# The pro-$p$ Iwahori Hecke algebra of a reductive $p$-adic group IV (Levi subgroup and central extension)

Vignéras Marie-France

November 21, 2017

**Abstract**

Let $R$ be a commutative ring and let $G$ be a connected reductive $p$-adic group. We compare the parahoric subgroups and the pro-$p$ Iwahori Hecke $R$-algebra of $G$ with those of groups naturally related to $G$, as a Levi subgroup $M$, a $z$-extension of $G$ (more generally a central extension $H$ of $G$), the derived group $G^{der}$ of $G$, the simply connected cover $G^{sc}$ of the derived group of $G$.

## Contents



Lu dans Lusztig Square: the image by $i : G_{sc}(F) \to G_{ad}(F)$ of an Iwahori subgroup of $G_{sc}(F)$ is called an Iwahori subgroup of $G_{ad}(F)$. This coincides with our definition if $i$ is surjective (Remark 6.6). Counter-example ?

Lu dans Gan-Savin metaplectic II. Suppose $p \neq 2$. Let $V^+$ and $V^-$ be two quadratic spaces of dimension $2n+1$, trivial discriminant, and trivial and non-trivial Hasse invariants, respectively. Then $SO(V^+)$ is a split, adjoint group of type $B_n$, while $SO(V^-)$ is its unique non-split inner form. Dans §3, description des sous-groupes ouverts compacts stabilisateurs de bons lattices, alcove poour le groupe symplectique.

The exceptional types are both simply connected and adjoint. $SL(n+1), Spin(2n+1), Sp(2n), Spin(2n)$ simply connected of types $A_n, B_n, C_n, D_n$ and $Spin(2n+1), Spin(2n)$ are double covers of $SO(2n+1), SO(2n)$ Section 1.11 of Carter's book Finite groups of Lie type.

# 1 Introduction

Let $F$ be a finite extension of the field of $p$-adic numbers or a field of Laurent series in one variable over a finite field of characteristic $p$. The residue field $k$ of $F$ is a finite field of characterictic $p$ and order $q$. Algebraic $F$-groups will be denoted by a bold capital letter and the group of their $F$-rational points by the same capital letter but not in bold. Let $\mathbf{G}$ be a connected reductive linear algebraic $F$-group and $G = \mathbf{G}(F)$ be the group of its $F$-rational points.

The parameters of the quadratic relations of the Iwahori-Matsumoto presentation of the (pro-$p$) Iwahori Hecke ring $\mathcal{H}_{\mathbb{Z}}(G, \mathfrak{U})$ of $G$ determine a priori the parameters of the quadratic relations in the (pro-$p$) Iwahori Hecke ring of a Levi subgroup $M$ of $G$, but the relation between the parameters for $G$ and for $M$ was not known, even for the complex Iwahori Hecke algebras of reductive split groups. The solution of this problem is simple: we extend the parameters to "parameter maps" and we show that the parameter maps of a Levi subgroup $M$ are the restrictions of the parameter maps for $G$. This is is new, even for the complex Iwahori Hecke algebras. A more elaborate comparison of the pro-$p$ Iwahori Hecke rings of $M$ and of $G$ with applications to the theory of parabolic induction for the Hecke algebras is given in [Vig5].

The main body of this article is the comparison of the pro-$p$ Iwahori Hecke rings of $G$ [Vig1] and of a central $F$-extension $H$ of $G$; for example, a $z$-extension, the simply connected extension $G_{sc}$ of the derived group $G_{der}$ of $G$. The property that an irreducible admissible $R$-representations of $G$ is supercuspidal if and only if its invariants by a pro-$p$-Iwahori subgroup $\mathfrak{U}$ is a supersingular $\mathcal{H}_R(G, \mathfrak{U})$-module, is reduced to the simplest case where $G$ is almost simple, simply connected and isotropic (a proof of this simple case is proved in [OV])

This work is motivated by the forthcoming articles [OV], [AHHV2] on irreducible $R$-representations of a reductive $p$-adic group $G$, and [Abe] on the classification of simple $\mathcal{H}_R(G, \mathfrak{U})$-modules, when $R$ is an algebraically closed field of characteristic $p$.

*Ackowledgements* I thank Abe, Henniart, Herzig, Ollivier for our discussions on the representations modulo $p$ of reductive $p$-adic groups or pro-$p$ Iwahori Hecke algebras, and the Mathematical Institute of Jussieu for a stimulating scientific environment.

