherian, the center $\mathcal{Z}_R(q_s, c_s)$ is a finitely generated $R$-algebra, and the $R$-algebra $\mathcal{H}_R(q_s, c_s)$ is noetherian.*

**Remark 1.4.** The action of $W$ on $\Lambda$ factorizes through $W_0$. The center of $\mathcal{H}_R(q_s)$ is $\mathcal{A}_o^{W_0}$ where $\mathcal{A}_o$ is the commutative subring of basis $E_o(\lambda)$ for $\lambda \in \Lambda$.

This theorem generalizes our results on the center for $G$ split [5]. In a sequel of this paper, when $C$ is an algebraically closed field of characteristic $p$, this theorem will be used to generalize to any reductive connected group $G$, some results of Ollivier on pro-$p$-Iwahori–Hecke $C$-algebra when $G$ is split: the embedding of the weighted spherical algebras in the pro-$p$ Iwahori–Hecke $R$-algebra $\mathcal{H}_C(1)$, the inverse of the Satake isomorphism considered in [2] and the classification of the supersingular $\mathcal{H}_C(1)$-modules.

## 2. Proofs

We denote by $T$ the maximal split torus of $G$, and by $X^*(T)$ and $X_*(T)$ the lattices of characters and of cocharacters of $T$. Let $\omega$ be the valuation of $F$ such that $\omega(F^\times) = \mathbb{Z}$. The kernel of the canonical map $t \mapsto (\chi \mapsto (\omega\circ\chi)(t))$ : $T \to \mathrm{Hom}(X^*(T), \mathbb{Z}) \simeq X_*(T)$ is the maximal compact subgroup $T_0$ of $T$. The choice of an uniformizer $p_F$ defines a splitting of the exact sequence

$$1 \to T_0 \to T \to X_*(T) \to 1$$

sending $\mu \in X_*(T)$ to $\mu(p_F)$. We have $Z_0 \cap T = T_0$ and $Z_0(1) \cap T = T_0(1)$ is the unique pro-$p$-Sylow of $T$. The quotient $T_k = T_0/T_0(1)$ is the group of $k$-points of a split $k$-torus. The splitting induces isomorphisms

$$T \simeq T_0 \times X_*(T),\ T/T_o(1) \simeq T_k \times X_*(T).$$

The commutative finitely generated group $T/T_0(1)$ is central of finite index in $\Lambda(1)$. We recall the homomorphism

(5) $\nu : Z \to V = \mathbb{R} \otimes_{\mathbb{Z}} Q(\Phi^\vee) \mid \alpha \circ \nu(t) = -\omega \circ \alpha(t)$ for $\alpha \in \Phi,\ t \in T$,

where $\Phi \subset X^*(T)$ is the set of roots of $T$ in $G$. The homomorphism $\nu$ factorizes through $\Lambda = Z/Z_0$ and $\Lambda(1) = Z/Z_0(1)$.

**Lemma 2.1.** *A conjugacy class of $\Lambda$ is finite.*

*A conjugacy class of $W$ is finite and contained in $\Lambda$, or infinite and disjoint from $\Lambda$.*

*The same is true for $W(1)$ and $\Lambda(1)$.*

*Proof.* A conjugacy class of $W$ is contained in $\Lambda$ or disjoint from $\Lambda$ because $\Lambda$ is normal in $W$. The same is true for $W(1)$ and $\Lambda(1)$.

The group $\Lambda$ is commutative and the conjugacy classes of $W$ contained in $\Lambda$ are the orbits of the finite Weyl group $W_0$ acting on $\Lambda$. They are of course finite.

The group $\Lambda(1)$ is not commutative but the conjugacy classes of $\Lambda(1)$ are finite, as the center of $\Lambda(1)$ contains $T/T_0(1)$ hence has a finite index. For $\lambda \in \Lambda(1)$ we denote by $c(\lambda)$ the conjugacy class of $\lambda \in \Lambda(1)$. We have $W(1) = \Lambda(1)W_0(1)$. Obviously $tc(\lambda)t^{-1} = c(\lambda)$ for $t \in Z_k$, and $W_0 = W_0(1)/Z_k$ acts on the conjugacy classes of $\Lambda(1)$, with orbits the conjugacy classes of $W(1)$ contained in $\Lambda(1)$. They are of course finite.

The image in $W$ of a conjugacy class of $W(1)$ not contained in $\Lambda(1)$ is a conjugacy class of $W$ not contained in $\Lambda$.

We show that a conjugacy class of $W$ not contained in $\Lambda$ is infinite. Let $\lambda \in \Lambda$, $w \in W_0$, $w \neq 1$. The conjugacy class of $\lambda w$ in $W$ is infinite because, for $x \in T/T_0$, $x\lambda w x^{-1} = x\lambda w x^{-1} w^{-1} w$ and $\nu(x\lambda w x^{-1} w^{-1}) = \nu(x) - w(\nu(x)) + \nu(\lambda)$, and the set of $\nu(x) - w(\nu(x))$ for $x \in T/T_0$ contains the set of $y - w(y)$ for $y$ in the subgroup $Q(\Phi^\vee)$ of $X_*(T)$ generated by the set $\Phi^\vee$ of coroots of $T$ in $G$. This latter set is infinite. ☐

It is now easy to see that the center of the group $R$-algebra $R[W(1)]$ is $R[\Lambda(1)]^{W(1)}$:

**Lemma 2.2.** *The center of $R[W(1)]$ is the free $R$-module of basis*

$$z_C = \sum_{\lambda \in C} \lambda,$$

*for all finite conjugacy classes $C$ of $W(1)$.*

*Proof.* Let $z = \sum_{u \in W(1)} z(u)u \in R[W(1)]$ where the function $z : W(1) \to R$ has finite support. The following properties are equivalent:

1. $zw = wz$ for all $w \in W(1)$,
2. $z(w^{-1}u) = z(uw^{-1})$ for all $u, w \in W(1)$,
3. $z(.)$ is constant on $C$ for all conjugacy classes $C$ of $W(1)$.

☐

We want now to determine the center of the $R$-algebra $\mathcal{H}_R(q_s, c_s)$, described in the first part of Theorem 1.3. The proof is long and we formulate the different steps as propositions, lemmas, and corollaries. We fix an orientation $o$ and we determine first the center of $\mathcal{A}_o(1)$.

