An integrality criterion for the homology of an equivariant coefficient system on the tree

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Four scores years ago ... Bon anniversaire, Jean-Pierre.

Résumé Soient F un corps p-adique, R un anneau commutatif de valuation discrète complet et $\mathcal{L}$ un système de coefficients $GL(2,F)$-équivariant de R-modules libres de type fini sur l'arbre de $PGL(2,F)$. On donne un critère nécessaire et suffisant pour que l'homologie de degré 0 de $\mathcal{L}$ soit un R-module libre. Ceci permet de construire des structures entières sur des représentations localement algébriques de $GL(2,F)$, et par réduction de montrer que des représentations de $GL(2,F)$ sur un corps fini de caractéristique p qui se relèvent à la caractéristique 0, sont isomorphes à l'homologie de degré 0 d'un système de coefficients. Par exemple, prenons un caractère modérément ramifié p-adique $\chi_1 \otimes \chi_2$ du tore diagonal $T(F)$ de $GL(2,F)$, tel que $\chi_1(p_F)\chi_2(p_F)$ soit une unité p-adique, $q\chi_1(p_F)$ et $\chi_2(p_F)$ soient des entiers p-adiques, p_F étant une uniformisante de F et q l'ordre du corps résiduel; alors la série principale de $GL(2,F)$ induite lisse non normalisée de $\chi_1 \otimes \chi_2$ est entière avec une structure entière remarquable explicite. Toute représentation irréductible de $GL(2,\mathbf{Q}_p)$ sur un corps fini de caractéristique $p \neq 2$, ayant un caractère central, s'obtient comme réduction d'une telle structure entière, et est égale à l'homologie de degré 0 d'un système de coefficients $GL(2,F)$-équivariant sur l'arbre.

Introduction

Let p be a prime number, q a power of p, let F be a local non archimedean field of characteristic 0 or p, with ring of integers O_F and residual field $k_F = \mathbf{F}_q$, let G be the group of F-points of a reductive connected F-group and let E/F is a finite extension. An irreducible locally algebraic E-representation of G is the tensor product $V_{sm} \otimes_E V_{alg}$ of a smooth one V_{sm} and of an algebraic one V_{alg}, uniquely determined ([Prasad] th.1). The problem of existence of integral structures in $V_{sm} \otimes_E V_{alg}$ (and their classification modulo commensurability) is crucial for the p-adic local Langlands correspondence expected to relate p-adic continuous finite dimensional E-representations of the absolute Galois group Gal_F and Banach admissible E-representations of G. The classification of irreducible representations of G over a finite field k of characteristic p remains a mystery when $G \neq GL(2,\mathbf{Q}_p)$, and the reduction of integral structures is a fundamental open problem. It is either "obvious" or "very hard" to see if $V_{sm} \otimes_E V_{alg}$ is integral or to determine the reduction of an integral structure. It is obvious that a non trivial algebraic representation V_{alg} is not integral, or that a smooth cuspidal irreducible representation V_{sm} with an integral central character is integral.

From now on, $G = GL(2,F)$.

We fix a local non archimedean field E of characteristic 0 and of residual field k_E of characteristic p and we suppose V_{alg} trivial when F is not contained in E, in particular when the characteristic of F is p. We will present a local integrality criterion for $V_{sm} \otimes_E V_{alg}$, by a purely representation theoretic method, not relying on the theory of (ϕ,Γ)-modules as in [Co04], [Co05], [BeBr] or on rigid analytic geometry as in [Br]. The idea is to realise $V_{sm} \otimes_E V_{alg}$ as the 0-homology of a G-equivariant coefficient system on the Serre's tree [Se77] (an easy generalization of a general result of Schneider and Stuhler [SS97] for complex finitely generated smooth representations).

Let $\mathcal{X}$ be the tree of $PGL(2,F)$ with the natural action of G [Se77]. The vertices of $\mathcal{X}$ are the similarity classes $[L]$ of O_F-lattices L in the 2-dimensional F-vector space F^2. Two vertices z_o, z_1 are related by an edge $\{z_o, z_1\}$ when they admit representatives L_o, L_1 such that $p_F L_o \subset L_1 \subset L_o$. The group $G = GL(2,F)$ acts naturally on the tree; a fundamental system consists of an edge σ_1 and of a vertex σ_o of σ_1. For $i = 0, 1$, we denote by K_i the stabilizer in G of σ_i; the intersection $K_o \cap K_1$ has index 2 in K_1, we choose $t \in K_1$ not in $K_o \cap K_1$, and we denote by ε the non trivial $\mathbf{Z}$-character of $K_1/(K_o \cap K_1)$.

We choose for σ_o the vertex defined by the O_E-module generated by the canonical basis of F^2 and for σ_1 the edge between σ_o and $t\sigma_o$ where

$$t = \begin{pmatrix} 0 & 1 \\ p_F & 0 \end{pmatrix}, \quad p_F \text{ uniformizer of } O_F.$$

Then $K_o = GL(2, O_F)Z$ and $K_1 =< IZ, t >$ is the group generated by IZ and t, where Z is the center of $GL(2, F)$, isomorphic to F^* diagonally embedded, and I is the Iwahori group of matrices of $GL(2, O_F)$ congruent modulo p_F to the upper triangular group $B(\mathbf{F}_q)$ of $GL(2, \mathbf{F}_q)$. The intersection $K_o \cap K_1$ is IZ. The element t normalizes the Iwahori subgroup I and its congruence subgroups $I(e)$ for $e \geq 1$.

Let R be a commutative ring. A G-equivariant coefficient system $\mathcal{L}$ of R-modules on $\mathcal{X}$ is determined by its restriction to the vertex σ_o and to the edge σ_1, i.e. by a diagram

$$r : L_1 \to L_o$$

where r is a $R(K_o \cap K_1)$-morphism from a representation of K_1 on an R-module L_1 to a representation of K_o on an R-module L_o. The word “diagram” was introduced by Paskunas [Pas] in his beautiful construction of supersingular irreducible representations of $GL(2, F)$ on finite fields of characteristic p. The boundary map from the oriented 1-chains to the 0-chains gives an exact sequence of RG-modules

$$0 \to H_1(\mathcal{L}) \to \mathrm{ind}_{K_1}^G (L_1 \otimes \varepsilon) \to \mathrm{ind}_{K_o}^G L_o \to H_o(\mathcal{L}) \to 0,$$

where the middle map associates to the function $[1, tv_1]$ supported on K_1 and value $tv_1 \in L_1$ at 1, the function $[1, r(tv_1)] - t[1, r(v_1)]$ supported on $K_o \cup K_o t^{-1}$ of value $r(tv_1)$ at 1 and $-r(v_1)$ at t^{-1}, and $H_i(\mathcal{L})$ is the i-th homology of $\mathcal{L}$ for $i = 0, 1$.

The natural RK_o-equivariant map $w_o : L_o \to H_o(\mathcal{L})$ is injective, and the natural map $w_o \circ r : L_1 \to L_o \to H_o(\mathcal{L})$ is K_1-equivariant (lemma 1.2).

0.1 Basic proposition: integrality local criterion.
1) $H_1(\mathcal{L}) = 0$ if and only if r is injective.
2) Suppose that
- R is a complete discrete valuation ring of fractions field S,
- L_o is a free R-module of finite rank,
- r is injective,
and let $\mathcal{V} := \mathcal{L} \otimes_R S, r_S := r \otimes_R \mathrm{id}_S : V_1 \to V_o$. Then, the map $H_o(\mathcal{L}) \to H_o(\mathcal{V})$ is injective and the R-module $H_o(\mathcal{L})$ is torsion free and contains no line Sv for $v \in H_o(\mathcal{V})$, when the equivalent conditions are satisfied :
a) $r_S(V_1) \cap L_o = r(L_1)$,
b) the map $V_1/L_1 \to V_o/L_o$ is injective.
c) $r(L_1)$ is a direct factor in L_o.

Let R as in 2). An S-representation V of G of countable dimension with a basis generating a G-stable R-submodule L, is called integral of R-integral structure L. When the properties of 2) are true, $H_o(\mathcal{L})$ is an R-integral structure of $H_o(\mathcal{V})$ such that (lemma 1.4.bis)

$$H_o(\mathcal{L}) \cap V_o = L_o.$$

0.2 Corollary *Let R as in 2). The S-representation $H_o(\mathcal{V})$ of G is R-integral if and only if there exists an R-integral structure L_o of the representation V_o of K_o such that $L_1 = L_o \cap V_1$ is stable by t (considering V_1 embedded in V_o).*

When this is true, the diagram $L_1 \to L_o$ defines an G-equivariant coefficient system $\mathcal{L}$ of R-modules on $\mathcal{X}$, and $H_o(\mathcal{L})$ is an R-integral structure of $H_o(\mathcal{V})$.

From now on, r is injective (and we forget r) and $V_o = K_o V_1$.

