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THE PRO-p-IWAHORI HECKE ALGEBRA OF A REDUCTIVE p-ADIC GROUP III (SPHERICAL HECKE ALGEBRAS AND SUPERSINGULAR MODULES)

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Abstract Let R be a large field of characteristic p. We classify the supersingular simple modules of the pro-p-Iwahori Hecke R-algebra $\mathcal{H}$ of a general reductive p-adic group G. We show that the functor of pro-p-Iwahori invariants behaves well when restricted to the representations compactly induced from an irreducible smooth R-representation ρ of a special parahoric subgroup K of G. We give an almost-isomorphism between the center of $\mathcal{H}$ and the center of the spherical Hecke algebra $\mathcal{H}(G, K, \rho)$, and a Satake-type isomorphism for $\mathcal{H}(G, K, \rho)$. This generalizes results obtained by Ollivier for G split and K hyperspecial to G general and K special.

Keywords: group theory and generalizations; number theory

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Contents

1. Introduction

Let p be a prime number, let F be a finite extension of $\mathbb{Q}_p$ or $\mathbb{F}_p((T))$, and let G be the group of rational points of a connected reductive F-group.

1.1.

The smooth representations of G over an algebraically closed field C of characteristic p have been the subject of many investigations in recent years, in the modulo p Langlands program. The pro-p-Iwahori invariant functor $V \mapsto V^{I(1)}$ relates the representations of G to the modules of the pro-p-Iwahori Hecke C-algebra $\mathcal{H} = \mathcal{H}_C(G, I(1))$ studied in [13–15]. The $I(1)$-invariant functor and the theory of $\mathcal{H}$-modules play an increasingly important role in the representation theory of G modulo p. They are the key to the proof of the change of weight in the recent classification of irreducible smooth C-representations of G in terms of supersingular ones (a forthcoming work by Abe *et al.* [1]). The supersingular smooth irreducible C-representations π of G and their $I(1)$-invariant remain mysterious, but the supersingular simple $\mathcal{H}$-modules are classified in this paper, and the supersingularity of $\pi^{I(1)}$ and of π are related. A variant of the modulo p Langlands program seems to exist for $\mathcal{H}$-modules. Grosse-Kloenne [5] constructed a functor from finite-dimensional $\mathcal{H}_C(GL(n, \mathbb{Q}_p), I(1))$-modules to finite-dimensional smooth C-representations of $\mathrm{Gal}_{\mathbb{Q}_p}$, inducing a bijection between the simple supersingular $\mathcal{H}_C(GL(n, F), I(1))$-modules of dimension n and the irreducible smooth C-representations of Gal_F (the absolute Galois group of F) of dimension n as in [9, 14].

In this paper, we prove that the $I(1)$-invariant functor behaves well when restricted to compactly induced representations $\text{c-Ind}_K^G \rho$, where ρ is an irreducible smooth C-representation of a special parahoric subgroup K of G. The vector space $\rho^{I(1)}$ has dimension 1, and the pro-p-Iwahori Hecke C-algebra $\mathfrak{h} = H_C(K, I(1))$ of K acts on $\rho^{I(1)}$ by a character η. The $\mathcal{H}$-module $(\text{c-Ind}_K^G \rho)^{I(1)}$ is isomorphic to $\eta \otimes_{\mathfrak{h}} \mathcal{H}$, and the spherical algebra $\mathrm{End}_{CG}(\text{c-Ind}_K^G \rho)$ is isomorphic to the algebra $\mathrm{End}_{\mathcal{H}}(\eta \otimes_{\mathfrak{h}} \mathcal{H})$. This paper is devoted to the study of the modules $\eta \otimes_{\mathfrak{h}} \mathcal{H}$ and of the spherical Hecke algebras $\mathrm{End}_{\mathcal{H}}(\eta \otimes_{\mathfrak{h}} \mathcal{H})$. In the last section, we transfer our results from $\mathcal{H}$ to the group G using the $I(1)$-invariant functor.

Let ρ be an irreducible smooth C-representation of K, and let η, η_1 be two arbitrary characters of $\mathfrak{h}$. We obtain the following:

(i) *Isomorphisms*

$$(\text{c-Ind}_K^G \rho)^{I(1)} \simeq \rho^{I(1)} \otimes_{\mathfrak{h}} \mathcal{H}, \quad \mathrm{End}_{CG}(\text{c-Ind}_K^G \rho) \simeq \mathrm{End}_{\mathcal{H}}(\rho^{I(1)} \otimes_{\mathfrak{h}} \mathcal{H}).$$

(ii) *A Satake-type isomorphism for the spherical Hecke algebra* $\mathcal{H}(\mathfrak{h}, \eta) = \mathrm{End}_{\mathcal{H}}(\eta \otimes_{\mathfrak{h}} \mathcal{H})$.

(iii) *A basis of the space of intertwiners* $\mathrm{Hom}_{\mathcal{H}}(\eta_1 \otimes_{\mathfrak{h}} \mathcal{H}, \eta \otimes_{\mathfrak{h}} \mathcal{H})$.

(iv) *An almost-isomorphism from the center of $\mathcal{H}$ to the center of $\mathcal{H}(\mathfrak{h}, \eta)$ (an isomorphism between finite index affine subalgebras).*

(v) *The classification of the supersingular simple $\mathcal{H}$-modules.*

When G is split and K hyperspecial, Ollivier proved (i), (ii), (iv) and (v). We follow her method. The alcove walk bases of $\mathcal{H}$ and the product formula [12, 15] allow us to simplify her method and to extend it to G general and K special. Analogs of 2, 3 were proved for G in [6, 7] and 5 for G remains a wide-open question.

In the rest of this introduction, we consider the content of 2, 3, 4, 5.

After [13, 14], a generalization of $\mathcal{H}_C(G, I(1))$ was introduced in [12] when G is split, and in [15] for G general, in order to study it. This is an algebra $\mathcal{H}_R(q_s, c_{\tilde{s}})$ over a commutative ring R with two sets of parameters (q_s), $(c_{\tilde{s}})$. The properties of this algebra are often proved by reduction to $(q_s) = (1)$ (this changes the parameters $(c_{\tilde{s}})$), and transferred to $\mathcal{H}_R(0, c_{\tilde{s}})$ by specialization to $(q_s) = (0)$. The algebra $\mathcal{H}_R(q_s, c_{\tilde{s}})$ contains a natural finite-dimensional subalgebra $\mathfrak{h}_R(q_s, c_{\tilde{s}})$.

In 1.2 and 1.3, we recall the basic properties of $\mathcal{H}_R(q_s, c_{\tilde{s}})$ used in this work and the dictionary between $\mathfrak{h}_R(q_s, c_{\tilde{s}}), \mathcal{H}_R(q_s, c_{\tilde{s}})$ and $\mathcal{H}_R(K, I(1)), \mathcal{H}_R(G, I(1))$ [15, 16]. Theorems 1.2, 1.3, 1.4, and 1.5 are proved for $\mathfrak{h}_R(0, c_{\tilde{s}}), \mathcal{H}_R(0, c_{\tilde{s}})$, and are given in 1.4. They apply to the algebras $\mathcal{H}_R(K, I(1)), \mathcal{H}_R(G, I(1))$ when R has characteristic p.

1.2.

Let $\mathcal{W} = (\Sigma, \Delta, \Omega, \Lambda, \nu, W, Z_k, W(1))$ be data consisting of the following:

(i) a reduced root system Σ of basis Δ associated with the finite Weyl Coxeter system (W_0, S) of an affine Weyl Coxeter system $(W^{\mathrm{aff}}, S^{\mathrm{aff}})$ acting on a real vector space V of dual of basis Δ, with a W_0-invariant scalar product;

(ii) three commutative groups, Ω and Λ finitely generated, and Z_k finite;

(iii) a group $W = W^{\mathrm{aff}} \rtimes \Omega = \Lambda \rtimes W_0$ which is a semi-direct product of subgroups in two different ways, Ω acting on $(W^{\mathrm{aff}}, S^{\mathrm{aff}})$ and W_0 on Λ. The length ℓ and the Bruhat order $\leqslant$ of $(W^{\mathrm{aff}}, S^{\mathrm{aff}})$ extend trivially to $W = W^{\mathrm{aff}} \rtimes \Omega$;

(iv) a W_0-equivariant homomorphism $\nu : \Lambda \to V$ such that the action of W^{aff} on V and the action of Λ on V by translation $v \mapsto v + \nu(\lambda)$ for $\lambda \in \Lambda, v \in V$, extend to an action of W by affine automorphisms permuting the set of affine hyperplanes $\mathfrak{H} = \{\mathrm{Ker}(\alpha + n), \mid \alpha + n \in \Sigma^{\mathrm{aff}} = \Sigma + \mathbb{Z}\}$;

(v) a system of the representatives of W_0 in Λ:

$$\Lambda^+ := \{\mu \in \Lambda \mid \nu(\mu) \in \overline{\mathfrak{D}}^+\},$$

where $\overline{\mathfrak{D}}^+ = \{x \in V \mid 0 \leqslant \alpha(x),\ \alpha \in \Delta\}$ is the dominant closed Weyl chamber;

(vi) an extension $1 \to Z_k \to W(1) \to W \to 1$.

Notation. The inverse image in $W(1)$ of a subset X of W is denoted by $X(1)$, and $\tilde{w}$ denotes an element of $W(1)$ of image $w \in W$. For $c \in R[Z_k]$, the conjugate of c by $\tilde{w}$ depends only on w, and is denoted $w \bullet c := \tilde{w} c \tilde{w}^{-1}$. The dominant Weyl chamber $\mathfrak{D}^+ = \{x \in V \mid 0 < \alpha(x), \alpha \in \Delta\}$ is open. The dominant alcove $\mathfrak{C}^+$ is the connected component $\mathfrak{D}^+ \cap (V - \bigcup_{H \in \mathfrak{H}} H)$ of vertex $0 \in V$. The set $\Sigma^{\mathrm{aff},+}$ of positive affine roots is the set of $\gamma \in \Sigma^{\mathrm{aff}}$ positive on $\mathfrak{C}^+$. The action of W on V defines by functoriality an action of W on Σ^{aff}.

We will often suppose that Λ contains a subgroup Λ_T satisfying the following.

(T1) $\Lambda = \bigsqcup_{y\in Y} \Lambda_T y$ for a finite set Y.

(T2) Λ_T is W_0-stable.

(T3) There exists a central subgroup $\tilde{\Lambda}_T$ of $\Lambda(1)$ normalized by $W_0(1)$ such that the quotient map $\Lambda(1) \to \Lambda$ induces a group isomorphism $\tilde{\Lambda}_T \xrightarrow{\simeq} \Lambda_T$ sending $\tilde{w}\tilde{\mu}\tilde{w}^{-1}$ to $w\mu w^{-1}$ if $\tilde{w} \in W_0(1)$ lifts $w \in W_0$ and $\tilde{\mu} \in \tilde{\Lambda}_T$ lifts $\mu \in \Lambda_T$.

Let $(q_{\tilde{s}}, c_{\tilde{s}})_{\tilde{s}\in S^{\text{aff}}(1)})$ be a set of elements in $R \times R[Z_k]$ satisfying $q_{\tilde{s}'} = q_{\tilde{s}}, c_{\tilde{s}'} = w \bullet c_{\tilde{s}}$ if $\tilde{s}' = \tilde{w}\tilde{s}\tilde{w}^{-1} \in S^{\text{aff}}(1)$, $\tilde{w} \in W(1)$, and $q_{t\tilde{s}} = q_{\tilde{s}}, c_{t\tilde{s}} = tc_{\tilde{s}}$ if $t \in Z_k$. As $q_{\tilde{s}}$ depends only on the image $s \in S^{\text{aff}}$ of $\tilde{s}$, we denote also $q_{\tilde{s}} = q_s$.

There is a unique R-algebra $\mathcal{H} = \mathcal{H}_R(\mathcal{W}, q_s, c_{\tilde{s}})$, free of basis $(T_{\tilde{w}})_{\tilde{w}\in W(1)}$, with product satisfying

(i) the braid relations:

$$T_{\tilde{w}}T_{\tilde{w}'} = T_{\tilde{w}\tilde{w}'}, \quad \text{if } \tilde{w}, \tilde{w}' \in W(1), \ell(w) + \ell(w') = \ell(ww'), \tag{1}$$

allowing one to identify $R[\Omega(1)]$ to a subalgebra of $\mathcal{H}$;

(ii) the quadratic relations:

$$T_{\tilde{s}}T^*_{\tilde{s}} = q_s\tilde{s}^2, \quad \text{if } \tilde{s} \in S^{\text{aff}}(1), T^*_{\tilde{s}} = T_{\tilde{s}} - c_{\tilde{s}}. \tag{2}$$

This is called the Iwahori–Matsumoto presentation of $\mathcal{H}_R(\mathcal{W}, q_s, c_{\tilde{s}})$.

The R-submodule of basis $(T_{\tilde{w}})_{\tilde{w}\in W_0(1)}$ is a finite subalgebra $\mathfrak{h} = \mathfrak{h}_R(\mathcal{W}, q_s, c_{\tilde{s}})$.

The R-submodule of basis $(T_{\tilde{w}})_{\tilde{w}\in W^{\text{aff}}(1)}$ is a subalgebra $\mathcal{H}^{\text{aff}}$. The R-algebra $\mathcal{H}^{\text{aff}}$ is an algebra like $\mathcal{H}$ with Ω trivial, and $\mathcal{H}$ is isomorphic to the twisted tensor product

$$x \otimes y \mapsto xy : \mathcal{H}^{\text{aff}} \otimes^t_{R[Z_k]} R[\Omega(1)] \to \mathcal{H} \tag{3}$$

of its subalgebras $R[\Omega(1)]$ and $\mathcal{H}^{\text{aff}}$. The algebra $\mathcal{H}$ admits an involutive R-automorphism ι, equal to the identity on $R[\Omega(1)]$ and such that [15, Proposition 4.23]

$$\iota(T_{\tilde{s}}) := -T^*_{\tilde{s}} \quad \text{for } s \in S^{\text{aff}}. \tag{4}$$

All the orientations that we consider are spherical [15]. For the orientation o associated to an (open) Weyl chamber $\mathfrak{D}_o$, the o-positive side of the affine hyperplane $\text{Ker}(\alpha + n)$ is the set of $x \in V$ where $\alpha(x) + n > 0$, if $\alpha \in \Sigma$ takes positive values on $\mathfrak{D}_o$. The dominant orientation o, denoted by o^+, is associated to the dominant Weyl chamber $\mathfrak{D}^+$, and the anti-dominant orientation, denoted by o^-, to the anti-dominant Weyl chamber $-\mathfrak{D}^+ = \mathfrak{D}^-$. The orientation associated to the Weyl chamber $w^{-1}(\mathfrak{D}_o)$, $w \in W_0$, is denoted by $o \bullet w$. For $w \in W$ of projection $w_0 \in W_0$, the orientation $o \bullet w_0$ is also denoted by $o \bullet w$. We have $o \bullet \lambda = o$ for $\lambda \in \Lambda$. We set

$$S^{\text{aff}}_o := \{s \in S^{\text{aff}} \mid \mathfrak{C}^+ \text{ is in the } o\text{-positive side of } H_s\}, \quad S_o := S \cap S^{\text{aff}}_o, \tag{5}$$

where H_s is the affine hyperplane of V fixed by s and $\mathfrak{C}^+$ the dominant alcove (Notation). There exists a unique set of bases $(E_o(\tilde{w}))_{\tilde{w}\in W(1)}$ of $\mathcal{H}$, parameterized by the orientations o, satisfying [15, § 5.3]

$$E_o(\tilde{s}) := T_{\tilde{s}} \text{ if } s \in S^{\mathrm{aff}} - S_o^{\mathrm{aff}}, \quad E_o(\tilde{s}) := T_{\tilde{s}}^* \text{ if } s \in S_o^{\mathrm{aff}}, \tag{6}$$

and the product formula, for $\tilde{w}, \tilde{w}' \in W(1)$,

$$E_o(\tilde{w}) E_{o \bullet w}(\tilde{w}') = E_o(\tilde{w}\tilde{w}') \quad \text{if } \ell(w) + \ell(w') = \ell(ww'). \tag{7}$$

In particular, for $\tilde{\lambda}, \tilde{\lambda}' \in \Lambda(1)$,

$$E_o(\tilde{\lambda}) E_o(\tilde{\lambda}') = E_o(\tilde{\lambda}\tilde{\lambda}') \quad \text{if } \nu(\lambda), \nu(\lambda') \text{ belong to a same closed Weyl chamber.} \tag{8}$$

We have $E_o(\lambda) = T_\lambda$ when $\nu(\lambda) \in \overline{\mathcal{D}_o}$.

The basis $(E_o(\tilde{w}))_{\tilde{w} \in W(1)}$ is called an alcove walk basis; the alcove walk bases generalize the integral Bernstein bases defined in [11, 14].

The R-submodule of basis $(E_o(\tilde{\lambda}))_{\tilde{\lambda} \in \Lambda(1)}$ is a subalgebra $\mathcal{A}_o$ of $\mathcal{H}$ containing the subalgebra $\mathcal{A}_o^+$ of basis $(E_o(\tilde{\lambda}))_{\tilde{\lambda} \in \Lambda^+(1)}$, isomorphic to $R[\Lambda^+(1)]$.

If $q_s = 0$ for all $s \in S^{\mathrm{aff}}$, then for $\tilde{w}, \tilde{w}' \in W(1)$ such that $\ell(w) + \ell(w') > \ell(ww')$ we have $E_o(\tilde{w}) E_{o \bullet w}(\tilde{w}') = 0$; in particular, $E_o(\tilde{\lambda}) E_o(\tilde{\lambda}') = 0$ if $\tilde{\lambda}, \tilde{\lambda}' \in \Lambda(1)$, and $\nu(\lambda), \nu(\lambda')$ do not belong to the same closed Weyl chamber.

1.3.

Let F be a local field of finite residue field k with q elements and of characteristic p, and p_F a generator of the maximal ideal of the ring of integers O_F of F. Let G, T, Z, and N be respectively the F-rational points of a connected reductive F-group, a maximal F-split subtorus, its centralizer, and its normalizer. Let $\mathfrak{C}^+$ be an open alcove of the semi-simple apartment of G defined by T, let x_0 be a special vertex of the closed alcove $\overline{\mathfrak{C}}^+$, and let $I, I(1), K$, be respectively the Iwahori subgroup of G fixing $\mathfrak{C}^+$, its pro-p-Sylow subgroup, and the parahoric subgroup of G fixing x_0.

We associate to $G, T, Z, N, I, I(1), K$ the data

$$(\mathcal{W} = (\Sigma, \Delta, \Omega, \Lambda, \nu, W, Z_k, W(1)); (q_s, c_{\tilde{s}})),$$

and a group Λ_T, satisfying the properties given in § 1.2 with $R = \mathbb{Z}$, as follows.

The apartment defined by T identifies with a Euclidean real vector space V. The set S^{aff} of orthogonal reflections with respect to the walls of $\mathfrak{C}^+$ generates an affine Coxeter system $(W^{\mathrm{aff}}, S^{\mathrm{aff}})$, given by a based reduced root system (Σ, Δ). The action of N on the apartment transfers to an action on V. The subgroup Z acts by translations $(z, x) \mapsto x + \nu_Z(z)$, $(z, x) \in Z \times V$, for an homomorphism $\nu_Z : Z \to V$ satisfying $\alpha \circ \nu_Z(t) = -\alpha(t)$ for $t \in T$ and α in the root system Φ of T in G. There is a surjective map $\alpha \mapsto e_\alpha \alpha : \Phi \to \Sigma$, where e_α is a positive integer for all $\alpha \in \Phi$.

Let $T_0 := T \cap K$ (the maximal compact subgroup of T), $Z_0 := K \cap Z$ (the parahoric subgroup of Z), and let $Z_0(1)$ be the pro-p-Sylow subgroup of Z_0. Then

$$\Lambda_T := T/T_0, \quad \Lambda := Z/Z_0, \quad \Lambda(1) := Z/Z_0(1), \quad Z_k := Z_0/Z_0(1),$$
$$W_0 := N/Z, \quad W := N/Z_0, \quad W(1) := N/Z_0(1).$$

The homomorphism ν_Z and the action of N on V are trivial on Z_0. They induce an homomorphism $\nu : \Lambda \to V$ and an action of W on N. The monoid Λ^+ represents the orbits of W_0 in Λ [7, 6.3] and the double cosets $K\backslash G/K$. The groups $W, W(1)$ represent the double cosets $I\backslash G/I, I(1)\backslash G/I(1)$. The group Ω is the W-stabilizer of the alcove $\mathfrak{C}^+$. We denote by $\tilde{w}$ an element of $W(1)$ of image w in W, and we call $\tilde{w}$ a lift of w.

For $s \in S^{\text{aff}}$, let K_s be the parahoric subgroup of G fixing the face of $\overline{\mathfrak{C}^+}$ fixed by s. The quotient of K_s by its pro-p-radical is the group $G_{s,k}$ of rational points of a k-reductive connected group of rank 1. The image of $I(1)$ in $G_{s,k}$ is the group $U_{s,k}$ of rational points of the unipotent radical of a k-Borel subgroup $Z_k U_{s,k}$ of opposite group $Z_k \overline{U}_{s,k}$. It is known that s admits a lift $n_s \in N \cap K_s$ of image in $G_{s,k}$ belonging to the group $\langle U_{s,k}, \overline{U}_{s,k}\rangle$ generated by $U_{s,k} \cup \overline{U}_{s,k}$. The image of n_s in $W(1)$ is called an admissible lift of s. We set $Z_{k,s} = Z_k \cap \langle U_{s,k}, \overline{U}_{s,k}\rangle$.

For $s \in S^{\text{aff}}$, $\tilde{s}$ an admissible lift of s, and $t \in Z_k$, let

$$q_s = [In_sI : I] \text{ is a power of } q, \quad c_s := (q_s - 1)|Z_{k,s}|^{-1} \sum_{z \in Z_{k,s}} z,$$

and $c_{t\tilde{s}} = \sum_{z \in Z_{k,s}} c_{\tilde{s}}(z) tz$, for positive integers $c_{\tilde{s}}(z) = c_{\tilde{s}}(z^{-1})$ of sum $q_s - 1$, constant on each coset modulo $\{xs(x)^{-1} \mid x \in Z_k\}$, and $c_{\tilde{s}} \equiv c_s \mod p$ as in [15, Theorem 2.2].

The cocharacter group $X_*(T)$ of T is isomorphic to Λ_T and embeds in $\Lambda(1)$ by the map $\mu \mapsto \mu(p_F)^{-1} : X_*(T) \to Z$ followed by the quotient maps of Z onto Λ and $\Lambda(1)$. Remembering the sign $-$ in the definition of ν,

$$\mu \in \Lambda_T^+ \Leftrightarrow \alpha(\mu(p_F)) \in O_F \quad \text{for all } \alpha \in \Delta.$$

We identify μ with its image in Λ_T, and $\tilde{\mu}$ denotes its image in $\Lambda(1)$.