**Proposition 2.3.** *The center of $\mathcal{A}_o(1)$ is a $R$-free module of basis*

$$E_o(c) = \sum_{\lambda \in c} E_o(\lambda),$$

*for all conjugacy classes $c$ of $\Lambda(1)$.*

*Proof.* Let $z = \sum_{x \in \Lambda(1)} z(x) E_o(x)$ be an element in $\mathcal{A}_o(1)$ where $z : \Lambda(1) \to R$ is a function of finite support, and let $\lambda \in \Lambda(1)$. Let ([7, Def. 4.14]):

$$q_{w,w'} = (q_w q_{w'} q_{ww'}^{-1})^{1/2} \quad (w, w' \in W(1)). \tag{6}$$

By [7, Cor. 5.28],

$$E_o(x) E_o(\lambda) = q_{x,\lambda} E_o(x\lambda)$$

and $q_{\lambda x \lambda^{-1}, \lambda} = q_{x,\lambda} = q_{\lambda,x}$ because $q_\lambda$ depends only on the image of $\lambda$ in $\Lambda$, and $\Lambda$ is commutative. Then $z$ commutes with $E_o(\lambda)$ if and only if

$$\sum_{x \in \Lambda(1)} z(x) q_{x,\lambda} E_o(x\lambda) = \sum_{x \in \Lambda(1)} z(x) q_{\lambda,x} E_o(\lambda x).$$

Replacing $x$ by $\lambda x \lambda^{-1}$, the left hand side is equal to

$$\sum_{x \in \Lambda(1)} z(\lambda x \lambda^{-1}) q_{\lambda,x} E_o(\lambda x).$$

Hence $z$ belongs to the center of $\mathcal{A}_o(1)$ if and only if

$$z(\lambda x \lambda^{-1}) q_{x,\lambda} = z(x) q_{x,\lambda} \text{ for all } x, \lambda \in \Lambda(1).$$

If $z(.)$ is constant on the conjugacy classes of $\Lambda(1)$, then $z$ central in $\mathcal{A}_o(1)$. For the converse, there is a problem when $q_{x,\lambda}$ is not invertible in $R$, but it can be raised. We have $q_{x,\lambda} = 1$ if and only if $\ell(\lambda x) = \ell(\lambda) + \ell(x)$. If $z$ central in $\mathcal{A}_o$, the next lemma implies that $z(.)$ is constant on the conjugacy classes of $\Lambda(1)$. Admitting the next lemma, the proposition is proved. □

**Lemma 2.4.** *Let $x, x'$ be two conjugate elements of $\Lambda(1)$. There exists $\lambda \in \Lambda(1)$ such that $\ell(\lambda x) = \ell(\lambda) + \ell(x)$ and $x' = \lambda x \lambda^{-1}$.*

*Proof.* We choose an arbitrary element $\lambda \in \Lambda(1)$ such that $x' = \lambda x \lambda^{-1}$. We choose $\lambda' \in T/T_o(1)$ such that $\nu(\lambda\lambda')$ and $\nu(x)$ belong to the same closed Weyl chamber. We have $\ell(\lambda\lambda' x) = \ell(\lambda\lambda') + \ell(x)$ (cp. [7, Ex. 5.12]) and $x' = (\lambda\lambda') x (\lambda\lambda')^{-1}$ because $\lambda'$ belongs to the center of $\Lambda(1)$. □

The orientation $o$ is associated to a Weyl chamber $\mathfrak{D}_o$ of $V$. We denote by $\Delta_o$ the set of reduced roots $\alpha \in \Phi$ positive on $\mathfrak{D}_o$ such that $\operatorname{Ker} \alpha$ is a wall of $\mathfrak{D}_o$, by $S_o \subset W_0$ the set of reflections $s_\alpha$ for $\alpha \in \Delta_o$. The opposite orientation associated to the opposite Weyl chamber $-\mathfrak{D}_o$ gives the same set $S_o$. The set $S = S^{\text{aff}} \cap W_0$ is associated to the dominant and antidominant Weyl chambers

$\mathfrak{D}^+$ and $-\mathfrak{D}^+$. For $s \in S_o(1)$, there exists a unique root $\alpha \in \Delta_o \cup -\Delta_o$ taking positive values on $\mathfrak{D}^+$ such that the image of $s$ in $S_o$ is $s_\alpha$. We say that the root $\alpha$ of $\Phi$ is associated to $(o, s)$.

For $\gamma \in \Phi$ let $e_\gamma \in \mathbb{N}_>0$ be the positive integer such that the image of $\Phi$ by the map $\gamma \mapsto e_\gamma \gamma$ is a reduced root system $\Sigma$ of affine Weyl group $W^{\mathrm{aff}}$. If $\alpha$ is the root of $\Phi$ is associated to $(o, s)$, the root $\beta = e_\alpha \alpha$ of $\Sigma$ is called associated to $(o, s)$.

Let $z_o$ be a central element of $\mathcal{A}_o(1)$. By Proposition 2.3:

$$z_o = \sum_c z_o(c) E_o(c) \quad (z_o(c) \in R). \tag{7}$$

Let $s \in S_o(1)$. In the $R$-basis $(E_o(w))_{w \in W(1)}$ of $\mathcal{H}_R(q_s, c_s)$, we write

$$-z_o E_o(s) + E_o(s) z_o = z'_o + z''_o$$

where $z''_o \in \mathcal{A}_o(1)$ and the component of $z'_o$ in $\mathcal{A}_o(1)$ is 0. The elements $z_o$ and $E_o(s)$ commute if and only if $z'_o = z''_o = 0$. We denote by $scs^{-1}$ the set of $s\lambda s^{-1}$ for $\lambda \in c$.

**Lemma 2.5.** *We have*

$$z''_o = E_o(s) \sum_c z_o(c)(E_o(c) - E_{o \bullet s}(c)),$$

$$z'_o = E_o(s) \sum_c (z_o(c) - z_o(scs^{-1})) E_{o \bullet s}(c) = \sum_c z'_o(sc) E_o(sc),$$

*with* $z'_o(sc) = q_{s,c}(z_o(c) - z_o(scs^{-1}))$.

*Proof.* The equality $q_{scs^{-1},s} = q_{s,c}$ deduced from (6), implies by the product formula (3)

$$E_o(scs^{-1}) E_o(s) = E_o(s) E_{o \bullet s}(c).$$

Writing $z_o E_o(s) = E_o(s) \sum_c z_o(c) E_{o \bullet s}(s^{-1} c s) = E_o(s) \sum_c z_o(scs^{-1}) E_{o \bullet s}(c)$ we obtain

$$-z_o E_o(s) + E_o(s) z_o = E_o(s) \sum_c z_o(c) E_o(c) - z_o(scs^{-1}) E_{o \bullet s}(c),$$

from which we deduce $-z_o E_o(s) + E_o(s) z_o = z'_o + z''_o$ with $z'_o, z''_o$ as in the lemma. The element $z''_o$ belong to $\mathcal{A}_o(1)$ by the Bernstein relations (4) and the component of

$$z'_o = \sum_c z'_o(sc) E_o(sc), \ z'_o(sc) = q_{s,c}(z_o(c) - z_o(scs^{-1}))$$

in $\mathcal{A}_o(1)$ is 0 because $s\Lambda(1) \cap \Lambda(1) = \varnothing$. □

**Proposition 2.6.** *Let $s \in S_o(1)$ and let $c$ be a conjugacy class in $\Lambda(1)$. Then*
*$E_o(s)$ and $E_o(c) + E_o(scs^{-1})$ commute.*
*$E_o(s)$ and $E_o(c)$ commute if and only if $c = scs^{-1}$.*

*Proof.* a) Let $\beta$ be the root of $\Sigma$ associated to $(o,s)$. We note that $\nu$ is constant on a conjugacy class of $\Lambda(1)$, $scs^{-1}$ is a conjugacy class of $\Lambda(1)$, $\nu(s_\beta \lambda s_\beta^{-1}) = s_\beta(\nu(\lambda))$ for $\lambda \in \Lambda$ and $s_\beta(x) = x - \beta(x)\beta^\vee$ for $x \in V$ where $\beta^\vee$ is the coroot of $\beta$. Therefore we have

$$\nu(scs^{-1}) = \nu(c) - \beta \circ \nu(c)\beta^\vee.$$

If $c = scs^{-1}$ then $\beta \circ \nu(c) = 0$.