When V_i, for $i = 0, 1$ identified with an element of $\mathbf{Z}/2\mathbf{Z}$, contains a R-integral structure M_i which is finitely generated R-submodule, one constructs inductively an increasing sequence of finitely generated R-integral structures $(z^n(M_i))_{n\geq 1}$ of V_i, called the zigzags of M_i, as follows.
The RK_{i+1}-module M_{i+1} defined by
- if $i = 1$, then $M_o = K_o M_1$,
- if $i = 0$, then $M_1 = (M_o \cap V_1) + t(M_o \cap V_1)$,

is an R-integral structure of the SK_{i+1}-module V_{i+1} (a finitely generated R-module is free if and only if it is torsion free and does not contain a line). We repeat this construction to get the first zigzag $z(M_i)$:
- if $i = 1$, then $z(M_1) = (K_o M_1 \cap V_1) + t(K_o M_1 \cap V_1)$,
- if $i = 0$, then $z(M_o) = K_o((M_o \cap V_1) + t(M_o \cap V_1))$.

0.3 Corollary *Let $i \in \mathbf{Z}/2\mathbf{Z}$ and let M_i be an R-integral structure of the SK_i-module V_i. The representation of G on $H_o(\mathcal{V})$ is R-integral if and only if the sequence of zigzags $(z^n(M_i))_{n\geq 0}$ is finite.*

Set $P_F = O_F p_F$. For an integer $e \geq 1$, the e-congruence subgroup $K(e) = \begin{pmatrix} 1+P_F^e & P_F^e \\ P_F^e & 1+P_F^e \end{pmatrix}$ normalized by K_o is contained in the group $I(e) = \begin{pmatrix} 1+P_F^e & P_F^{e-1} \\ P_F^e & 1+P_F^e \end{pmatrix}$ normalized by K_1 and generated by $K(e)$ and $tK(e)t^{-1}$. The pro-p-Iwahori subgroup of I is $I(1)$.

0.4 Proposition *Let V_{alg} be an irreducible algebraic E-representation of G (hence $F \subset E$ if V_{alg} is not trivial), let V_{sm} be a finite length smooth E-representation of G and let e be an integer ≥ 1 such that V_{sm} is generated by its $K(e)$-invariants .*

1) The locally algebraic E-representation $V := V_{sm} \otimes_E V_{alg}$ of G is isomorphic to the 0-th homology $H_o(\mathcal{V})$ of the coefficient system $\mathcal{V}$ associated to the inclusion

$$V_{sm}^{I(e)} \otimes_E V_{alg} \to V_{sm}^{K(e)} \otimes_E V_{alg}.$$

2) The representation of G on V is O_E-integral if and only if there exists an O_E-integral structure L_o of the representation of K_o on $V_{sm}^{K(e)} \otimes_E V_{alg}$ such that $L_1 = L_o \cap (V_{sm}^{I(e)} \otimes_E V_{alg})$ is invariant by t. Then the 0-th homology L of the G-equivariant coefficient system on $\mathcal{X}$ defined by the diagram $L_1 \to L_o$ is an O_E-structure of V.

We have $L_o = L \cap (V_{sm}^{K(e)} \otimes_E V_{alg})$ in 2) by the lemma 1.4bis; when $(V_{sm}^{K(e)} \otimes_E V_{alg}) = K_o(V_{sm}^{I(e)} \otimes_E V_{alg})$, one can suppose $L_o = K_o L_1$ in 2) by the corollary 0.3.

We define the contragredient $\tilde{V} = \tilde{V}_{sm} \otimes_E V'_{alg}$ of $V = V_{sm} \otimes_E V_{alg}$ by tensoring the smooth contragredient $\tilde{V}_{sm}$ of V_{sm} and the linear contragredient V'_{alg} of V_{alg}.

0.5 Corollary *A finite length locally algebraic E-representation of G is O_E-integral if and only if its contragredient is O_E-integral.*

0.6 Remark A “moderately ramified” diagram:
- an R-representation L_o of K_o trivial on $K(1)$ with Z acting by a character ω,
- an R-representation L_1 of K_1 trivial on $I(1)$ and semi-simple as a SI-module,
- an RIZ-inclusion $L_1 \to L_o$,

is equivalent by “inflation” to a data:
- an R-representation Y_o of $GL(2, \mathbf{F}_q)$ with $Z(\mathbf{F}_q)$ acting by a character,
- a semi-simple R-representation Y_1 of $T(\mathbf{F}_q)$,

- an $RT(\mathbf{F}_q)$-inclusion $Y_1 \to Y_o$ with image contained in $Y_o^{N(\mathbf{F}_q)}$,
- an operateur τ on Y_1 such that:

τ^2 is the multiplication by a scalar $a \in E^*$,

τ permutes the χ-isotypic part and the χs isotypic part of Y_1 for any character χ of $T(\mathbf{F}_q)$.

The action of $GL(2, O_F)$ on L_o inflates the action of Y_o, the action of I on L_1 inflates the action of Y_1, the action of t on L_1 is given by τ, and $a = \omega(p_F)$.

Reduction An R-integral finitely generated S-representation V of G contains an R-integral structure L_{ft} which is finitely generated as a RH-module; two finitely generated R-integral structures L_{ft}, L'_{ft} of V are commensurable: there exists $a \in R$ non zero such that $aL_{ft} \subset L'_{ft}, aL'_{ft} \subset L_{ft}$.

Let x be an uniformizer of R and $k = R/xR$. When the reduction $\overline{L}_{ft} := L_{ft}/xL_{ft}$ is a finite length kG-module, the reduction $\overline{L}$ of an R-integral structure L of V commensurable to L_{ft} has finite length and the same semi-simplification than $\overline{L}_{ft}$. See [Vig96] I.9.5 Remarque, and [Vig96] I.9.6).

0.7 Lemma *If the reduction $\overline{L}_{ft}$ is an irreducible k-representation of H, then the R-integral structures of V are the multiples of L_{ft}.*

In the integrality criterion 0.1, when the properties of 2) are true, the reduction of the R-integral structure $H_o(\mathcal{L})$ of the S-representation $H_o(\mathcal{V})$ of G is the 0-th homology of the G-equivariant coefficient system defined by the diagram $\overline{L}_1 \to \overline{L}_o$. We have the exact sequences of SG-modules:

$$0 \to \operatorname{ind}_{K_1}^G V_1 \otimes \varepsilon \to \operatorname{ind}_{K_o}^G V_o \to H_o(\mathcal{V}) \to 0,$$

of free RG-modules:

$$0 \to \operatorname{ind}_{K_1}^G L_1 \otimes \varepsilon \to \operatorname{ind}_{K_o}^G L_o \to H_o(\mathcal{L}) \to 0,$$

of kG-modules:

$$0 \to \operatorname{ind}_{K_1}^G \overline{L}_1 \otimes \varepsilon \to \operatorname{ind}_{K_o}^G \overline{L}_o \to \overline{H}_o(\mathcal{L}) \to 0.$$

We will explicit the integral structures constructed in the proposition 0.4 when V_{sm} is a Steinberg representation, and when $V = V_{sm}$ is a moderately ramified principal series.

The Steinberg representation Let $B = NT$ be the upper triangular subgroup of G, with unipotent radical N and diagonal torus T. The Steinberg R-representation St_R of G is the R-module of B-left invariant locally constant functions $f : G \to R$ modulo the constant functions, with G acting by right translations. By [BS] 2.6, we have

$$\mathrm{St}_{\mathbf{Z}} \otimes_{\mathbf{Z}} R = \mathrm{St}_R\,.$$

The Steinberg representation over any field of characteristic p is irreducible [BL], [Vig06]. The same definition and the same property hold for the Steinberg R-representation st_R of the finite group $GL(2, \mathbf{F}_q)$ [CE] 6.13, ex. 6. From the lemma 0.7 and [SS91] th.8, one obtains:

0.8 Proposition *The Steinberg R-representation St_R of G is the 0-th homology of the moderately ramified diagram inflated from the Steinberg R-representation st_R of $GL(2, \mathbf{F}_q)$, the trivial character of $T(\mathbf{F}_q)$ on $\mathrm{st}_R^{N(\mathbf{F}_q)} \simeq R$, and the multiplication by -1 on $\mathrm{st}_R^{N(\mathbf{F}_q)}$ (remark 0.6).*

The S-Steinberg representation St_S of G is integral, all R-integral structures are multiple of St_R.

The irreducible algebraic representation Sym^k of G of dimension $k+1$ is realized in the space $F[X,Y]_k$ of homogeneous polynomials in X, Y of degree $k \geq 0$ with coefficients in F; it has a central character $z \to z^k$. We denote by $|?|$ the absolute value on an algebraic closure F^{ac} of F normalized by $|p| = p^{-1}$. Over any finite extension E of $F[p_F^{k/2}]$ contained in F^{ac}, the representation

$$\mathrm{Sym}^k \otimes_E |\det(?)|^{k/2}$$

of G has an O_E-integral central character. Let us consider the O_E-module M_1 generated in $O_E[X,Y]_k$ by the monomials

$$X^iY^j \text{ if } i \leq j \text{ and } p_F^{(-i+j)/2}X^iY^j \text{ if } i > j,$$

and the image ϕ_{BI} in St_{O_E} of the characteristic function of BI. One sees that the O_E-integral structure $L_1 = O_E\phi_{BI} \otimes_{O_E} M_1$ of the representation $\mathrm{St}^{I(1)} \otimes_E \mathrm{Sym}^k \otimes_E |\det(?)|^{k/2}$ of K_1 is equal to its zigzag $z(L_1) = K_oL_1 \cap (\mathrm{St}^{I(1)} \otimes_E \mathrm{Sym}^k \otimes_E |\det(?)|^{k/2})$, using $tX = p_F^{1/2}uY, tY = p_F^{-1/2}uX$ where $u \in O_F^*$ and a small computation.