For a commutative ring R, the pro-p-Iwahori Hecke R-algebra $\mathcal{H}_R(G, I(1))$ is isomorphic to the algebra $\mathcal{H}_R(q_s, c_{\tilde{s}})$ associated to this data.

The pro-p-Iwahori Hecke R-algebra $\mathcal{H}_R(K, I(1))$ of K is a subalgebra of $\mathcal{H}_R(G, I(1))$ isomorphic to the finite subalgebra $\mathfrak{h}(q_s, c_{\tilde{s}})$ of $\mathcal{H}$.

The Iwahori Hecke R-algebra $\mathcal{H}_R(G, I)$ is an algebra $\mathcal{H}$ associated to the same data except that $Z_k = \{1\}$, $W(1) = W$, $c_s = q_s - 1$.

The group G is split $\Leftrightarrow T = Z \Rightarrow \Lambda_T = \Lambda$. The group G is quasi-split $\Leftrightarrow Z$ is the F-points of an F-torus $\Rightarrow \Lambda(1)$ is commutative. The group G is semi-simple $\Leftrightarrow \operatorname{Ker} \nu$ is finite $\Rightarrow \Omega$ is finite and ν is injective on Λ_T.

The quotient of K by its pro-p-radical $K(1)$ is the group G_k of k-rational points of a connected reductive k-group. The images in G_k of T_0, Z_0, I, and $I(1)$ are the groups T_k, Z_k, B_k, and U_k of k-rational points of a maximal k-split torus, its centralizer (a k-torus), a Borel k-subgroup containing the maximal k-split torus, and its unipotent radical.

The finite Hecke algebras $\mathcal{H}_R(K, I(1))$ and $\mathcal{H}_R(G_k, U_k)$ are isomorphic.

The condition $q_s = 0$ for all $s \in S^{\text{aff}}$ means that the characteristic of R is p. Then,

$$c_{t\tilde{s}} = -|Z_{k,s}|^{-1} \sum_{z \in Z_{k,s}} tz,$$

and the irreducible smooth R-representations ρ of K are trivial on $K(1)$; they identify with the irreducible R-representations of G_k, in bijection with the characters of $\mathcal{H}_R(G_k, U_k)$ by the U_k-invariant functor $\rho \mapsto \rho^{U_k}$ for R as in 1.4.

1.4.

For the remainder of this article, unless otherwise specified, we are in the setting of § 1.2 with $q_s = 0$ for all $s \in S^{\text{aff}}$, and R is a field containing a root of unity of order the exponent of Z_k.

Notation. We denote by $\hat{Z}_k$ the group of R-characters of Z_k. For a character $\chi \in \hat{Z}_k$, a character η of $\mathfrak{h}$, and a character Ξ of $\mathcal{H}^{\text{aff}}$, we set

$$S_\chi^{\text{aff}} := \{s \in S^{\text{aff}} \mid \chi(c_{\tilde{s}}) \neq 0\}, \quad S_\chi := S_\chi^{\text{aff}} \cap S, \tag{9}$$

$$S_\eta := \{s \in S \mid \eta(T_{\tilde{s}}) \neq 0\}, \quad S_\Xi^{\text{aff}} := \{s \in S^{\text{aff}} \mid \Xi(T_{\tilde{s}}) \neq 0\}. \tag{10}$$

These sets are independent of the choice of the lift $\tilde{s}$ of s. For $(\tilde{w}, \chi) \in W(1) \times \hat{Z}_k$ we denote by $\chi^w \in \hat{Z}_k$ the character $\chi^w(t) = \chi(\tilde{w} t \tilde{w}^{-1})$ for $t \in Z_k$. The subgroup generated by a subset X of a group is denoted by $\langle X \rangle$. For $\lambda \in \Lambda$ we set

$$\Delta_\lambda := \{\alpha \in \Delta \mid \alpha \circ \nu(\lambda) = 0\}, \quad S_\lambda := \{s_\alpha \mid \alpha \in \Delta_\lambda\}. \tag{11}$$

We recall from § 1.2 the R-algebra $\mathfrak{h}$ associated to the finite Coxeter system (W_0, S) and the extension $1 \to Z_k \to W_0(1) \to W_0 \to 1$, of basis $(T_{\tilde{w}})_{\tilde{w} \in W_0(1)}$ satisfying the braid relations and the quadratic relations $T_{\tilde{s}}(T_{\tilde{s}} - c_{\tilde{s}}) = 0$ for $\tilde{s} \in S(1)$.

Theorem 1.1 (The characters of $\mathfrak{h}$)**.** (a) *The characters η of $\mathfrak{h}$ are in bijection with the pairs (χ, J), where $\chi \in \hat{Z}_k$ and $J \subset S_\chi$, $\chi = \eta|_{Z_k}$, and $J = S_\eta$.*

(b) *For any η, there exists an orientation o such that the equivalent properties $S_\eta = S_\chi \cap S_o \Leftrightarrow \eta(E_o(\tilde{s})) = 0$, for all $s \in S$, hold true. We set $\chi_o := \eta$.*

(c) *For two characters η_1, η of $\mathfrak{h}$, there exists an orientation o such that $\eta_1 = (\chi_1)_o$, $\eta = \chi_o$ if and only if*

$$S_\eta \cap S_{\chi_1} = S_{\eta_1} \cap S_\chi.$$

For a reduced decomposition of $\tilde{w} = \tilde{s}_1 \dots \tilde{s}_r$ of $W(1)$, the element $c_{\tilde{w}} = c_{\tilde{s}_1} \dots c_{\tilde{s}_r}$ of $R[Z_k]$ does not depend on the choice of the reduced decomposition [15, Propositions 4.13(ii) and 4.22].

Theorem 1.2 (A basis of the intertwiners)**.** *Let η_1, η be two characters of $\mathfrak{h}$ of restrictions χ_1, χ to Z_k.*

(a) *η_1 is contained in $\eta \otimes_{\mathfrak{h}} \mathcal{H}$ (is a submodule) if and only if*

$$\chi_1 = \chi^\lambda, \quad S_{\eta_1} \cap S_\lambda = S_\eta \cap S_\lambda, \quad \textit{for some } \lambda \in \Lambda^+.$$

(b) *For $\lambda \in \Lambda^+$ satisfying (a), there exists a non-zero $\mathcal{H}$-intertwiner*

$$\Phi_{\tilde{\lambda}} : 1 \otimes 1 \mapsto 1 \otimes \mathcal{E}_{\tilde{\lambda}} : \eta_1 \otimes_{\mathfrak{h}} \mathcal{H} \to \eta \otimes_{\mathfrak{h}} \mathcal{H}, \quad \mathcal{E}_{\tilde{\lambda}} := \sum_{w_0 \in Y_\lambda} \chi_1(c_{\tilde{w}_0})^{-1} \otimes T_{\tilde{\lambda}\tilde{w}_0},$$

where $Y_\lambda = \{w_0 \in \langle S_{\chi_1} - S_{\eta_1} \rangle \mid \chi_1^{w_0} = \chi_1, \ell(\lambda w_0) = \ell(\lambda) - \ell(w_0)\}$, and $\tilde{w}_0$ is a lift of w_0; note that $\chi_1(c_{\tilde{w}_0})^{-1} \otimes T_{\tilde{\lambda}\tilde{w}_0}$ does not depend on the choice of the lift.
$(\Phi_{\tilde{\lambda}})$, for $\lambda \in \Lambda^+$ satisfying (a), is a basis of $\mathrm{Hom}_{\mathcal{H}}(\eta_1 \otimes_{\mathfrak{h}} \mathcal{H}, \eta \otimes_{\mathfrak{h}} \mathcal{H})$.

(c) *If o satisfies (d) and $\lambda \in \Lambda^+$ satisfies (a), there exists a non-zero $\mathcal{H}$-intertwiner*

$$\Phi_{o,\tilde{\lambda}} : 1 \otimes 1 \mapsto 1 \otimes E_o(\tilde{\lambda}) : (\chi_1)_o \otimes_{\mathfrak{h}} \mathcal{H} \to \chi_o \otimes_{\mathfrak{h}} \mathcal{H}.$$

$(\Phi_{o,\tilde{\lambda}})$*, for* $\lambda \in \Lambda^+$ *satisfying (a), is a basis of* $\mathrm{Hom}_{\mathcal{H}}((\chi_1)_o \otimes_{\mathfrak{h}} \mathcal{H}, \chi_o \otimes_{\mathfrak{h}} \mathcal{H})$.

We note that $\chi_1(c_{\tilde{w}_0})^{-1} \otimes T_{\tilde{\lambda}\tilde{w}_0} \in \eta \otimes_{\mathfrak{h}} \mathcal{H}$ does not depend on the choice of the lift $\tilde{w}_0$ of $w_0 \in Y_\lambda$. We set

$$\Lambda_\chi := \{\lambda \in \Lambda \mid \chi^\lambda = \chi\}, \quad \text{resp. } \Lambda_\chi^+ := \Lambda^+ \cap \Lambda_\chi. \tag{12}$$

The idempotent $e_\chi := |Z_k|^{-1} \sum_{t \in Z_k} \chi(t)^{-1} t$ of $R[Z_k]$ is central in $R[\Lambda_\chi(1)]$, and the R-linear map

$$\chi \otimes_{R[Z_k]} R[\Lambda_\chi(1)] \to e_\chi R[\Lambda_\chi(1)] \quad 1 \otimes \tilde{\lambda} \mapsto e_\chi \tilde{\lambda} \quad (\lambda \in \Lambda_\chi) \tag{13}$$

is an isomomorphism. Any R-algebra A with a basis $(a_{\tilde{\lambda}})_{\lambda \in \Lambda_\chi^+}$ satisfying

$$a_{\tilde{\lambda}} a_{\tilde{\lambda}'} = \chi(t) a_{\tilde{\lambda}''} \quad \text{for } \lambda, \lambda', \lambda'' \in \Lambda_\chi^+, t \in Z_k, \tilde{\lambda}\tilde{\lambda}' = t\tilde{\lambda}'', \tag{14}$$

is canonically isomorphic to the algebra $e_\chi R[\Lambda_\chi^+(1)]$ with its natural basis $(e_\chi \tilde{\lambda})_{\lambda \in \Lambda_\chi^+}$.

For an orientation o, the R-submodule $\mathcal{A}_{o,\chi}^+$ of basis $(E_o(\tilde{\lambda}))_{\tilde{\lambda} \in \Lambda_\chi^+(1)}$ is a subalgebra of $\mathcal{H}$. The algebra $\chi \otimes_{R[Z_k]} \mathcal{A}_{o,\chi}^+$ of basis $(1 \otimes E_o(\tilde{\lambda}))_{\lambda \in \Lambda_\chi^+}$ is an R-algebra with a basis satisfying (14).

A spherical Hecke algebra is the algebra of $\mathcal{H}$-intertwiners of a right $\mathcal{H}$-module $\eta \otimes_{\mathfrak{h}} \mathcal{H}$ induced from a character η of $\mathfrak{h}$, by analogy with the reductive p-adic groups

$$\mathcal{H}(\mathfrak{h}, \eta) := \mathrm{End}_{\mathcal{H}}(\eta \otimes_{\mathfrak{h}} \mathcal{H}).$$

Theorem 1.2 with $\eta_1 = \eta$ becomes the following.

Theorem 1.3 (A Satake-type isomorphism for the spherical algebra)**.** (a) *A basis of the spherical Hecke algebra* $\mathcal{H}(\mathfrak{h}, \eta)$ *is* $(\Phi_{\tilde{\lambda}})_{\lambda \in \Lambda_\chi^+}$*, where*

$$\Phi_{\tilde{\lambda}} : 1 \otimes 1 \mapsto 1 \otimes \mathcal{E}_{\tilde{\lambda}} : \eta \otimes_{\mathfrak{h}} \mathcal{H} \to \eta \otimes_{\mathfrak{h}} \mathcal{H}, \quad \mathcal{E}_{\tilde{\lambda}} := \sum_{w_0 \in Y_\lambda} \chi(c_{\tilde{w}_0}) \otimes T_{\tilde{\lambda}\tilde{w}_0},$$

$$Y_\lambda = \{w_0 \in \langle S_\chi - S_\eta \rangle \mid \chi^{w_0} = \chi, \ell(\lambda w_0) = \ell(\lambda) - \ell(w_0)\}.$$

(b) *Let o be an orientation such that $\eta = \chi_o$. For $\lambda \in \Lambda_\chi^+$, there exists an injective $\mathfrak{h}$-intertwiner*

$$\Phi_{o,\tilde{\lambda}} : 1 \otimes 1 \mapsto 1 \otimes E_o(\tilde{\lambda}) : \eta \otimes_{\mathfrak{h}} \mathcal{H} \to \eta \otimes_{\mathfrak{h}} \mathcal{H}.$$

$(\Phi_{o,\tilde{\lambda}})_{\lambda \in \Lambda_\chi^+}$ *is a basis of the spherical Hecke algebra* $\mathcal{H}(\mathfrak{h}, \eta)$ *satisfying (14), inducing an isomorphism*

$$\mathcal{H}(\mathfrak{h}, \eta) \simeq e_\chi R[\Lambda_\chi^+(1)].$$

We suppose now that Λ_T exists. The center $\mathcal{Z}$ of $\mathcal{H}$ is the algebra $\mathcal{A}_o^{W(1)}$ of $W(1)$-invariants of $\mathcal{A}_o$, and is a free R-module of basis

$$E(\tilde{C}) = \sum_{\tilde{\lambda} \in \tilde{C}} E_o(\tilde{\lambda}) \tag{15}$$

($E(\tilde{C})$ is independent of the choice of o) for all finite conjugacy classes $\tilde{C}$ of $W(1)$. We denote by $\tilde{C}(\mu)$ the $W(1)$-conjugacy class of $\tilde{\mu}$ for $\mu \in \Lambda_T^+$. The R-subspace of basis $(E(\tilde{C}(\mu)))_{\mu\in\Lambda_T^+}$ is a central subalgebra $\mathcal{Z}_T$ of $\mathcal{H}$ which has better properties than $\mathcal{Z}$.

A central element $x \in \mathcal{Z}$ induces naturally a $\mathcal{H}$-intertwiner of $\eta \otimes_{\mathfrak{h}} \mathcal{H}$:

$$\Phi_x : 1 \otimes h \mapsto 1 \otimes xh = 1 \otimes hx \quad \text{for } h \in \mathcal{H}. \tag{16}$$

It is straightforward to check that Φ_x belongs to the center $\mathcal{Z}(\eta, \mathfrak{h})$ of $\mathcal{H}(\eta, \mathfrak{h})$. The R-subspace of basis $(\Phi_{E(\tilde{C}(\mu))})_{\mu\in\Lambda_T^+}$ is a central subalgebra $\mathcal{Z}(\eta, \mathcal{H})_T$ of the spherical algebra $\mathcal{H}(\eta, \mathfrak{h})$.

Theorem 1.4 (Almost-isomorphism between the centers of $\mathcal{H}$ and $\mathcal{H}(\eta, \mathfrak{h})$). *We suppose that Λ_T exists. Let η be a character of $\mathfrak{h}$.*

(a) *$\mathcal{Z}_T$ is a finitely generated central R-subalgebra of $\mathcal{H}$, and $\mathcal{H}$ is a finitely generated $\mathcal{Z}_T$-module. This is also true for $(\mathcal{Z}(\eta, \mathcal{H})_T, \mathcal{H}(\eta, \mathfrak{h}))$ instead of $(\mathcal{Z}_T, \mathcal{H})$.*

(b) *$\Phi_{E(\tilde{C}(\mu))} = \Phi_{o,\tilde{\mu}}$ for $\mu \in \Lambda_T^+$ and any orientation o such that $\eta = \chi_o$.*
The linear map $\tilde{\mu} \mapsto \Phi_{E(\tilde{C}(\mu))} : R[\tilde{\Lambda}_T^+] \to \mathcal{Z}(\eta, \mathcal{H})_T$ is an algebra isomorphism.

(c) *The map $x \mapsto \Phi_x : \mathcal{Z} \to \mathcal{Z}(\eta, \mathcal{H})$ restricts to an isomorphism $\mathcal{Z}_T \to \mathcal{Z}(\eta, \mathcal{H})_T$.*

We prove (a) over any commutative ring R.

We transfer these results to the group G. The spherical Hecke algebra $\mathcal{H}_R(G, K, \rho) = \mathrm{End}_{RG}\, \text{c-Ind}_K^G\, \rho$ of an irreducible smooth representation ρ of K with $\mathcal{H}_R(K, I(1))$ acting by η on $\rho^{I(1)}$ is isomorphic to $\mathcal{H}(\eta, \mathfrak{h})$ by the pro-p-Iwahori invariant functor. We denote by $\mathcal{Z}_R(G, K, \rho)_T$ the algebra corresponding to $\mathcal{Z}(\eta, \mathcal{H})_T$. We denote by $\mathcal{H}_R(Z^+, Z_0, \chi)$ the R-algebra of elements in the Hecke algebra $\mathcal{H}_R(Z^+, Z_0, \chi)$ with support contained in the monoid Z^+ of $z \in Z$ with $\nu_Z(z)$ dominant.

From Theorem 1.3 we obtain an algebra isomomorphism

$$\mathcal{S}_o : \mathcal{H}_R(G, K, \rho) \to \mathcal{H}_R(Z^+, Z_0, \chi) \tag{17}$$

for each orientation o such that $\eta = \chi_o$. This isomorphism restricts to an isomorphism, independent of the choice of o,

$$\mathcal{S}_T : \mathcal{Z}_R(G, K, \rho)_T \to \mathcal{H}_R(T^+, T_0, \chi). \tag{18}$$

Let π be a smooth R-representation of G such that $\mathrm{Hom}_R(\rho, \pi)$ contains a $\mathcal{Z}_R(G, K, \rho)_T$-eigenvector A of eigenvalue ξ, seen as an homomorphism $\tilde{\Lambda}_T^+ \to R$ (Theorem 1.4). From Theorem 1.4, for $v \in \rho^{I(1)}$ non-zero and $\mu \in \Lambda_T^+$,

$$\xi(\tilde{\mu})A(v) = A(v)E_o(\tilde{\mu}) = A(v)E(\tilde{C}(\mu)).$$

Theorem 1.5 (Supersingularity in G and in $\mathcal{H}$). *The eigenvalue ξ of the $\mathcal{Z}_R(G, K, \rho)_T$-eigenvector $A \in \mathrm{Hom}_R(\rho, \pi)$ is supersingular if and only if the submodule $A(v)\mathcal{H}$ of $\pi^{I(1)}$ is supersingular.*

We recall that an homomorphism $\tilde{\Lambda}_T^+ \to R$ is called supersingular if it vanishes on the non-invertible elements, and that a simple right $\mathcal{H}$-module M is called supersingular if $ME(\tilde{C}) = 0$ for all finite conjugacy classes $\tilde{C}$ in $W(1)$ with positive length [13, Definition 1]. This is equivalent to $ME(\tilde{C}(\mu)) = 0$ for all non-invertible $\mu \in \tilde{\Lambda}_T^+$.

In a forthcoming article, we will study the parabolic induction for $\mathcal{H}$-modules; we hope to prove that the isomorphism $\mathcal{S}_o$ (17) is the Satake isomorphism of [7] for a good choice of o such that $\eta = \chi_o$ (this was proved by Ollivier [10, Theorem 5.5]), when G is split with a simply connected derived group, and K is hyperspecial; as $Z = T$, we have $\mathcal{S}_o = \mathcal{S}_T$, and that an irreducible smooth admissible representation π is supersingular if and only if $\pi^{I(1)}$ contains a supersingular module (this was proved by Ollivier for $G = GL(n, F)$ and $PGL(n, F)$ [11, Theorem 5.26]).

Finally, we classify the supersingular simple finite-dimensional $\mathcal{H}$-modules (proved by Ollivier when G is split, and K is hyperspecial [11, Corollary 5.15]).

For a character Ξ of $\mathcal{H}^{\mathrm{aff}}$, the R-subalgebra $\mathcal{H}_\Xi$ of $\mathcal{H}$ generated by $\mathcal{H}^{\mathrm{aff}}$ and the $\Omega(1)$-fixator of Ξ,

$$\Omega(1)_\Xi := \{u \in \Omega(1) \mid \Xi(uhu^{-1}) = \Xi(h) \text{ for } h \in \mathcal{H}^{\mathrm{aff}}\},$$

is identified by (3) with the twisted tensor product $\mathcal{H}^{\mathrm{aff}} \otimes_{R[Z_k]} R[\Omega(1)_\Xi] \to \mathcal{H}_\Xi$. For a simple finite-dimensional R-representation σ of $\Omega(1)_\Xi$ equal to Ξ on Z_k, let

$$M(\Xi, \sigma) := (\Xi \otimes \sigma) \otimes_{\mathcal{H}_\Xi} \mathcal{H} \tag{19}$$

be the right $\mathcal{H}$-module induced from the right $\mathcal{H}_\Xi$-module $\Xi \otimes \sigma$. The induced module $M(\Xi, \sigma)$ is finite dimensional. Two pairs $(\Xi_1, \sigma_1), (\Xi_2, \sigma_2)$ are called conjugate by an element $u \in \Omega(1)$ if

$$\Xi_1(uhu^{-1}) = \Xi_2(h), \sigma_1(uvu^{-1}) = \sigma_2(v) \quad \text{for } (h, v) \in \mathcal{H}^{\mathrm{aff}} \times u^{-1}\Omega_\Xi(1)u.$$

The affine Coxeter system $(W^{\mathrm{aff}}, S^{\mathrm{aff}})$ is the direct product of the irreducible affine Coxeter systems $(W_i^{\mathrm{aff}}, S_i^{\mathrm{aff}})_{1 \leqslant i \leqslant r}$ associated to the irreducible components $(\Sigma_i, \Delta_i)_{1 \leqslant i \leqslant r}$ of the based reduced root system (Σ, Δ). The R-submodule of basis $(T_{\tilde{w}})_{\tilde{w}_i \in W_i^{\mathrm{aff}}(1)}$ is a subalgebra $\mathcal{H}_i^{\mathrm{aff}}$ of $\mathcal{H}^{\mathrm{aff}}$. The algebras $\mathcal{H}_i^{\mathrm{aff}}$ are called the irreducible components of $\mathcal{H}^{\mathrm{aff}}$.

Theorem 1.6 (Supersingular simple modules)**.** (a) *The characters Ξ of $\mathcal{H}^{\mathrm{aff}}$ are in bijection with the pairs (χ, J), where $\chi \in \hat{Z}_k$ and $J \subset S_\chi^{\mathrm{aff}}$, $\chi = \Xi|_{Z_k}$, and $J = S_\Xi^{\mathrm{aff}}$ (10). When $S_\Xi^{\mathrm{aff}} = S^{\mathrm{aff}}$, Ξ is called a sign character, and the character $\Xi \circ \iota$ (4) is called a trivial character.*

(b) *A character Ξ of $\mathcal{H}^{\mathrm{aff}}$ is supersingular if and only if it is not a sign or trivial character on each irreducible component of $\mathcal{H}^{\mathrm{aff}}$.*

(c) *A finite-dimensional right $\mathcal{H}$-module is supersingular if and only if it is isomorphic to $M(\Xi, \sigma)$, where Ξ is a supersingular character of $\mathcal{H}^{\mathrm{aff}}$ and σ is a simple finite-dimensional R-representation σ of $\Omega(1)_\Xi$ equal to Ξ on Z_k.*

(d) *$M(\Xi_1, \sigma_1) \simeq M(\Xi_2, \sigma_2)$ if and only if $(\Xi_1, \sigma_1), (\Xi_2, \sigma_2)$ are $\Omega(1)$-conjugate.*

2. The characters of $\mathfrak{h}$ and $\mathcal{H}^{\mathrm{aff}}$

Proposition 2.1. *A simple $\mathfrak{h}$-module has dimension* 1.