If $\beta \circ \nu(c) = 0$ then $q_{s,c} = q_{c,s} = 1$ and $E_{o\bullet s}(c) = E_o(c)$. This implies

$$E_o(s)E_o(c) - E_o(c)E_o(s) = E_o(s)E_{o\bullet s}(c) - E_o(c)E_o(s) = E_o(sc) - E_o(cs).$$

We deduce that $E_o(s)$ and $E_o(c)$ commute when $c = scs^{-1}$, and that $E_o(s)$ and $E_o(c)$ do not commute when $c \neq scs^{-1}$ and $\beta \circ \nu(c) = 0$.

b) We suppose $\beta \circ \nu(c) \neq 0$. We prove that either $E_o(c)$ or $E_o(scs^{-1})$ does not commutes with $E_o(s)$.

When $\ell(sc) = 1 + \ell(c)$, take $z_o = E_o(c)$ in Lemma 2.5. The coefficient of $z'_o$ on $E_o(sc)$ is $q_{s,c} = (q_s q_c q_{sc}^{-1})^{1/2} = 1$. We have $z'_o \neq 0$ hence $E_o(c)$ does not commute with $E_o(s)$.

When $\ell(sc) = -1 + \ell(c)$ we have $\ell(cs) = 1 + \ell(c)$. Take $z_o = E_o(scs^{-1})$. We have $E_o(scs^{-1}) = E_o(s^{-1}cs)$ because $s^2 \in Z_k$. We have $\beta \circ \nu(c) = -\beta \circ \nu(scs^{-1}) \neq 0$. The coefficient of $z'_o$ on $E_o(cs)$ is $q_{s,s^{-1}cs} = (q_s q_{s^{-1}cs} q_{cs}^{-1})^{1/2} = 1$. We have $z'_o \neq 0$ hence $E_o(scs^{-1})$ does not commute with $E_o(s)$.

c) We show that $E_o(c) + E_o(scs^{-1})$ commutes with $E_o(s)$. In Lemma 2.5 we take $z_o = E_o(c) + E_o(scs^{-1})$. We have obviously $z'_o = 0$.

When the $q_s = 1$ we show that $z''_o = 0$. By symetry, we can suppose $\beta \circ \nu(c) < 0$ and $\beta \circ \nu(scs^{-1}) > 0$. The Bernstein relations (4) give an explicit element $B_{o,n} \in \mathcal{A}_o(1)$ and different signs $\epsilon_{o\bullet s}(1,s) \neq \epsilon_o(1,s)$ such that

$E_o(s)(E_{o\bullet s}(c) - E_o(c)) = \epsilon_{o\bullet s}(1,s)B_{o,n}E_o(c)$,

$E_o(s)(E_{o\bullet s}(scs^{-1}) - E_o(scs^{-1})) = \epsilon_o(1,s)E_o(c)B_{o,n}$.

As $E_o(c)$ is central in $\mathcal{A}_o$,

$$-z''_o = E_o(s)(E_{o\bullet s}(c) - E_o(c) + E_{o\bullet s}(scs^{-1}) - E_o(scs^{-1})) = 0.$$

We proved that $E_o(s)$ commutes with $E_o(c) + E_o(scs^{-1})$ when the $q_s$ are 1.

We choose indeterminates $\mathfrak{q}_s$ for $s \in S^{\mathrm{aff}}/\sim$ of square $\mathfrak{q}_s^2 = \mathbf{q}_s$. By Lemma 5.43 in [7], we deduce that $\mathfrak{q}_s^{-1}E_o(s)$ commutes with $\mathfrak{q}_c^{-1}E_o(c) + \mathfrak{q}_{scs^{-1}}^{-1}E_o(scs^{-1})$ in the algebra $\mathcal{H}_{R[(\mathfrak{q}_s,\mathfrak{q}_s^{-1})]}(\mathbf{q}_s, c_s)]$. We have $\mathfrak{q}_c = \mathfrak{q}_{scs^{-1}}$ (cp. [7, Prop. 5.13]). Hence $E_o(s)$ commutes with $E_o(c) + E_o(scs^{-1})$ in the generic subalgebra $\mathcal{H}_{R[(\mathfrak{q}_s)]}(\mathbf{q}_s, c_s)]$. We specialize $\mathbf{q}_s \mapsto q_s$ hence $E_o(s)$ commutes with $E_o(c) + E_o(scs^{-1})$ in $\mathcal{H}_R(q_s, c_s)$.

d) We deduce from b) and c) that both $E_o(c)$ and $E_o(scs^{-1})$ do not commutes with $E_o(s)$ when $\beta \circ \nu(c) \neq 0$.

□

**Proposition 2.7.** *For any conjugacy class $C$ of $W(1)$ contained in $\Lambda(1)$, the element*

$$E(C) = E_o(C) = \sum_{\lambda \in C} E_o(\lambda),$$

*does not depend on the choice of the orientation $o$.*

*The $R$-algebra $\mathcal{A}_o(1)^{W(1)}$ does not depend on the choice of $o$, is $R$-free of basis $(E(C))$ for the conjugacy classes $C$ of $W(1)$ contained in $\Lambda(1)$, and is contained in the center $\mathcal{Z}_R(q_s, c_s)$.*

*Proof.* Let $s \in S_o(1)$. By Proposition 2.6 and its proof, $z_o = E_o(C)$ commutes with $E_o(s)$ and $z''_o = E_o(s)(E_{o\bullet s}(C) - E_o(C)) = 0$. When $q_s = 1$ for all $s$, $E_o(s)$ is invertible and $z''_o = 0$ implies $E_{o\bullet s}(C) = E_o(C)$. As $\mathfrak{q}_{\mathbf{C}} = \mathfrak{q}_{\mathbf{w}}$ is constant on $C$ we have $E_{o\bullet s}(C) = E_o(C)$ in the generic algebra $\mathcal{H}_{R[(\mathfrak{q}_s, \mathfrak{q}_s^{-1})]}(\mathbf{q}_s, c_s)]$, hence in the generic subalgebra $\mathcal{H}_{R[(\mathfrak{q}_s)]}(\mathbf{q}_s, c_s)]$. This is valid for any $s \in S_o(1)$, and the set $S_o(1)$ generates the group $W_0(1)$. The spherical orientations are $o \bullet w$ for $w \in W_0(1)$. Hence $E_{o\bullet w}(C) = E_o(C)$ for any $w \in W_0(1)$. We denote this element by $E(C)$.

We show that $E(C)$ belongs to the center of $\mathcal{H}_{R[(\mathfrak{q}_s)]}(\mathbf{q}_s, c_s)$. Indeed, for the orientation $o$ associated to the antidominant Weyl chamber $-\mathfrak{D}^+$, we have: $S_o = S^{\text{aff}} \cap W_0$, $E_o(s) = T_s$ for $s \in S_o(1)$, and $E(C)$ commutes with $\mathcal{A}_o(1)$ and $T_s$ for $s \in S_o(1)$. By the braid relations (1), $E(C)$ commutes with $T_w$ for $w \in W_0(1)$. We have $W(1) = \Lambda(1)W_0(1)$ with $Z_k = \Lambda(1) \cap W_0(1)$. The product formula (3) implies that $\mathcal{H}_{R[(\mathfrak{q}_s)]}(\mathbf{q}_s, c_s)]$ is generated by $\mathcal{A}_o(1)$ and $T_w$ for $w \in W_0(1)$.