# 2 Main definitions and results

## 2.1 Admissible datum

The structure of (pro-$p$) Iwahori Hecke rings of connected reductive $p$-adic groups inspired the notions of an admissible datum $\mathcal{W}$, of a parameter map $\mathfrak{c}$ of $(\mathcal{W}, R)$ where $R$ is a commutative ring, and of a splitting of $\mathcal{W}$; they give rise to $R$-algebras allowing flexibility to study (pro-$p$) Iwahori Hecke rings.

**Definition 2.1.** *[Vig3, §1.2] An admissible datum is a datum*

$$\mathcal{W} = (\Sigma, \Delta, \Omega, \Lambda, \nu, W, Z_k, W_1) \tag{1}$$

*consisting of:*

(i) *A reduced root system $\Sigma$ with basis $\Delta$.* We denote by $(V, \mathfrak{H}, \mathfrak{D}, \mathfrak{C})$ a real vector space $V$ of dual of basis $\Delta$ with a scalar product invariant by the finite Weyl group $W_0$ of $\Delta$, the set $\mathfrak{H}$ of affine hyperplanes of $V$ associated to the affine roots of $\Sigma$, $\mathfrak{H}_0 \subset \mathfrak{H}$ the set of hyperplanes containing 0, the dominant open Weyl chamber $\mathfrak{D}$, the alcove $\mathfrak{C} \subset \mathfrak{D}$ of $(V, \mathfrak{H})$ of vertex 0, $(W_0, S) \subset (W^{aff}, S^{aff})$ the finite and affine Weyl Coxeter systems, $H_s \in \mathfrak{H}$ the affine hyperplane fixed by $s \in S^{aff}$, $s_\alpha \in S$ the reflection with respect to $\operatorname{Ker} \alpha \in \mathfrak{H}$ for $\alpha \in \Delta$.

(ii) *Three abelian groups $\Omega, \Lambda, Z_k$ with $\Omega, \Lambda$ finitely generated and $Z_k$ finite.*

(iii) *A group with two semidirect product decompositions $W = \Lambda \rtimes W_0 = W^{aff} \rtimes \Omega$.*

(iv) *An exact sequence $1 \to Z_k \to W_1 \to W \to 1$.*

(v) *A $W_0$-equivariant homomorphism $\nu : \Lambda \to V$ giving an action of $\Lambda$ by translation on $(V, \mathfrak{H})$, and extending to an action of $W$ on $(V, \mathfrak{H})$, compatible with the action of $W^{aff}$ and where the action of $\Omega$ normalizes $\mathfrak{C}$.*

We denote by $\ell$ the length of $W$ and of $W_1$ inflating the length of the affine Weyl Coxeter system $(W^{aff}, S^{aff})$, by $\tilde{w}$ a lift in $W_1$ of an element $w \in W$ and by $X(1)$ the inverse image in $W_1$ of a subset $X \subset W$ as in [Vig1], [Vig2], [Vig3], [Vig5]. changer pour $X_1$ ou $W_1 = W(1)$ But in this article, if $X \subset W$ is a subgroup we will write often $X_1$ instead of $X(1)$ (in [AHHV], we write ${}_1X$), for example $W_1$. The set of elements of length 0 is $\Omega$ in $W$, and $\Omega_1$ in $W_1$. The set $S^{aff}$ is stable by conjugation by $\Omega$, the same holds true for $S^{aff}(1)$ and $\Omega_1$.

**Example 2.2.** If the reduced root system $\Sigma$ is trivial, there is no $(\Sigma, \Delta, \nu)$ and $W = \Omega = \Lambda$; we denote $\mathcal{W} = (\Lambda, Z_k, \Lambda_1)$.

The product of $\mathcal{W}$ (Definition 2.1) and of $\mathcal{W}' = (\Lambda', Z'_k, \Lambda'_1)$ with a trivial reduced root system, is an admissible datum with the same based root system $(\Sigma, \Delta)$:

$$\mathcal{W} \times \mathcal{W}' = (\Sigma, \Delta, \Omega \times \Lambda', \Lambda \times \Lambda', \nu \circ p, W \times \Lambda', Z_k \times Z'_k, W_1 \times \Lambda'_1)$$

where $\Lambda \times \Lambda' \xrightarrow{p} \Lambda$ is the first projector.