Proof. The finite-dimensional R-algebra $\mathfrak{h}$ is generated by Z_k and $T_{\tilde{s}}$ for all $s \in S$. By the hypothesis on R (§ 1.4), a right simple $\mathfrak{h}$-module is finite dimensional and contains an eigenvector v of Z_k. Following the argument of [4, Theorem 6.10], we choose w in the finite group W_0 of maximal length such that $vT_{\tilde{w}} \neq 0$, and we show that $RvT_{\tilde{w}}$ is $\mathfrak{h}$-stable.

$RvT_{\tilde{w}}$ is stable by T_t, because $T_{\tilde{w}}T_t = (w \bullet t)T_{\tilde{w}}$ for $t \in Z_k$.

$RvT_{\tilde{w}}$ is stable by $T_{\tilde{s}}$, because

- if $\ell(ws) = \ell(w) + 1$, $vT_{\tilde{w}}T_{\tilde{s}} = vT_{w\tilde{s}}$ and by the hypothesis on w, $vT_{\tilde{w}\tilde{s}} = 0$;
- if $\ell(ws) = \ell(w) - 1$, $T_{\tilde{w}}T_{\tilde{s}} = T_{\tilde{w}\tilde{s}^{-1}}T_{\tilde{s}}^2 = T_{\tilde{w}\tilde{s}^{-1}}c_{\tilde{s}}T_{\tilde{s}} = T_{w\tilde{s}^{-1}}T_{\tilde{s}}c_{\tilde{s}} = (w \bullet c_{\tilde{s}})T_{\tilde{w}}$. We used that $T_{\tilde{s}}$ and $c_{\tilde{s}}$ commute. $\square$

Proposition 2.2. *The characters η of $\mathfrak{h}$ are in bijection with the pairs (χ, J), where $\chi \in \hat{Z}_k$ and $J \subset S_\chi$ (9), by the recipe*

$$\eta|_{Z_k} = \chi, \quad S_\eta = \{s \in S \mid \eta(T_{\tilde{s}}) \neq 0\} = J.$$

We have $\eta(T_{\tilde{s}}) = \chi(c_{\tilde{s}})$ if $s \in J$.

The characters Ξ of $\mathcal{H}^{\mathrm{aff}}$ are in bijection with the pairs (χ, J), where $\chi \in \hat{Z}_k$ and $J \subset S_\chi^{\mathrm{aff}}$, by the recipe

$$\Xi|_{Z_k} = \chi, \quad S_\Xi^{\mathrm{aff}} = \{s \in S^{\mathrm{aff}} \mid \Xi(T_{\tilde{s}}) \neq 0\} = J.$$

We have $\Xi(T_{\tilde{s}}) = \chi(c_{\tilde{s}})$ if $s \in J$.

The set J is independent of the choice of the lift $\tilde{s}$ of s. We call (χ, J) the parameters of the character. The restriction to $\mathfrak{h}$ of the character Ξ of $\mathcal{H}^{\mathrm{aff}}$ with parameters $(\chi, S_\Xi^{\mathrm{aff}})$ is the character of parameters $(\chi, S_\Xi^{\mathrm{aff}} \cap S)$.

Proof. The proposition follows from the Iwahori–Matsumoto presentation in both cases. If $\eta|_{Z_k} = \chi$, we have

$$\eta(T_{\tilde{s}})(\eta(T_{\tilde{s}}) - \chi(c_{\tilde{s}})) = 0$$

for $s \in S$. We can replace η, S by Ξ, S^{aff}. $\square$

The involutive automorphism ι of $\mathcal{H}$ (4) has the property for $s \in S$ that

$$\eta(T_{\tilde{s}}) = 0 \Leftrightarrow \eta \circ \iota(T_{\tilde{s}}) = \eta(c_{\tilde{s}}).$$

The same holds for (Ξ, S^{aff}) instead of (η, S).

Lemma 2.3. *Let η be a character with parameters (χ, S_η) of $\mathfrak{h}$. Then $\eta \circ \iota$ is a character of $\mathfrak{h}$ with parameters $(\chi, S_\chi - S_\eta)$. We can replace $\eta, S, \mathfrak{h}$ by $\Xi, S^{\mathrm{aff}}, \mathcal{H}^{\mathrm{aff}}$.*

Let o be an orientation. We recall the notation (5), (6), (9), (10).

Lemma 2.4. *Let η be a character of $\mathfrak{h}$ with parameters (χ, S_η). Then $S_\eta = S_\chi \cap S_o \Leftrightarrow \eta(E_o(\tilde{s})) = 0$ for all $s \in S$. When this holds true, we denote $\eta = \chi_o$.*

We can replace $(\eta, \mathfrak{h}, S, \chi_o)$ by $(\Xi, \mathcal{H}^{\mathrm{aff}}, S^{\mathrm{aff}}, \chi_o^{\mathrm{aff}})$.

Proof. We compare the values of $E_o(\tilde{s})$ and $\eta(T_{\tilde{s}})$ for $s \in S$:

$$\begin{aligned} E_o(\tilde{s}) &= T_{\tilde{s}} \Leftrightarrow s \in S - S_o, \\ &= T_{\tilde{s}} - c_{\tilde{s}} \Leftrightarrow s \in S_o, \\ \eta(T_{\tilde{s}}) &= 0 \Leftrightarrow s \in S - S_\eta, \\ &= \chi(c_{\tilde{s}}) \neq 0 \text{ if } s \in S_\eta. \end{aligned}$$

We see that

if $s \in S - S_\chi$, then $\eta(E_o(\tilde{s})) = \eta(T_{\tilde{s}}) = \chi(c_{\tilde{s}}) = 0$;

if $s \in S_\chi - S_\eta$, then $\eta(E_o(\tilde{s})) = \eta(T_{\tilde{s}}) = 0 \Leftrightarrow s \notin (S_\chi - S_\eta) \cap S_o$;

if $s \in S_\eta$, then $\eta(E_o(\tilde{s})) = 0 \Leftrightarrow s \in S_\eta \cap S_o$.

Hence we obtain the lemma for η. The proof is the same for Ξ. □

Example 2.5. For the dominant orientation o^+, $S^{\mathrm{aff}}_{o^+} = S$, and the parameters of χ_{o^+} and of $\chi^{\mathrm{aff}}_{o^+}$ are (χ, S_χ).

For the anti-dominant orientation o^-, $S^{\mathrm{aff}}_{o^-} = S^{\mathrm{aff}} - S$, and the parameters of χ_{o^-} are $(\chi, \emptyset)$, while those of $\chi^{\mathrm{aff}}_{o^-}$ are $(\chi, S^{\mathrm{aff}}_\chi - S_\chi)$.

Lemma 2.6. (i) *Any subset of S is equal to S_o for some orientation o.*

A character η of $\mathfrak{h}$ of restriction χ to Z_k is equal to χ_o for some orientation o, and

$$\eta = \chi_o \Leftrightarrow S_o \cap S_\chi = S_\eta.$$

(ii) *Two R-characters η_1, η of $\mathfrak{h}$ of parameters $(\chi_1, S_{\eta_1}), (\chi, S_\eta)$ are equal to $(\chi_1)_o, \chi_o$ for some orientation o if and only if*

$$S_{\eta_1} \cap S_\chi = S_\eta \cap S_{\chi_1}.$$

In this case, $\eta_1 = (\chi_1)_o$ and $\eta = \chi_o \Leftrightarrow S_o \cap (S_{\chi_1} \cup S_\chi) = S_{\eta_1} \cup S_\eta$.

Proof. (i) Let $w_o \in W_0$. For $\alpha \in \Delta$, the root in $\{\alpha, -\alpha\}$ positive on $w_o^{-1}(\mathfrak{D}^+)$ is equal to $\alpha_o = \alpha$ if $w_o(\alpha) > 0$ and $\alpha_o = -\alpha$ if $w_o(\alpha) < 0$; hence

$$s_\alpha \in S_o \Leftrightarrow w_o(\alpha) > 0.$$

For a subset X of S, we have $X = S_o$ for the orientation $o = o^+ \bullet w_o$ of Weyl chamber $\mathfrak{D}_o = w_o^{-1}(\mathfrak{D}^+)$, where w_o is the longest element of the group $\langle S - X \rangle$ ($w = 1$ if $S = X$).

(ii) $S_o \cap S_{\chi_1} = S_{\eta_1}$ and $S_o \cap S_\chi = S_\eta$ imply that $S_o \cap S_{\chi_1} \cap S_\chi = S_{\eta_1} \cap S_\chi = S_\eta \cap S_{\chi_1}$. If $S_{\eta_1} \cap S_\chi = S_\eta \cap S_{\chi_1}$, then $S_o \cap (S_{\chi_1} \cup S_\chi) = S_{\eta_1} \cup S_\eta$ implies that $S_o \cap S_{\chi_1} = S_{\eta_1}$ and $S_o \cap S_\chi = S_\eta$. □

Definition 2.7. A character of $\mathfrak{h}$ not vanishing on $T_{\tilde{s}}$ for all $s \in S$ is called a twisted sign character, and its image by the involution ι is called a twisted trivial character.

We make the same definition for $\mathcal{H}^{\mathrm{aff}}$, S^{aff} replacing $\mathfrak{h}$, S.

The twisted sign characters η are never 0 on $T_{\tilde{w}}$ for $w \in W_0$. The algebra $\mathfrak{h}$ admits no twisted sign or trivial characters when $c_{\tilde{s}} = 0$ for some $s \in S$. They are equal to χ_{o^+}, where $\chi \in \hat{Z}_k$ satisfies $S_\chi = S$.

The twisted trivial characters η vanish on $T_{\tilde{w}}$ for all $w \in W_0$. They are equal to χ_{o^-}, where $\chi \in \hat{Z}_k$ satisfies $S_\chi = S$.

The same remarks can be made for $\mathcal{H}^{\mathrm{aff}}$, $(W^{\mathrm{aff}}, S^{\mathrm{aff}})$ replacing $\mathfrak{h}$, (W_0, S).

3. Distinguished representatives of $W_0\backslash W$

We recall a well-known lemma for the affine Coxeter system $(W^{\mathrm{aff}}, S^{\mathrm{aff}})$ extended to the group $W = W^{\mathrm{aff}} \rtimes \Omega$.

For $s \in S^{\mathrm{aff}}$, we denote by A_s the unique positive affine root such that $s(A_s)$ is negative. We have $s(A_s) = -A_s$ [8, 1.3.11]. When $s \in S$ we write $A_s = \alpha_s$.

Lemma 3.1. (1) *For $(s, w) \in S^{\mathrm{aff}} \times W$, we have*

$$\ell(ws) = 1 + \ell(w) \Leftrightarrow w(\alpha_s) > 0.$$

(2) *For $v \leqslant w$ in W and $s \in S^{\mathrm{aff}}$, we have*

(a) *either $sv \leqslant w$ or $sv \leqslant sw$;*

(b) *either $v \leqslant sw$ or $sv \leqslant sw$.*

Proof. We recall that $W = W^{\mathrm{aff}} \rtimes \Omega$. Let $(s, u, w) \in S^{\mathrm{aff}} \times \Omega \times W^{\mathrm{aff}}$.

(1) We have $\ell(uws) = \ell(ws), \ell(uw) = \ell(w)$, and $\ell(ws) = \ell(w) + 1 \Leftrightarrow w(\alpha_s) > 0$ [8, 1.13.13]. By definition (§ 1.2) an affine root is positive if and only if it is positive on the dominant alcove $\mathfrak{C}^+$. As the group Ω normalizes $\mathfrak{C}^+$, it normalizes the set of positive affine roots, in particular $w(\alpha_s) > 0 \Leftrightarrow (uw)(\alpha_s) > 0$.

(2) Let $(v, u') \in W^{\mathrm{aff}} \times \Omega$. By definition of the Bruhat–Chevalley partial order [14, Ap. 2], $vu' \leqslant wu$ is equivalent to $u' = u, v \leqslant w$. In W^{aff} [8, 1.3.19],

(a) either $sv \leqslant w$ or $sv \leqslant sw$;

(b) either $v \leqslant sw$ or $sv \leqslant sw$.

We multiply (a) and (b) by u on the right without changing $\leqslant$. □

Remark 3.2. As $\ell(w) = \ell(w^{-1})$ and $v \leqslant w \Leftrightarrow v^{-1} \leqslant w^{-1}$, in Lemma 3.1(1) we also have $\ell(sw) = 1 + \ell(w) \Leftrightarrow w^{-1}(\alpha_s) > 0$, and in Lemma 3.1(2), (a) and (b) can be replaced by

(c) either $vs \leqslant w$ or $vs \leqslant ws$;

(d) either $v \leqslant ws$ or $vs \leqslant ws$.

We introduce now a distinguished set $\mathcal{D}$ of representatives of $W_0\backslash W$.

Proposition 3.3. *The three sets*

$$\mathcal{D}_1 = \{d \in W \mid d^{-1}(\alpha) > 0 \textit{ for all } \alpha \in \Sigma^+\},$$

$$\mathcal{D}_2 = \{\lambda w_0 \mid (\lambda, w_0) \in \Lambda^+ \times W_0, \ell(\lambda w_0) = \ell(\lambda) - \ell(w_0)\},$$
$$\mathcal{D}_3 = \{d \in W \mid \ell(w_0 d) = \ell(w_0) + \ell(d) \text{ for all } w_0 \in W_0\},$$

are equal, and will be denoted by $\mathcal{D}$. The cosets $W_0 d$, for $d \in \mathcal{D}$, are disjoint of union W.

Proof. The set $\mathcal{D}_1$ is also equal to

$$\{d \in W \mid \ell(sd) = \ell(d) + 1 \text{ for all } s \in S\}, \tag{20}$$

because one can restrict to $\alpha \in \Delta$ in the definition of $\mathcal{D}_1$ and, for $s \in S$, $d^{-1}(\alpha_s) > 0 \Leftrightarrow \ell(sd) = \ell(d) + 1$ (Remark 3.2). Let $w \in W$ not in $\mathcal{D}_1$. There exists $s \in S$ with $\ell(sw) = \ell(w) - 1$. Then $w_1 = sw$ satisfies $\ell(w) = 1 + \ell(w_1)$. We reiterate, and after finitely many steps we obtain $(w_0, d) \in W_0 \times \mathcal{D}_1$ such that $w = w_0 d$, $\ell(w) = \ell(w_0) + \ell(d)$. The pair (w_0, d) is unique. Indeed, for d, d' in $\mathcal{D}_1$ with $d'd^{-1} \in W_0$, for all $\alpha \in \Delta$ we have $d'd^{-1}(\alpha) = \gamma \in \Sigma$, and $d^{-1}(\alpha) = d'^{-1}(\gamma)$ is positive as $d \in \mathcal{D}_1$; hence $\gamma > 0$ as $d' \in \mathcal{D}_1$. This implies $d = d'$. We deduce that $\mathcal{D}_1$ is a set of representatives of $W_0 \backslash W$, that $d \in \mathcal{D}_1$ is the unique element of minimal length in $W_0 d$, and that $\mathcal{D}_1 \subset \mathcal{D}_3$. This implies that $\mathcal{D}_1 = \mathcal{D}_3$.

We now compare the sets $\mathcal{D}_1$ and $\mathcal{D}_2$. For $(\lambda, w_0) \in \Lambda \times W_0$, we deduce from Lemma 3.1 (see [15, Corollary 5.11]) that

$$\ell(\lambda w_0) = \ell(\lambda) - \ell(w_0) \Leftrightarrow \alpha \circ \nu(\lambda) > 0 \quad \text{for all } \alpha \in \Sigma^+ \cap w_0(\Sigma^-). \tag{21}$$

On the other hand, for all $\alpha \in \Sigma^+$, $(\lambda w_0)^{-1}(\alpha) = w_0^{-1}(\alpha) + \alpha \circ \nu(\lambda)$ is positive if and only if

$$w_0^{-1}(\alpha) > 0, \alpha \circ \nu(\lambda) \geqslant 0 \quad \text{or} \quad w_0^{-1}(\alpha) < 0, \alpha \circ \nu(\lambda) > 0 \tag{22}$$

[15, (36)]. Comparing (21) and (22), we deduce that $\mathcal{D}_1 = \mathcal{D}_2$. □

Remark 3.4. (i) The distinguished set $\mathcal{D}^{\text{aff}}$ of representatives of $W_0 \backslash W^{\text{aff}}$ given by Proposition 3.3 applied to W^{aff} is equal to $\mathcal{D}^{\text{aff}} = \mathcal{D} \cap W^{\text{aff}}$, and $\mathcal{D} = \mathcal{D}^{\text{aff}} \Omega$.

(ii) The distinguished set $\mathcal{D}$ of representatives of $W_0 \backslash W^{\text{aff}}$ can be inductively constructed: it is the set of $\lambda w_0 \in \mathcal{D}$ for $\lambda \in \Lambda^+$ and $w_0 \in W_0$, such that $w_0 = 1$ or w_0 has a reduced decomposition $w_0 = s_1 \dots s_r$ ($s_i \in S$), such that

$$\ell(\lambda s_1 \dots s_{i+1}) = \ell(\lambda s_1 \dots s_i) - 1 \quad \text{for } 1 \leqslant i \leqslant r.$$

Note that $\lambda s \in \mathcal{D} \Leftrightarrow \alpha_s \circ \nu(\lambda) > 0$ when $s \in S$.

We denote by w_1 the unique element of maximal length in the finite Weyl group W_0.

Lemma 3.5. *Let $\lambda, \mu \in \Lambda^+$. The double W_0-coset $W_0 \lambda W_0$ has a unique element w_λ of maximal length,*

$$w_\lambda = w_1 \lambda, \quad \ell(w_\lambda) = \ell(w_1) + \ell(\lambda) \quad \textit{and} \quad \lambda \leqslant \mu \Leftrightarrow w_\lambda \leqslant w_\mu.$$

The set $W_0 \lambda W_0 \cap \mathcal{D}$ is equal to $\mathcal{D}(\lambda) = \{\lambda w_0 \mid w_0 \in W_0, \ell(\lambda w_0) = \ell(\lambda) - \ell(w_0)\}$.

Proof. The coset $W_0 d$ of $d \in \mathcal{D}$ contains a unique element of maximal length, equal to $w_1 d$, $\ell(w_1 d) = \ell(w_1) + \ell(d)$. For $\lambda \in \Lambda^+$, the set $\mathcal{D} \cap W_0 \lambda W_0$ contains a unique element of maximal length, equal to λ (Remark 3.4(ii)). Hence $W_0 \lambda W_0$ contains a unique element w_λ of maximal length, equal to $w_1 \lambda$ and $\ell(w_\lambda) = \ell(w_1) + \ell(\lambda)$. As $w_\mu = w_1 \mu$, $\ell(w_\mu) = \ell(w_1) + \ell(\mu)$, the equivalence $\lambda \leqslant \mu \Leftrightarrow w_1 \lambda \leqslant w_1 \mu$ is clear. We have $\mathcal{D}(\lambda) = \lambda W_0 \cap \mathcal{D}$ (Proposition 3.3), and $\mu \in W_0 \lambda W_0 \Leftrightarrow \mu = w \lambda w^{-1}$ for some $w \in W_0 \Leftrightarrow \mu = \lambda$, as Λ^+ represents the orbits of W_0 in Λ [7, 6.3]. □

Lemma 3.6. *Let $(\lambda, w_0) \in \Lambda^+ \times W_0$, $d = \lambda w_0 \in \mathcal{D}$, and let $\mu \in \Lambda^+$.*

(1) *For $s \in S^{\mathrm{aff}}$, $ds \notin \mathcal{D} \Leftrightarrow dsd^{-1} \in S \Rightarrow \ell(ds) = \ell(d) + 1$.*

(2) *For $s \in S$ and $ds \in \mathcal{D}$, we have $\ell(ds) = \ell(d) + 1 \Leftrightarrow \ell(w_0 s) = \ell(w_0) - 1$.*

(3) *For $(w, d') \in W_0 \times \mathcal{D}$, we have $d \leqslant wd' \Rightarrow d \leqslant d'$.*

(4) *For $s \in S$ such that $ds \in \mathcal{D}$, we have $d \leqslant \mu \Rightarrow ds \leqslant \mu$.*

(5) *We have $d \leqslant w_\mu \Leftrightarrow d \leqslant \mu \Leftrightarrow \lambda \leqslant \mu$.*

Proof. (1) Let $s \in S^{\mathrm{aff}}$. By (20) and Remark 3.2,

$$ds \notin \mathcal{D} \Leftrightarrow (ds)^{-1}(\alpha) < 0 \quad \text{for some } \alpha \in \Delta.$$

As $d^{-1}(\beta) > 0$ for all $\beta \in \Sigma^+$, and $dsd^{-1} \in W^{\mathrm{aff}}$, we have

$$s((d^{-1}(\alpha))) < 0 \Leftrightarrow d^{-1}(\alpha) = A_s \Leftrightarrow \alpha = d(A_s) \Leftrightarrow s_\alpha = dsd^{-1}.$$

We have $\ell(ds) = \ell(d) + 1$ by Lemma 3.1(1).

(2) Let $s \in S$ with $ds \in \mathcal{D}$. Then

$$\ell(ds) = \ell(d) + 1 \Leftrightarrow \ell(\lambda) - \ell(w_0 s) = \ell(\lambda) - \ell(w_0) + 1 \Leftrightarrow \ell(w_0 s) = \ell(w_0) - 1.$$

(3) $d \leqslant wd'$ and $s \in S$ imply that $d \leqslant swd'$ or $sd \leqslant swd'$ by Lemma 3.1(2); as $d < sd$, we obtain

$$d \leqslant wd' \Rightarrow d \leqslant swd'.$$

If $w \neq 1$, we choose s such that $sw < w$. Repeating the procedure, we obtain $d \leqslant d'$ by induction on the length of $w \in W_0$.