The proposition is proved in $\mathcal{H}_{R[(\mathfrak{q}_s)]}(\mathbf{q}_s, c_s)$. By specialization $\mathbf{q}_s \mapsto q_s$ the proposition is true in $\mathcal{H}_R(q_s, c_s)$. ☐

The following proposition implies that the intersection of $\mathcal{A}_o(1)$ with $\mathcal{Z}_R(q_s, c_s)$ is contained in $\mathcal{A}_o(1)^{W(1)}$. We recall the reduced root system $\Sigma = \{e_\alpha \alpha \mid \alpha \in \Phi\}$ attached to $W^{\text{aff}}$ [7].

**Proposition 2.8.** *Let $s \in S_o(1)$, let $\beta \in \Sigma$ be the root associated to $(o, s)$ and let $z_o$ be a central element of $\mathcal{A}_o(1)$ as in (7). Then $E_o(s)$ commutes with $z_o$ if and only if $z_o(c) = z_o(scs^{-1})$ for all $c$ such that $\beta \circ \nu(c) \neq 0$.*

*Proof.* We suppose that $z_0 \neq 0$, and thanks to Proposition 2.6, that $z_o(c) \neq 0$ implies $z_o(scs^{-1}) = 0$. We will prove that $z'_o$ or $z''_o$ does not vanish. This implies that $z_o$ does not commute with $E_o(s)$ by Lemma 2.5.

If there exists $c$ with $z_o(c) \neq 0$ and $\beta \circ \nu(c) = 0$, then $\ell(sc) = \ell(c) + 1$, $q_{s,c} = 1$ and $z'_o(sc) = z_o(c)$. Hence $z'_o \neq 0$.

We suppose, as we may, that $z_o(c) \neq 0$ implies $\beta \circ \nu(c) \neq 0$ and $z_o(scs^{-1}) = 0$. We prove this time that $z''_o \neq 0$.

We analyze $z''_o = -\sum_c z_o(c) \sum_{\lambda \in c} E_o(s)(E_{o\bullet s}(\lambda) - E_o(\lambda))$ using the Bernstein relations (4) (cp. [7, Thm. 5.45 and Rem. 5.46]) for $\lambda \in c$:

$$E_o(s)(E_{o\bullet s}(\lambda) - E_o(\lambda)) = \epsilon_o(1, s)\epsilon_\beta(c) \sum_{k=0}^{n(c)-1} q(k, \lambda)c(k, \lambda)E_o(\mu(k, \lambda))$$

where $\beta \circ \nu(c) = \epsilon_\beta(c)n(c)$ with $\epsilon_\beta(c) \in \{\pm 1\}$ and $n(c) > 0$, and $\beta \circ \nu(\mu(k, \lambda)) = 2k - n(c)$.

Let $m = \max\{n(c) \mid z_o(c) \neq 0\}$. To prove the proposition it suffices to show that the component $z''_{o,m}$ of $z''_o$ in $\oplus_{|\beta\circ\nu(\lambda)|=m} RE_o(\lambda)$ is not 0.

Only the terms with $k = 0$ in the expansion of $E_o(s)(E_{o\bullet s}(\lambda) - E_{o\bullet s}(\lambda))$ for $\lambda \in c$ with $n(c) = m$ and $z_o(c) \neq 0$ contribute to $z''_{o,m}$. Up to multiplication by a sign, $z''_{o,m}$ is equal to

$$\sum_{c,n(c)=m} z_o(c)\epsilon_\beta(c) \sum_{\lambda\in c} q(0,\lambda)c(0,\lambda)E_o(\mu(0,\lambda)).$$

By [7, Rem. 5.46], $q(0,\lambda)c(0,\lambda)E_o(\mu(0,\lambda))$ is equal to

$$c_s E_o(\lambda) \text{ if } \epsilon_\beta(c) = -1, \quad E_o(s\lambda s^{-1})c_s \text{ if } \epsilon_\beta(c) = 1.$$

We deduce

$$\pm z''_{o,m} = \sum_{c,\beta\circ\nu(c)=-m} z_o(c)\epsilon_\beta(c)c_s E_o(c) + \sum_{c,\beta\circ\nu(c)=m} z_o(c)\epsilon_\beta(c)E_o(scs^{-1})c_s.$$

As $\beta\circ\nu(c) = -\beta\circ\nu(scs^{-1})$, and $z_o(c) \neq 0$ implies $z_o(scs^{-1}) = 0$, we obtain $z''_{o,m} \neq 0$. □

**Corollary 2.9.** *The intersection of $\mathcal{A}_o(1)$ with $\mathcal{Z}_R(q_s, c_s)$ is $\mathcal{A}_o(1)^{W(1)}$.*

*Proof.* Proposition 2.8 and 2.7. □

The last step of the proof of the equality $\mathcal{Z}_R(q_s, c_s) = \mathcal{A}_o(1)^{W(1)}$ is given by:

**Proposition 2.10.** *When $o$ is the orientation associated to the anti-dominant Weyl chamber $-\mathfrak{D}^+$, an element of $\mathcal{H}_R(q_s, c_s)$ which commutes with each element of $\mathcal{A}_o(1)$ is contained in $\mathcal{A}_o(1)$.*

*Proof.* We pick a nonzero element in $\mathcal{H}_R(q_s, c_s) - \mathcal{A}_o(1)$,

$$z = \sum_{x\in W(1)} z(x)E_o(x),$$

which commute with $E_o(\lambda)$ for all $\lambda \in \Lambda(1)$. We pick an element $w \in W(1)$ of maximal length in the nonempty set of $x \in W(1) - \Lambda(1)$ with $z(x) \neq 0$. We will show that there exists $\lambda \in \Lambda(1)$ such that

$$\ell(w) < \ell(\lambda w\lambda^{-1}),\ \ell(\lambda w) = \ell(\lambda) + \ell(w). \tag{8}$$

We have $q_{\lambda,w} = 1$ (cp. [7, Lemma 4.13]) and $E_o(\lambda w) = E_o(\lambda)E_o(w)$. We have $E_o(\lambda)z = zE_o(\lambda)$ and the coefficient of $E_o(\lambda)z$ on $E_o(\lambda w)$ is $z(w) \neq 0$.

We will show that the coefficient of $E_o(\lambda w)$ in $zE_o(\lambda)$ is $z(\lambda w\lambda^{-1})q_{\lambda w\lambda^{-1},\lambda}$. We have $z(\lambda w\lambda^{-1}) = 0$ because $\lambda w\lambda^{-1}$ does not belong to $\Lambda(1)$ and $\ell(w) < \ell(\lambda w\lambda^{-1})$. Our hypothesis was absurd and the proposition is proved if we admit the existence of $\lambda$ and the value of the coefficient of $E_o(\lambda w)$ in $zE_o(\lambda)$, given by the next two lemmas. □

**Lemma 2.11.** *Let $w \in W$ not in $\Lambda$. There exists $\lambda \in \Lambda$ such that*

$$\ell(w^{-1}\lambda w\lambda^{-1}) > 2\ell(w),\ \ell(\lambda w) = \ell(\lambda) + \ell(w) > \ell(w). \tag{9}$$

The lemma is stronger than the claim in the proof of Proposition 2.10 because (9) implies $2\ell(w) < \ell(w^{-1}\lambda w\lambda^{-1}) \leq \ell(w) + \ell(\lambda w\lambda^{-1})$ hence $\ell(w) < \ell(\lambda w\lambda^{-1})$.