0.9 Proposition *The locally algebraic representation $St_E \otimes_E \mathrm{Sym}^k \otimes_E |\det(?)|^{k/2}$ is O_E-integral for any integer $k \geq 0$; the 0-th homology of the G-equivariant coefficient system on the tree defined by the diagram*

$$L_1 = O_E\phi_{BI} \otimes_{O_E} M_1 \to L_o = K_oL_1$$

is an integral O_E-structure.

When $F = \mathbf{Q}_p$, there are other four different non trivial proofs of the integrality, Teitelbaum [T], Grosse-Klönne [GK1], Breuil [Br] (with some restrictions), Colmez [Co4].

Principal series Let $\chi_1 \otimes \chi_2 : T \to E^*$ be an E-character of T inflated to B. The principal series $\mathrm{ind}_B^G(\chi_1 \otimes \chi_2)$ is the set of functions $f : G \to E$ satisfying $f(hgk) = \chi_1(a)\chi_2(d)f(g)$ for all $g \in G,\ h \in B$ with diagonal (a,d), and k in a small open subgroup of G depending on f, with the group G acting by right translation.

When $\chi_1 \otimes \chi_2$ is moderately ramified, i.e. trivial on $T(1+P_F)$, its restriction to $T(O_F)$ is the inflation of an E-character $\eta_1 \otimes \eta_2$ of $T(\mathbf{F}_q)$, and the principal series is the 0-th homology of the G-equivariant coefficient system defined by the moderately ramified diagram

$$(\mathrm{ind}_B^G(\chi_1 \otimes \chi_2))^{I(1)} \to (\mathrm{ind}_B^G(\chi_1 \otimes \chi_2))^{K(1)}$$

inflated (Remark 0.6) from the inclusion

$$(\mathrm{ind}_{B(\mathbf{F}_q)}^{G(\mathbf{F}_q)}(\eta_1 \otimes \eta_2))^{N(\mathbf{F}_q)} \to \mathrm{ind}_{B(\mathbf{F}_q)}^{G(\mathbf{F}_q)}(\eta_1 \otimes \eta_2)$$

and the operator τ on $(\mathrm{ind}_{B(\mathbf{F}_q)}^{G(\mathbf{F}_q)}(\eta_1 \otimes \eta_2))^{N(\mathbf{F}_q)} = E\phi_1 \oplus E\phi_s$ such that

$$\tau\phi_1 = \chi_1(p_F)\phi_s \text{ and } \tau^2 \text{ is the multiplication by } \chi_1(p_F)\chi_2(p_F),$$

where ϕ_1, ϕ_s have support $B(\mathbf{F}_q), B(\mathbf{F}_q)sN(\mathbf{F}_q)$ and value 1 at id, s. Clearly,

$$Y_1 = O_E\phi_1 \oplus O_E\chi_1(p_F)\phi_s,$$

is an O_E-integral structure of $(\mathrm{ind}_{B(\mathbf{F}_q)}^{G(\mathbf{F}_q)}(\eta_1 \otimes \eta_2))^{N(\mathbf{F}_q)}$ stable by τ and

$$Y_o := GL(2,\mathbf{F}_q)Y_1$$

is an O_E-integral structure of $\mathrm{ind}_{B(\mathbf{F}_q)}^{G(\mathbf{F}_q)}(\eta_1 \otimes \eta_2)$. When the central character $\chi_1\chi_2$ is integral, (Y_o, Y_1, τ) inflates to a moderately ramified diagram $L_{Y_1} \to L_{Y_o} := K_oL_{Y_1}$ defining a G-equivariant coefficient system $\mathcal{L}$ of free O_E-modules of finite rank on $\mathcal{X}$. An $H_E(G, I(1))$-module is called O_E-integral when it contains an E-basis which generates an O_E-module stable by $H_{O_E}(G, I(1))$.

0.10 Theorem *We suppose that the E-character $\chi_1 \otimes \chi_2$ is moderately ramified, that $\chi_1(p_F)\chi_2(p_F) \in O_E^*$ is a unit, and that E contains a p-root of 1. The following properties are equivalent:*

a) the principal series $\mathrm{ind}_B^G(\chi_1 \otimes \chi_2)$ is O_E-integral,

b) the $H_E(G, I(1))$-module $(\mathrm{ind}_B^G(\chi_1 \otimes \chi_2))^{I(1)}$ is O_E-integral,

c) $\chi_2(p_F), \chi_1(p_F)q$ *are integral,*
d) $Y_o^{N(\mathbf{F}_q)} = Y_1$,
e) $L := H_o(\mathcal{L})$ *is an* O_E*-integral structure of* $\mathrm{ind}_B^G(\chi_1 \otimes \chi_2)$.
When they are satisfied, we have $L^{K(1)} = L_{Y_o}$ *and* $L^{I(1)} = L_{Y_1}$ *generates the* $O_E G$*-module* L.

When $\chi_1\chi_2^{-1}$ is moderately unramified, one reduces to $\chi_1 \otimes \chi_2$ moderately ramified by twist by a character. When $F = \mathbf{Q}_p$ and $\chi_1\chi_2^{-1}$ is unramified, i.e. trivial on O_F^*, the equivalence between c) and a) has been proved by [Br1].

0.11 Remarks (i) In the theorem 0.10, $\chi_1(p_F)$ is a unit if and only if the character $\chi_1 \otimes \chi_2$ of T is O_E-integral. Using [Vig04] th. 4.10, L is the natural O_E-integral structure of functions in $\mathrm{ind}_B^G(\chi_1 \otimes \chi_2)$ with values in O_E. The reduction of L is the k_E-principal series of G induced from the reduction $\overline{\chi}_1 \otimes \overline{\chi}_2$ of $\chi_1 \otimes \chi_2$; when $\overline{\chi}_1 \neq \overline{\chi}_2$ it is irreducible [BL], [Vig04], [Vig06] and each O_E-integral structure of $\mathrm{ind}_B^G(\chi_1 \otimes \chi_2)$ is a multiple of L, by the lemma 0.7.

When $\overline{\chi}_1 = \overline{\chi}_2$, then $\mathrm{ind}_B^G(\overline{\chi}_1 \otimes \overline{\chi}_2)$ has length 2; are the O_E-integral structures of $\mathrm{ind}_B^G(\chi_1 \otimes \chi_2)$ $O_E G$-finitely generated ? When $F = \mathbf{Q}_p$, compare with [BeBr] 5.4.4 and [Co05] 8.5.

(ii) The module of B is $|?|_F \otimes |?|_F^{-1}$ where $|p_F|_F = 1/q$. The contragredient of $\mathrm{ind}_B^G(\chi_1 \otimes \chi_2)$ is $\mathrm{ind}_B^G(\chi_1^{-1}|?|_F \otimes \chi_2^{-1}|?|_F^{-1})$ ([Vig96] I.5.11), hence the proposition 0.10 and the corollary 0.5 are compatible.

The representation
$$\mathrm{ind}_B^G(\chi_1 \otimes \chi_2) \simeq \mathrm{ind}_B^G(\chi_1|?|_F \otimes \chi_2|?|_F^{-1})$$
is irreducible when $\chi_1 \neq \chi_2, \chi_2|?|_F^2$ (the induction is not normalized); the isomorphism is compatible with the theorem 0.10.

(iii) Theoretically, there is no reason to restrict to the moderately ramified smooth case, but the computations become harder when the level increases or when one adds an algebraic part.

(iv) One should see c) as the limit at ∞ of the integrality local criterion. For $\mathrm{ind}_B^G(\chi_1 \otimes \chi_2) \otimes \mathrm{Sym}^k \otimes |\det(?)|^{k/2}$, one should replace c) by:

$$\chi_2(p_F)q^{-k/2}, \chi_1(p_F)q^{1-k/2} \quad \text{are integral.}$$

This condition is automatic when the representation is integral; this can be seen either via Hecke algebras or via exponents [E]. The representation of T on the 2-dimensional space of N-coinvariants $(V_{sm})_N$ of the smooth part $V_{sm} = |\det(?)|^{k/2} \otimes \mathrm{ind}_B^G(\chi_1 \otimes \chi_2)$ is a direct sum of two characters

$$|\det(?)|^{k/2}[(\chi_1 \otimes \chi_2) \oplus (\chi_2|?|_F \otimes \chi_1|?|_F^{-1})].$$

The N-invariants V_{alg}^N of the algebraic part $V_{alg} = \mathrm{Sym}^k$ has dimension 1 and T acts V_{alg}^N by $?^k \otimes 1$. The representation of T on $(V_{sm})_N \otimes_E V_{alg}^N$ is the direct sum of two characters called the exponents of V,

$$(\chi_1 ?^k |?|_F^{k/2} \otimes \chi_2|?|_F^{k/2}) \oplus (\chi_2 ?^k |?|_F^{k/2+1} \otimes \chi_1|?|_F^{k/2-1}).$$

They are integral on the element
$$\begin{pmatrix} 1 & 0 \\ 0 & p_F \end{pmatrix}$$
which dilates N if and only if $\chi_2(p_F)q^{-k/2}$ and $\chi_1(p_F)q^{1-k/2}$ are integral.