(4) As $d \leqslant \mu$, $ds \leqslant \mu$ or $ds \leqslant \mu s$ by Lemma 3.1(2). When $\mu s < \mu$, we obtain $ds \leqslant \mu$. Suppose that $\mu s > \mu$ and $ds \leqslant \mu s$. By Lemma 3.1(1),

$$\ell(\mu s) = \ell(mu) + 1 \Leftrightarrow \mu(\alpha_s) = \alpha_s - \alpha_s \circ \nu(\mu) > 0 \Leftrightarrow \alpha_s \circ \nu(\mu) \leqslant 0,$$
$$\Leftrightarrow \alpha_s \circ \nu(\mu) = 0 \Leftrightarrow \nu(\mu) \text{ fixed by } s \Leftrightarrow \mu s = s\mu u, u \in \Lambda \cap \Omega.$$

We deduce that $ds \leqslant s\mu u$. By (3), $ds \leqslant \mu u$, because $ds, \mu u \in \mathcal{D}$. As Λ is commutative, $ds \leqslant u\mu$. For $w \in W$, there is a unique element $u_w \in \Omega$ such that $w \in u_w W^{\mathrm{aff}}$. By the definition of the Bruhat–Chevalley order, $d \leqslant \mu, ds \leqslant u\mu$ imply that $u_d = u_\mu = uu_\mu$. We deduce that $u = 1, ds \leqslant \mu$.

(5) The implications $d \leqslant w_\mu \Leftarrow d \leqslant \mu \Leftarrow \lambda \leqslant \mu$ are obvious, because $d \leqslant \lambda, \mu \leqslant w_\mu$. The implication $d \leqslant w_\mu \Rightarrow d \leqslant \mu$ follows from (3), because $w_\mu = w_1 \mu$ (Lemma 3.5) and $\mu \in \mathcal{D}$. The implication $d \leqslant \mu \Rightarrow \lambda \leqslant \mu$ follows from (4) reiterated finitely many times for $s \in S$ such that $\ell(ds) = \ell(d) + 1$ if $d \neq \lambda$ (Remark 3.4(ii)). □

Remark 3.7. Results similar to Proposition 3.3 and Lemma 3.6 are already in [9, Proposition 2.5, Lemma 2.6, Proposition 2.7], [10, Lemma 2.4], [11, Proposition 1.3], when W is the Iwahori Weyl group of a split reductive p-adic group G.

Lemma 3.8. *In Lemma 3.6, for $s \in S$ and Δ_λ as in (11),*

$$ds \notin \mathcal{D} \Leftrightarrow dsd^{-1} = w_0 s w_0^{-1} \in S_\lambda \Leftrightarrow w_0(\alpha_s) \in \Delta_\lambda \Leftrightarrow w_0(\alpha_s) \in \Sigma^+, w_0(\alpha_s) \circ \nu(\lambda) = 0.$$

This implies that $\ell(w_0 s) = \ell(w_0) + 1$ and $\ell(ds) = \ell(d) + 1 = \ell(\lambda) - \ell(w_0 s) + 2$.

Proof. By Lemma 3.6(1), $ds \notin \mathcal{D} \Leftrightarrow d(\alpha_s) = \lambda w_0(\alpha_s) = w_0(\alpha_s) - w_0(\alpha_s) \circ \nu(\lambda) \in \Delta \Leftrightarrow w_0(\alpha_s) \in \Delta, w_0(\alpha_s) \circ \nu(\lambda) = 0 \Leftrightarrow w_0(\alpha_s) \in \Delta_\lambda$. In the proof of Lemma 3.6(1), we saw that $dsd^{-1} = s_{w_0(\alpha_s)} = w_0 s w_0^{-1}$. Note that $ds \notin D$ implies that $\ell(ds) = \ell(d) + 1 = \ell(\lambda) - \ell(w_0) + 1 \neq \ell(\lambda) - \ell(w_0 s)$. Hence $\ell(w_0 s) = \ell(w_0) + 1, \ell(ds) = \ell(\lambda w_0 s) = \ell(\lambda) - \ell(w_0 s) + 2$.

By (22), $ds \in \mathcal{D} \Leftrightarrow \alpha \circ \nu(\lambda) > 0$ for all $\alpha \in \Sigma^+ \cap w_0 s(\Sigma^-)$. We have

$$\begin{aligned}\Sigma^+ \cap w_0 s(\Sigma^-) &= (\Sigma^+ \cap w_0(\Sigma^-)) - \{w_0(-\alpha_s)\} \quad \text{if } w_0(\alpha_s) \in \Sigma^-,\\ &= (\Sigma^+ \cap w_0(\Sigma^-)) \cup \{w_0(\alpha_s)\} \quad \text{if } w_0(\alpha_s) \in \Sigma^+,\end{aligned}$$

because, for $\gamma \in \Sigma^+$, we have $s w_0^{-1}(\gamma) < 0$ if and only if $\gamma \in \{w_0(\alpha_s)\} \cup (w_0(\Sigma^-) - \{w_0(-\alpha_s)\})$, as recalled at the beginning of this section. As $d \in \mathcal{D}$, we have $\alpha \circ \nu(\lambda) > 0$ for all $\alpha \in \Sigma^+ \cap w_0(\Sigma^-)$. We deduce that $ds \notin \mathcal{D} \Leftrightarrow w_0(\alpha_s) \in \Sigma^+, w_0(\alpha_s) \circ \nu(\lambda) = 0$. $\square$

4. $\mathfrak{h}$-eigenspace in $\eta \otimes_{\mathfrak{h}} \mathcal{H}$

Proposition 4.1. *For any choice of lift $\tilde{d}$ of $d \in \mathcal{D}$ in $\mathcal{D}(1)$, the left $\mathfrak{h}$-module $\mathcal{H}$ is free of basis $(T_{\tilde{d}})_{d \in \mathcal{D}}$, and the right $\mathfrak{h}$-module $\mathcal{H}$ is free of basis $(T_{\tilde{d}^{-1}})_{d \in \mathcal{D}}$.*

Proof. To the set $\mathcal{D}$ of distinguished representatives of the right W_0-cosets in W is associated a disjoint union $W(1) = \bigsqcup_{d \in \mathcal{D}} W_0(1)\tilde{d}$. Hence $\mathcal{H}$ admits the R-bases

$$(T_{w\tilde{d}})_{w \in W_0(1), d \in \mathcal{D}} \quad \text{and} \quad (T_{\tilde{d}^{-1}w})_{w \in W_0(1), d \in \mathcal{D}}.$$

A basis of $\mathfrak{h}$ is $(T_w)_{w \in W_0(1)}$. By the braid relations, $T_{w\tilde{d}} = T_w T_{\tilde{d}}$ and $T_{\tilde{d}^{-1}w} = T_{\tilde{d}^{-1}} T_w$, because $\ell(wd) = \ell(w) + \ell(d)$. $\square$

Remark 4.2. An element of $\mathcal{H}$ can be written as a sum $\sum_{d \in \mathcal{D}} h_{\tilde{d}} T_{\tilde{d}}$, where $h_{\tilde{d}} \in \mathfrak{h}$, and, for $t \in Z_k$,

$$h_{\tilde{d}} T_{\tilde{d}} = h_{t\tilde{d}} T_{t\tilde{d}} = h_{t\tilde{d}} h_t T_{\tilde{d}}, \quad h_{\tilde{d}} = h_{t\tilde{d}} h_t.$$

The monoid Λ^+ represents the orbits of W_0 in Λ, and the double (W_0, W_0)-cosets of W, because $W = \Lambda \rtimes W_0$. The $(\mathfrak{h}, \mathfrak{h})$-module $\mathcal{H}$ is the direct sum

$$\mathcal{H} = \bigoplus_{\lambda \in \Lambda^+} \mathfrak{h}(\lambda) \tag{23}$$

of the $(\mathfrak{h}, \mathfrak{h})$-submodules $\mathfrak{h}(\lambda)$ of R-basis $(T_w)_{w \in W_0(1)\tilde{\lambda} W_0(1)}$. We set $\mathcal{D}(\lambda) := W_0 \lambda W_0 \cap \mathcal{D}$.

Corollary 4.3. *Let $\lambda \in \Lambda^+$. The left $\mathfrak{h}$-module $\mathfrak{h}(\lambda)$ is free of basis $(T_{\tilde{d}})_{d \in \mathcal{D}(\lambda)}$, and the right $\mathfrak{h}$-module $\mathfrak{h}(\lambda)$ is free of basis $(T_{\tilde{d}^{-1}})_{d \in \mathcal{D}(\lambda^{-1})}$.*

Let η be a character of $\mathfrak{h}$ of parameters (χ, S_η). Let $\lambda \in \Lambda^+$. By Corollary 4.3, an R-basis of $\eta \otimes_{\mathfrak{h}} \mathfrak{h}(\lambda)$ is

$$(1 \otimes T_{\tilde{d}})_{d \in \mathcal{D}(\lambda)}. \tag{24}$$

When the algebra $\mathcal{H}$ arises from a split reductive p-adic group G, Ollivier proved that the right $\mathfrak{h}$-module $\eta \otimes_{\mathfrak{h}} \mathfrak{h}(\lambda)$ has multiplicity 1 (private communication by email March 2014). This property is general, and the characters of $\mathfrak{h}$ contained in $\eta \otimes_{\mathfrak{h}} \mathfrak{h}(\lambda)$ admit the following description.

Proposition 4.4. *Let η_1 be a character of $\mathfrak{h}$ of parameters (χ_1, S_{η_1}). The η_1-eigenspace of $\eta \otimes_{\mathfrak{h}} \mathfrak{h}(\lambda)$ is not 0 if and only if (η_1, η, λ) satisfies*

$$\chi_1 = \chi^\lambda, \quad S_{\eta_1} \cap S_\lambda = S_\eta \cap S_\lambda.$$

When (η_1, η, λ) satisfies these conditions, the η_1-eigenspace of $\eta \otimes_{\mathfrak{h}} \mathfrak{h}(\lambda)$ has dimension 1 and is generated by $1 \otimes \mathcal{E}_{\tilde{\lambda}}$ (defined in Theorem 1.2).

Proof. Let $\mathcal{E} \in \eta \otimes_{\mathfrak{h}} \mathfrak{h}(\lambda)$. We write (24) $\mathcal{E} = \sum_{d \in \mathcal{D}(\lambda)} a_{\tilde{d}} \otimes T_{\tilde{d}}$, where $a_{\tilde{d}} \in R$, and, for $t \in Z_k$,

$$a_{\tilde{d}} \otimes T_{\tilde{d}} = a_{t\tilde{d}} \otimes T_{t\tilde{d}} = \chi(t) a_{t\tilde{d}} \otimes T_{\tilde{d}}, \quad a_{\tilde{d}} = \chi(t) a_{t\tilde{d}}.$$

For $(w, t) \in W \times Z_k$ and a lift $\tilde{w}$ of w in $W(1)$, using the notation of §§ 1.2 and 1.4,

$$(1 \otimes T_{\tilde{w}}) T_t = 1 \otimes (w \bullet t) T_{\tilde{w}} = \chi^w(t) \otimes T_{\tilde{w}}. \tag{25}$$

Using Proposition 2.2 and (25), $\mathcal{E}$ is an $\mathfrak{h}$-eigenvector of $\eta \otimes_{\mathfrak{h}} \mathfrak{h}(\lambda)$ with eigenvalue η_1 if and only if $\mathcal{E}$ satisfies

$$\mathcal{E} = \sum_{d \in \mathcal{D}(\lambda), \chi^d = \chi_1} a_{\tilde{d}} \otimes T_{\tilde{d}} \neq 0, \tag{26}$$

$$\mathcal{E} T_{\tilde{s}} = 0 \quad \text{for } s \in S - S_{\eta_1}, \quad \mathcal{E} T_{\tilde{s}} = \chi_1(c_{\tilde{s}}) \mathcal{E} \quad \text{for } s \in S_{\eta_1}. \tag{27}$$

The space $\eta \otimes_{\mathfrak{h}} \mathfrak{h}(\lambda)$ does not contain a $\mathfrak{h}$-eigenvector with eigenvalue η_1 when the set $X = \{d \in \mathcal{D}(\lambda), \chi^d = \chi_1\}$ is empty, and the proposition is obviously true. When $\nu(\lambda) = 0$, we have $\mathcal{D}(\lambda) = \{\lambda\}$ by Lemma 3.5, and the proposition is true, because it is clearly true when $X = \{\lambda\}$.

We suppose that $\nu(\lambda) \neq 0$. For $s \in S$, the set X is the disjoint union of the subsets

$X_1(s) = \{d \in X \mid \ell(ds) = \ell(d) + 1, ds \in \mathcal{D}\}$,

$X_2(s) = \{d \in X \mid ds \notin \mathcal{D}\}$,

$X_3(s) = \{d \in X \mid \ell(ds) = \ell(d) - 1\}$.

In $\eta \otimes_{\mathfrak{h}} \mathfrak{h}(\lambda)$, we have

$$(1 \otimes T_{\tilde{d}}) T_{\tilde{s}} = 1 \otimes T_{\tilde{d}} T_{\tilde{s}} = \begin{cases} 1 \otimes T_{\tilde{d}\tilde{s}} & (d \in X_1(s)) \\ \eta(T_{\tilde{d}\tilde{s}\tilde{d}^{-1}}) \otimes T_{\tilde{d}} & (d \in X_2(s)) \\ \chi_1(c_{\tilde{s}}) \otimes T_{\tilde{d}} & (d \in X_3(s)). \end{cases}$$

Indeed, if $\ell(ds)=\ell(d)+1$, the braid relations imply that $T_{\tilde d}T_{\tilde s}=T_{\tilde d\tilde s}$. If $ds\notin\mathcal{D}$, by Lemma 3.6, $dsd^{-1}\in S, T_{\tilde d\tilde s}=T_{\tilde d\tilde s\tilde d^{-1}\tilde d}=T_{\tilde d\tilde s\tilde d^{-1}}T_{\tilde d}$. If $\ell(ds)=\ell(d)-1$, the braid and quadratic relations imply that $T_{\tilde d}T_{\tilde s}=T_{\tilde d\tilde s^{-1}}T_{\tilde s}^2=T_{\tilde d\tilde s^{-1}}c_{\tilde s}T_{\tilde s}=\tilde d c_{\tilde s}\tilde d^{-1}T_{\tilde d\tilde s^{-1}}T_{\tilde s}=\tilde d c_{\tilde s}\tilde d^{-1}T_{\tilde d}$.

Multiplying (26) by $T_{\tilde s}$ on the right,

$$\mathcal{E}T_{\tilde s}=\sum_{d\in X_1(s)}a_{\tilde d}\otimes T_{\tilde d\tilde s}+\sum_{d\in X_2(s)}\eta(T_{\tilde d\tilde s\tilde d^{-1}})a_{\tilde d}\otimes T_{\tilde d}+\sum_{d\in X_3(s)}\chi_1(c_{\tilde s})a_{\tilde d}\otimes T_{\tilde d}.$$

As $X_1(s)s=X_3(s)$, the expansion of $\mathcal{E}T_{\tilde s}$ in the basis (24) of $\eta\otimes_{\mathfrak{h}}\mathfrak{h}(\lambda)$ is

$$\mathcal{E}T_{\tilde s}=\sum_{d\in X_2(s)}\eta(T_{\tilde d\tilde s\tilde d^{-1}})a_{\tilde d}\otimes T_{\tilde d}+\sum_{d\in X_3(s)}(a_{\tilde d(\tilde s)^{-1}}+\chi_1(c_{\tilde s})a_{\tilde d})\otimes T_{\tilde d}. \tag{28}$$

Relations (27) are equivalent to the following.
For $d\in X_2(s)$,

$$\eta(T_{\tilde d\tilde s\tilde d^{-1}})a_{\tilde d}=0\quad\text{if } s\in S-S_{\eta_1},\qquad \eta(T_{\tilde d\tilde s\tilde d^{-1}})a_{\tilde d}=\chi_1(c_{\tilde s})a_{\tilde d}\quad\text{if } s\in S_{\eta_1}. \tag{29}$$

For $d\in X_1(s)$,

$$0=\chi_1(c_{\tilde s})a_{\tilde d}\quad\text{if } s\in S_{\eta_1}.$$

For $d\in X_3(s)$,

$$a_{\tilde d(\tilde s)^{-1}}=\chi_1(c_{\tilde s})a_{\tilde d}\quad\text{if } s\in S-S_{\eta_1},\qquad a_{\tilde d(\tilde s)^{-1}}=0\quad\text{if } s\in S_{\eta_1}.$$

The relations for $d\in X_3(s)=X_1(s)s^{-1}$ are equivalent to the following.
For $d\in X_1(s)$,

$$a_{\tilde d}=\chi_1(c_{\tilde s})a_{\tilde d\tilde s}\quad\text{if } s\in S-S_{\eta_1},\qquad a_{\tilde d}=0\quad\text{if } s\in S_{\eta_1}.$$

The relations associated to $\bigcup_{s\in S}(X_1(s)\cup X_3(s))$ are equivalent to

$$a_{\tilde d}=0\quad\text{if } d\in\bigcup_{s\in S_{\eta_1}}X_1(s). \tag{30}$$

$$a_{\tilde d}=\chi_1(c_{\tilde s})a_{\tilde d\tilde s}\quad\text{if } d\in\bigcup_{s\in S-S_{\eta_1}}X_1(s). \tag{31}$$

As $\nu(\lambda)\neq 0$, we have $X=\bigcup_{s\in S}(X_1(s)\cup X_3(s))$, because $d=\lambda w_0\in\mathcal{D}(\lambda)$, $w_0\in W_0$ (Lemma 3.5), satisfies $ds\notin\mathcal{D}(\lambda)$ for all $s\in S$ if and only if $w_0(\alpha_s)\in\Sigma^+$, $w_0(\alpha_s)\circ\nu(\lambda)=0$ for all $s\in S$ (Lemma 3.8), and this is equivalent to $\nu(\lambda)=0$.

For $d=\lambda w_0\in X$ and $\tilde d=\tilde\lambda\tilde w_0$, the relations (30), (31) are equivalent to

$$a_{\tilde d}=\chi_1(c_{\tilde w_0})^{-1}a_{\tilde\lambda}\quad\text{if } w_0 \text{ in } \langle S_{\chi_1}-S_{\eta_1}\rangle,\qquad a_{\tilde d}=0\text{ otherwise.} \tag{32}$$

With the notation $\mathcal{E}_{\tilde\lambda}, Y_\lambda$ introduced in Theorem 1.2, (32) implies that $\mathcal{E}=a_{\tilde\lambda}\otimes\mathcal{E}_{\tilde\lambda}$. If η_1 is contained in $\eta\otimes_{\mathfrak{h}}\mathfrak{h}(\lambda)$, then $a_{\tilde\lambda}\neq 0$, the multiplicity of η_1 is 1, and $\chi_1=\chi^\lambda$.

To end the proof of the proposition, we show that the conditions associated to $\bigcup_{s\in S}X_2(s)$ on $\mathcal{E}=1\otimes\mathcal{E}_{\tilde\lambda}$ are

$$S_\lambda-S_{\eta_1}=S_\lambda-S_\eta. \tag{33}$$

Relation (29) for $d \in X_2(s)$ is always true if $a_{\tilde{d}} = 0$. For $\mathcal{E} = 1 \otimes \mathcal{E}_{\tilde{\lambda}}$, we have $a_{\tilde{d}} \neq 0 \Leftrightarrow d \in \lambda Y_\lambda$. By Lemma 3.8, $d \in \lambda Y_\lambda \cap X_2(s) \Leftrightarrow d = \lambda w_0$, where

$$w_0 \in \langle S_{\chi_1} - S_{\eta_1} \rangle, \quad \ell(\lambda w_0) = \ell(\lambda) - \ell(w_0), \quad \chi_1^{w_0} = \chi_1, \quad dsd^{-1} = w_0 s w_0^{-1} \in S_\lambda.$$

For $s_d = dsd^{-1} \in S_\lambda$ and $\tilde{s}_d = \tilde{d}\tilde{s}\tilde{d}^{-1}$, we have $\chi(c_{\tilde{s}_d}) = \chi(\tilde{d} c_{\tilde{s}} \tilde{d}^{-1}) = \chi^d(c_{\tilde{s}}) = \chi_1(c_{\tilde{s}})$. The conditions associated to $\bigcup_{s \in S} X_2(s)$ are as follows: for all $d \in \lambda Y_\lambda \cap X_2(s)$,

$$s_d \in S - S_\eta \quad \text{if } s \in S - S_{\eta_1} \quad \text{and} \quad s_d \in S_\eta \quad \text{if } s \in S_{\eta_1}; \tag{34}$$

that is, $s \in S_{\eta_1} \Leftrightarrow s_d \in S_\eta$ when $s \in S, d \in \lambda Y_\lambda \cap X_2(s)$. They are equivalent to (33); that is, $s \in S_{\eta_1} \Leftrightarrow s \in S_\eta$ when $s \in S_\lambda$, because, for $d \in \lambda Y_\lambda \cap X_2(s)$, we have $s_d \in S_\lambda$, and $\langle s, S_{\chi_1} - S_{\eta_1} \rangle = \langle s_d, S_{\chi_1} - S_{\eta_1} \rangle$; hence $s_d \in S_{\eta_1} \Leftrightarrow s \in S_{\eta_1}$. $\square$

Let η, η_1 be two characters of $\mathfrak{h}$ of parameters $(\chi, S_\eta), (\chi_1, S_{\eta_1})$, and let o, o_1 be an orientation such that $\eta = \chi_o, \eta_1 = (\chi_1)_{o_1}$.

By the decomposition (23), the $\mathfrak{h}$-module $\eta \otimes_{\mathfrak{h}} \mathcal{H}$ is a direct sum of $\mathfrak{h}$-submodules:

$$\eta \otimes_{\mathfrak{h}} \mathcal{H} = \bigoplus_{\lambda \in \Lambda^+} \eta \otimes_{\mathfrak{h}} \mathfrak{h}(\lambda). \tag{35}$$

Proposition 4.5. *The character η_1 of $\mathfrak{h}$ is contained in $\eta \otimes_{\mathfrak{h}} \mathcal{H}$ if and only if there exists λ such that (η, η_1, λ) satisfies*

$$\lambda \in \Lambda^+, \quad \chi_1 = \chi^\lambda, \quad S_{\eta_1} \cap S_\lambda = S_\eta \cap S_\lambda.$$

The η_1-eigenspace of $\eta \otimes_{\mathfrak{h}} \mathcal{H}$ admits the R-basis $(1 \otimes \mathcal{E}_{\tilde{\lambda}})$ for all λ such that (η, η_1, λ) satisfies these conditions.

For (η, η_1, λ) as in Proposition 4.5, we denote by $\Phi_{\tilde{\lambda}}$ the $\mathcal{H}$-intertwiner

$$\Phi_{\tilde{\lambda}} : 1 \otimes 1 \mapsto 1 \otimes \mathcal{E}_{\tilde{\lambda}} : \eta_1 \otimes_{\mathfrak{h}} \mathcal{H} \to \eta \otimes_{\mathfrak{h}} \mathcal{H}.$$

Corollary 4.6. *An R-basis of* $\mathrm{Hom}_{\mathcal{H}}(\eta_1 \otimes_{\mathfrak{h}} \mathcal{H}, \eta \otimes_{\mathfrak{h}} \mathcal{H})$ *is $(\Phi_{\tilde{\lambda}})$ for all λ such that (η, η_1, λ) satisfies the conditions of Proposition 4.5.*

Taking $\eta = \eta_1$, and recalling the Λ^+-fixator Λ_χ^+ of χ (12), we obtain the following.