*Proof.* Replacing $\lambda$ by $w^{-1}\lambda w$ which has the same length, (9) is replaced by

$$\ell(\lambda w\lambda^{-1}w^{-1}) > 2\ell(w),\ \ell(w\lambda) = \ell(\lambda) + \ell(w) > \ell(w). \tag{10}$$

We show first that there exists $\lambda \in \Lambda$ satisfying the weaker property

$$\ell(\lambda w\lambda^{-1}w^{-1}) > 0,\ \ell(w\lambda) = \ell(\lambda) + \ell(w) > \ell(w). \tag{11}$$

We write $w = w_0\lambda_0 \in W_0 \ltimes \Lambda$ with $w_0 \neq 1$. The length formula [7, Cor. 5.10] shows that $\ell(w\lambda) = \ell(w) + \ell(\lambda)$ is equivalent to

1) $(\gamma \circ \nu)(\lambda)$, $(\gamma \circ \nu)(\lambda_0)$ have the same sign when $\gamma \in \Sigma^+$, $w_0(\gamma) > 0$,
2) $(\gamma \circ \nu)(\lambda)$, $1 + (\gamma \circ \nu)(\lambda_0)$ have the same sign when $\gamma \in \Sigma^+$, $w_0(\gamma) < 0$.

We have:

$\ell(\lambda w\lambda^{-1}w^{-1}) = \ell(\lambda w_0\lambda^{-1}w_0^{-1})$ because $\Lambda$ is commutative,

$\ell(\lambda w_0\lambda^{-1}w_0^{-1}) > 0 \Leftrightarrow (\gamma \circ \nu)(\lambda w_0\lambda^{-1}w_0^{-1}) \neq 0$ for some $\gamma \in \Sigma$, by the length formula [7, Cor. 5.10],

$\Leftrightarrow \nu(\lambda w_0\lambda^{-1}w_0^{-1}) \neq 0 \Leftrightarrow \nu(\lambda) \neq w_0(\nu(\lambda))$.

The existence of $\lambda \in \Lambda$ satisfying (11) is equivalent to the existence of $\lambda \in \Lambda$ satisfying

$$0 \neq \nu(\lambda) \neq w_0(\nu(\lambda)),\ \ell(\lambda w) = \ell(\lambda) + \ell(w). \tag{12}$$

It is obvious that there are infinitely many $\lambda \in \Lambda$ satisfying these conditions.

In fact, the existence of $\lambda \in \Lambda$ satisfying (11) is not a weaker property than the existence of $\lambda \in \Lambda$ satisfying (10). Indeed, let $\lambda \in \Lambda$ satisfying (11), we show that, for a large odd integer $n$, $\lambda^n$ satisfies (10).

We have $(\lambda w\lambda^{-1}w^{-1})^n = \lambda^n w\lambda^{-n}w^{-1}$ because $\Lambda$ is commutative. If $n$ is large and $\ell(\lambda w\lambda w^{-1}) > 0$ then $\ell(\lambda^n w\lambda^{-n}w^{-1}) = n\ell(\lambda w\lambda w^{-1}) > 2\ell(w)$. In particular, $\ell(w\lambda) = \ell(\lambda) + \ell(w)$ implies $\ell(w\lambda^n) = \ell(\lambda^n) + \ell(w)$ for odd $n > 0$. Obviously $\ell(\lambda) + \ell(w) > \ell(w)$ implies $\ell(\lambda^n) + \ell(w) > \ell(w)$. $\square$

**Lemma 2.12.** *The coefficient of $E_o(\lambda w)$ in $zE_o(\lambda)$ is $z(\lambda w\lambda^{-1})q_{\lambda w\lambda^{-1},\lambda}$, for $z, \lambda, w$ in the proof of Proposition 2.10.*

*Proof.* We write $E_o(x)E_o(\lambda)$ for $x \in W(1)$, as

$$E_o(x)E_{o\bullet x}(\lambda)+E_o(x)(E_o(\lambda)-E_{o\bullet x}(\lambda)) = q_{x,\lambda}E_o(x\lambda)+E_o(x)(E_o(\lambda)-E_{o\bullet x}(\lambda)),$$

and $zE_o(\lambda) = z_1 + z_2$ where

$$z_1 = \sum_{x\in W(1)} z(x)q_{x,\lambda}E_o(x\lambda),\ z_2 = \sum_{x\in W(1)} z(x)E_o(x)(E_o(\lambda) - E_{o\bullet x}(\lambda)).$$

The coefficient of $z_1$ on $E_o(\lambda w)$ is $z(\lambda w\lambda^{-1})q_{\lambda w\lambda^{-1},\lambda}$ as $x\lambda = \lambda w \Leftrightarrow x = \lambda w\lambda^{-1}$. The coefficient of $z_2$ on $E_o(\lambda w)$ is 0 because

a) if $x \in \Lambda(1)$ we have $o \bullet x = o$.

b) if $x \in W(1) - \Lambda(1)$, we have (cp. [7, Thm. 4.5, Cor. 5.26])

$$\begin{aligned} E_o(x) &\in T_x + \sum_{y<x} RT_y, \\ E_o(\lambda) - E_{o\bullet x}(\lambda) &\in \sum_{u<\lambda} RT_u, \\ T_y T_u &\in \sum_{v \leq yu} RT_v, \\ E_o(x)(E_o(\lambda) - E_{o\bullet x}(\lambda)) &\in \sum_{v\leq yu, u<\lambda, y<x} RT_v, \end{aligned}$$

and from $v \leq yu, u < \lambda, y < x$, we have $\ell(v) < \ell(\lambda) + \ell(x) \leq \ell(w) + \ell(\lambda) = \ell(\lambda w)$. □

This ends the proof of the first part of Theorem 1.3 describing the center $\mathfrak{Z}_R(q_s, c_s)$. The second part generalizes the following finiteness properties:

**Proposition 2.13.** *The group algebra $R[W]$ is a finitely generated module over its center. If the ring $R$ is noetherian, the center is a finitely generated algebra, and $R[W]$ is a noetherian algebra.*

*The group algebra $R[W(1)]$ satisfies the same properties.*

*Proof.* i) $T/T_0(1)$ is a free, finitely generated commutative group which is normalized by $W(1)$. The action of $W(1)$ by conjugation on $T/T_0(1)$ factorizes by $W_0$. By a general theorem ([1, AC V.1.9 Thm. 2 p. 29]) for any finitely generated commutative $R$-algebra with an action of a group with finite orbits, $R[T/T_0(1)]$ is a finitely generated $R[T/T_0(1)]^{W_0}$-module; moreover, if $R$ is noetherian, $R[T/T_0(1)]^{W_0}$ is a finitely generated $R$-algebra.

ii) The center $R[\Lambda(1)]^{W(1)}$ of $R[W(1)]$ contains $R[T/T_0(1)]^{W_0}$, $R[W(1)]$ is a finitely generated $R[\Lambda(1)]$-module and $R[\Lambda(1)]$ is a finitely generated $R[T/T_0(1)]$-module, because the index of $T/T_0(1)$ in $\Lambda(1)$ and the index of $\Lambda(1)$ in $W(1)$ are finite.

iii) $R[W(1)]$ is finitely generated over $R[\Lambda(1)]^{W(1)}$ and over $R[T/T_0(1)]^{W_0}$. If the ring $R$ is noetherian, then $R[W(1)]$ is noetherian, $R[\Lambda(1)]^{W(1)}$ is a finitely generated $R[T/T_0(1)]^{W_0}$-module hence a finitely generated $R$-algebra. □

The proof of these finiteness properties for the $R$-algebra $\mathcal{H}_R(q_s, c_s)$ (the second part of Theorem 1.3) follows the same pattern. We pick a spherical orientation $o$. The $R$-module $\mathcal{A}_o(T/T_0(1))$ generated by $E_o(\lambda)$ for $\lambda \in T/T_0(1)$ is a commutative algebra with an action of $W(1)$ factorizing through $W_0$. We claim:

a) The $R$-algebra $\mathcal{A}_o(T/T_0(1))$ is finitely generated.