***k*-representations of *G*.** Let k be a finite field of characteristic p. The theorem 0.10 and the remark 0.11 (i)) imply that a principal series of G over k is the 0-th homology of a G-equivariant coefficient system.

Let $\mu_1 \otimes \mu_2$ be a k-character of T; its restriction to $T(O_F)$ is the inflation of a k-character $\eta_1 \otimes \eta_2$ of $T(\mathbf{F}_q)$. As before, $(\mathrm{ind}_{B(\mathbf{F}_q)}^{G(\mathbf{F}_q)}(\eta_1 \otimes \eta_2))^{N(\mathbf{F}_q)} = E\phi_1 \oplus E\phi_s$ where ϕ_1, ϕ_s have support $B(\mathbf{F}_q), B(\mathbf{F}_q)sN(\mathbf{F}_q)$ and value 1 at id, s.

0.12 Proposition *The principal series* $\mathrm{ind}_B^G(\mu_1 \otimes \mu_2)$ *is the* 0*-th homology of the* G*-equivariant coefficient system defined by the moderately ramified diagram*

$$(\mathrm{ind}_B^G(\mu_1 \otimes \mu_2))^{I(1)} \to (\mathrm{ind}_B^G(\mu_1 \otimes \mu_2))^{K(1)}$$

inflated (Remark 0.6) from the inclusion

$$(\mathrm{ind}_{B(\mathbf{F}_q)}^{G(\mathbf{F}_q)}(\eta_1 \otimes \eta_2))^{N(\mathbf{F}_q)} \to \mathrm{ind}_{B(\mathbf{F}_q)}^{G(\mathbf{F}_q)}(\eta_1 \otimes \eta_2)$$

and the operator τ *on* $(\mathrm{ind}_{B(\mathbf{F}_q)}^{G(\mathbf{F}_q)}(\eta_1 \otimes \eta_2))^{N(\mathbf{F}_q)}$ *such that*

$$\tau\phi_1 = \chi_1(p_F)\phi_s \ \text{ and } \ \tau^2 \text{ is the multiplication by } \chi_1(p_F)\chi_2(p_F).$$

Supersingular representations

The theorem 0.10 and [Vig04] imply:

0.13 Proposition *The simple supersingular modules of the Hecke* k*-algebra* $H_{k_E}(G, I(1))$ *are the reductions of the integral structures of the* $I(1)$*-invariants of integral principal series of* G *induced from non integral moderately ramified characters of* B.

Recall that a simple $H_{k_E}(G, I(1))$-module is supersingular if the action of the center of $H_{k_E}(G, I(1))$ is "null" [Vig04].

0.14 Proposition *Let* L *be the* O_E*-integral structure of the theorem 0.10, of an* O_E*-integral principal series of* G *induced from non integral moderately ramified* E*-characters* $\chi_1 \otimes \chi_2$ *of* B*, and let* $\overline{L}$ *be its reduction.*

a) If the reduction of $L^{I(1)}$ *is equal to* $(\overline{L})^{I(1)}$*, the* k_E*-representation* $\overline{L}$ *of* G *is irreducible and supersingular.*

b) The reduction of $L^{I(1)}$ *is equal to* $(\overline{L})^{I(1)}$ *when* $F = \mathbf{Q}_p, p \neq 2$.

From [Vig04], the map $V \to V^{I(1)}$ is a bijection between the irreducible k-representations of $GL(2, \mathbf{Q}_p)$ and the simple $H_k(G, I(1))$-modules, when the central element p_F acts by a fixed scalar. The proposition 0.14 implies:

0.15 Corollary *When* $F = \mathbf{Q}_p, p \neq 2$*, any irreducible representation of* G *over a finite field of characteristic* p*, with a central character, is the reduction of a moderately ramified integral principal series of* G*, and is the* 0*-th homology of a coefficient system on the tree.*

These results were presented in Tel-Aviv and Montreal (2005) and in Luminy (2006). One can hope that the new techniques introduced by Elmar Grosse-Klönne on the Bruhat-Tits building of $PGL(n, F)$ [GK2] will allow to generalize the local integrality criterion to $GL(n, F)$ for $n \geq 2$.

1 Coefficient system on the tree

Let R be a commutative ring. A G-equivariant coefficient system of R-modules $\mathcal{V}$ on the tree $\mathcal{X}$ consists of

- an R-module V_σ for each simplex σ of $\mathcal{X}$,
- a restriction linear map $r_\sigma^\tau : V_\tau \to V_\sigma$ for each edge τ containing the vertex σ,
- linear maps $g_\sigma : V_\sigma \to V_{g\sigma}$ for each simplex σ of $\mathcal{X}$ and each $g \in G$, compatible with the product of G and with the restriction: $(gg')_\sigma = g_{g'\sigma} g'_\sigma, \ g_\sigma r_\sigma^\tau = r_{g\sigma}^{g\tau} g_\tau$.

The stabilizer in G of a simplex σ acts on V_σ and the restrictions $r_\sigma^\tau, r_{\sigma'}^\tau$ are equivariant by the intersection of the stabilizers of the vertices σ, σ' of τ.

We denote by $\mathcal{X}(o)$ the set of vertices, by $\mathcal{X}(1)$ the set of oriented edges (σ, σ') (with origin σ) and by $\mathcal{X}_1$ the set of non oriented edges $\{\sigma, \sigma'\}$.

The R-module $C_o(\mathcal{V})$ of 0-chains is the set of functions $\phi : \mathcal{X}(o) \to \prod_{\sigma \in \mathcal{X}(o)} V_\sigma$ with finite support such that $\phi(\sigma) \in V_\sigma$ for any vertex σ.

The R-module $C_1(\mathcal{V})$ of oriented 1-chains is the set of functions $\omega : \mathcal{X}(1) \to \prod_{\{\sigma,\sigma'\} \in \mathcal{X}_1} V_{\{\sigma,\sigma'\}}$ with finite support such that $\omega(\sigma, \sigma') = -\omega(\sigma', \sigma) \in V_{\{\sigma,\sigma'\}}$ for any edge $\{\sigma, \sigma'\}$.

The boundary $\partial : C_1(\mathcal{V}) \to C_0(\mathcal{V})$ is the R-linear map sending an oriented 1-chain ω supported on one edge $\tau = \{\sigma, \sigma'\}$ to the 0-chain $\partial\omega$ supported on the vertices σ, σ', with

$$\partial\omega(\sigma) = r_\sigma^\tau \omega(\sigma, \sigma'), \quad \partial\omega(\sigma') = r_{\sigma'}^\tau \omega(\sigma', \sigma).$$

The group G acts on the R-module of oriented $*$-chains, for $* = 0, 1$, by

$$(g\omega)(g\sigma) = g(\omega(\sigma))$$

for any $g \in G$ and any oriented $*$-chain ω; the boundary ∂ is G-equivariant; the 0-homology

$$H_o(\mathcal{V}) = \frac{C_o(\mathcal{V})}{\partial C_1(\mathcal{V})}$$

and the 1-homology $H_1(\mathcal{V}) = \operatorname{Ker} \partial$ are R-representations of G.

For any oriented edge (σ, σ') there exists $g \in G$ such that $g(\sigma, \sigma') = (\sigma_o, t\sigma_o)$.

This property is equivalent to the fact that a G-equivariant coefficient system is determined by its restriction to the vertex σ_o and to the edge $\sigma_1 := (\sigma_o, t\sigma_o)$, i.e. by a diagram

$$r : V_1 \to V_o$$

where V_o is an R-representation on K_o, V_1 is an R-representation of K_1 and r is an R-linear map which is $K_o \cap K_1$-equivariant,

$$V_o := V_{\sigma_o}, \ V_1 := V_{\sigma_1}, \ r := r_{\sigma_o}^{\sigma_1}.$$

Conversely, any diagram defines a G-equivariant coefficient system [Pas]. The representations of G on $C_o(\mathcal{V})$ and on $C_1(\mathcal{V})$ are isomorphic to the compactly induced representations $\operatorname{ind}_{K_o}^G V_o$ and $\operatorname{ind}_{K_1}^G (V_1 \otimes \varepsilon)$ where $\varepsilon : K_1 \to R^*$ is the R-character of K_1 trivial on $K_o \cap K_1$ such that $\varepsilon(t) = -1$.

For $v_1 \in V_1$, if ω is the oriented 1-chain with support the edge $\sigma_1 = \{\sigma_o, t\sigma_o\}$ such that $\omega(\sigma_o, t\sigma_o) = tv_1$, the boundary map $\partial : C_1(\mathcal{V}) \to C_0(\mathcal{V})$ is the linear G-equivariant map such

(boundaryformula) $$\partial(\omega)(\sigma_o) = r(tv_1), \quad \partial(\omega)(t\sigma_o) = -r_{t\sigma_o}^{\sigma_1} tv_1 = -t_{\sigma_o} r(v_1).$$

The combinatorial distance on $\mathcal{X}$ is the number of edges between two vertices; the action of the group G respects the distance. For any integer $n \geq 0$, we denote by S_n the sphere of vertices of distance n to σ_o and by B_n the ball of radius n. For any chain $\omega \neq 0$, let $n(\omega)$ be the integer such that the support of ω is contained in the ball $B_{n(\omega)}$ and not in $B_{n(\omega)-1}$. When ω is a 1-chain we have $n(\omega) \geq 1$.