**Example 2.3.** We say that $\mathcal{W}$ is affine if the abelian group $\Omega$ is trivial, because $W = W^{aff}$; then $\mathcal{W} = (\Sigma, \Delta, \Lambda, \nu, W, Z_k, W_1)$ is determined by $(\Sigma, \Delta, Z_k, W_1)$.

We say that $\mathcal{W}$ is Iwahori if the finite abelian group $Z_k$ is trivial, because $W = W_1$; we denote $\mathcal{W} = (\Sigma, \Delta, \Omega, \Lambda, \nu, W)$.

If the two abelian groups $\Omega, Z_k$ are trivial, then $\mathcal{W} = (\Sigma, \Delta, \Lambda, \nu, W)$ is determined by the based reduced root system $(\Sigma, \Delta)$.

An admissible datum $\mathcal{W}$ (Definition 2.1) determines an affine admissible datum $\mathcal{W}^{aff}$, an Iwahori one $\mathcal{W}^{Iw}$ and an affine, Iwahori one $\mathcal{W}^{aff,Iw} = \mathcal{W}^{Iw,aff}$ with the same based reduced root system $(\Sigma, \Delta)$:

$\mathcal{W}^{aff} = (\Sigma, \Delta, \Lambda^{aff}, \nu^{aff}, W^{aff}, Z_k, W_1^{aff})$ with $\Lambda^{aff} = \Lambda \cap W^{aff}$ isomorphic to the coroot lattice in $V$ (generated by the set $\Sigma^\vee$ of coroots of $\Sigma$) with its natural action on $V$ by translation.

$\mathcal{W}^{Iw} = (\Sigma, \Delta, \Omega, \Lambda, \nu, W)$.

$\mathcal{W}^{aff,Iw} = \mathcal{W}^{Iw,aff} = (\Sigma, \Delta, \Lambda^{aff}, \nu^{aff}, W^{aff})$.

We denote by $\mathfrak{S} \subset W^{aff}$ the subset of elements $W^{aff}$-conjugate to an element of $S^{aff}$; it is stable by conjugation by $W$. Its inverse image $\mathfrak{S}(1)$ in $W_1$ is stable by conjugation by $W_1$. The finite abelian subgroup $Z_k$ of $W_1$ acts by by left and right multiplication on $\mathfrak{S}(1)$ and on itself.

**Let $\mathcal{W}$ be an admissible datum (Definition 2.1) and $R$ a commutative ring.**

**Definition 2.4.** *An $R$-parameter map $\mathfrak{c}$ of $\mathcal{W}$ is a $W_1 \times Z_k$-equivariant map $\mathfrak{S}(1) \xrightarrow{\mathfrak{c}} R[Z_k]$:*

$$\mathfrak{c}(\tilde{s}t) = \mathfrak{c}(t\tilde{s}) = t\mathfrak{c}(\tilde{s}), \quad \tilde{w}\mathfrak{c}(\tilde{s})(\tilde{w})^{-1} = \mathfrak{c}(\tilde{w}\tilde{s}(\tilde{w})^{-1}) \quad \text{for } t \in Z_k,\ \tilde{w} \in W_1.$$

*An $R$-parameter map of $\mathcal{W}^{Iw}$ (Example 2.3) is a $W$-equivariant map $\mathfrak{S} \xrightarrow{\mathfrak{q}} R$. Its inflation is the map $\mathfrak{S}(1) \xrightarrow{\tilde{\mathfrak{q}}} R$ satisfying*

$$\tilde{\mathfrak{q}}(\tilde{s}) = \tilde{\mathfrak{q}}(\tilde{s}t) = \tilde{\mathfrak{q}}(t\tilde{s}) = \tilde{\mathfrak{q}}(\tilde{w}\tilde{s}(\tilde{w})^{-1}) \quad \text{for } t \in Z_k,\ \tilde{w} \in W_1.$$

An $R$-parameter map $\mathfrak{S}(1) \xrightarrow{\mathfrak{c}} R[Z_k]$ of $\mathcal{W}$ is also an $R$-parameter map of $\mathcal{W}^{aff}$ , but not conversely because $W^{aff}(1) \neq W_1$.