Corollary 4.7. *$(\Phi_{\tilde{\lambda}})_{\lambda \in \Lambda_\chi^+}$ is a basis of the spherical Hecke algebra $\mathcal{H}(\eta, \mathfrak{h})$.*

To obtain a basis of the spherical Hecke algebra satisfying (14), for an orientation o we construct $\mathfrak{h}$-eigenvectors of the form

$$1 \otimes E_o(\tilde{\lambda}) \in \chi_o \otimes_{\mathfrak{h}} \mathcal{H}$$

with $\tilde{\lambda} \in \Lambda^+(1)$, where, as in § 1.2, $(E_o(\tilde{w}))_{\tilde{w} \in W(1)}$ is the alcove walk basis of $\mathcal{H}$ associated to o [15, § 5.3 Corollary 5.26], and the character χ_o of $\mathfrak{h}$ is as in Lemma 2.4.

Lemma 4.8. *Let $\lambda \in \Lambda$. We have, in $\chi_o \otimes_{\mathfrak{h}} \mathcal{H}$,*

$$1 \otimes E_o(\tilde{\lambda}) - 1 \otimes T_{\tilde{\lambda}} \in \sum_d R \otimes T_{\tilde{d}},$$

where d runs over the elements of $\mathcal{D}$ satisfying $d < \lambda$ and $\chi^d = \chi^\lambda$. If $\lambda \in \Lambda^+$, then $1 \otimes E_o(\tilde{\lambda}) \neq 0$ is a Z_k-eigenvector of eigenvalue χ^λ.

Proof. For $t \in Z_k$, we have [15, Example 5.30] $E_o(\tilde{\lambda})T_t = T_{\lambda(t)}E_o(\tilde{\lambda})$, $T_{\tilde{\lambda}}T_t = T_{\lambda(t)}T_{\tilde{\lambda}}$; hence $1 \otimes E_o(\tilde{\lambda})T_t = \chi^{\lambda}(t) \otimes E_o(\tilde{\lambda})$, $(1 \otimes T_{\tilde{\lambda}})T_t = \chi^{\lambda}(t) \otimes T_{\tilde{\lambda}}$. With the disjoint decomposition $W(1) = \bigcup_{d \in \mathcal{D}} W_0(1)\tilde{d}$ and the triangular decomposition of $E_o(\tilde{\lambda})$ in the basis $(T_{\tilde{w}})_{\tilde{w} \in W(1)}$ of $\mathcal{H}$ [15, Corollary 5.26], if $1 \otimes E_o(\tilde{\lambda}) \neq 0$ is a Z_k-eigenvector of eigenvalue χ^{λ}, we have

$$1 \otimes E_o(\tilde{\lambda}) - 1 \otimes T_{\tilde{\lambda}} \in \sum_{d \in \mathcal{D}, \chi^d = \chi^{\lambda}} \sum_{\tilde{w} \in W_0(1), wd < \lambda} R \otimes T_{\tilde{w}\tilde{d}}.$$

As $\ell(wd) = \ell(w) + \ell(d)$, by the braid relations, $1 \otimes T_{\tilde{w}\tilde{d}} = 1 \otimes T_{\tilde{w}}T_{\tilde{d}} = \eta(T_{\tilde{w}}) \otimes T_{\tilde{d}}$,

$$\sum_{\tilde{w} \in W_0(1), wd < \lambda} R(1 \otimes T_{\tilde{w}\tilde{d}}) = R(1 \otimes T_{\tilde{d}}).$$

As $d < wd$ for $w \in W_0$, we deduce that

$$1 \otimes E_o(\tilde{\lambda}) - 1 \otimes T_{\tilde{\lambda}} \in \sum_{d \in \mathcal{D}, \chi^d = \chi^{\lambda}, d < \lambda} R \otimes T_{\tilde{d}}.$$

For $\lambda \in \Lambda^+$, $1 \otimes E_o(\tilde{\lambda})$ is not 0, because $\Lambda^+ \subset \mathcal{D}$, and $(1 \otimes T_{\tilde{d}})_{d \in \mathcal{D}}$ is a basis of $\eta \otimes_{\mathfrak{h}} \mathcal{H}$ (Proposition 4.1). $\square$

Lemma 4.9. *Let $\lambda \in \Lambda$. Then $1 \otimes E_o(\tilde{\lambda}) \in \chi_o \otimes_{\mathfrak{h}} \mathcal{H}$ is a $\mathfrak{h}$-eigenvector of eigenvalue $(\chi_1)_{o_1}$ if and only $1 \otimes E_o(\tilde{\lambda}) \neq 0$ and*

$$\chi_1 = \chi^{\lambda}, \quad 1 \otimes E_o(\tilde{\lambda})E_{o_1}(\tilde{s}) = 0 \quad \textit{for all } s \in S.$$

Proof. By Lemma 4.8(ii), $1 \otimes E_o(\tilde{\lambda})$ is a $\mathfrak{h}$-eigenvector with eigenvalue η_1 if and only if $1 \otimes E_o(\tilde{\lambda}) \neq 0$, and $\chi_1 = \chi^{\lambda}$, $(1 \otimes E_o(\tilde{\lambda}))E_{o_1}(\tilde{s}) = 0$ for all $s \in S$ (Lemma 2.4). We have $(1 \otimes E_o(\tilde{\lambda}))E_{o_1}(\tilde{s}) = 1 \otimes E_o(\tilde{\lambda})E_{o_1}(\tilde{s})$. $\square$

Lemma 4.10. *Let $\lambda \in \Lambda^+$. Then $1 \otimes E_o(\tilde{\lambda})$ is a $\mathfrak{h}$-eigenvector of eigenvalue $(\chi^{\lambda})_o$ if and only if $\eta(E_o(\tilde{s})) = 0$ for all $s \in S$ such that $\ell(\lambda s) = 1 + \ell(\lambda)$.*

Proof. Let $s \in S$.

If $\ell(\lambda s) = \ell(\lambda) - 1$, then $E_o(\tilde{\lambda})E_o(\tilde{s}) = 0$ by the product formula.

If $\ell(\lambda s) = \ell(\lambda) + 1$, then $E_o(\tilde{\lambda})E_o(\tilde{s}) = E_o(\lambda\tilde{s}) = E_o(\tilde{s}\tilde{s}^{-1}\lambda\tilde{s}) = E_o(\tilde{s})E_{o \bullet s}(\tilde{s}^{-1}\lambda\tilde{s})$.

The latter equality follows from the fact that the length is constant on a W_0-orbit in Λ. It implies that $1 \otimes E_o(\tilde{s})E_{o \bullet s}(\tilde{s}^{-1}\lambda\tilde{s}) = \eta(E_o(\tilde{s})) \otimes E_o(\tilde{\lambda})$. Apply Lemmas 4.8 and 4.9. $\square$

Proposition 4.11. *Let $\lambda \in \Lambda^+$. Then,*

$1 \otimes E_o(\tilde{\lambda})$ is a $\mathfrak{h}$-eigenvector in $\chi_o \otimes_{\mathfrak{h}} \mathcal{H}$ of eigenvalue $(\chi^{\lambda})_o$, and

$\mathcal{E}_{\tilde{\lambda}}$ is the component of $1 \otimes E_o(\tilde{\lambda})$ in $\chi_o \otimes_{\mathfrak{h}} \mathfrak{h}(\lambda)$.

Proof. Use Lemmas 2.4 and 4.10 for the first assertion. The non-zero components of $1 \otimes E_o(\tilde{\lambda})$ in the direct decomposition (35) are $\mathfrak{h}$-eigenvectors of eigenvalue $(\chi^{\lambda})_o$. Apply Proposition 4.4 and Lemma 4.8 for the second assertion. $\square$

Corollary 4.12. *If $o = o_1$ (Lemma 2.6), an R-basis of $\mathrm{Hom}_{\mathcal{H}}((\chi_1)_o \otimes_{\mathfrak{h}} \mathcal{H}, \chi_o \otimes_{\mathfrak{h}} \mathcal{H})$ is $(1 \otimes E_o(\tilde{\lambda}))$ for all λ such that $(\chi_o, (\chi_1)_o, \lambda)$ satisfies the conditions of Proposition 4.5.*

Proposition 4.13. *For each $\lambda \in \Lambda_\chi^+$, we have an injective $\mathcal{H}$-intertwiner*

$$\Phi_{o,\tilde{\lambda}} : 1 \otimes 1 \mapsto 1 \otimes E_o(\tilde{\lambda}) : \chi_o \otimes_{\mathfrak{h}} \mathcal{H} \to \chi_o \otimes_{\mathfrak{h}} \mathcal{H}.$$

$(\Phi_{o,\tilde{\lambda}})_{\lambda \in \Lambda_\chi^+}$ is an R-basis satisfying (14) of the spherical Hecke algebra $\mathcal{H}(\chi_o, \mathfrak{h})$.

Proof. By Corollary 4.12 and the product formula (8), $(\Phi_{o,\tilde{\lambda}})_{\lambda \in \Lambda_\chi^+}$ is an R-basis of $\mathcal{H}(\chi_o, \mathfrak{h})$ satisfying (14).

If $\Phi_{o,\tilde{\lambda}}$ is not injective, $\mathrm{Ker}\, \Phi_{o,\tilde{\lambda}}$ contains a simple character η_1 of $\mathfrak{h}$, and $\Phi_{o,\tilde{\lambda}} \circ \Phi_1 = 0$ for some non-zero $\Phi_1 \in \mathrm{End}_{\mathfrak{h}}(\eta_1 \otimes_{\mathfrak{h}} \mathcal{H}, \eta \otimes_{\mathfrak{h}} \mathcal{H})$.

Expanding $\Phi_1(1 \otimes 1) = \sum_{\mu \in \Lambda^+} a_{\tilde{\mu}} \otimes E_o(\tilde{\mu})$, $a_{\tilde{\mu}} \in R$, in the basis $(1 \otimes E_o(\tilde{\mu}))_{\mu \in \Lambda^+}$ of $\eta \otimes_{\mathfrak{h}} \mathcal{H}$, and using the product formula $E_o(\tilde{\lambda}) E_o(\tilde{\mu}) = E_o(\tilde{\lambda}\tilde{\mu})$, the decomposition of $(\Phi_{o,\tilde{\lambda}} \circ \Phi_1)(1 \otimes 1)$ in this basis is

$$\sum_{\mu \in \Lambda^+} \Phi_{o,\tilde{\lambda}}(a_{\tilde{\mu}} \otimes E_o(\tilde{\mu})) = \sum_{\mu \in \Lambda^+} a_{\tilde{\mu}} \otimes E_o(\tilde{\lambda}) E_o(\tilde{\mu}) = \sum_{\mu \in \Lambda^+} a_{\tilde{\mu}} \otimes E_o(\tilde{\lambda}\tilde{\mu}).$$

We have $\Phi_1 \neq 0 \Leftrightarrow \Phi_1(1 \otimes 1) \neq 0 \Leftrightarrow a_{\tilde{\mu}} \neq 0$ for some $\mu \in \Lambda^+ \Leftrightarrow (\Phi_{o,\tilde{\lambda}} \circ \Phi_1)(1 \otimes 1) \neq 0 \Leftrightarrow \Phi_{o,\tilde{\lambda}} \circ \Phi_1 \neq 0$. $\square$

Corollary 4.14. $1 \otimes E_o(\tilde{\lambda}) = 0$ *in* $\chi_o \otimes_{\mathfrak{h}} \mathcal{H}$ *if* $\lambda \in \Lambda - \Lambda^+$.

Proof. Let $\lambda \in \Lambda - \Lambda^+$. We choose $\mu \in \Lambda_\chi^+$ not 0. Then $\Phi_{o,\tilde{\mu}}$ of $\mathrm{End}_{\mathfrak{h}}\, \eta \otimes_{\mathfrak{h}} \mathcal{H}$ is injective (Proposition 4.13) and $\Phi_{o,\tilde{\mu}}(1 \otimes E_o(\tilde{\lambda})) = 1 \otimes E_o(\tilde{\mu}) E_o(\tilde{\lambda})$. As μ, λ belong to different closed Weyl chambers, $E_o(\tilde{\mu}) E_o(\tilde{\lambda}) = 0$; hence $1 \otimes E_o(\tilde{\lambda}) = 0$. $\square$

More generally, if $(\chi_o, (\chi_1)_o, \lambda)$ satisfies the conditions of Proposition 4.5, we have the non-zero $\mathcal{H}$-intertwiner

$$\Phi_{o,\tilde{\lambda}} : 1 \otimes 1 \mapsto 1 \otimes E_o(\tilde{\lambda}) : (\chi_1)_o \otimes_{\mathfrak{h}} \mathcal{H} \to \chi_o \otimes_{\mathfrak{h}} \mathcal{H}.$$

An R-basis of $\mathrm{Hom}_{\mathcal{H}}((\chi_1)_o \otimes \mathcal{H}, \chi_o \otimes \mathcal{H})$ is $(\Phi_{o,\tilde{\lambda}})$ for all λ such that $(\chi_o, (\chi_1)_o, \lambda)$ satisfies the conditions of Proposition 4.5.

We fix $x_1 \in \Lambda$ such that $\chi_1 = \chi^{x_1}$. For $\lambda \in \Lambda$, $\chi_1 = \chi^{\lambda x_1} \Leftrightarrow \lambda \in \Lambda_\chi$. We embed $\mathrm{Hom}_{\mathcal{H}}(\eta_1 \otimes_{\mathfrak{h}} \mathcal{H}, \eta \otimes_{\mathfrak{h}} \mathcal{H})$ into the algebra $e_\chi R[\Lambda_\chi]$ (§ 1.4) by the R-linear map

$$S_{\eta_1,\eta,\tilde{x}_1} : \mathrm{Hom}_{\mathcal{H}}(\eta_1 \otimes_{\mathfrak{h}} \mathcal{H}, \eta \otimes_{\mathfrak{h}} \mathcal{H}) \to e_\chi R[\Lambda_\chi], \tag{36}$$

$$\Phi_{o,\tilde{\lambda}\tilde{x}_1} \mapsto e_\chi \tilde{\lambda} \quad (\lambda \in \Lambda_\chi \cap \Lambda^+ x_1^{-1}), \tag{37}$$

where $\tilde{\lambda}, \tilde{x}_1 \in \Lambda(1)$ lift λ, x_1. If $\eta = \eta_1$ and $\tilde{x}_1 = 1$, the map $S_{\eta,\eta,1} = S_{\eta,\eta}$ embeds the spherical Hecke algebra $\mathcal{H}(\eta, \mathfrak{h}) = \mathrm{End}_{\mathcal{H}}(\eta \otimes_{\mathfrak{h}} \mathcal{H})$ into the algebra $e_\chi R[\Lambda_\chi]$

$$S_{\eta,\eta} : \mathcal{H}(\eta, \mathfrak{h}) \to e_\chi R[\Lambda_\chi]. \tag{38}$$

Lemma 4.15. *The composition*

$$(A, B) \mapsto B \circ A : \mathrm{Hom}_{\mathcal{H}}(\eta_1 \otimes_{\mathfrak{h}} \mathcal{H}, \eta \otimes_{\mathfrak{h}} \mathcal{H}) \times \mathrm{End}_{\mathcal{H}}(\eta \otimes_{\mathfrak{h}} \mathcal{H}) \to \mathrm{Hom}_{\mathcal{H}}(\eta_1 \otimes_{\mathfrak{h}} \mathcal{H}, \eta \otimes_{\mathfrak{h}} \mathcal{H}),$$

corresponds to the product $S_{\eta_1,\eta,\tilde{x}_1}(A \circ B) = S_{\eta,\eta}(B) S_{\eta_1,\eta,\tilde{x}_1}(A)$ *in* $e_\chi R[\Lambda_\chi]$.

Proof. For $\lambda \in \Lambda_\chi^+$ and $\lambda_1 \in \Lambda^+$, $\chi^{\lambda_1} = \chi_1$, $S_{\eta_1} \cap S_{\lambda_1} = S_\eta \cap S_{\lambda_1}$, we have

$$\Phi_{o,\tilde{\lambda}} \circ \Phi_{o,\tilde{\lambda}_1}(1 \otimes 1) = \Phi_{o,\tilde{\lambda}}(1 \otimes E_o(\tilde{\lambda}_1)) = 1 \otimes E_o(\tilde{\lambda}) E_o(\tilde{\lambda}_1) = 1 \otimes E_o(\tilde{\lambda}\tilde{\lambda}_1),$$

by the product formula (8). Hence $\Phi_{o,\tilde{\lambda}} \circ \Phi_{o,\tilde{\lambda}_1} = \Phi_{o,\tilde{\lambda}\tilde{\lambda}_1}$ and $S_{\eta_1,\eta,\tilde{x}_1}(\Phi_{o,\tilde{\lambda}} \circ \Phi_{o,\tilde{\lambda}_1}) = e_\chi \tilde{\lambda}\tilde{\lambda}_1 (\tilde{x}_1)^{-1}$. As e_χ is a central idempotent of $R[\Lambda_\chi]$, we have $e_\chi \tilde{\lambda}\tilde{\lambda}_1(\tilde{x}_1)^{-1} = e_\chi \tilde{\lambda} e_\chi \tilde{\lambda}_1 (\tilde{x}_1)^{-1} = S_{\eta,\eta}(\Phi_{o,\tilde{\lambda}}) S_{\eta_1,\eta,\tilde{x}_1}(\Phi_{o,\tilde{\lambda}_1})$. □

5. Centers

We make the same hypotheses as in § 1.2, and we suppose that Λ_T exists.

As $\tilde{\Lambda}_T$ is central in $\Lambda(1)$, the action of $W(1)$ on $\tilde{\Lambda}_T$ factorizes through an action of W_0, and the R-module $\mathcal{A}_o(\Lambda_T)$ of basis $(E_o(\tilde{\mu}))_{\mu \in \Lambda_T}$ is a W_0-stable subalgebra of the center $\mathcal{Z}_o$ of $\mathcal{A}_o$, for any orientation o. The quotient map $\Lambda_T(1) \to \Lambda_T$ of splitting $\mu \mapsto \tilde{\mu}$ is W_0-equivariant. For $\mu \in \Lambda_T$ of W_0-conjugacy class $C(\mu)$, and $\tilde{C}(\mu)$ the W_0-conjugacy class of $\tilde{\mu}$, the set $\nu(C(\mu))$ contains a single element in the dominant closed Weyl chamber, and

$$\ell(\mu) = 0 \Leftrightarrow \nu(\mu) = 0 \Leftrightarrow \mu \in \Lambda_T^{W_0} \Leftrightarrow \tilde{C}(\mu) = \tilde{\mu}. \tag{39}$$

By axiom (T1) (1.2), a $W(1)$-conjugacy class $\tilde{C}$ is finite if and only if $\tilde{C} \subset \Lambda(1)$.

In the following theorem, R is any commutative ring.

Theorem 5.1. *The center* $\mathcal{Z}$ *of* $\mathcal{H}_R(q_s, c_{\tilde{s}})$ *is the algebra* $\mathcal{A}_o^{W(1)}$ *of* $W(1)$*-invariants of* $\mathcal{A}_o$*, equal to the algebra* $\mathcal{Z}_o^{W_0}$ *of the* W_o*-invariants of the center* $\mathcal{Z}_o$ *of* $\mathcal{A}_o$*. The center* $\mathcal{Z}$ *is a free* R*-module of basis (independent of the choice of the orientation* o*)*

$$E(\tilde{C}) = \sum_{\tilde{\lambda} \in \tilde{C}} E_o(\tilde{\lambda}) \quad \text{for } \tilde{C} \text{ running through the finite conjugacy classes of } W(1).$$

The involution ι *of* $\mathcal{H}$ *satisfies, for any finite conjugacy class* $\tilde{C}$ *of* $W(1)$*,*

$$\iota(E(\tilde{C})) = (-1)^{\ell(C)} E(\tilde{C}). \tag{40}$$

The algebra $\mathcal{Z}_T = \mathcal{A}_o(\Lambda_T)^{W_0}$ *of* W_0*-invariants of* $\mathcal{A}_o(\Lambda_T)$ *is a central subalgebra of* $\mathcal{H}$*, and a free* R*-module of basis* $(E(\tilde{C}(\mu))_{\mu \in \Lambda_T^+}$.

The $\mathcal{Z}_T$*-modules* $\mathcal{Z}$ *and* $\mathcal{H}_R(q_s, c_{\tilde{s}})$ *are finitely generated.*

When the ring R *is noetherian, the* R*-algebras* $\mathcal{Z}_T, \mathcal{Z}$*, and* $\mathcal{H}_R(q_s, c_{\tilde{s}})$ *are finitely generated.*

Proof. The steps of the proof are as follows.

(1) The center $\mathcal{Z}_o$ of $\mathcal{A}_o$ is a free R-module of basis $E_o(\tilde{c}) = \sum_{\tilde{\lambda}\in\tilde{c}} E_o(\tilde{\lambda})$ for all conjugacy classes $\tilde{c}$ of $\Lambda(1)$.

(2) $\sum_{\tilde{\lambda}\in\tilde{C}} E_o(\tilde{\lambda})$ does not depend on the orientation o, and the center $\mathcal{Z}$ is equal to $\mathcal{A}_{o^-}^{W(1)}$ for the anti-dominant orientation o^-.

(3) (a) The $\mathcal{A}_o(\Lambda_T)^{W_0}$-module $\mathcal{A}_o(\Lambda_T)$ is finitely generated, and if R is noetherian the algebra $\mathcal{A}_o(\Lambda_T)^{W_0}$ is finitely generated.

(b) The left $\mathcal{A}_o(\Lambda_T)$-module $\mathcal{A}_o$ is finitely generated.

(c) The left $\mathcal{A}_o$-module $\mathcal{H}_R(q_s, c_{\tilde{s}})$ is finitely generated.

The theorem is proved for the pro-p-Iwahori Hecke algebra $\mathcal{H}_R(G, I(1))$, where the assertions on $\mathcal{Z}_T$ are not formulated but are implicit in the proof. Properties (1), (2), (3)(a), (b) and (40) admit exactly the same proofs as in [16, Propositions 2.3, 2.7, Lemma 2.15 and Proposition 3.3]. The same is true for the property (3)(c) [16, Lemma 2.17], once we have strengthened the finiteness property [14, 1.6.3], [16, Lemma 2.16]. This is done in Lemma 5.3 below. As in [16, added in proof], this is a variant of the finiteness of the set of minimal elements in a subset L of $\mathbb{Z}^n$ $(n > 0)$ [12, Lemma 4.2.18]. □

Let L be a group isomorphic to $\mathbb{Z}^n$. For $a = (a_i), b = (b_i) \in \mathbb{Z}^n$, we write $b \leqslant a$ if $|a_i| = |b_i| + |a_i - b_i|$ for all i. We write $b < a$ if $a \neq b, b \leqslant a$; we say that $a \in L$ is minimal if $b \in L, b \leqslant a$ implies that $b = a$.

Lemma 5.2. (1) *Let $a \in L$. There exists $b \in L$ minimal such that $b \leqslant a$.*

(2) *The set L_{min} of minimal elements in L is finite.*

Proof. We have $|a_i| = |b_i| + |a_i - b_i| \Leftrightarrow b_i = 0$ or $a_i b_i > 0, |b_i| \leqslant |a_i|$.