For any vertex $\sigma \in S_n$ with $n \geq 1$, the neighbours of σ belong to S_{n+1} except one neighbour which belongs to S_{n-1}; let τ_σ be the unique oriented edge starting from σ and pointing toward the origin σ_o. For any oriented 1-chain ω,

(key formula) $$\partial\omega(\sigma) = r_\sigma^{\tau_\sigma} \omega(\tau_\sigma) \quad \text{for all } \sigma \in S_{n(\omega)}.$$

We identify naturally $v_o \in V_o$ with a 0-chain with support on the single vertex σ_o; we consider the natural K_o-equivariant linear map

$$w_o : V_o \to H_o(\mathcal{X}, \mathcal{V})$$

and the $K_o \cap K_1$-equivariant map

$$w_o \circ r : V_1 \to V_o \to H_o(\mathcal{X}, \mathcal{V}).$$

1.2 Lemma *The map w_o is injective when r is injective and the map $w_o \circ r$ is K_1-equivariant.*

Proof. There is no non zero 1-chain ω with $\partial\omega$ supported on the single vertex σ_o because $n(\omega) \geq 1$ and $\partial\omega$ is not zero on $S_{n(\omega)}$ by the key formula, because r is injective.

The boundary formula gives the K_1-equivariance.

When the boundary map ∂ is injective, r must be injective. By the key formula, the converse is true.

1.3 Lemma *∂ is injective if the map r is injective.*

Proof. Let $\omega \neq 0$ be any oriented 1-chain and let $\sigma \in S_{n(\omega)}$; the edge τ_σ belongs to the support of ω. By the key formula $\partial(\omega)(\sigma) = r_\sigma^\tau \omega(\tau_\sigma)$ does not vanish because r_σ^τ is injective if r is injective, by the properties of the action of G.

We suppose from now on that the map $r : V_1 \to V_o$ is injective.

Descent *Let $\phi \neq 0$ be a 0-chain not supported on the origin. There exists an oriented 1-chain ω such that $n(\phi - \partial\omega) < n(\phi)$ if and only if $\phi(\sigma)$ belongs to $r_\sigma^{\tau_\sigma} V_{\tau_\sigma}$ for all $\sigma \in S_{n(\phi)}$.*

Proof. Let ω be any oriented 1-chain. By the key formula, $n(\phi - \partial\omega) < n(\phi)$ is equivalent to $n(\omega) = n(\phi)$ and

$$\phi(\sigma) = r_\sigma^{\tau_\sigma} \omega(\tau_\sigma)$$

for all $\sigma \in S_{n(w)}$. When the necessary condition $\phi(\sigma) = r_\sigma^{\tau_\sigma}(v_{\tau_\sigma}),\ v_{\tau_\sigma} \in V_{\tau_\sigma}$ for all $\sigma \in S_{n(\phi)}$ is satisfied, the oriented 1-chain ω_ϕ supported on

$$\cup_{\sigma \in S_{n(\phi)}} \tau_\sigma$$

with value v_{τ_σ} on τ_σ, satisfies $n(\phi - \partial\omega_\phi) < n(\phi)$. The oriented 1-chains satisfying $n(\phi - \partial\omega) < n(\phi)$ are $\omega_\phi + \omega'$ with $n(\omega') \leq n(\phi) - 1$.

When the R-module $r(V_1)$ has a supplementary in $V_o = W_o \oplus r(V_1)$, then the R-module $r_\sigma^{\tau_\sigma} V_{\tau_\sigma}$ has a (non canonical) supplementary in $V_\sigma = W_\sigma \oplus r_\sigma^\tau(V_\tau)$; we can find an oriented 1-chain ω supported on τ_σ such that $(\phi - \partial\omega)(\sigma) \in W_\sigma$ for any $\sigma \in S_{n(\phi)}$. By induction on $n(\phi)$, *any non zero element of $H_o(\mathcal{V})$ has a representative ϕ either supported on the origin, or such that $\phi(\sigma) \in W_\sigma$ for any $\sigma \in S_{n(\phi)}$.*

1.4 Proof of the basic proposition 0.1.

1) Lemma 1.3.

2) As r is injective, we can reduce r to an inclusion $V_1 \to V_o$.

Equivalence of the properties a), b), c). It is obvious that $V_1 \cap L_o = L_1$ is equivalent to: the kernel of $V_1 \to V_o/L_o$ is L_1, is equivalent to: the quotient of L_o by L_1 is torsion free, and is equivalent to L_1 is a direct factor of L_o because L_o is a free module of finite rank over the principal ring R.

The R-module $H_o(\mathcal{L})$ embeds in the S-vector space $H_o(\mathcal{V})$ because the map $V_1/L_1 \to V_o/L_o$ is injective by b) hence $H_1(\mathcal{V}/\mathcal{L}) = 0$ by 1) and the sequence $H_1(\mathcal{V}/\mathcal{L}) \to H_o(\mathcal{L}) \to H_o(\mathcal{V})$ is exact.

Let v be a non zero element of $H_o(\mathcal{L})$. Suppose that the line Sv is contained in $H_o(\mathcal{L})$. We choose

- a representative $\phi \in C_o(\mathcal{L})$ of v such that ϕ is supported on σ_o or such that $\phi(\sigma) \in W_\sigma$ for any $\sigma \in S_{n(\phi)}$,
- a vertex $\sigma' \in S_{n(\phi)}$ such that $\phi(\sigma') \neq 0$,
- an integer $n \geq 1$ such that $\phi(\sigma')$ does not belong to $x^n L_{\sigma'}$.

As $Sv \subset H_o(\mathcal{L})$, there exists an integral oriented 1-cocycle $\omega \in C_1(\mathcal{L})$ such that $(\phi + \partial(\omega))(\sigma) \in x^n L_\sigma$ for any vertex σ of the tree.

We may suppose $n(\omega) \leq n(\phi)$ by the following argument. If $n(\omega) > n(\phi)$, the key formula implies that $\omega(\tau_\sigma) \in x^n L(\tau_\sigma)$, for any vertex $\sigma \in S_{n(\omega)}$ because $r_\sigma^{\tau_\sigma} L_{\tau_\sigma} \cap x^n L_\sigma = r_\sigma^{\tau_\sigma}(x^n L_{\tau_\sigma})$ by a) and the injectivity of $r_\sigma^{\tau_\sigma}$. Let ω_{ext} be the integral oriented 1-cocycle supported on

$$\cup_{\sigma \in S_{n(\omega)}} \tau_\sigma$$

and equal to ω on this set. We may replace ω by $\omega - \omega_{ext}$; as $n(\omega - \omega_{ext}) < n(\omega)$ we reduce to $n(\omega) \leq n(\phi)$ by decreasing induction.

If ϕ is supported on σ_o, then $\omega = 0$ and $\phi(\sigma_o) \in x^n L_o$ which is false.

If $n_\phi \geq 1$, we have $\phi(\sigma) + \omega(\tau_\sigma) \in x^n L_\sigma$ for any $\sigma \in S_{n(\phi)}$ by the key formula. As $\phi(\sigma) \in W_\sigma$ and $\omega(\tau_\sigma) \in r_\sigma^{\tau_\sigma}(V_{\tau_\sigma})$, this is impossible.

As R is a local complete principal ring, $H_o(\mathcal{L})$ is R-free.

1.4bis Lemma *Let ϕ be a 0-chain with support on the single vertex σ_o and let ω be an oriented 1-chain such that $\phi + \partial(\omega)$ is integral. Then ϕ is integral.*

Proof. As $n(\omega) \geq 1$, the restriction of ω on $S_{n(\omega)}$ is integral by the key formula. By a decreasing induction on $n(\omega)$, ϕ is integral.

1.5 Proof of the corollary 0.2

Sufficient. When L_o is an R-integral structure of V_o such that $L_1 = L_o \cap V_1$ is stable by t, then L_1 is an R-integral structure of V_1; the map r induces an injective diagram $L_1 \to L_o$. By the integrality criterion 0.1, $H_o(\mathcal{V})$ is R-integral.

Necessary. Suppose that L is an R-integral structure of $H_o(\mathcal{V})$. We apply the lemma 1.2. The inverse image L_o of $w_o(V_o) \cap L$ in V_o by w_o is an R-integral structure of the representation of K_o on V_o, and the inverse image L_1 of $w_o(V_1) \cap L$ is an R-integral structure of V_1, of course stable by t, equal to $L_o \cap V_1$.

1.6 Proof of the corollary 0.3

1) When the sequence of zigzags is finite, there exists a finitely generated R-integral stucture M_i of V_i equal to its first zigzag $z(M_i) = M_i$, for $i = 0$ or $i = 1$. If $z(M_1) = M_1$, set $L_o = K_o M_1$. If $z(M_o) = M_o$, set $L_o = M_o$. In both cases, L_o is a finitely generated R-integral structure of V_o and $L_1 = L_o \cap V_1$ is stable by t. By the corollary 0.2, $H_o(\mathcal{V})$ is R-integral.

2) When $H_o(\mathcal{V})$ is R-integral, there exists an R-integral structure L_o of V_o such that $L_1 = V_1 \cap L_o$ is t-invariant by the corollary 0.2.