**Remark 2.5.** *If $R[Z_k] \xrightarrow{\epsilon} R$ denotes the augmentation map, then $\epsilon \circ \mathfrak{c}$ is the inflation of an $R$-parameter map of $\mathcal{W}^{Iw}$, that we denote also by $\epsilon \circ \mathfrak{c}$.*

**Let $\mathfrak{q}$ be an $R$-parameter map of $\mathcal{W}^{Iw}$ and $\mathfrak{c}$ an $R$-parameter map of $\mathcal{W}$ (Definition 2.4).**

**Definition 2.6.** *[Vig1, Theorem 2.4, 4.7] The $R$-algebra $\mathcal{H}_R(\mathcal{W}, \mathfrak{q}, \mathfrak{c})$ is the free $R$-module of basis $(T_{\tilde{w}})_{\tilde{w} \in W_1}$ with a product satisfying the relations generated by:*

(i) *The braid relations $T_{\tilde{w}} T_{\tilde{w}'} = T_{\tilde{w}\tilde{w}'}$ for $\tilde{w}, \tilde{w}' \in W_1$ if $\ell(w) + \ell(w') = \ell(ww')$.*

(ii) *The quadratic relations $T_{\tilde{s}}^2 = \mathfrak{q}(s) T_{\tilde{s}^2} + \mathfrak{c}(\tilde{s}) T_{\tilde{s}}$ for $\tilde{s} \in S^{aff}(1)$ (we identify the $R$-algebra $R[\Omega_1]$ to a subalgebra $\mathcal{H}_R(\mathcal{W}, \mathfrak{q}, \mathfrak{c})$ via the linear map $z \mapsto T_z$ for $z \in \Omega_1$).*

**Example 2.7.** When the root system is trivial (Example 2.2) the parameter maps $\mathfrak{q}, \mathfrak{c}$ are the trivial maps $\{1\} \to R$; the corresponding algebra is the group algebra $R[\Lambda_1]$.

**Definition 2.8.** *The affine subalgebra of $\mathcal{H}_R(\mathcal{W}, \mathfrak{q}, \mathfrak{c})$ is $\mathcal{H}_R(\mathcal{W}^{aff}, \mathfrak{q}, \mathfrak{c})$.*

The intersection $R[\Omega_1] \cap \mathcal{H}_R(\mathcal{W}^{aff}, \mathfrak{q}, \mathfrak{c})$ is the commutative subalgebra $R[Z_k]$. The algebra $\mathcal{H}_R(\mathcal{W}, \mathfrak{q}, \mathfrak{c})$ identifies with the twisted tensor product of $R[Z_k]$ and of its affine subalgebra:

$$\mathcal{H}_R(\mathcal{W}, \mathfrak{q}, \mathfrak{c}) \simeq \mathcal{H}_R(\mathcal{W}^{aff}, \mathfrak{q}, \mathfrak{c}) \rtimes_{R[Z_k]} R[\Omega_1] \simeq R[\Omega_1] \rtimes_{R[Z_k]} \mathcal{H}_R(\mathcal{W}^{aff}, \mathfrak{q}, \mathfrak{c}). \tag{2}$$

**Definition 2.9.** *The Iwahori quotient algebra of $\mathcal{H}_R(\mathcal{W}, \mathfrak{q}, \mathfrak{c})$ is $\mathcal{H}_R(\mathcal{W}^{Iw}, \mathfrak{q}, \epsilon \circ \mathfrak{c})$.*

The Iwahori quotient algebra $\mathcal{H}_R(\mathcal{W}^{Iw}, \mathfrak{q}, \epsilon \circ \mathfrak{c})$ identifies with the tensor product by the augmentation map $R[Z_k] \xrightarrow{\epsilon} R$, of $\mathcal{H}_R(\mathcal{W}, \mathfrak{q}, \mathfrak{c})$:

$$\mathcal{H}_R(\mathcal{W}^{Iw}, \mathfrak{q}, \epsilon \circ \mathfrak{c}) \simeq R \rtimes_{R[Z_k], \epsilon} \mathcal{H}_R(\mathcal{W}, \mathfrak{q}, \mathfrak{c}) \simeq \mathcal{H}_R(\mathcal{W}, \mathfrak{q}, \mathfrak{c}) \rtimes_{R[Z_k], \epsilon} R \tag{3}$$

As a particular case of (2), $\mathcal{H}_R(\mathcal{W}^{Iw}, \mathfrak{q}, \epsilon \circ \mathfrak{c}) \simeq \mathcal{H}_R(\mathcal{W}^{Iw, aff}, \mathfrak{q}, \epsilon \circ \mathfrak{c}) \rtimes_R R[\Omega]$.