(1) If a is not minimal in L, we choose $b < a$ and we reiterate. The processes stops after finitely many steps, because $b < a$ implies that $|b_i| \leqslant |a_i|$ for $1 \leqslant i \leqslant n$, and $|b_i| \in \mathbb{N}$.

(2) Suppose that L_{min} is infinite. If the set $\{a_i \mid a \in L_{min}\}$ is finite, a_i is constant for a in an infinite subset of L_{min}. If the set $\{a_i \mid a \in L_{min}\}$ is infinite, L_{min} contains a sequence $(a(m))_{m\in\mathbb{N}}$ such that $(a(m)_i)_{m\in\mathbb{N}}$ is strictly increasing positive or strictly decreasing negative. Hence L_{min} contains a sequence $(a(m))_{m\in\mathbb{N}}$ such that, for all $1 \leqslant i \leqslant n$, $(a(m)_i)_{m\in\mathbb{N}}$ is either constant, or strictly increasing positive or strictly decreasing negative. For all i in the non-empty set where $(a(m)_i)_{m\in\mathbb{N}}$ is not constant, we have $a(m)_i a(m+1)_i > 0, |a(m)_i| < |a(m+1)_i|$ for all $m \in \mathbb{N}$. Hence $a(m) < a(m+1)$ for all $m \in \mathbb{N}$. This contradicts the minimality of the $a(m)$. □

By axiom (T1), $W = \bigsqcup_{(y,w_0)\in Y\times W_0} \Lambda_T y w_0$. For $(y, w_0) \in Y \times W_0$, let

$$L(y, w_0) = \{\vec{\ell}(w) = (\ell_\gamma(w))_{\gamma\in\Sigma^+} \mid w \in \Lambda_T y w_0\},$$

where $\ell(w) = \sum_{\gamma\in\Sigma^+} |\ell_\gamma(w)|$ and $\ell_\gamma(w)$ as in [15, Propositions 5.7 and 5.9]. By Lemma 5.2, the set $L(y, w_0)_{min}$ is finite. Let $X_*(y, w_0)$ be a finite subset of Λ_T such that

$$L(y, w_0)_{min} = \{\vec{\ell}(w) \mid w \in X_*(y, w_0) y w_0\}.$$

Let X be the finite subset $\bigcup_{(y,w_0)\in Y\times W_0} X_*(y,w_0)y$ of Λ. We have

$$\ell(w)=\ell(ww'^{-1})+\ell(w') \quad \text{for } w,w'\in \Lambda w_0, \quad \vec{\ell}(w')\leqslant \vec{\ell}(w),$$

[16, Proof of Lemma 2.16(18)]. This implies the following.

Lemma 5.3. *For any $(\lambda,w_0)\in\Lambda\times W_0$ there exists $x\in X$ such that*

$$\lambda x^{-1}\in\Lambda_T, \quad \ell(\lambda w_0)=\ell(\lambda x^{-1})+\ell(xw_0).$$

For a central element x of $\mathcal{H}$, the $\mathcal{H}$-intertwiner

$$\Phi_x : 1\otimes h\mapsto 1\otimes xh = 1\otimes hx \quad \text{for } h\in\mathcal{H}. \tag{41}$$

is central in $\mathcal{H}(\chi_o,\mathfrak{h})$ by Proposition 4.13 and

$$\begin{aligned}\Phi_x\circ\Phi_{o,\tilde\lambda}(1\otimes 1) &= \Phi_x(1\otimes E_{o,\tilde\lambda}) = 1\otimes xE_{o,\tilde\lambda}\\ &= 1\otimes E_{o,\tilde\lambda}x = \Phi_{o,\tilde\lambda}(1\otimes x)=\Phi_{o,\tilde\lambda}\circ\Phi_x(1\otimes 1).\end{aligned}$$

We denote by $\mathcal{Z}(\chi_o,\mathfrak{h})$ the center of $\mathcal{H}(\chi_o,\mathfrak{h})$. The homomorphism

$$x\mapsto \Phi_x : \mathcal{Z}\to\mathcal{Z}(\chi_o,\mathfrak{h}) \tag{42}$$

may be not injective or not surjective.

Proposition 5.4. (1) *For $\mu\in\Lambda_T^+$, we have $1\otimes E(\tilde C(\mu))=1\otimes E_o(\tilde\mu)$ and $\Phi_{E(\tilde C(\mu))}=\Phi_{o,\tilde\mu}$.*

(2) *$(\Phi_{o,\tilde\mu})_{\mu\in\Lambda_T^+}$ is a basis, independent of o, satisfying (14) of a central subalgebra $\mathcal{Z}_T(\eta,\mathfrak{h})$ of the spherical algebra $\mathcal{H}(\eta,\mathfrak{h})$, and $\mathcal{H}(\eta,\mathfrak{h})$ is a finitely generated $\mathcal{Z}_T(\eta,\mathfrak{h})$-module.*

Proof. (1) From Corollary 4.14,

$$1\otimes E(\tilde C(\mu))=\sum_{\tilde\lambda\in\tilde C(\mu)\cap\Lambda^+(1)} 1\otimes E_o(\tilde\lambda) \quad \text{in } \chi_o\otimes\mathcal{H}.$$

For $\mu\in\Lambda_T^+$ we have $\tilde C(\mu)\cap\Lambda^+(1)=\{\tilde\mu\}$. Hence $1\otimes E(\tilde C(\mu))=1\otimes E_o(\tilde\mu)$ and $\Phi_{E(\tilde C(\mu))}=\Phi_{o,\tilde\mu}$.

(2) The canonical isomorphism $\mathcal{H}(\eta,\mathfrak{h})\to e_\chi R[\Lambda_\chi^+]$ associated to the basis $(\Phi_{o,\tilde\lambda})_{\lambda\in\Lambda_\chi^+}$ (Proposition 4.13) sends $\mathcal{Z}_T(\eta,\mathfrak{h})$ to $e_\chi R[\Lambda_T^+]$, and $e_\chi R[\Lambda_\chi^+]$ is a finitely generated $e_\chi R[\Lambda_T^+]$-module. □

6. Supersingular $\mathcal{H}$-modules

We make the same hypotheses as in § 1.2 and we suppose that Λ_T exists. We construct different filtrations of $\mathcal{H}$ which are all equivalent when the ring R is noetherian.

Lemma 6.1. *The R-module $\mathcal{F}_{o,n}$ of basis $\{E_o(\tilde w)\mid \tilde w\in W(1), \ell(w)\geqslant n\}$ for $n\in\mathbb{N}$ is a right ideal of $\mathcal{H}$, for any orientation o.*

Proof. We have $\mathcal{F}_{o,n}\mathcal{H} \subset \mathcal{F}_{o,n}$, because, for $\tilde{w} \in W(1)$, a basis of $\mathcal{H}$ is $(E_{o\bullet w}(\tilde{w}'))_{\tilde{w}' \in W(1)}$, and $E_o(w)E_{o\bullet w}(\tilde{w}') = E_o(\tilde{w}\tilde{w}')$ if $\ell(w) + \ell(w') = \ell(ww')$ and 0 otherwise. □

The length is constant on the projection C in W of a finite $W(1)$-conjugacy class $\tilde{C}$, and is denoted by $\ell(\tilde{C}) = \ell(C)$.

Lemma 6.2. *The R-module $\mathcal{Z}_{\ell>0}$ of basis $E(\tilde{C})$ for the finite $W(1)$-conjugacy classes $\tilde{C}$ of positive length is an ideal of the center $\mathcal{Z}$ of $\mathcal{H}$, stable by the involutive R-automorphism ι (4).*

Proof. Let $\tilde{C}_1, \tilde{C}_2$ be two finite $W(1)$-conjugacy classes. They are contained in $\Lambda(1)$. By the product formula,

$$E(\tilde{C}_1)E(\tilde{C}_2) = \sum_{\tilde{C}} a_{\tilde{C}} E(\tilde{C}), \tag{43}$$

where $\tilde{C}$ runs over finite conjugacy classes with $\ell(C) = \ell(C_1) + \ell(C_2)$. The stability by ι follows from (40). □

It is more convenient to replace the center $\mathcal{Z}$ of $\mathcal{H}$ by the central subalgebra $\mathcal{Z}_T$ of basis $(E(\tilde{C}(\mu)))_{\mu \in X_*^+(T)}$ which admits better properties.

Lemma 6.3. *We have*

$$\mathcal{Z}_T = \mathcal{R}_T \oplus \mathcal{Z}_{T,\ell>0},$$

where $\mathcal{R}_T$ is the algebra of basis $(T_{\tilde{\mu}})_{\mu \in \Lambda_T^{W_0}}$, isomorphic to $R[\Lambda_T^{W_0}]$, and $\mathcal{Z}_{T,\ell>0}$ is the ideal of $\mathcal{Z}_T$ of basis $(E(\tilde{C}(\mu)))_{\mu \in \Lambda_T^+, \ell(\mu)>0}$.
The algebras $\mathcal{R}_T$ and $\mathcal{Z}_{T,\ell>0}$ are stable by the involutive automorphism ι.

Proof. The proof is straightforward. □

The R-module $\mathcal{F}_{T,o,n}$ of basis $(E_o(\tilde{\mu}))_{\mu \in \Lambda_T, \ell(\mu) \geqslant n}$ is contained in $\mathcal{F}_{o,n}$ and contains $(\mathcal{Z}_{T,\ell>0})^n$.

Proposition 6.4. *When R is noetherian, the filtrations of $\mathcal{H}$*

$$((\mathcal{Z}_{T,\ell>0})^n \mathcal{H})_{n\in\mathbb{N}}, \quad ((\mathcal{Z}_{\ell>0})^n \mathcal{H})_{n\in\mathbb{N}}, \quad (\mathcal{F}_{T,o,n})_{n\in\mathbb{N}}\mathcal{H}, \quad (\mathcal{F}_{o,n})_{n\in\mathbb{N}},$$

are equivalent.

We have $(\mathcal{Z}_{T,\ell>0})^n \mathcal{H} \subset (\mathcal{Z}_{\ell>0})^n \mathcal{H} \subset \mathcal{F}_{o,n}$. The last inclusion uses the product formula, the equality $\tilde{E}(C) = \tilde{E}_o(C)$, and that $(E_o(w))_{w \in W(1)}$ is a basis of $\mathcal{H}$. The noetherianity of R is used only for the proof (Lemma 6.7) of the property (which implies the proposition):

$$\text{for } n \in \mathbb{N} \text{ there exists } n' \in \mathbb{N} \text{ such that } \mathcal{F}_{o,n'} \subset (\mathcal{Z}_{T,\ell>0})^n \mathcal{H}.$$

This property follows from the next three lemmas.

Lemma 6.5. *$E(\tilde{C}(\mu))^n E_o(\tilde{\mu}) = E_o(\tilde{\mu}^{n+1})$ for $\mu \in \Lambda_T$ and $n > 0$.*

Proof. By the product formula, $E(C(\tilde\mu))E_o(\tilde\mu) = E_o(\tilde\mu^2)$, because $\tilde\mu$ is the only element of $\tilde C(\mu)$ sent by ν in the same closed Weyl chamber as $\nu(\mu)$. By induction on n,

$$\begin{aligned} E(\tilde C(\mu))^{n+1}E_o(\tilde\mu) &= E(\tilde C(\mu))E(\tilde C(\mu))^n E_o(\tilde\mu) = E(\tilde C(\mu))E_o(\tilde\mu^{n+1}) \\ &= E(\tilde C(\mu))E_o(\tilde\mu)E_o(\tilde\mu^n) = E_o(\tilde\mu^2)E_o(\tilde\mu^n) = E_o(\tilde\mu^{n+2}). \end{aligned}$$ □

Lemma 6.6. *There exists a positive integer a such that, for any positive integer n,*

$$E_o(\mu) \in \mathcal{Z}^n_{T,\ell>0}\mathcal{A}_o$$

if $\mu \in \Lambda_T$ satisfies $\ell(\mu) \geqslant na$.

Proof. Let $\overline{\mathfrak{D}}$ be a closed Weyl chamber. We choose $\mu_1, \ldots, \mu_r$ in $\Lambda_T - \Lambda_T^{W_0}$ such that $\nu(\mu_1), \ldots, \nu(\mu_r)$ generate the monoid $\nu(\Lambda_T) \cap \overline{\mathfrak{D}}$. We show that

$$E_o(\mu) \in \mathcal{Z}^n_{T,\ell>0}\mathcal{A}_o,$$

if $\mu \in \Lambda_T$, $\nu(\mu) \in \overline{\mathfrak{D}}$ and $\ell(\mu) > n(\ell(\mu_1) + \cdots + \ell(\mu_r))$. Clearly, this implies the lemma.

Let $\mu = \mu_1^{n_1} \ldots \mu_r^{n_r} u$ with $u \in (\Lambda_T)^{W_0}$, $n_1, \ldots, n_r$ in $\mathbb{N}$. We have $\ell(\mu_i) \neq 0$ for $1 \leqslant i \leqslant r$ and $\ell(\mu) = n_1\ell(\mu_1) + \cdots + n_r\ell(\mu_r)$. Changing the numerotation, we suppose that $n_1 > n$, and obtain

$$E_o(\mu) = E_o(\mu_1)^{n_1}h, \quad h = E_o(\mu_2)^{n_2} \ldots E_o(\mu_r)^{n_r} T_u \in \mathcal{A}_o.$$

By Lemma 6.5, $E_o(\mu_1)^{n_1} = E(\tilde C(\mu_1))^{n_1-1}E_o(\mu_1)$. Hence $E_o(\mu) \in E(\tilde C(\mu_1))^n\mathcal{A}_o \subset \mathcal{Z}^n_{T,\ell>0}\mathcal{A}_o$. □

Lemma 6.7. *When R is noetherian, for every positive integer $n > 0$ there exists a positive integer $n' > 0$ such that $\mathcal{F}_{o,n'} \subset (\mathcal{Z}_{T,\ell>0})^n\mathcal{H}$.*

Proof. By Lemma 5.3, we can choose a finite subset $X \subset \Lambda$ such that, for $(\lambda, w_0) \in \Lambda \times W_0$, we have $\ell(\lambda w_0) = \ell(\lambda x^{-1}) + \ell(x w_0)$ for some $x \in X$ with $\mu = \lambda x^{-1} \in \Lambda_T$. By the product formula , $E_o(\lambda w_0) = E_o(\mu)E_o(x w_0)$. If

$$\ell(\lambda w_0) \geqslant n' = na + \max\{\ell(xw) \mid (x, w) \in X \times W_0\},$$

we have $\ell(\mu) \geqslant na$. Taking a as in Lemma 6.6, $E_o(\mu) \in (\mathcal{Z}_{T,\ell>0})^n\mathcal{A}_0$; hence $E_o(\lambda w_0) \in (\mathcal{Z}_{T,\ell>0})^n\mathcal{H}$. As (λ, w_0) was arbitrary, we get the lemma. □

We define $\mathcal{F}^{\mathrm{aff}}_{o,n}$ as $\mathcal{F}_{o,n}$, with $W(1)$ replaced by $W^{\mathrm{aff}}(1)$. The isomorphism (3) restricts to an isomorphism

$$\mathcal{F}^{\mathrm{aff}}_{o,n} \otimes_{R[Z_k]} R[\Omega(1)] \simeq \mathcal{F}_{o,n}. \tag{44}$$

The based root system (Φ, Δ) is the finite disjoint union of irreducible based root systems (Φ_i, Δ_i) for $1 \leqslant i \leqslant r$, the Coxeter affine Weyl group $(W^{\mathrm{aff}}, S^{\mathrm{aff}})$ is the product of the irreducible Coxeter affine Weyl groups $(W_i^{\mathrm{aff}}, S_i^{\mathrm{aff}})$, and $W^{\mathrm{aff}}(1)$ is an extension

$$1 \to Z_k \to W^{\mathrm{aff}}(1) \to \prod_i W_i^{\mathrm{aff}} \to 1.$$

The algebras $\mathcal{H}_i^{\mathrm{aff}}$ defined by (Φ_i, Δ_i) identify with the subalgebras of basis $(T_w)_{w \in W_i^{\mathrm{aff}}(1)}$ of $\mathcal{H}^{\mathrm{aff}}$, called the irreducible components of $\mathcal{H}^{\mathrm{aff}}$.

Lemma 6.8. *The filtrations of $\mathcal{H}^{\mathrm{aff}}$*

$$(\mathcal{F}^{\mathrm{aff}}_{o,n})_{n\in\mathbb{N}}, \qquad \Big(\sum_i \mathcal{F}^{\mathrm{aff}}_{i,o,n}\mathcal{H}^{\mathrm{aff}}\Big)_{n\in\mathbb{N}}$$

are equivalent.

Proof. The length of $w_i \in W_i^{\mathrm{aff}}$ seen as an element of $(W_i^{\mathrm{aff}}, S_i^{\mathrm{aff}})$ or of $(W^{\mathrm{aff}}, S^{\mathrm{aff}})$ is the same; hence

$$\mathcal{F}^{\mathrm{aff}}_{i,o,n} \subset \mathcal{F}^{\mathrm{aff}}_{o,n}.$$

For $w \in W^{\mathrm{aff}}$ of components $w_i \in W_i^{\mathrm{aff}}$, we have $\ell(w) = \sum_i \ell(w_i)$ and $E_o(w) = \prod_i E_o(w_i)$ by the product formula, and the factors $E_o(w_i)$ commute. If $\ell(w) \geqslant nr$, at least one component w_i satisfies $\ell(w_i) \geqslant n$; hence

$$\mathcal{F}^{\mathrm{aff}}_{o,n} \subset \sum_i \mathcal{F}^{\mathrm{aff}}_{i,o,n}\mathcal{H}^{\mathrm{aff}}.$$ □

Proposition 6.9. *Let M be a right $\mathcal{H}$-module, and let o be an orientation. The following properties are equivalent.*

(1) *There exists a positive integer n such that $M\mathcal{F}_{o,n} = 0$.*

(2) *There exists a positive integer n such that $M(\mathcal{Z}_{\ell>0})^n = 0$.*

(3) *There exists a positive integer n such that $M(\mathcal{Z}_{T,\ell>0})^n = 0$.*

(4) *There exists a positive integer n such that $M\mathcal{F}_{T,o,n} = 0$.*

(5) *There exists a positive integer n such that $M\mathcal{F}^{\mathrm{aff}}_{o,n} = 0$.*

(6) *There exists a positive integer n such that $M\mathcal{F}^{\mathrm{aff}}_{i,o,n} = 0$ for $1 \leqslant i \leqslant r$.*

Proof. The isomorphism (44) shows that $M\mathcal{F}_{o,n} = 0 \Leftrightarrow M\mathcal{F}^{\mathrm{aff}}_{o,n} = 0$, because the action of $\Omega(1)$ is invertible. Applying Proposition 6.4 and Lemma 6.8, the properties are equivalent. □

Definition 6.10. A right $\mathcal{H}$-module M is called supersingular if it is not 0 and satisfies the properties of Proposition 6.9.

For future reference, we present the properties of the supersingular right $\mathcal{H}$-modules M deduced easily from Proposition 6.9 and Lemma 6.3, as a proposition. For a right $\mathcal{H}$-module M, we have the right $\mathcal{H}$-module $\iota(M)$, equal to M with $h \in \mathcal{H}$ acting by $\iota(h)$.

Proposition 6.11. (1) *The category of supersingular right $\mathcal{H}$-modules is stable by subquotients, by extensions, and by finite sums.*

(2) *A right $\mathcal{H}$-module is supersingular if and only it is supersingular as a right $\mathcal{H}^{\mathrm{aff}}$-module.*

(3) *A right $\mathcal{H}$-module generated by a supersingular right $\mathcal{H}^{\mathrm{aff}}$-submodule is supersingular.*

(4) *A right $\mathcal{H}^{\mathrm{aff}}$-module is supersingular if and only if it is supersingular as a right $\mathcal{H}_i^{\mathrm{aff}}$-module for all the irreducible components $\mathcal{H}_i^{\mathrm{aff}}$ of $\mathcal{H}^{\mathrm{aff}}$.*

(5) *A right $\mathcal{H}$-module M is supersingular if and only if $\iota(M)$ is supersingular.*

(6) *A simple right $\mathcal{H}$-module M is supersingular if and only if $M\mathcal{Z}_{\ell>0} = 0 \Leftrightarrow M\mathcal{Z}_{T,\ell>0} = 0$.*

The properties in (vi) are also equivalent to $M\mathcal{F}_{T,o,1} = 0$. See Remark 6.16.

The classification of the supersingular simple $\mathcal{H}$-modules reduces to the classification of the supersingular characters of $\mathcal{H}^{\mathrm{aff}}$. For the algebra $\mathcal{H}(G, I(1))$, this was a conjecture for $G = GL(n, F)$ [13] proved in [11, Proposition 5.10] for G split.

Proposition 6.12. *A supersingular right $\mathcal{H}$-module M contains a character of $\mathcal{H}^{\mathrm{aff}}$.*

Proof. A non-zero element of M generates a right $\mathfrak{h}$-module containing a character of $\mathfrak{h}$ (Proposition 2.1). We choose a $\mathfrak{h}$-eigenvector $v \in M$ of eigenvalue η. Let (χ, S_η) be the parameters of η (Proposition 2.2). As M is supersingular, there exists a positive integer n such that $M\mathcal{F}_{o,n} = 0$. We choose $d \in \mathcal{D}$ of maximal length satisfying $vE_o(\tilde{d}) \neq 0$ (Proposition 3.3). We show that $vE_o(\tilde{d})$ is a $\mathcal{H}^{\mathrm{aff}}$-eigenvector. Let $(t, s) \in Z_k \times S^{\mathrm{aff}}$.

We have $vE_o(\tilde{d})T_t = vT_{dtd^{-1}}E_o(\tilde{d}) = \chi(dtd^{-1})vE_o(\tilde{d}) = \chi^d(t)vE_o(\tilde{d})$.

For the computation of $vE_o(\tilde{d})T_{\tilde{s}}$, we distinguish three cases.

(1) $\ell(ds) = \ell(d) - 1$. Then $E_o(\tilde{d}) = T_t E_o(\tilde{d}\tilde{s})E_o(\tilde{s})$, where $t \in Z_k, t\tilde{d}\tilde{s}^2 = \tilde{d}$.

If $E_o(\tilde{s}) = T_{\tilde{s}} - c_{\tilde{s}}$, we have $E_o(\tilde{s})T_{\tilde{s}} = (T_{\tilde{s}} - c_{\tilde{s}})T_{\tilde{s}} = 0$.

If $E_o(\tilde{s}) = T_{\tilde{s}}$, we have $E_o(\tilde{s})T_{\tilde{s}} = T_{\tilde{s}}^2 = c_{\tilde{s}}T_{\tilde{s}} = c_{\tilde{s}}E_o(\tilde{s})$; as $E_o(\tilde{d}\tilde{s})c_{\tilde{s}} = (ds \bullet c_{\tilde{s}})E_o(\tilde{d}\tilde{s}) = d \bullet c_{\tilde{s}}E_o(\tilde{d}\tilde{s})$, we deduce that $vE_o(\tilde{d})T_{\tilde{s}} = 0$ or $\chi(d \bullet c_{\tilde{s}})vE_o(\tilde{d}) = \chi^d(c_{\tilde{s}})vE_o(\tilde{d})$.