- Let M_1 be an R-integral structure of V_1. Replacing L by a multiple, we suppose $M_1 \subset L_1$. Then $K_o M_1 \subset L_o$ and $z(M_1) \subset L_1$. The sequence of zizgags of M_1 is contained in L_1 and increasing, hence finite because L_1 is a finitely generated R-module and R is noetherian.

- Let M_o be an R-integral structure of V_o. We apply the above argument to $M_1 = (V_1 \cap M_o) + t(V_1 \cap M_o)$; the sequence of zizgags of M_1 is finite, and also the sequence of zizgags of M_o.

1.7 Proof of the proposition 0.4

1)The exactness of the sequence

$$0 \to \mathrm{ind}_{K_1}^G(V_{sm}^{I(e)} \otimes_E V_{alg} \otimes_E \varepsilon) \to \mathrm{ind}_{K_o}^G(V_{sm}^{K(e)} \otimes_E V_{alg}) \to V_{sm} \otimes_E V_{alg} \to 0$$

follows from the following facts.

The assertion is true when V_{alg} is trivial if E is replaced by the field $\mathbf{C}$ of complex numbers by [SS97] II.3.1; this is also true for E because the scalar extension $\otimes_E \mathbf{C}$ commutes with the invariants by an open compact subgroup and with the compact induction from an open subgroup. The tensor product by $\otimes_E V_{alg}$ of an exact sequence of EG-representations remains exact and commutes with the compact induction from an open subgroup.

2) The finite length representation V_{sm} is admissible; this is known for complex representations and remain true for E-representations because $V_{sm} \otimes_E \mathbf{C}$ has finite length [Vig96] II.43.c, and $? \otimes_E \mathbf{C}$ commutes

with the $K(e)$-invariant functor. The E-vector space $V_{sm}^{K(e)} \otimes_E V_{alg}$ is finite dimensional. Apply the corollary 0.2.

1.8 Proof of the corollary 0.5

Let V_{sm} be a non zero smooth E-representation of G of finite length; there exists an integer $e \geq 1$ such that each non zero irreducible subquotient of V_{sm} contains a non zero $K(e)$-invariant vector. The E-vector space $(\tilde{V}_{sm})^{K(e)}$ isomorphic to the dual $(V_{sm}^{K(e)})'$; the irreducible subquotients of the contragredient $\tilde{V}_{sm}$ are the contragredients of the irreducible subquotients of V_{sm}. Hence V_{sm} and $\tilde{V}_{sm}$ are generated by their $K(e)$-invariants.

Suppose that $V = V_{sm} \otimes_E V_{alg}$ is O_E-integral. We choose an O_E-integral structure L_o of the representation of K_o on $V_{sm}^{K(e)} \otimes_E V_{alg}$ such that $L_1 := L_o \cap (V_{sm}^{I(e)} \otimes_E V_{alg})$ is t-stable (proposition 0.4), and we take the linear dual $L'_o = \mathrm{Hom}_{O_E}(L_o, O_E)$ of L_o. It is clear that L'_o is an O_E-integral structure of the representation of K_o on

$$(V_{sm}^{K(e)} \otimes_E V_{alg})' \simeq (V_{sm}^{K(e)})' \otimes_E (V_{alg})' \simeq (\tilde{V}_{sm})^{K(e)} \otimes_E V'_{alg}.$$

We take the intersection $L'_o \cap ((\tilde{V}_{sm})^{I(e)} \otimes_E V'_{alg}) = L'_o \cap (V_{sm}^{I(e)} \otimes_E V_{alg})'$. The O_E-module L_1 is a direct factor of L_o hence its linear dual L'_1 is equal to this intersection; it is clearly invariant by t. By the proposition 0.4, $\tilde{V}$ is O_E-integral.

The length of V_{sm} and the E-dimension of V_{alg} are finite, hence V is isomorphic to the contragredient of $\tilde{V}$. If $\tilde{V}$ is O_E-integral then V is O_E-integral.

1.9 Proof of the lemma 0.7

Let L be an R-integral structure of V which is different from L_{ft}. Taking a multiple of L_{ft}, we reduce to $L_{ft} \subset L$ and L_{ft} not contained in xL. The inclusions

$$xL_{ft} \subset (xL \cap L_{ft}) \subset L_{ft},$$

the right one beeing strict, and the irreducibility of L_{ft}/xL_{ft} imply $xL \cap L_{ft} = xL_{ft}$, equivalent to $L = L_{ft}$ because there exists no $v \in L$ and $v \notin L_{ft}$ such that $v \in x^{-1}L_{ft}$.

2 The Steinberg representation

The proposition 0.8 results from the remarkable properties of the Steinberg representations that we recall below and from the lemma 0.7.

2.1 St_R is the 0-th homology of the G-coefficient system associated to the inclusion

$$\mathrm{St}_R^{I(1)} \to \mathrm{St}_R^{K(1)}$$

by [SS91] th. 8.

2.2 $\mathrm{St}_R = \mathrm{St}_{\mathbf{Z}} \otimes_{\mathbf{Z}} R$ is an R-free module, isomorphic as an R-representation of B to $C_c^\infty(N, R)$, where N acts by translations and T by conjugation, by the map [BS] 3.7:

$$f \to \phi_f(n) = f(n) - f(sn) \ \text{ for } n \in N \ \text{ and } s = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

One can check that the image of $\mathrm{St}_R^{K(1)}$ is $C_c^\infty(N(0), R)^{N(1)}$ where $N(0) = N \cap GL(2, O_F)$ and $N(1) = N \cap K(1)$.

2.3 When R is a field of characteristic p, the action of the monoid generated by N and $\begin{pmatrix} p_F & 0 \\ 0 & 1 \end{pmatrix}$ on $C_c^\infty(N, R)$ is irreducible [Vig06]. The unique projective irreducible R-representation of $GL(2, \mathbf{F}_q)$ is st_R [CE] 6.12.

2.4 The Steinberg R-representation St_R of G is the highest cohomology with compact supports of the tree by [BS] 5.6, and by [SS91] cor. 17,

$$\mathrm{St}_R = R[G/I] \otimes_{H_R(G,I)} \mathrm{sign}\,.$$

where sign is the character of the Hecke algebra $H_R(G,I)$ of the Iwahori subgroup I on $\mathrm{St}_R^I = \mathrm{St}_R^{I(1)}$.

2.5 Let ϕ_{BI} be the characteristic function of BI modulo the constants. We have

$$\mathrm{St}_R^{I(1)} = R\phi_{BI}, \quad K_o R\phi_{BI} = \mathrm{St}_R^{K(1)}, \quad t\phi_{BI} = -\phi_{BI},$$

by the same proof of [BL] lemma 26 for the first equality, because the characteristic function of $B(\mathbf{F}_q)$ generates $\mathrm{ind}_{B(\mathbf{F}_q)}^{G(\mathbf{F}_q)} 1_R$ for the second equality (see also the lemma 2.7), and because $\phi_{BI}(t) = -\phi_{BsI}(t) = -1$ for the third equality.

The representation of K_o on $\mathrm{St}_R^{K(1)}$ is the inflation of the Steinberg representation $\mathrm{st}_R = \mathrm{st}_{\mathbf{Z}} \otimes_{\mathbf{Z}} R$ of $GL(2,\mathbf{F}_q)$ which is a free R-module of rank q.

A system of representatives of $K_o/ZI \simeq GL(2,\mathbf{F}_q)/B(\mathbf{F}_q)$ is $s, (v_x)_{x\in\mathbf{F}_q}$ where $v_x = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}$, because $GL(2,\mathbf{F}_q) = B(\mathbf{F}_q) \cup N(\mathbf{F}_q) s B(\mathbf{F}_q)$. One embeds $\mathbf{F}_q$ in O_F by the Teichmüller map. If M is an RZI-module, then

$$K_o M = sM + \sum_{x\in\mathbf{F}_q} v_x M.$$

2.6 Lemma *$s\phi_{BI} + \sum_{x\in\mathbf{F}_q} v_x\phi_{BI} = 0$. When R is a field, the q elements $s\phi_{BI}, v_x\phi_{BI}$ $(x \in \mathbf{F}_q^*)$ form an R-basis of $\mathrm{St}_R^{K(1)}$.*

Proof. $RK_o\phi_{BI} = \mathrm{St}_R^{K(1)}$ is R-free of rank q (2.5), the sum $s\phi_{BI} + \sum_{x\in\mathbf{F}_q} v_x\phi_{BI}$ is K_o-invariant and $\mathrm{St}_R^{K_o} = 0$.

2.7 Proof of the integrality of $\mathrm{St}\otimes S_k$ (proposition 0.9).

We can suppose $k \geq 1$. We have

$$L_o = K_o L_1 = (s\phi_{BI} \otimes sM_1) + \sum_{x\in\mathbf{F}_q} (v_x\phi_{BI} \otimes v_x M_1).$$

The first zizgag $z(L_1) = K_o L_1 \cap (St_E^{I(1)} \otimes_E \mathrm{Sym}^k \otimes_E |\det(?)|^{k/2})$ of L_1 is, by (2.5) and the lemma 2.6,

$$z(L_1) = O_E\phi_{BI} \otimes_{O_E} (M_1 + N),$$

where N is the intersection of sM_1 with $\cap_{x\in\mathbf{F}_q^*} v_x M_1$.