$T_w^*$

The algebra $\mathcal{H}_R(\mathcal{W}, \mathfrak{q}, \mathfrak{c})$ posseses other important bases, called the alcove walk bases. They are a generalization of the bases given in [], which themselves generalize the Bernstein basis given in []. They are parametrized by the Weyl chambers of $(V, \mathfrak{H}_0)$, or equivalently by the orientations of $(V, \mathfrak{H})$ defined by the alcoves with vertex the origin.

An orientation $o$ of alcove $\mathfrak{C}_o$ with vertex the origin, allows to distinguish the two sides of the affine hyperplanes in $\mathfrak{H}$. An affine hyperplane $H \in \mathfrak{H}$ is uniquely written as $H = \mathrm{Ker}_V(\alpha_o + n_o)$ for $\alpha_o \in \Sigma$ positive on $\mathfrak{C}_o$, $n_o \in \mathbb{Z}$; the $o$-negative side of $H$ is $(V-H)^{o,-} = \{x \in V \mid \alpha_o(x) + n_o < 0\}$. For $\tilde{s} \in S^{aff}(1)$ fixing $H_s \in \mathfrak{H}$ and $w \in W^{aff}$ such that $\ell(ws) > \ell(w)$, we set:

$$T_{\tilde{s}}^{\epsilon_o(w,s)} = \begin{cases} T_{\tilde{s}} & \text{if } w(\mathfrak{C}) \subset (V - H_s)^{o,-}, \\ T_{\tilde{s}} - \mathfrak{c}(\tilde{s}) & \text{otherwise .} \end{cases} \tag{4}$$

**Let $o$ be an orientation of $(V, \mathfrak{H})$.**

**Definition 2.10.** *[Vig1, Theorem 2.7] For $\tilde{w} \in W_1$,*

$$E_o(\tilde{w}) := T_{\tilde{s}_1}^{\epsilon_o(1,s_1)} \dots T_{\tilde{s}_r}^{\epsilon_o(s_1 \dots s_{r-1}, s_r)} T_{\tilde{u}},$$

*where $\tilde{w} = \tilde{s}_1 \dots \tilde{s}_r \tilde{u}$ with $\tilde{s}_i \in S^{aff}(1), r = \ell(w), \tilde{u} \in \Omega_1$, is a reduced decomposition, depends only on $\tilde{w}$. The alcove walk basis of $\mathcal{H}_R(\mathcal{W}, \mathfrak{q}, \mathfrak{c})$ associated to $o$ is $(E_o(\tilde{w}))_{\tilde{w} \in W_1}$.*

The Bernstein basis was introduced to the study the center of the Iwahori Hecke algebras. Our aim is now to describe the center of $\mathcal{H}_R(\mathcal{W}, \mathfrak{q}, \mathfrak{c})$ using the alcove walk basis when $\mathcal{W}$ admits a splitting.

**Definition 2.11.** *A splitting of $\mathcal{W}$ is $W_0$-equivariant splitting $\Lambda^\flat \xrightarrow{\iota} \Lambda_1^\flat$ of the quotient map $\Lambda_1 \to \Lambda$ on a $W_0$-stable finite index subgroup $\Lambda^\flat \subset \Lambda$ with $W_0$-fixed set $(\Lambda^\flat)^{W_0} = \Omega \cap \Lambda^\flat$, of image $\iota(\Lambda^\flat) = \Lambda_1^\flat$ central in $\Lambda_1$.*

Note that $\Lambda_1^\flat$ is not the inverse image $(\Lambda^\flat)_1$ of $\Lambda^\flat$ in $\Lambda_1$.

The definition is motivated by the properties of the finite conjugacy classes of $W_1$. A conjugacy class of $W_1$ is finite if and only it is contained in the normal subgroup $\Lambda_1$ of $W_1$. On a finite conjugacy class $C_1$ of $W_1$, the length is constant, denoted by $\ell(C_1)$, and

$$E(C_1) := \sum_{\tilde{\lambda} \in C_1} E_o(\tilde{\lambda})$$

does not depend on the orientation $o$. The group $\Lambda$ is commutative and the action of $W$ on $\Lambda$ by conjugation is trivial on $\Lambda$ hence factorizes by the natural action of $W_0$. The group $\Lambda_1$ is not commutative, but its center of $\Lambda_1$ is stable by conjugation by $W_1$, and the action of $W_1$ on it is trivial on $\Lambda_1$, hence defines an action of $W_0$. For a central element $\tilde{\mu} \in \Lambda_1$ lifting $\mu \in \Lambda$, the quotient map $\Lambda_1 \to \Lambda$ induces a surjective $W_0$-equivariant map from the $W_1$-conjugacy class $C_1(\tilde{\mu})$ onto the $W$-conjugacy class $C(\mu)$ of $\mu$.