(2) $\ell(ds) = \ell(d) + 1$ and $ds \in Z_k\mathcal{D}$. Either $E_o(\tilde{d})T_{\tilde{s}} = E_o(\tilde{d})E_o(\tilde{s}) = E_o(\tilde{d}\tilde{s})$ or $E_o(\tilde{d})T_{\tilde{s}} = E_o(\tilde{d})(E_o(\tilde{s}) + c_{\tilde{s}}) = E_o(\tilde{d}\tilde{s}) + (d \bullet c_{\tilde{s}})E_o(\tilde{d})$. By the maximality of $\ell(d)$, $vE_o(\tilde{d}\tilde{s}) = 0$ and $vE_o(\tilde{d})T_{\tilde{s}} = 0$ or $\chi(d \bullet c_{\tilde{s}})vE_o(\tilde{d}) = \chi^d(c_{\tilde{s}})vE_o(\tilde{d})$.

(3) $\ell(ds) = \ell(d) + 1$ and $ds \notin Z_k\mathcal{D}$. Let $s_d \in S$ such that $\tilde{d}\tilde{s} = \tilde{s}_d\tilde{d}$ (Lemma 3.8). Either $E_o(\tilde{d})T_{\tilde{s}} = E_o(\tilde{d})E_o(\tilde{s}) = E_o(\tilde{d}\tilde{s}) = E_o(\tilde{s}_d\tilde{d}) = E_o(\tilde{s}_d)E_o(\tilde{d})$ or $E_o(\tilde{d})T_{\tilde{s}} = E_o(\tilde{d})(E_o(\tilde{s}) + c_{\tilde{s}}) = E_o(\tilde{d}\tilde{s}) + E_o(\tilde{d})c_{\tilde{s}} = (E_o(\tilde{s}_d) + d \bullet c_{\tilde{s}})E_o(\tilde{d})$. Hence $vE_o(\tilde{d})T_{\tilde{s}} = \eta(E_o(\tilde{s}_d))vE_o(\tilde{d})$ or $\eta(E_o(\tilde{s}_d) + d \bullet c_{\tilde{s}})vE_o(\tilde{d}) = (\eta(E_o(\tilde{s}_d)) + \chi(d \bullet c_{\tilde{s}}))vE_o(\tilde{d}) = (\eta(E_o(\tilde{s}_d)) + \chi^d(c_{\tilde{s}}))vE_o(\tilde{d})$. □

The compatibility of supersingularity for $\mathcal{H}$ and $\mathcal{H}^{\mathrm{aff}}$ (Proposition 6.9) and Proposition 6.12 imply the following.

Corollary 6.13. (1) *A simple supersingular right $\mathcal{H}^{\mathrm{aff}}$-module has dimension* 1.

(2) *A simple right $\mathcal{H}$-module is supersingular*

if and only if it contains a supersingular character of $\mathcal{H}^{\mathrm{aff}}$;

if and only if any simple right $\mathcal{H}^{\mathrm{aff}}$-submodule is a supersingular character of $\mathcal{H}^{\mathrm{aff}}$.

The classification of the supersingular characters of $\mathcal{H}^{\mathrm{aff}}$, given in Theorem 6.15 after technical Lemma 6.14, follows from the classification of the characters of $\mathcal{H}^{\mathrm{aff}}$ (Proposition 2.2). The classification was done for $\mathcal{H}(G, I(1))$ in [13] for $G = GL(n, F)$ and in [11, Lemma 5.11 and Theorem 5.13] for G split.

Let Ξ be a character of $\mathcal{H}^{\mathrm{aff}}$, χ a character of Z_k, and o an orientation such that $\Xi|_{\mathfrak{h}} = \chi_o$ (Lemma 2.4). Let $w_o \in W_0$ such that the Weyl chamber of o is $w_o^{-1}(\mathfrak{D}^+)$. For a subset J of S, let w_J be the longest element of the subgroup of W_0 generated by J.

Lemma 6.14. (1) $\Xi(E(\tilde{C}(\mu)) = \Xi(E_o(\tilde{\mu}))$ *for* $\mu \in \Lambda_T^+$.

(2) *If* $S^{\mathrm{aff}} - S = \{s_0\}$ *and* $\lambda \in \Lambda^+$ *has positive length, we have*

(i) $\ell(s_0\lambda) = -1 + \ell(\lambda)$;

(ii) $E_o(\tilde{s}_0) = T_{\tilde{s}_0} \Leftrightarrow w_o(\alpha_0) \in \Sigma^+$, *where* α_0 *is the highest root of* Σ^+;

(iii) $E_o(\tilde{\lambda}) = T_{\tilde{s}_0} E_{o\bullet s_0}(\tilde{s}_0^{-1}\tilde{\lambda}))$ *if* $w_o(\alpha_0) \in \Sigma^+$;

(iv) $w_J(\alpha_0) \in \Sigma^+ \Leftrightarrow J \neq S$.

Proof. (1) The character ξ factorizes through the canonical homomorphism

$$h \mapsto 1 \otimes h : \mathcal{H}^{\mathrm{aff}} \to \xi|_{\mathfrak{h}} \otimes_{\mathfrak{h}} \mathcal{H}^{\mathrm{aff}},$$

and $1 \otimes E(\tilde{C}(\mu)) = 1 \otimes E_o(\tilde{\mu})$ in $\chi_o \otimes_{\mathfrak{h}} \mathcal{H}^{\mathrm{aff}}$ by Proposition 5.4.

(2) The hypothesis means that the root system Σ is irreducible. The highest positive root $\alpha_0 \in \Sigma^+$ has the following well-known properties: $-\alpha_0 + 1$ is a simple affine root and $s_0 = s_{-\alpha_0+1}$, $0 < -\alpha_0(x) + 1 < 1$ for $x \in \mathfrak{C}^+$.

(i) $\ell(s_0\lambda) = -1 + \ell(\lambda) \Leftrightarrow \mathfrak{C}^+$ and $\mathfrak{C}^+ + \nu(\lambda)$ are not on the same side of $\mathrm{Ker}(-\alpha_0 + 1)$ [15, Example 5.4]. This is equivalent to $-\alpha_0(x + \nu(\lambda)) + 1 = -\alpha_0(x) + 1 - \alpha_0 \circ \nu(\lambda)$ is negative for $x \in \mathfrak{C}^+ \Leftrightarrow \alpha_0 \circ \nu(\lambda) \geqslant 1$, which is true, because $\alpha_0 \circ \nu(\lambda) \in \mathbb{N}_{>0}$ as $\lambda \in \Lambda^+$ has positive length [15, Corollary 5.11].

(ii) By (6), $E_o(\tilde{s}_0) = T_{\tilde{s}_0} \Leftrightarrow \mathfrak{C}^+$ is on the o-negative side of $\mathrm{Ker}(-\alpha_0 + 1)$. By [15, Definition 5.16], this means that $-\alpha_0$ is o-negative, because $-\alpha_0 + 1$ is positive on $\mathfrak{C}^+$. The root $-\alpha_0$ is o-negative if and only if α_0 is positive on the Weyl chamber $w_o^{-1}(\mathfrak{D}^+)$ of o. This is true if and only if $w_o(\alpha_0) \in \Sigma^+$.

(iii) For any orientation o, $E_o(\tilde{\lambda}) = E_o(\tilde{s}_0) E_{o\bullet s_0}(\tilde{s}_0^{-1}\tilde{\lambda})$ by the product formula and $\ell(\lambda) = 1 + \ell(s_0\lambda)$ (i). Apply (ii).

(iv) Let $S = J \cup J'$. We have $\alpha_0 = (\sum_{s\in J} n_s\alpha_s) + (\sum_{s\in J'} n_s\alpha_s)$ with $n_s \in \mathbb{N}_{>0}$, and $w_J(\alpha_0) = -(\sum_{s\in J} n_s\alpha_s) + (\sum_{s\in J'} n_s w_J(\alpha_s))$. If $J' = \emptyset$, then $w_J(\alpha_0) \notin \Sigma^+$. If $J' \neq \emptyset$, for any $s \in J'$, the root $w_J(\alpha_s)$ is positive and does not belong to the group generated by J. The decomposition of $w_J(\alpha_0)$ on the basis $(\alpha_s)_{s\in S}$ has a positive coefficient; i.e., $w_J(\alpha_0) \in \Sigma^+$. □

Theorem 6.15. *A character of* $\mathcal{H}^{\mathrm{aff}}$ *is supersingular if and only if its restriction to each irreducible component of* $\mathcal{H}^{\mathrm{aff}}$ *is not a twisted sign or trivial character.*

Proof. The involutive automorphism ι of $\mathcal{H}^{\mathrm{aff}}$ respects supersingularity and exchanges a twisted sign character and a twisted trivial character (Definition 2.7). For $s \in S^{\mathrm{aff}}$ and a character ξ of $\mathcal{H}^{\mathrm{aff}}$, ξ vanishes on T_s or $\iota(T_s)$ (Proposition 2.2). Let $\mu \in \Lambda_T^+$ of positive length. We have $\xi(E(\tilde{C}(\mu))) = \xi(E_o(\tilde{\mu}))$ for any orientation o (Lemma 6.14) and $E_o(\tilde{\mu}) = T_\mu$ when o is dominant [15, Example 5.30].

(i) A twisted sign character is not 0 on T_w for all $w \in W(1)$ of positive length; hence it is not 0 on $E(\tilde{C}(\mu))$, and it is not supersingular. Applying ι, a twisted trivial character is not supersingular.

(ii) It remains to prove that, when $\mathcal{H}^{\mathrm{aff}}$ is irreducible, i.e., $S^{\mathrm{aff}} - S = \{s_0\}$, a character ξ of $\mathcal{H}^{\mathrm{aff}}$ different from a twisted sign or trivial character is supersingular.

Applying ι, it suffices to prove it when $\xi(T_{\tilde{s}_0}) = 0$. The set $J = S - \{s \in S \mid \xi(T_{\tilde{s}}) \neq 0\}$ is different from S, because ξ is not a twisted sign character. Let o be the orientation of Weyl chamber $w_J^{-1}(\mathfrak{D}^+)$. By Lemma 2.6, the restriction of ξ to $\mathfrak{h}$ is of the form χ_o, because $S_o = \{s \in S \mid \xi(T_{\tilde{s}}) \neq 0\}$ (5). Applying Lemma 6.14, we obtain, for any $\mu \in \Lambda_T^+$ of positive length,

$$E_o(s_0) = T_{\tilde{s}_0}, \quad E_o(\tilde{\mu}) = T_{\tilde{s}_0} E_{o \bullet s_0}((\tilde{s}_0)^{-1}\tilde{\mu}), \quad \xi(E(C(\tilde{\mu}))) = \xi(E_o(\tilde{\mu})) = 0.$$

Hence ξ is supersingular. □

Remark 6.16. We can complete Proposition 6.11(6): a simple $\mathcal{H}$-module M is supersingular if and only if $M\mathcal{F}_{T,o,1} = 0$. This follows from Corollary 6.13 and part (ii) in the proof of Theorem 6.15.

Clifford's theory studies classically the induction of representations from normal subgroups. We give a "Clifford's theory style" proposition to describe the simple finite-dimensional $\mathcal{H}$-modules containing a character of $\mathcal{H}^{\mathrm{aff}}$, as in [13, Proposition 3], [11, Lemma 5.12] for the algebra $\mathcal{H}(G, I(1))$ when G is split.

Let R be a field, and let A be an R-algebra of the form $A = JB$, where J is an ideal of A and B a subalgebra of A equal to the R-algebra $R[G]$ of a group G.

Let $\Xi : J \to R$ be a character of J with a G-fixator $G_\Xi = \{g \in G \mid \Xi^g = \Xi\}$ of Ξ of finite index in G, where Ξ^g is the character $j \mapsto \Xi^g(j) = \Xi(gjg^{-1})$ of J.

Let V be a finite-dimensional right $R[G_\Xi]$-module, where the group $J \cap G$ acts by $\Xi|_{J\cap G}$. For $g \in G$, we denote by V^g the right $R[g^{-1}G_\Xi g]$-module V, where $g^{-1}hg$ acts by h for $h \in G_\Xi$.

We extend V to a right $A_\Xi = JR[G_\Xi]$-module, where J acts by Ξ, denoted by $\Xi \otimes V$. We induce $\Xi \otimes V$ to a right A-module

$$I(\Xi, V) = (\Xi \otimes V) \otimes_{A_\Xi} A.$$

Proposition 6.17. *Let R, A, J, G, Ξ, V be as above. We suppose V to be simple. We have the following.*

(i) *$I(\Xi, V)$ is finite dimensional and is a simple right A-module.*

(ii) *A finite-dimensional simple right A-module containing Ξ as a J-module is isomorphic to $I(\Xi, V)$ for some V.*

(iii) *$I(\Xi_1, V_1) \simeq I(\Xi_2, V_2)$ if and only if $(\Xi_2, V_2) = (\Xi_1^g, V_1^g)$, for some element $g \in G$.*

Proof [11, Lemma 5.12]. $\Xi \otimes V$ is finite dimensional and is a simple A_Ξ-module, because its restriction to the subalgebra $R[G_\Xi]$ satisfies these properties. The left A_Ξ-module $A = \bigoplus_{g \in G_\Xi \backslash G} A_\Xi g$ is free of finite rank. The restriction of $I(\Xi \otimes V)$ to J is isomorphic to a direct sum $\bigoplus^{\dim_R V} \bigoplus_{g\in G_\Xi\backslash G} \Xi^g$, and $I(\Xi, V) = \bigoplus_{g\in G_\Xi\backslash G}(\Xi^g \otimes V^g)$ is equal to the direct sum of all the conjugates of $\Xi\otimes V$ by G. The dimension of $I(\Xi\otimes V)$ is finite, equal to $[G : G_\Xi] \dim_R V$. The restriction to J of a non-zero A-submodule of $I(\Xi\otimes V)$ contains a submodule isomorphic to $\bigoplus_{g\in G_\Xi\backslash G} \Xi^g$; hence its Ξ-isotypic component is not 0. The Ξ-isotypic component of $I(\Xi\otimes V)$ is the simple A_Ξ-module $\Xi\otimes V$. Therefore $I(\Xi\otimes V)$ is a simple A-module.

Let M be a finite-dimensional simple right A-module with a non-zero Ξ-isotypic component as a J-module. The Ξ-isotypic component is an A_Ξ-module of the form $\Xi\otimes V'$ for some finite-dimensional right $R[G_\Xi]$-module V'. The non-zero $R[G_\Xi]$-module V' contains a simple submodule V. The module $\Xi\otimes V$ is isomorphic to an A_Ξ-submodule of M, and $I(\Xi\otimes V)$ is isomorphic to an A-submodule of M. As M is simple, $M = I(\Xi, V)$.

The restriction of $I(\Xi\otimes V)$ to J shows that $I(\Xi\otimes V)$ determines the G-orbit of Ξ, the Ξ-isotypic part of $I(\Xi\otimes V)$ determines V, and the Ξ^g-isotypic part of $I(\Xi\otimes V)$ is $\Xi^g\otimes V^g$ for $g\in G$. This implies that $I(\Xi_1, V_1) \simeq I(\Xi_2, V_2)$ if and only if $(\Xi_2, V_2) = (\Xi_1^g, V_1^g)$, for some $g \in G$. □

We can apply Proposition 6.17 to the R-algebra $A = \mathcal{H}$, its ideal $J = \mathcal{H}^{\mathrm{aff}}$, the group $G = \Omega(1)$, an arbitrary character Ξ of $\mathcal{H}^{\mathrm{aff}}$, and a finite-dimensional simple right $R[\Omega(1)]$-module V such that Z_k acts on V by the character $\Xi|_{Z_k}$. As a subgroup of Ω of finite index acts trivially on V, the fixator $\Omega(1)_\Xi$ of Ξ has a finite index in $\Omega(1)$.

Corollary 6.13, Theorem 6.15, and Proposition 6.17 imply the following.

Theorem 6.18. *The supersingular simple finite-dimensional right $\mathcal{H}$-modules are isomorphic to the $\mathcal{H}$-modules $I(\Xi, V)$, where*

(i) *Ξ is a character of $\mathcal{H}^{\mathrm{aff}}$ different from a twisted sign or trivial character on each irreducible component of $\mathcal{H}^{\mathrm{aff}}$,*

(ii) *V is a simple finite-dimensional right $R[\Omega(1)_\Xi]$-module, where Z_k acts by $\Xi|_{Z_k}$.*

Two $\mathcal{H}$-modules $I(\Xi_1, V_1), I(\Xi_2, V_2)$ are isomorphic if and only if the pairs $(\Xi_1, V_1), (\Xi_2, V_2)$ are $\Omega(1)$-conjugate.

7. Pro-p-Iwahori invariants and compact induction

We use the notation of 1.3, and R is as in 1.4. The algebras $\mathcal{H}$ and $\mathfrak{h}$ denote the pro-p-Iwahori Hecke algebra $\mathcal{H}_R(G, I(1))$ and $\mathcal{H}_R(K, I(1))$.

Let ρ be an irreducible smooth R-representation of K, let $v\in\rho^{I(1)}$ not 0, let η be the character of $\mathfrak{h}$ on $\rho^{I(1)}$, let χ be the restriction of η to Z_k, and let o be an orientation such that $\eta = \chi_o$ (Lemma 2.4).

We show that the pro-p-Iwahori invariant functor behaves well on compact induced representations of G, generalizing the results of Ollivier [10, Corollary 3.14] proved when G is split.

By Cabanes [3, Theorem 2], the $I(1)$-invariant functor $\rho\mapsto\rho^{I(1)}$ gives an equivalence

- from the category of finite-dimensional R-representations ρ of K trivial on $K(1)$, such that ρ and its dual ρ^* are generated by $I(1)$;
- to the category of finite-dimensional right $\mathfrak{h}$-modules.

Remark 7.1. For $n \in N \cap K$ of image $w \in W_o(1)$, the action on $\rho^{I(1)}$ of the basis element $T_w \in \mathfrak{h}$ is

$$vT_w = \sum_{\gamma \in I(1)\backslash I(1)nI(1)} \gamma^{-1} v = \eta(T_w)v.$$

The action of Z_k on $\rho^{I(1)}$ arising from the action of $Z_0 \subset I$ normalizing $I(1)$ and the action of Z_k embedded in the Hecke algebra $\mathfrak{h}$ on $\rho^{I(1)}$ are *inverse* from each other.

Let

$$\text{c-Ind}_K^G \rho$$

be the compactly induced representation of G by right translations on the space of compactly supported functions $f : G \to V(\rho)$ satisfying $f(k_1 g) = \rho(k_1) f(g)$ for $k_1 \in K$ and $g \in G$. Let

$$[1, v]_K \in (\text{c-Ind}_K^G \rho)^{I(1)}$$

be the function of support K and value v at 1. The representation $\text{c-Ind}_K^G \rho$ is generated by $[1, v]_K$, and $\dim_R \rho^{I(1)} = 1$.

Proposition 7.2. *The $\mathcal{H}$-equivariant linear map*

$$\rho^{I(1)} \otimes_{\mathfrak{h}} \mathcal{H} \to (\text{c-Ind}_K^G \rho)^{I(1)} \quad 1 \otimes 1 \mapsto [1, v]_K$$

is an isomorphism.

We explain the strategy of the proof, which reduces the proposition to the next lemma.

The disjoint union of W into W_0-cosets corresponds to a disjoint union of G into (K, I)-cosets. A (K, I)-coset is equal to a $(K, I(1))$-coset. We have

$$G = \bigcup_{d \in \mathcal{D}} KdI = \bigcup_{d \in \mathcal{D}} K\tilde{d}I(1), \tag{45}$$

where, for d in the distinguished set $\mathcal{D}$ of representatives of $W_0\backslash W$ (Proposition 3.3), $KdI = K\tilde{d}I(1)$ denotes the double coset $Kn_{\tilde{d}}I = Kn_{\tilde{d}}I(1)$, $\tilde{d} \in \mathcal{D}(1)$ lifting d, and $n_{\tilde{d}} \in N$ lifting $\tilde{d}$, with $n_1 = 1$. The space $(\text{c-Ind}_K^G \rho)^{I(1)}$ is the direct sum

$$(\text{c-Ind}_K^G \rho)^{I(1)} = \bigoplus_{d \in \mathcal{D}} (\text{c-Ind}_K^{KdI} \rho)^{I(1)} \tag{46}$$

of the subspaces of functions in $(\text{c-Ind}_K^G \rho)^{I(1)}$ with support contained in KdI, for $d \in \mathcal{D}$. The pro-p-Iwahori Hecke algebra is the direct sum

$$\mathcal{H} = \bigoplus_{d \in \mathcal{D}} \mathfrak{h} T_{\tilde{d}} \tag{47}$$

of the left $\mathfrak{h}$-modules $\mathfrak{h}T_{\tilde{d}}$ of functions in $\mathcal{H}$ with support contained in KdI, for $d \in \mathcal{D}$. We denote by η the character of $\mathfrak{h}$ on $\rho^{I(1)}$, and by $f_{\tilde{d}}$ the function in $(\text{c-Ind}_K^G \rho)^{I(1)}$ of support KdI with $f(n_{\tilde{d}}) = v$, for $d \in \mathcal{D}$. We have $f_1 = [1, v]_K$. The proposition follows from the following lemma.

Lemma 7.3. (i) *For $d \in \mathcal{D}$, we have $K(1)(K \cap n_{\tilde{d}} I(1) n_{\tilde{d}}^{-1}) = I(1)$.*

(ii) *A basis of* $(\text{c-Ind}_K^G \rho)^{I(1)}$ *is* $(f_{\tilde{d}})_{d \in \mathcal{D}}$.

(iii) $f_{\tilde{d}} = f_1 T_{\tilde{d}}$ *for* $d \in \mathcal{D}$.

(iv) f_1 *is a* $\mathfrak{h}$*-eigenvector in* $(\text{c-Ind}_K^G \rho)^{I(1)}$ *of eigenvalue* η.

Proof. (1) We denote by I' the subgroup of $I(1)$ generated by $U \cap I = U \cap K$ and $U^- \cap I$. We have $I(1) = Z_0(1) I'$ and $Z_0(1) = K(1) \cap Z_0$. The lemma follows from the inclusion

$$U \cap I \subset n_{\tilde{d}} I' n_{\tilde{d}}^{-1},$$

because $K(1)(K \cap n_{\tilde{d}} I(1) n_{\tilde{d}}^{-1}) = K(1)(K \cap n_{\tilde{d}} I' n_{\tilde{d}}^{-1})$ is a pro-p-subgroup of K and $I(1) = K(1)(U \cap I)$ is a pro-p-Sylow subgroup of K. The group $U \cap I$ is the product of the groups $U_{\alpha,0} = U_\alpha \cap K$ for all α in the set Φ^+_{red} of positive reduced roots of associated root subgroup U_α. By Proposition 3.3 and § 1.3, $d^{-1}(e_\alpha \alpha)$ is positive on $\mathfrak{C}^+$. As e_α is a positive integer, $d^{-1}(\alpha)$ is positive on $\mathfrak{C}^+$. By [15, §§ 3.3 and 3.5], $n_{\tilde{d}}^{-1} U_{\alpha,0} n_{\tilde{d}} = U_{d^{-1}(\alpha)}$. As $d^{-1}(\alpha)$ is positive on $\mathfrak{C}^+$, $U_{d^{-1}(\alpha)} \subset I'$. Hence $U_{\alpha,0} \subset n_{\tilde{d}} I' n_{\tilde{d}}^{-1}$.