It is clear that $O_E[X,Y]_k \subset N$ because $O_E[X,Y]_k$ is stable by K_o and contained in M_1. The key of the proof is to check the opposite inclusion, because $N = O_E[X,Y]_k$ implies $z(L_1) = L_1$ and one can apply the corollary 0.3.

As $L_o \subset M_1$, one deduces from $N = L_o$ that L_1 is equal to its first zigzag $z(L_1)$. We apply the corollaries 0.3 and 0.2 and the proposition 0.9 is proved. Let us check the opposite inclusion. A basis of sM_1 is X^iY^j if $i \geq j$ and $p_F^{(-j+i)/2} X^iY^j$ if $i < j$ for $i,j \in \mathbf{N}, i+j = k$; a basis of $v_x M_1$ is $(X+xY)^iY^j$ if $i \leq j$ and $p_F^{(-i+j)/2}(X+xY)^iY^j$ if $i > j$ for $i,j \in \mathbf{N}, i+j = k$. Suppose that

$$\sum_{i+j=k} c_{i,j} X^iY^j = \sum_{i+j=k} d_{i,j}(x)(X+xY)^iY^j \qquad (c_{i,j}, d_{i,j}(x) \in E, x \in \mathbf{F}_q^*)$$

belongs to N. Modulo $O_E[X,Y]_k$, we can forget the $c_{i,j}$ with $i \geq j$ and $d_{i,j}(x)$ with $i \leq j$, and we have

$$\sum_{i<j} c_{i,j} X^i Y^j \equiv \sum_{j<i} d_{i,j}(x)(X+xY)^i Y^j \mod O_E[X,Y]_k.$$

When $k=2u$ is even and $i \geq u$, X^i does not appear on the left side. By decreasing induction on i, we can show that $d_{k,o}(x), d_{k-1,1}(x)\ldots, d_{u,u}(x) \in O_E$. When $k=2u+1$ is odd and $i \geq u+1$, X^i does not appear on the left side, and we can show that the $d_{i,j}(x)$ for $j<i$ belong to O_E. Hence $N \subset O_E[X,Y]_k$.

3 Principal series. Proof of the theorem 0.10

A moderately ramified character of O_F^* is the inflation of a character of $\mathbf{F}_q^*$, that we denote by the same letter; we use the Teichmüller embedding $\mathbf{F}_q \to O_F$.

As $G = BGL(2,O_F)$, the restriction to $GL(2,O_F)$ of $\mathrm{ind}_B^G(\chi_1 \otimes \chi_2)$ is isomorphic to

$$\mathrm{ind}_{B(O_F)}^{GL(2,O_F)}(\eta_1 \otimes \eta_2)$$

and the representation of $GL(2,O_F)$ on $(\mathrm{ind}_B^G(\chi_1 \otimes \chi_2))^{K(1)}$ is the inflation of the principal series

$$\mathrm{ind}_{B(\mathbf{F}_q)}^{GL(2,\mathbf{F}_q)}(\eta_1 \otimes \eta_2).$$

A system of representatives of $B(O_F)\backslash GL(2,O_F)/K(1) \simeq B(\mathbf{F}_q)\backslash GL(2,\mathbf{F}_q)$ is

$$1, su_x \text{ for } x \in \mathbf{F}_q, \quad u_x = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$

by the decomposition $GL(2,\mathbf{F}_q) = B(\mathbf{F}_q) \cup B(\mathbf{F}_q)sN(\mathbf{F}_q)$. Let L_o be the O_E-integral structure of the E-representation of K_o on $(\mathrm{ind}_{B(O_F)}^{GL(2,O_F)}(\eta_1 \otimes \eta_2))^{K(1)}$ given by the functions with values in O_E. We denote by $f_g \in L_o$ the function of support $B(O_F)gK(1)$ and value 1 at g. An R-basis of L_o is $\{f_1, (f_{su_x})_{x\in\mathbf{F}_q}\}$. The $O_E K_o$-module L_o is cyclic generated by f_1 because

$$u_x s f_1 = f_{su_{-x}} \quad \text{for } x \in \mathbf{F}_q.$$

Modulo the first congruence group $K(1)$, the pro-p-Iwahori $I(1)$ is represented by $(u_x)_{x\in\mathbf{F}_q}$. A basis of $L_o^{I(1)}$ is

$$f_1, \ \sum_{x\in\mathbf{F}_q} f_{su_x}.$$

It is convenient to write $t = sh = shss$ where $h = \begin{pmatrix} p_F & 0 \\ 0 & 1 \end{pmatrix}$. It is obvious that $tf_1(1) = f_1(t) = 0$, $tf_1(s) = f_1(st) = \chi(h) = \chi_1(p_F)$, hence

$$tf_1 = \chi_1(p_F) \sum_{x\in\mathbf{F}_q} f_{su_x}$$

and because $t^2 = p_F \,\mathrm{id}$,

$$t \sum_{x\in\mathbf{F}_q} f_{su_x} = \chi_2(p_F) f_1.$$

The O_E-module $L_1 := L_o^{I(1)} + tL_o^{I(1)}$ is equal to

$$L_1 = (O_E + \chi_2(p_F)O_E)f_1 \oplus (O_E + \chi_1(p_F)O_E \sum_{x\in\mathbf{F}_q} f_{su_x}.$$

We see that the module $L_o^{I(1)}$ is stable by t if and only if $\chi_1(p_F)$ and $\chi_2(p_F)$ belong to O_E, i.e. are units because their product is a unit. When $\chi_1(p_F)$ and $\chi_2(p_F)$ are units, $L_1 = L_{Y_1}$, $L_o = L_{Y_o}$, $L_o^{I(1)} = L_1$.

As $L_o = RK_of_1$, the zigzag $z(L_o) = K_oL_1$ contains $\chi_2(p_F)L_o$; if the sequence of zigzags $(z^n(L_o))_{n\geq 0}$ is finite, then $\chi_2(p_F) \in O_E$. By the corollary 0.3, if $\chi_2(p_F)$ does not belong to O_E then $\mathrm{ind}_B^G(\chi_1 \otimes \chi_2)$ is not integral.

Suppose $\chi_2(p_F) \in O_E$ and $\chi_1(p_F) \notin O_E$. Then

$$L_1 = O_Ef_1 + \chi_1(p_F)O_E \sum_{c\in\mathbf{F}_q} f_{su_c} = L_{Y_1}, \quad L_{Y_o} = K_oL_1 = L_o + \chi_1(p_F)K_oO_E \sum_{c\in\mathbf{F}_q} f_{su_c}.$$

A system of representatives of $K_o/ZI(1) \simeq GL(2,\mathbf{F}_q)/Z(\mathbf{F}_q)N(\mathbf{F}_q)$ is

$$\{d_\lambda,\ d_\lambda u_x s \text{ for all } x \in \mathbf{F}_q, \lambda \in \mathbf{F}_q^*\}, \quad d_\lambda = \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}.$$

We compute the O_EK_o-module M_o generated by $\sum_{c\in\mathbf{F}_q} f_{su_c}$. As $su_cd_\lambda = sd_\lambda ssu_{\lambda^{-1}c}$ we have

$$d_\lambda f_1 = \eta_1(\lambda)f_1, \quad d_\lambda f_{su_c} = \eta_2(\lambda)f_{su_{\lambda c}}.$$

As $\eta_2(\lambda)$ is a unit, we have

$$O_Ed_\lambda \sum_{c\in\mathbf{F}_q} f_{su_c} = O_E \sum_{c\in\mathbf{F}_q} f_{su_c}.$$

As

$$su_{x^{-1}}s = \begin{pmatrix} -x & 1 \\ 0 & x^{-1} \end{pmatrix} su_x, \text{ if } x \neq 0,$$

we have, if $c \in \mathbf{F}_q^*$,

$$u_xsf_s = f_1, \quad u_xsf_{su_c} = \eta_1(-1)\theta(c)f_{su_{c^{-1}-x}}, \quad \theta := \eta_1\eta_2^{-1},$$

and

$$F_x := u_xs \sum_{c\in\mathbf{F}_q} f_{su_c} = f_1 + \eta_1(-1) \sum_{c\in\mathbf{F}_q} \theta^{-1}(x+c)f_{su_c}$$

where the character θ^{-1} of $\mathbf{F}_q^*$ is extended to a function on $\mathbf{F}_q$ vanishing on 0. We have

$$d_\lambda u_xs \sum_{c\in\mathbf{F}_q} f_{su_c} = \eta_1(\lambda)f_1 + \eta_2(\lambda)\eta_1(-1) \sum_{c\in\mathbf{F}_q} \theta^{-1}(x+c)f_{su_{\lambda c}} = \eta_1(\lambda)F_{\lambda x}.$$

As $\eta_1(\lambda)$ is a unit, we have

$$O_Ed_\lambda u_xs \sum_{c\in\mathbf{F}_q} f_{su_c} = O_EF_{\lambda x}.$$

We deduce that M_o is the O_E-module generated by

$$\sum_{c\in\mathbf{F}_q} f_{su_c},\ (F_x)_{x\in\mathbf{F}_q}.$$

The sum $\sum_{x\in\mathbf{F}_q} F_x$ is $qf_1 + \eta_1(-1)(q-1)\sum_{c\in\mathbf{F}_q} f_{su_c}$ if θ is the trivial character, and qf_1 if θ is not trivial. Hence M_o contains qf_1; beeing K_o-stable, M_o contains qL_o. The zizgag $z(L_o) = K_oL_1 = L_o + \chi_1(p_F)M_o$ contains $q\chi_1(p_F)L_o$. If the sequence of zigzags $(z^n(L_o))_{n\geq 0}$ is finite, then $q\chi_1(p_F) \in O_E$. By the corollary 0.3, if $q\chi_1(p_F)$ does not belong to O_E then $\mathrm{ind}_B^G(\chi_1 \otimes \chi_2)$ is not integral.