The homomorphism $\nu : \Lambda \to V$ is $W_0$-equivariant of kernel $\mathrm{Ker}\, \nu = \Omega \cap \Lambda$ and $V^{W_0} = \cap_{\alpha \in \Sigma} \mathrm{Ker}\, \alpha = \{0\}$. Therefore $\Lambda^{W_0}$ is contained in $\Omega \cap \Lambda$. The maximal subgroup of the dominant monoid $\Lambda^+$ (the set of $\mu \in \Lambda$ such that $\nu(\mu)$ belongs to the dominant closed Weyl chamber $\overline{\mathfrak{D}}$) is $\Lambda^{W_0}$.

**We suppose now that $\mathcal{W}$ admits a splitting $\Lambda^\flat \xrightarrow{\iota} \Lambda_1^\flat$**

**Definition 2.12.** *Let $\mathcal{Z}_R(\mathcal{W}, \mathfrak{q}, \mathfrak{c})$ be the $R$-submodule of $\mathcal{H}_R(\mathcal{W}, \mathfrak{q}, \mathfrak{c})$ of basis $E(C_1)$ for all conjugacy classes $C_1$ of $W_1$ contained in $\Lambda_1$, and $\mathcal{Z}_R(\mathcal{W}, \mathfrak{q}, \mathfrak{c})^\iota$ the $R$-submodule of basis $E(C_1)$ for all conjugacy classes $C_1$ of $W_1$ contained in $\Lambda_1^\flat$.*

The submodules where we restrict to the $C_1$ with $\ell(C_1) = 0$ are denoted with an index $\ell = 0$; those with $\ell(C_1) > 0$ with an index $\ell > 0$.

The maximal subgroup of the dominant monoid $\Lambda^{\flat,+} = \Lambda^+ \cap \Lambda^\flat$ is $(\Lambda^\flat)^{W_0}$. The commutative groups $\Lambda^b, (\Lambda^\flat)^{W_0}$ are finitely generated and the monoid $\Lambda^{\flat,+} \setminus (\Lambda^\flat)^{W_0}$ is finitely generated (see Lemma 3.5) with no non trivial invertible element.

By [Vig2, Theorem 1.3], [Vig3, Theorem 5.1, Lemma 6.3, Proposition 6.4] and Lemma 3.9, check the proofs we have:

**Proposition 2.13.** (i) *$\mathcal{Z}_R(\mathcal{W}, \mathfrak{q}, \mathfrak{c})$ is the center of $\mathcal{H}_R(\mathcal{W}, \mathfrak{q}, \mathfrak{c})$ and $\mathcal{Z}_R(\mathcal{W}, \mathfrak{q}, \mathfrak{c})^\iota$ is a subalgebra of $\mathcal{Z}_R(\mathcal{W}, \mathfrak{q}, \mathfrak{c})$.*

(ii) *$\mathcal{Z}_R(\mathcal{W}, \mathfrak{q}, \mathfrak{c})^\iota_{\ell=0}$ is isomorphic to the group algebra $R[(\Lambda^\flat)^{W_0}]$ and $\mathcal{Z}_R(\mathcal{W}, \mathfrak{q}, \mathfrak{c})^\iota_{\ell>0}$ is an ideal of $\mathcal{Z}_R(\mathcal{W}, \mathfrak{q}, \mathfrak{c})^\iota$.*

(iii) *When the ring $R$ is noetherian, the filtrations $((\mathcal{Z}_R(\mathcal{W}, \mathfrak{q}, \mathfrak{c})^\iota_{\ell>0})^n \mathcal{H}_R(\mathcal{W}, \mathfrak{q}, \mathfrak{c}))_{n\in\mathbb{N}}$ and $((\mathcal{Z}_R(\mathcal{W}, \mathfrak{q}, \mathfrak{c})_{\ell>0})^n \mathcal{H}_R(\mathcal{W}, \mathfrak{q}, \mathfrak{c}))_{n\in\mathbb{N}}$ are equivalent.*