(2) By (46) and $f_{n_{\tilde{d}}} \in (\text{c-Ind}_K^{KdI(1)} \rho)^{I(1)}$, it suffices to prove that the dimension of $(\text{c-Ind}_K^{Kn_{\tilde{d}}I(1)} \rho)^{I(1)}$ is 1. The value at $n_{\tilde{d}}$ gives a linear map

$$(\text{c-Ind}_K^{KdI(1)} \rho)^{I(1)} \to \rho^{K \cap n_{\tilde{d}} I(1) n_{\tilde{d}}^{-1}},$$

because $kf(n_{\tilde{d}}) = f(kn_{\tilde{d}}) = f(n_{\tilde{d}} n_{\tilde{d}}^{-1} k n_{\tilde{d}})$ for $k \in K$. The map is clearly injective, and $\rho^{K \cap n_{\tilde{d}} I(1) n_{\tilde{d}}^{-1}} = \rho^{I(1)}$, because ρ is trivial on $K(1)$ and (1). As $\dim_R \rho^{I(1)} = 1$, we have $\dim_R (\text{c-Ind}_K^{Kn_{\tilde{d}}I(1)} \rho)^{I(1)} = 1$.

(3) We show that the support of the function $f_1 T_{n_{\tilde{d}}}$ is contained in $KdI(1)$ and that the value at $n_{\tilde{d}}$ of $f_1 T_{n_{\tilde{d}}}$ is v.

For $g \in G$, we have

$$f_1 T_g = \sum_{\gamma \in I(1) \backslash I(1) g I(1)} \gamma^{-1} f_1,$$

and $\gamma^{-1} f_1(x) = f_1(x \gamma^{-1})$ for $x \in G$. The support of f_1 is K, and the support of $f_1 T_g$ is contained in $KgI(1)$.

In particular, the support of the function $f_1 T_{n_{\tilde{d}}}$ is contained in $KdI(1)$. We have

$$(f_1 T_{n_{\tilde{d}}})(n_{\tilde{d}}) = \sum_{\gamma \in I(1) \backslash I(1) n_{\tilde{d}} I(1)} f_1(n_{\tilde{d}} \gamma^{-1}) = \sum_{u \in (K \cap n_{\tilde{d}} I(1) n_{\tilde{d}}^{-1})/(I(1) \cap n_{\tilde{d}} I(1) n_{\tilde{d}}^{-1})} f_1(u).$$

By (1), this is equal to $f_1(1) = v$.

(4) For $k \in K$, the support of the function $f_1 T_k$ is contained in K, and

$$(f_1 T_k)(1) = \sum_{\gamma \in I(1) \backslash I(1) k I(1)} f_1(\gamma^{-1}) = \sum_{\gamma \in I(1) \backslash I(1) k I(1)} \gamma^{-1} f_1(1) = \eta(T_k) v.$$

Therefore $f_1 T_k = \eta(T_k) f_1$ for $k \in K$. □

Remark 7.4. For $\lambda \in \Lambda$, the isomorphism of Proposition 7.2 restricts to a right $\mathfrak{h}$-module isomorphism

$$\rho^{I(1)} \otimes_{\mathfrak{h}} \mathfrak{h}(\lambda) \to (\text{c-Ind}_K^{K\lambda K} \rho)^{I(1)}.$$

Proposition 7.5. *Let ρ_1, ρ be two irreducible smooth R-representations of K. The $I(1)$-invariant map*

$$\operatorname{Hom}_{RG}(\text{c-Ind}_K^G \rho_1, \text{c-Ind}_K^G \rho) \to \operatorname{Hom}_{\mathcal{H}}((\text{c-Ind}_K^G \rho_1)^{I(1)}, (\text{c-Ind}_K^G \rho)^{I(1)})$$

is an isomorphism.

To explain the strategy of the proof, we recall the adjunction isomorphisms

$$\operatorname{Hom}_{RG}(\text{c-Ind}_K^G \rho_1, \pi) \simeq \operatorname{Hom}_{RK}(\rho_1, \pi) = \operatorname{Hom}_{RK}(\rho_1, \pi^{K(1)}),$$
$$\operatorname{Hom}_{\mathcal{H}}(\rho_1^{I(1)} \otimes_{\mathfrak{h}} \mathcal{H}, \pi^{I(1)}) \simeq \operatorname{Hom}_{\mathfrak{h}}(\rho_1^{I(1)}, \pi^{I(1)}),$$

for any smooth R-representation π of G. The $I(1)$-invariant map

$$\operatorname{Hom}_{RG}(\text{c-Ind}_K^G \rho_1, \pi) \to \operatorname{Hom}_{\mathcal{H}}((\text{c-Ind}_K^G \rho_1)^{I(1)}, \pi^{I(1)})$$

is an isomorphism if and only if the $I(1)$-invariant map

$$\operatorname{Hom}_K(\rho_1, \pi^{K(1)}) \to \operatorname{Hom}_{\mathfrak{h}}(\rho_1^{I(1)}, \pi^{I(1)}) \tag{48}$$

is an isomorphism, by Proposition 7.2. The map (48) is injective, because $\rho^{I(1)}$ generates ρ, but it is not surjective in general. The proposition says that the map (48) is surjective if $\pi = \text{c-Ind}_K^G \rho$.

The dominant monoid Λ^+ represents the cosets $K\backslash G/K$ (see 1.3). The anti-dominant monoid Λ^- has the same property and is more convenient now. The representation of K on $\text{c-Ind}_K^G \rho$ is a direct sum

$$\text{c-Ind}_K^G \rho = \bigoplus_{\lambda \in \Lambda^-} \text{c-Ind}_K^{K\lambda K} \rho,$$

where $\text{c-Ind}_K^{K\lambda K} \rho$ is the space of functions in $\text{c-Ind}_K^G \rho$ with support in $K\lambda K$. We will prove that, for all $\lambda \in \Lambda^-$, the $I(1)$-invariant map

$$\operatorname{Hom}_K(\rho_1, (\text{c-Ind}_K^{K\lambda K} \rho)^{K(1)}) \to \operatorname{Hom}_{\mathfrak{h}}(\rho_1^{I(1)}, (\text{c-Ind}_K^{K\lambda K} \rho)^{I(1)}) \tag{49}$$

is an isomorphism. A representation of K trivial on $K(1)$ generated by its $I(1)$-invariant vectors identifies with a representation of the finite reductive group G_k generated by its U_k-invariant vectors (using the notation of § 1.3). We describe $(\text{c-Ind}_K^{K\lambda K} \rho)^{K(1)}$ as a representation of G_k. Let $z \in Z^-$ lifting λ. We have $KzK = K\lambda K$ and by [7, Proposition 6.13] the group

$$K_\lambda = K(1)(K \cap z^{-1}Kz)$$

is the inverse image in K of a parabolic subgroup $P_k = M_k N_k$ of G_k containing B_k, of unipotent radical N_k equal to the image in G_k of $\langle \bigcup_{\alpha \in \Phi^+, \alpha\circ\nu(z)<0} U_{\alpha,0}\rangle$ as $\nu(z)$ is anti-dominant and $\langle \alpha, z\rangle = \langle \alpha, -\nu(z)\rangle$ in the notation of [7, 6.11]; it is a parahoric subgroup of G of pro-p-radical $K_\lambda(1) = K(1)(K \cap z^{-1}K(1)z)$. Let ρ_z be the representation of $K \cap z^{-1}Kz$ on the space $V(\rho)$ of ρ such that $\rho_z(k) = \rho(zkz^{-1})$. The map $f \mapsto \phi : k \mapsto f(zk) : \mathrm{Ind}_K^{KzK} \rho \to \mathrm{Ind}_{K\cap z^{-1}Kz}^K \rho_z$ is a K-equivariant isomorphism. It restricts to a K-equivariant isomorphism

$$(\mathrm{Ind}_K^{KzK} \rho)^{K(1)} \to (\mathrm{Ind}_{K\cap z^{-1}Kz}^K \rho_z)^{K(1)} = \mathrm{Ind}_{K_\lambda}^K (\rho_z^{K(1)\cap z^{-1}Kz}),$$

where the natural representation of $K \cap z^{-1}Kz$ on $\rho_z^{K(1)\cap z^{-1}Kz}$ is extended to a representation of K_λ trivial on $K(1)$. The representation $\rho_z^{K(1)\cap z^{-1}Kz}$ of K_λ identifies to the representation $\rho_z^{N_k}$ of P_k on the space $V(\rho^{N_k})$ of ρ^{N_k} such that $\rho_z(m) = \rho(zmz^{-1})$ for m in the group $M_0 = \langle Z_0, \bigcup_{\alpha\in\Phi, \alpha\circ\nu(z)=0} U_{\alpha,0}\rangle$. The representation $\mathrm{Ind}_{K_\lambda}^K (\rho_z^{K(1)\cap z^{-1}Kz})$ identifies to $\mathrm{Ind}_{P_k}^{G_k}(\rho_z^{N_k})$. The representation $\rho_z^{N_k}$ of P_k is irreducible [2]. The U_k-invariant functor

$$\mathrm{Hom}_{G_k}(\rho_1, \mathrm{Ind}_{P_k}^{G_k}(\rho_z^{N_k})) \to \mathrm{Hom}_{\mathfrak{h}}(\rho_1^{U_k}, (\mathrm{Ind}_{P_k}^{G_k}(\rho_z^{N_k}))^{U_k}) \tag{50}$$

is an isomorphism, by Cabanes's equivalence recalled at the beginning of this section, because $\mathrm{Ind}_{P_k}^{G_k}(\rho_z^{N_k})$ and its contragredient are generated by their U_k-invariant vectors. This is a general property proved in the next lemma.

Lemma 7.6. *Let τ be an irreducible R-representation of P_k trivial on N_k. The representation $\mathrm{Ind}_{P_k}^{G_k} \tau$ of G_k and its contragredient are isomorphic to a subrepresentation and to a quotient of $\mathrm{Ind}_{U_k}^{G_k} 1$. In particular, they are generated by their U_k-invariant vectors. Their socle and their heads are multiplicity free.*

Proof. A representation of G_k is generated by its U_k-invariant vectors if and only if it is a quotient of a direct sum of representations isomorphic to $\mathrm{Ind}_{U_k}^{G_k} 1$.

The representation $\mathrm{Ind}_{P_k}^{G_k} \tau$ is a quotient of $\mathrm{Ind}_{U_k}^{G_k} 1$, because it is generated by a U_k-invariant vector (a function in $\mathrm{Ind}_{P_k}^{G_k} \tau$ of support P_k with non-zero value in $\tau^{U_k\cap M_k}$).

The inflation of τ to P_k is contained in $\mathrm{Ind}_{U_k}^{P_k} 1$. By transitivity of the induction, $\mathrm{Ind}_{P_k}^{G_k} \tau$ is contained in $\mathrm{Ind}_{U_k}^{G_k} 1$.

The contragredient representation $(\mathrm{Ind}_{P_k}^{G_k} \tau)^*$ is a subrepresentation and a quotient of $\mathrm{Ind}_{U_k}^{G_k} 1$, because $\mathrm{Ind}_{U_k}^{G_k} 1$ is isomorphic to its contragredient, the contragredient permutes the irreducible R-representations of M_k, and it commutes with the parabolic induction.

The socle of a subrepresentation of $\mathrm{Ind}_{U_k}^{G_k} 1$ is contained in the socle of $\mathrm{Ind}_{U_k}^{G_k} 1$. The socle of $\mathrm{Ind}_{U_k}^{G_k} 1$ is multiplicity free, because $\dim \rho_{U_k} = 1$, and by adjunction $\mathrm{Hom}_{G_k}(\rho, \mathrm{Ind}_{U_k}^{G_k} 1) \simeq \mathrm{Hom}_{U_k}(\rho_{U_k}, 1)$ for any irreducible R-representation ρ of G_k of U_k-coinvariants ρ_{U_k}.

The contragredient of the socle is the head of the contragredient. □

With (51) and the $I(1)$-invariant functor (Proposition 7.5 for $\rho_1 = \rho$), we transfer our results on the spherical algebra $\mathcal{H}(\eta, \mathfrak{h})$ to the spherical algebra $\mathcal{H}_R(G, K, \rho)$, which is the convolution algebra of compactly supported functions

$$\phi : G \to \mathrm{End}_R(V(\rho)) \text{ satisfying } \phi(k_1 g k_2) = \rho(k_1)\phi(g)\rho(k_2) \quad \text{for } k_1, k_2 \in K, g \in G.$$

It is isomorphic to the algebra $\mathrm{End}_{RG}\,\text{c-Ind}_K^G\,\rho$ by the map sending ϕ to the RG-intertwiner E_ϕ of $\text{c-Ind}_K^G\,\rho$ defined by

$$E_\phi(f_1)(g) = \phi(g)(v) \quad (g \in G). \tag{51}$$

The spherical Hecke R-algebra $\mathcal{H}_R(G, K, \rho)$ admits a natural basis [7, 7.3] $(\mathcal{F}_{\tilde{\lambda}})_{\lambda\in\Lambda_\chi^+}$, where

$$\mathcal{F}_{\tilde{\lambda}} \text{ has support } K\lambda K \quad \text{and} \quad \mathcal{F}_{\tilde{\lambda}}(\tilde{\lambda})(v) = v. \tag{52}$$

The basis $(\mathcal{F}_{\tilde{\lambda}})_{\lambda\in\Lambda_\chi^+}$ does not satisfy (14) in general. The basis (52) for the spherical Hecke algebra $\mathcal{H}_R(Z, Z_0, \chi)$ is denoted by $(\tau_{\tilde{\lambda}})_{\lambda\in\Lambda_\chi}$,

$$\tau_{\tilde{\lambda}} \text{ has support } Z_0\lambda \quad \text{and} \quad \tau_{\tilde{\lambda}}(\tilde{\lambda})(v) = v.$$

The basis (52) for the central spherical Hecke subalgebra $\mathcal{H}_R(T, T_0, \rho^{I(1)})$ is $(\tau_{\tilde{\mu}})_{\mu\in\Lambda_T}$, and the $\mathcal{H}_R(T, T_0, \rho^{I(1)})$-module $\mathcal{H}_R(Z, Z_0, \rho^{I(1)})$ is finitely generated. We denote by $\mathcal{H}_R(T^+, T_0, \rho^{I(1)}) \subset \mathcal{H}_R(Z^+, Z_0, \rho^{I(1)})$ the subalgebras of bases $(\tau_{\tilde{\mu}})_{\mu\in\Lambda_T^+}$ and $(\tau_{\tilde{\lambda}})_{\lambda\in\Lambda_\chi^+}$. The basis $(\tau_{\tilde{\lambda}})_{\lambda\in\Lambda_\chi^+}$ satisfies (14).

Theorem 7.7. *The R-algebras*

$$\mathcal{H}_R(G, K, \rho) \simeq \mathrm{End}_{RG}\,\text{c-Ind}_K^G\,\rho \simeq \mathrm{End}_{\mathcal{H}}(\eta \otimes_{\mathfrak{h}} \mathcal{H}) = \mathcal{H}(\eta, \mathfrak{h})$$

are isomorphic via (51) and the $I(1)$-invariant functor (Proposition 7.5).

The basis $(\mathcal{F}_{\tilde{\lambda}})_{\lambda\in\Lambda_\chi^+}$ of $\mathcal{H}_R(G, K, \rho)$ (52) corresponds to the basis $(\mathcal{E}_{\tilde{\lambda}})_{\lambda\in\Lambda_\chi^+}$ of $\mathcal{H}(\eta, \mathfrak{h})$ (Proposition 4.4).

The basis $(\phi_{o,\tilde{\lambda}})_{\lambda\in\Lambda_\chi^+}$ of $\mathcal{H}_R(G, K, \rho)$ corresponding to the basis $(\Phi_{o,\tilde{\lambda}})_{\lambda\in\Lambda_\chi^+}$ of $\mathcal{H}(\eta, \mathfrak{h})$ (Proposition 4.13) satisfies (14).

For $\mu \in \Lambda_T^+$, $\phi_{\tilde{\mu}} = \phi_{o,\tilde{\mu}}$ does not depend on the choice of o.

$(\phi_{\tilde{\mu}})_{\mu\in\Lambda_T^+}$ is a basis of a central subalgebra $\mathcal{Z}_R(G, K, \rho)_T$ of $\mathcal{H}_R(G, K, \rho)$, and $\mathcal{H}_R(G, K, \rho)$ is a finitely generated $\mathcal{Z}_R(G, K, \rho)_T$-module (Proposition 5.4).

Remark 7.8. The RG-endomorphism of $\text{c-Ind}_K^G\,\rho$ corresponding to $\phi_{\tilde{\mu}}$ sends $[1, v]_K$ to $[1, v]_K\,E_o(\tilde{\mu})$ for any orientation o such that $\eta = \chi_o$ (Propositions 7.2 and 4.13).

We denote by $\mathcal{A}_{o,T}^+$ the R-algebra of basis $(1 \otimes E_o(\tilde{\mu}))_{\mu\in\Lambda_T^+}$.

Corollary 7.9. *We have an R-algebra isomorphism*

$$(\phi_{o,\tilde{\lambda}})_{\lambda\in\Lambda_\chi^+} \mapsto (\tau_{\tilde{\lambda}})_{\lambda\in\Lambda_\chi^+} : \mathcal{H}_R(G, K, \rho) \xrightarrow{\mathcal{S}_o} \mathcal{H}_R(Z^+, Z_0, \chi)$$

restricting to an isomorphism $\mathcal{Z}_R(G, K, \rho)_T \xrightarrow{\mathcal{S}_T} \mathcal{H}_R(T^+, T_0, \chi)$ independent of o. We have the R-algebra isomorphisms

$$\mathcal{Z}_T \to \mathcal{Z}_R(G, K, \rho)_T \xrightarrow{\mathcal{S}_T} \mathcal{H}_R(T^+, T_0, \chi) \to \mathcal{A}_{o,T}^+ \to R[\tilde{\Lambda}_T^+] \to R[\Lambda_T^+]$$
$$(E(\tilde{C}(\mu)))_{\mu\in\Lambda_T^+} \to (\phi_{\tilde{\mu}})_{\mu\in\Lambda_T^+} \to (\tau_{\tilde{\mu}})_{\mu\in\Lambda_T^+} \to (E_o(\tilde{\mu}))_{\mu\in\Lambda_T^+} \to (\tilde{\mu})_{\mu\in\Lambda_T^+} \to (\mu)_{\mu\in\Lambda_T^+}.$$

When the group G is split, $(Z^+, Z_0) = (T^+, T_0)$ and $\mathcal{Z}_R(G, K, \rho)_T = \mathcal{H}_R(G, K, \rho)$.

Theorem 1.5 in §1 follows from Corollary 7.9 and the next proposition. The R-characters ξ of Λ_T^+ identify with the characters of the R-algebras isomorphic to $R[\tilde{\Lambda}_T^+]$ in Corollary 7.9. We write

$$\xi(\tau_{\tilde{\mu}}) = \xi(E(\tilde{C}(\mu))) = \xi(\phi_{\tilde{\mu}}) = \xi(E_o(\tilde{\mu})) = \xi(\tilde{\mu}) = \xi(\mu)$$

for $\mu \in \Lambda_T^+$. Let π be a smooth R-representation of G. We suppose that $\pi|_K$ contains ρ.

Proposition 7.10. *Let $A \in \mathrm{Hom}_{RK}(\rho, \pi)$ be non-zero, and let $\mu \in \Lambda_T^+$. We have*

$$(A\phi_{\tilde{\mu}})(v) = A(v)E_o(\tilde{\mu}) = A(v)E(\tilde{C}(\mu)).$$

In particular, if A is a $\mathcal{Z}_R(G, K, \rho)_T$-eigenvector in $\mathrm{Hom}_{RK}(\rho, \pi)$ of eigenvalue ξ,

$$\xi(\tilde{\mu})A(v) = A(v)E_o(\tilde{\mu}) = A(v)E(\tilde{C}(\mu)).$$

Proof. By the adjunction isomorphism, A and $A\phi_{\tilde{\mu}}$ correspond to the RG-intertwiners $\text{c-Ind}_K^G \rho \to \pi$ sending $[1, v]_K$ to $A(v)$ and to $A(v)E_o(\tilde{\mu})$ (Remark 7.8). We deduce that $(A\phi_{\tilde{\mu}})(v) = A(v)E_o(\tilde{\mu})$.

The $\mathcal{H}$-isomorphism $(\text{c-Ind}_K^G \rho)^{I(1)} \to \chi_o \otimes_{\mathfrak{H}} \mathcal{H}$ of Proposition 7.2 sends $[1, v]_K E(\tilde{C}(\mu))$ to $1 \otimes E(\tilde{C}(\mu))$. By Proposition 5.4, $1 \otimes E(\tilde{C}(\mu)) = 1 \otimes E_0(\tilde{\mu})$. Hence $[1, v]_K E(\tilde{C}(\mu)) = [1, v]_K E_o(\tilde{\mu})$. Applying the $\mathcal{H}$-intertwiner $(\text{c-Ind}_K^G \rho)^{I(1)} \to \pi^{I(1)}$ corresponding to A sending $[1, v]_K$ to $A(v)$, we deduce that $A(v)E_o(\tilde{\mu}) = A(v)E(\tilde{C}(\mu))$.

If A is a $\mathcal{Z}_R(G, K, \rho)_T$-eigenvector in $\mathrm{Hom}_{RK}(\rho, \pi)$ of eigenvalue ξ (Corollary 7.9), we have $A\phi_{\tilde{\mu}} = \xi(\phi_{\tilde{\mu}})A$ for $\mu \in \Lambda_T^+$ (Theorem 7.7). □

For $J \subset \Delta$, we denote by μ_J an element of Λ_T^+ such that $\alpha \circ \nu(\mu_J) > 0$ for all $\alpha \in \Delta - J$.

Remark 7.11. Let ξ be an R-character of Λ_T^+. The character ξ is called supersingular if it satisfies the following three equivalent properties.

(1) $\xi(\mu) = 0$ for all $\mu \in \Lambda_T^+$ non-invertible in Λ_T^+.

(2) $\xi(\mu_J) = 0$ for any $J \neq \Delta$.

(3) For some $n \geqslant 1$, $\xi(\mu) = 0$ for all $\mu \in \Lambda_T^+$ with $\ell(\mu) > n$.

In Proposition 7.10, the eigenvalue ξ of A is supersingular if and only if the module $A(v)\mathcal{H}$ is supersingular (Definition 6.10).

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