Suppose $\chi_1(p_F) \notin O_E$ and $q\chi_1(p_F) \in O_E$. To go further, we need a lemma. For a function $a : \mathbf{F}_q \to \chi_1(p_F)O_E$ and a character $\theta : \mathbf{F}_q^* \to O_E^*$ we consider the function $(a * \theta) : \mathbf{F}_q \to \chi_1(p_F)O_E$ the function defined by

$$(a*\theta)(y) := \sum_{x\in\mathbf{F}_q} a(-x)\theta(y+x) \text{ where } \theta(0) := 0;$$

we says that $a*\theta$ is constant modulo O_E if there exists $z\in E$ such that $(a*\theta)(y)-z\in O_E$ for all $y\in \mathbf{F}_q$.

3.1 Lemma *$\sum_{x\in\mathbf{F}_q} a(x)\in O_E$ if $a*\theta$ is constant modulo O_E.*

Proof. When the character θ is trivial, the function $a*\theta+a=\sum_{c\in\mathbf{F}_q} a(c)$ is constant. If $a*\theta$ is constant modulo O_E, then a is constant modulo O_E and $\sum_{x\in\mathbf{F}_q} a(x)\in q\chi_1(p_F)O_E\subset O_E$.

When the character θ is trivial, we use Fourier transform; we replace E by a finite extension in order to find a non trivial character $\psi:\mathbf{F}_q\to O_E$ to define the Fourier transform

$$\hat f(?)=\sum_{x\in\mathbf{F}_q}\psi(x?)f(x)$$

of a function $f:\mathbf{F}_q\to E$. We denote by $\mathcal{R}$ the space of integral functions $f:\mathbf{F}_q\to O_E$, by $\hat{\mathcal{R}}$ the image of $\mathcal{R}$ by Fourier transform, by $\delta_o\in\mathcal{R}$ the characteristic function of 0 and by $\Delta\in\mathcal{R}$ the constant function $\Delta(?)=1$. The remarkable properties of the Fourier transform give

$$\hat{\hat f}=qf,\ \hat\Delta=q\delta_o,\ \hat\delta_o=\Delta,\ \hat\theta(0)=0,$$

$$\hat\theta(x)\text{ is a Gauss sum and }\hat\theta(x)(\hat{\theta^{-1}})(x)=q\theta(-1)\ \text{ if }\ x\in\mathbf{F}_q^*;$$

the Fourier transform of a convolution product $f*g$ is the product of the Fourier transforms

$$f*g(x)=\sum_{y,z\in\mathbf{F}_q,y+z=x}f(y)g(z),\quad \widehat{f*g}=\hat f\hat g.$$

The lemma says that $\hat a(0)\in O_E$ for all $a\in\chi_1(p_F)\mathcal{R}$ such that $a*\theta\in O_E\Delta+\mathcal{R}$.

By Fourier transform $a*\theta\in O_E\Delta+\mathcal{R}$ is equivalent to $\hat a\hat\theta\ \in O_Eq\delta_o+\hat{\mathcal{R}}$. Multiplying by $(\hat{\theta^{-1}})$ vanishing only at 0, this is equivalent to $q\hat a=q\hat a(0)\delta_o+(\hat{\theta^{-1}})\hat\phi$ for some $\phi\in\mathcal{R}$. The function $b=qa$ belongs to $\mathcal{R}$ because $q\chi_1(p_F)\in O_E$. We have $\hat b=\hat b(0)\delta_o+(\hat{\theta^{-1}})\hat\phi$ and by Fourier transform $b=\lambda\Delta+\theta^{-1}*\phi$ where $b(0)=\lambda+(\theta^{-1}*\phi)(0)$. We have $\lambda\in O_E$ and $\hat a(0)=\lambda$.

We return to the proof of the theorem 0.10. The O_E-module $z(L_o)=L_o+\chi_1(p_F)M_o$ is generated by

$$L_o,\ \chi_1(p_F)\sum_{c\in\mathbf{F}_q}f_{su_c},\ (\chi_1(p_F)F_x)_{x\in\mathbf{F}_q},$$

the O_E-module $(z(L_o))^{I(1)}$ is generated by L_1 and by

$$\sum_{x\in\mathbf{F}_q}a(-x)f_1+\eta_1(-1)(a*\theta^{-1})(0)\sum_{c\in\mathbf{F}_q}f_{su_c}$$

for all functions $a:\mathbf{F}_q\to\chi_1(p_F)O_E$ such that $a*\theta^{-1}$ is constant modulo O_E. As $\eta_1(-1)(a*\theta^{-1})(0)\in\chi_1(p_F)O_E$ and $\sum_{x\in\mathbf{F}_q}a(-x)\in O_E$ by the lemma 3.1, we obtain

$$(z(L_o))^{I(1)}=L_1.$$

This is equivalent to $z(L_1)=L_1$, and also to $L_{Y_o}^{I(1)}=L_{Y_1}$. We summarize what we proved in the following proposition.

3.2 Proposition *1) $L_o=L_{Y_o},\ L_o^{I(1)}=L_{Y_1}$ if and only if $\chi_1(p_F),\chi_2(p_F)$ belongs to O_E^*.*

2) $\mathrm{ind}_B^G(\chi_1\otimes\chi_2)$ is integral if and only if $q\chi_1(p_F),\chi_2(p_F)$ belong to O_E.

3) When $\chi_1(p_F)\notin O_E,\ q\chi_1(p_F)\in O_E$, then $L_{Y_1}=L_o^{I(1)}+tL_o^{I(1)},\ L_{Y_o}^{I(1)}=L_{Y_1}$ if θ is trivial or if O_E contains a p-root of 1.

We prove now the theorem 0.10. By [Vig04 prop. 4.4], the properties b), c) are equivalent. By the proposition 3.2 2) the properties a), c), d) are equivalent and $L_{Y_o}^{I(1)} = L_{Y_1}$. By the corollary 0.2, d) and e) are equivalent. By the lemma 1.4bis, $L_{Y_o} = L^{K(1)}$. As L_{Y_1} generates the $O_E K_o$-module L_{Y_o} which generates the $O_E G$-module L, by transitivity the the $O_E G$-module L is generated by $L_{Y_1} = L^{I(1)}$.

We prove now the remark 0.11 (i). By [Vig04] th.4.10, the natural O_E-integral structure of $\mathrm{ind}_B^G(\chi_1 \otimes \chi_2)$ of functions with values in O_E is $O_E G$-generated by the function with support BI and value 1 at 1, which is contained in L_{Y_o}. As L_{Y_o} embeds in the functions in $\mathrm{ind}_B^G(\chi_1 \otimes \chi_2)$ with values in O_E and generates L, the natural O_E-integral structure is equal to L.

4 k-representations

Proof of the proposition 0.12.

Let $\mu_1 \otimes \mu_2 : T \to k^*$ be a continuous character. There exists a moderately ramified continuous character $\chi_1 \otimes \chi_2 : T \to O_E^*$ lifting $\mu_1 \otimes \mu_2$. Apply the theorem 0.10 and the remark 0.11 (i).

Proof of the proposition 0.13.

Theorem 0.10 and [Vig04] proposition 3.2, théorème 4.2 and proposition 4.4; by [Vig04] §2.4, one may need to take a ramified extension of E with residual field $k = k_E$.

Proof of the proposition 0.14.

By the proposition 0.13 and the Brauer-Nesbitt property, the reductions of the O_E-integral structures of $V := (\mathrm{ind}_B^G(\chi_1 \otimes \chi_2)^{I(1)}$ are simple and isomorphic $H_{k_E}(G, I(1))$-modules. This implies that the reduction of $L^{I(1)}$ is a simple supersingular $H_{k_E}(G, I(1))$-module; it generates the $k_E G$-module $\overline{L}$ because $L^{I(1)}$ generates the $O_E G$-module L.

A k_E-representation of G generated by its $I(1)$-invariants is irreducible if the $I(1)$-invariants is a simple $H_{k_E}(G, I(1))$-module (criterion 4.5 in [Vig04]). This implies the property a).

When $F = \mathbf{Q}_p, p \neq 2$, the following remarkable property

$$M \otimes_{H_k(G,I(1))} \mathrm{ind}_{I(1)}^G 1_k$$

is irreducible of $I(1)$-invariants $M \simeq M \otimes 1$, for any simple $H_k(G, I(1))$-module, well known for complex representations, remains true over a field k of characteristic p [Ollivier], and implies:

4.1 Lemma *A k-representation V of $G = GL(2, \mathbf{Q}_p), p \neq 2$, generated by a simple $H_k(G, I(1))$-submodule M of $V^{I(1)}$ is irreducible and $M = V^{I(1)}$.*

Proof. V is a quotient of $M \otimes_{H_k(G,I(1))} \mathrm{ind}_{I(1)}^G 1_k$.

This implies the property b).

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