Therefore, $\mathcal{A}_o(T/T_0(1))$ is a finitely generated $\mathcal{A}_o(T/T_0(1))^{W_0}$-module; moreover, if $R$ is noetherian, $\mathcal{A}_o(T/T_0(1))^{W_0}$ is a finitely generated $R$-algebra (cp. [1, AC V.1.9 Thm. 2 p. 29]). The center $\mathcal{A}_o^{W(1)} = \mathcal{Z}_R(q_s, c_s)$ contains $\mathcal{A}_o(T/T_0(1))^{W_0}$. Theorem 1.3 follows from:

b) The left $\mathcal{A}_o(1)$-module $\mathcal{H}_R(q_s, c_s)$ is finitely generated,
c) The left $\mathcal{A}_o(T/T_0(1))$-module $\mathcal{A}_o(1)$ is finitely generated.

It remains to prove the claims a), b) and c). When the homomorphism $\nu : \Lambda(1) \to V$ defined by (5) sends $\lambda, \lambda' \in \Lambda(1)$ to the same closed Weyl chamber, we have $\ell(\lambda) + \ell(\lambda') = \ell(\lambda\lambda')$ and the product formula (3) is simply

$$E_o(\lambda)E_o(\lambda') = E_o(\lambda\lambda').$$

We denote by $\Lambda(1)_{\mathfrak{D}}$ the inverse image by $\nu$ of the closure of a Weyl chamber $\mathfrak{D}$ of $V$. The maximal subgroup of the monoid $\Lambda(1)_{\mathfrak{D}}$ is the kernel $\Omega(1) \cap \Lambda(1)$ of $\nu$.

We denote $L = T/T_0(1)$ and $L_{\mathfrak{D}} = L \cap \Lambda(1)_{\mathfrak{D}}$. The $R$-module of basis $E_o(\lambda)$ for $\lambda \in \Lambda(1)_{\mathfrak{D}}$ is a subalgebra $\mathcal{A}_{o,\mathfrak{D}}$ and the $R$-algebra $\mathcal{A}_o(L_{\mathfrak{D}})$ of basis $E_o(\lambda)$ for $\lambda \in L_{\mathfrak{D}}$ satisfy

$$\mathcal{A}_{o,\mathfrak{D}} \simeq R[\Lambda(1)_{\mathfrak{D}}], \ \mathcal{A}_o(L_{\mathfrak{D}}) \simeq R[L_{\mathfrak{D}}], \tag{13}$$

$$\mathcal{A}_o(1) = \cup_{\mathfrak{D}} \mathcal{A}_{o,\mathfrak{D}}, \ \mathcal{A}_o(L) = \cup_{\mathfrak{D}} \mathcal{A}_o(L_{\mathfrak{D}}), \tag{14}$$

where $\mathfrak{D}$ runs over all the Weyl chambers of $V$.

**Lemma 2.14.** *$L_{\mathfrak{D}}$ is a finitely generated monoid of finite index in $\Lambda(1)_{\mathfrak{D}}$.*

*Proof.* We have exact sequences

$$1 \to L \cap \Omega(1) \to L_{\mathfrak{D}} \to \nu(L) \cap \mathfrak{D} \to 1,$$

$$1 \to \Lambda(1) \cap \Omega(1) \to \Lambda(1)_{\mathfrak{D}} \to \nu(\Lambda(1)) \cap \mathfrak{D} \to 1.$$

The monoid $\nu(L)\cap\mathfrak{D}$ is finitely generated of finite index in the monoid $\nu(\Lambda(1))\cap \mathfrak{D}$. The commutative group $L\cap\Omega(1)$ is finitely generated because any subgroup of $L$ is free and finitely generated. The index of $L \cap \Omega(1)$ in $\Lambda(1) \cap \Omega(1)$ is finite because $\Lambda(1)/L$ is finite. □

We deduce the claims a) and c):

**Lemma 2.15.** *The $R$-algebra $\mathcal{A}_o(T/T_0(1))$ is finitely generated and $\mathcal{A}_o(1)$ is a finitely generated left and right $\mathcal{A}_o(T/T_0(1))$-module. In particular, the $R$-algebra $\mathcal{A}_o(1)$ is finitely generated.*

*Proof.* By the Lemma 2.14, $\mathcal{A}_o(L_{\mathfrak{D}})$ is a finitely generated $R$-algebra and $\mathcal{A}_{o,\mathfrak{D}}$ is a finitely generated $\mathcal{A}_o(L_{\mathfrak{D}})$-module. □

**Lemma 2.16.** *$\Lambda$ contains a finite set $X$ such that, for any $(\lambda, w) \in \Lambda \times W_0$, there exists $\mu \in X$ such that*

$$\ell(\lambda w) = \ell(\lambda\mu^{-1}) + \ell(\mu w).$$

*Proof.* It suffices to prove the lemma for an arbitrary fixed element of the finite group $W_0$. Let $w \in W_0$ and let $\lambda, \mu \in \Lambda$. By the length formula [7, Cor. 5.10] $\ell(\lambda w)$ is equal to the sum of $|\alpha \circ \nu(\lambda)|$ over the positive roots $\alpha$ such that $w^{-1}(\alpha)$ is positive, plus the sum of $|\beta \circ \nu(\lambda) - 1|$ over the positive roots $\beta$ such that $w^{-1}(\beta)$ is negative. We deduce that $\ell(\lambda w) = \ell(\lambda\mu^{-1}) + \ell(\mu w)$ if and only if

$\alpha\circ\nu(\mu)>0$ implies $\alpha\circ\nu(\lambda)\geq\alpha\circ\nu(\mu)$,
$\alpha\circ\nu(\mu)<0$ implies $\alpha\circ\nu(\lambda)\leq\alpha\circ\nu(\mu)$,
$\beta\circ\nu(\mu)>1$ implies $\beta\circ\nu(\lambda)\geq\beta\circ\nu(\mu)$,
$\beta\circ\nu(\mu)<1$ implies $\beta\circ\nu(\lambda)\leq\beta\circ\nu(\mu)$.
for all $\alpha,\beta$ as above. It is clear that, for any $\lambda\in\Lambda$, there exists a positive integer $N$ such that $\ell(\lambda w)=\ell(\lambda\mu^{-1})+\ell(\mu w)$ for some $\mu$ with $|\gamma\circ\nu(\mu)|\leq N$, for all positive roots $\gamma$. There are finitely many choices of $x(\mu)=(\gamma\circ\nu(\mu))_{\gamma>0}$ with $|\gamma\circ\nu(\mu)|\leq N$ for all positive roots $\gamma$, and $\mu\in\Lambda$. We deduce that there are finitely many elements $\mu_1,\dots,\mu_r$ such that, for any $\lambda\in\Lambda$, there exists $\mu_i$ such that $\ell(\lambda w)=\ell(\lambda\mu_i^{-1})+\ell(\mu_i w)$. □

We choose a finite set $X\subset\Lambda$ as in Lemma 2.16, and we denote by $X(1)$ and $XW_0(1)$ the inverse image in $W(1)$ of the finite sets $X$ and $\{\mu w\mid\mu\in X,\ w\in W_0\}$.

**Lemma 2.17.** *The left $\mathcal{A}_o(1)$-module $\mathcal{H}_R(q_s,c_s)$ is generated by the elements $E_o(w)$ for $w$ in the finite set $XW_0(1)$, and the $R$-algebra $\mathcal{H}_R(q_s,c_s)$ is finitely generated.*

*Proof.* $W(1)=\Lambda(1)W_0(1)$ and for $w=\lambda w_0\in W(1)$ with $\lambda\in\Lambda(1),w_0\in W_0(1)$, there exists $\mu\in X(1)$ such that $\ell(\lambda w_0)=\ell(\lambda\mu^{-1})+\ell(\mu w_0)$. Hence $q_{\lambda\mu^{-1},\mu w_0}=1$ and

$$E_o(w)=E_o(\lambda w_0)=E_o(\lambda\mu^{-1})E_o(\mu w_0).$$

We deduce the first part of the lemma. As the $R$-algebra $\mathcal{A}_o(1)$ is finitely generated by Lemma 2.15, the same is true for the $R$-algebra $\mathcal{H}_R(q_s,c_s)$. □

We deduce the claim b). The proof of Theorem 1.3 is complete.

## 3. Remarks

**3.1.** When the order of the finite commutative group $Z_k$ is invertible in $R$, and when $R$ contains a root of unity of order the least common multiple of the orders of the elements of $Z_k$ (we say that $R$ splits $Z_k$), the idempotents of $R[Z_k]$

$$e_\chi=|Z_k|^{-1}\sum_{t\in Z_k}\chi^{-1}(t)t \tag{15}$$

for all $R$-characters $\chi$ of $Z_k$, are orthogonal of sum 1. When $X$ is a $W_0$-orbit of characters of $Z_k$, the idempotent $e_X=\sum_{\chi\in X}e_\chi$ is central in $\mathcal{H}_R(q_s,c_s)$.

**Lemma 3.2.** *When $R$ splits $Z_k$, the $R$-algebra $\mathcal{H}_R(q_s,c_s)$ is the direct sum of the subalgebras $e_X\mathcal{H}_R(q_s,c_s)$ when $X$ runs over the $W_0$-orbits of characters of $Z_k$. As a $R$-module,*

$$e_X\mathcal{H}_R(q_s,c_s)=\oplus_{\chi\in X}\ e_\chi\mathcal{H}_R(q_s,c_s),\quad e_\chi\mathcal{H}_R(q_s,c_s)\simeq\chi\otimes_{R[Z_k]}\mathcal{H}_R(q_s,c_s).$$

**3.3.** When $q_s = 1$ for $s \in S^{\mathrm{aff}}$,

$$E_o(\lambda)E_o(w) = E_o(\lambda w) \text{ for } \lambda \in \Lambda(1) \text{ and } w \in W(1).$$

The linear map $\lambda \mapsto E_o(\lambda)$ from $R[T/T_o(1)] \subset R[\Lambda(1)]$ to $\mathcal{A}_o(T/T_o(1)) \subset \mathcal{A}_o(1)$ are algebra isomorphisms and

$$\mathcal{H}_R(1, c_s) = \sum_{w \in W_0(1)} \mathcal{A}_o(1) E_o(w).$$

**3.4.** Set $\mathcal{J}_o(1) = \sum_{s \in S^{\mathrm{aff}}/\sim} q_s \mathcal{A}_o(1)$. In the generic algebra $\mathcal{H}_{R[(\mathbf{q}_s]}(\mathbf{q}_s, c_s)$, for any orientation $o$, we have the product formula (3)

$$E_o(\lambda)E_o(\lambda') = \mathbf{q}_{\lambda,\lambda'} E_o(\lambda\lambda'),$$

and $\mathbf{q}_{\lambda,\lambda'} = 1$ if and only if $\lambda, \lambda' \in \Lambda(1)_{\mathfrak{D}}$ for some Weyl chamber $\mathfrak{D}$. By specialization of the indeterminates $\mathbf{q}_s$ to $q_s$, we obtain in $\mathcal{H}_R(q_s, c_s)$,

$$\begin{aligned} &E_o(\lambda)E_o(\lambda') = E_o(\lambda\lambda') \text{ if } \lambda, \lambda' \in \Lambda(1)_{\mathfrak{D}} \text{ for some } \mathfrak{D}, \\ &E_o(\lambda)E_o(\lambda') \in \mathcal{J}_o(1) \text{ otherwise.} \end{aligned}$$

We denote by $W_0(\mu)$ the $W_0$-orbit of $\mu$ in $X_*(T)$ and by $X_{*,\mathfrak{D}}(T)$ the monoid of $\mu \in X_*(T)$ such that $\nu(\mu(p_F))$ belongs to the closure of $\mathfrak{D}$. We denote by $C(\mu)$ the conjugacy class in $W(1)$ of the image $\lambda$ of $\mu(p_F)$ in $\Lambda(1)$, and we set $E_o(\mu) = E_o(\lambda)$. We have

$$E(C(\mu)) = \sum_{\mu' \in W_0(\mu)} E_o(\mu').$$

**Proposition 3.5.** *Let $\mu_1, \mu_2 \in X_{*,\mathfrak{D}}(T)$. In $\mathcal{H}_R(q_s, c_s)$ we have*

$$E(C(\mu_1))E(C(\mu_2)) \in E(C(\mu_1 + \mu_2)) + \mathcal{J}_o(1).$$

*Proof.* We fix the Weyl chamber $\mathfrak{D}$. For another Weyl chamber $\mathfrak{D}'$ we denote by $w_{\mathfrak{D}'}$ the unique element of $W_0$ such that $w_{\mathfrak{D}'}(\mathfrak{D}) = \mathfrak{D}'$. For $\mu \in X_{*,\mathfrak{D}}(T)$, $\mu_{\mathfrak{D}'} = w_{\mathfrak{D}'}(\mu)$ belongs to $X_{*,\mathfrak{D}'}(T)$. We have $\mu_{1,\mathfrak{D}'} + \mu_{2,\mathfrak{D}'} = (\mu_1 + \mu_2)_{\mathfrak{D}'}$ and

$$E(C(\mu)) = \sum_{\mu_{\mathfrak{D}'}} E_o(\mu_{\mathfrak{D}'})$$

(the sum is not over the Weyl chambers $\mathfrak{D}'$, but over the distinct elements $\mu_{\mathfrak{D}'}$).

$E(C(\mu_1))E(C(\mu_2))$ is equal modulo $\mathcal{J}_o(1)$ to the sum of $E_o(\mu'_1 + \mu'_2)$ over the pairs $(\mu'_1, \mu'_2) \in W_0(\mu_1) \times W_0(\mu_2)$ which belong to $X_{*,\mathfrak{D}'}(T)$ for the same Weyl chamber $\mathfrak{D}'$.

□

**3.6.** We recall the involutive $R$ automorphism of $\mathcal{H}_R(q_s, c_s)$ defined by

$$\iota(T_w) = (-1)^{\ell(w)} T_w^* \text{ for } w \in W(1),$$

where $T_w^* = (T_{s_1} - c_{s_1}) \dots (T_{s_r} - c_{s_r}) T_u$ if $w = s_1 \dots s_r u$ is a reduced decomposition of $w$, $s_i \in S^{\mathrm{aff}}(1)$, $u \in \Omega(1)$, $\ell(w) = r$ (cp. [7, Prop. 4.23]). Let $o$ be a spherical orientation attached to a Weyl chamber $\mathfrak{D}$, and $\overline{o}$ the spherical orientation attached to the opposite Weyl chamber $-\mathfrak{D}$. We recall (cp. [7, Lemma 5.31])

$$\iota(E_o(w)) = (-1)^{\ell(w)} E_{\overline{o}}(w) \text{ for } w \in W(1).$$

**Proposition 3.7.** *If $C$ is a finite conjugacy class of $W(1)$, we have*

$$\iota(E(C)) = (-1)^{\ell(C)} E(C).$$

*Proof.* The length is constant on $C$ hence

$$\iota(E(C)) = \sum_{\mu \in C} \iota(E_o(\mu)) = (-1)^{\ell(C)} \sum_{\mu \in C} E_{\overline{o}}(\mu) = (-1)^{\ell(C)} E(C),$$

as $E(C) = E_o(C)$ does not depend on the choice of the orientation $o$. □

**Remark 3.8.** When $\mu \in X_*(T)$, $\ell(C(\mu))$ is an even number (cp. [6]), hence $E(C(\mu))$ is fixed by $\iota$.

## ADDED IN PROOF ON AUGUST 14, 2014

1) The Bernstein relations (4), therefore also Lemma 2.5, Proposition 2.6, and Proposition 2.8, are valid only for $s \in (S \cap S_o)(1)$ where $S = W_0 \cap S^{\mathrm{aff}}$ and not for $s \in S_o(1)$ [7, Thm. 5.45], hence the arguments for $E_o(C)$ being independent of the orientation $o$ in Proposition 2.7 are not valid.

*Proof that $E_o(C)$ is independent of $o$.* We reduce to $q_s = 1$ for $s \in S^{\mathrm{aff}}$. Then, this results from the formula

$$E_o(w)^{-1} E_o(\lambda) E_o(w) = E_{o \bullet w}(w^{-1} \lambda w) \quad (w \in W_0(1), \lambda \in \Lambda(1)), \tag{16}$$

for the anti-dominant orientation $o$, because $E_o(C)$ is central and $W_0(1)$ acts transitively on the (spherical) orientations.

2) The proof of Lemma 2.16 must be replaced by:

*Proof of Lemma 2.16.* Let $L = \{\vec{\ell}(w) : \gamma \mapsto \ell_\gamma(w) : \Sigma^+ \to \mathbb{Z} \mid w \in W\}$ where for $(\lambda, w_0) \in \Lambda \times W_0$, $\ell_\gamma(\lambda w_0)$ is equal to ([7, Cor. 5.9])

$$\alpha \circ \nu(\lambda) \text{ if } \gamma \in w_0(\Sigma^+), \quad \gamma \circ \nu(\lambda) - 1 \text{ if } \gamma \in w_0(\Sigma^-).$$

For $w, w' \in W$, we write $\vec{\ell}(w) \leq \vec{\ell}(w')$ if $|\ell_\gamma(w)| = |\ell_\gamma(w')| + |\ell_\gamma(w) - \ell_\gamma(w')|$ for all $\gamma \in \Sigma^+$. We say that $\vec{\ell}(w)$ is minimal if $\Lambda w = \Lambda w'$ and $\vec{\ell}(w') \leq \vec{\ell}(w)$ implies $\vec{\ell}(w') = \vec{\ell}(w)$. As in [4, Lem. 4.2] one shows that the set $L_{min}$ of minimal elements of $L$ is finite. The finite subset $X = \cup_{w_0 \in W_0} X(w_0)$ of $\Lambda$ where $L_{min} = \cup_{w_0 \in W_0} X(w_0) w_0$ satisfies Lemma 2.16 because

$$\Lambda w = \Lambda w' \ \text{ and } \vec{\ell}(w') \leq \vec{\ell}(w) \Rightarrow \ell(w) = \ell(w') + \ell(w w'^{-1}). \tag{17}$$

In the left hand side of (17), $ww'^{-1} \in \Lambda$ implies $\ell_\gamma(ww'^{-1}) = \ell_\gamma(w) - \ell_\gamma(w')$ for all $\gamma \in \Sigma^+$; with $\vec{\ell}(w') \leq \vec{\ell}(w)$ we have $|\ell_\gamma(w)| = |\ell_\gamma(w')| + |\ell_\gamma(w) - \ell_\gamma(w')| = |\ell_\gamma(w')| + |\ell_\gamma(ww'^{-1})|$. Apply the length formula [7, Prop. 5.7]

$$\ell(w) = \sum_{\gamma \in \Sigma^+} |\ell_\gamma(w)|$$

to obtain (17).

## References


[1] N. Bourbaki, *Éléments de mathématique. Algèbre commutative. Chapitres 5–7*, reprint of the 1983 original, Actualités Scientifiques et Industrielles, No. 1308/1314, Hermann, Paris 1964/1965. Springer, Berlin 2006.

[2] G. Henniart and M.-F. Vignéras, A Satake isomorphism for representations modulo $p$ of reductive groups over local fields. J. Reine Angew. Math. Published Online: 05/22/2013.

[3] G. Lusztig, Affine Hecke algebras and their graded version, J. Amer. Math. Soc. **2** (1989), no. 3, 599–635. MR0991016 (90e:16049)

[4] Nicolas Alexander Schmidt, Generische pro-$p$-algebren. Diplomarbeit Berlin 2009.

[5] M.-F. Vignéras, Erratum: "Pro-$p$-Iwahori Hecke ring and supersingular $\overline{\mathbf{F}}_p$-representations" [Math. Ann. **331** (2005), no. 3, 523–556; 2122539], Math. Ann. **333** (2005), no. 3, 699–701. MR2198804 (2007i:22024)

[6] M.-F. Vignéras, Algèbres de Hecke affines génériques, Represent. Theory **10** (2006), 1–20 (electronic). MR2192484 (2006i:20005)

[7] M.-F. Vignéras, The pro-$p$-Iwahori-Hecke algebra of a reductive $p$-adic group I. Preprint 2013.


Received January 10, 2014; accepted January 13, 2014; Errata added April 27, 2014

Marie-France Vignéras
Université de Paris 7, UMR 7586, Institut de mathématiques de Jussieu
4, place Jussieu, Case 247, 75005 Paris, France
E-mail: vigneras@math.jussieu.fr
URL: http://www.math.jussieu.fr/~vigneras/