(iv) *The $\mathcal{Z}_R(\mathcal{W}, \mathfrak{q}, \mathfrak{c})^\iota$-module $\mathcal{H}_R(\mathcal{W}, \mathfrak{q}, \mathfrak{c})$ is finitely generated.*

(v) *Assume that $\mathfrak{q} = 0$. Then $\mathcal{Z}_R(\mathcal{W}, 0, \mathfrak{c})^\iota$ isomorphic to the monoid algebra $R[\Lambda^{\flat,+}]$ and $\mathcal{Z}_R(\mathcal{W}, 0, \mathfrak{c})^\iota_{\ell>0}$ to $R[\Lambda^{\flat,+} \setminus (\Lambda^\flat)^{W_0}]$.*

The central subalgebra $\mathcal{Z}_R(\mathcal{W}, \mathfrak{q}, \mathfrak{c})^\iota$ can often replace the center and is easier to manipulate.

**Definition 2.14.** *Assume that $\mathfrak{q} = 0$. Let $\mathcal{M}$ be a right $\mathcal{H}_R(\mathcal{W}, 0, \mathfrak{c})$-module.*

*An non-zero element of $\mathcal{M}$ is called supersingular if it is killed by $(\mathcal{Z}_R(\mathcal{W}, 0, \mathfrak{c})_{\ell>0})^n$ for some positive integer $n$.*

*$\mathcal{M}$ is called supersingular if all its non-zero elements are supersingular.*

When $R$ is noetherian, we can replace $\mathcal{Z}_R(\mathcal{W}, 0, \mathfrak{c})_{\ell>0}$ by $\mathcal{Z}_R(\mathcal{W}, 0, \mathfrak{c})^\iota_{\ell>0}$ in the definition (Proposition 2.13 (iii)).

## 2.2 Reductive groups

We consider now a reductive connected $F$-group $\mathbf{G}$ [Borel, Chapter V] which is not anisotropic modulo its center and we fix a triple $(\mathbf{T}, \mathbf{B}, \varphi)$, where $\mathbf{T}$ is a maximal $F$-split subtorus of $\mathbf{G}$, $\mathbf{B}$ is a minimal parabolic $F$-subgroup of $\mathbf{G}$ of Levi decomposition $\mathbf{B} = \mathbf{ZU}$ where $\mathbf{Z}$ is the $\mathbf{G}$-centralizer of $\mathbf{T}$, and $\varphi$ is a special discrete valuation of the root datum of $G$ associated to $B$, compatible with the valuation $\omega$ of $F$ normalized by $\omega(F) = \mathbb{Z}$. We choose an uniformizer $p_F$ of the ring of integers $O_F$ of $F$. For an open compact subgroup $\mathfrak{K} \subset G$, the Hecke ring $\mathcal{H}_\mathbb{Z}(G, \mathfrak{K})$ is the module of functions $G \to \mathbb{Z}$, constant on the double classes modulo $\mathfrak{K}$, endowed with the convolution product. We associate to $(G, T, B, \varphi, p_F)$ an admissible datum, a $\mathbb{Z}$-parameter map and a splitting; they are implicit in [Vig1, §3, §4], [Vig3, §1.3].

**Theorem 2.15.** *To $(G, T, B, \varphi, p_F)$ is associated*

(i) *an admissible datum $\mathcal{W} = \mathcal{W}(G, T, B, \varphi) = (\Sigma, \Delta, \Omega, \Lambda, \nu, W, Z_k, W_1)$ with a parameter map $\mathfrak{c} = \mathfrak{c}(G, T, B, \varphi)$,*

(ii) *an Iwahori subgroup $\mathfrak{B} = \mathfrak{B}(G, T, B, \varphi)$ of pro-p Iwahori subgroup $\mathfrak{U} = \mathfrak{U}(G, T, B, \varphi)$ with Hecke rings*

$$\mathcal{H}_\mathbb{Z}(G, \mathfrak{B}) \simeq \mathcal{H}_\mathbb{Z}(\mathcal{W}^{Iw}, \mathfrak{q}, \mathfrak{q} - 1), \quad \mathcal{H}_\mathbb{Z}(G, \mathfrak{U}) \simeq \mathcal{H}_\mathbb{Z}(\mathcal{W}, \mathfrak{q}, \mathfrak{c}), \quad \mathfrak{q} = \epsilon \circ \mathfrak{c} + 1.$$

(iii) *a splitting $\iota = \iota(G, T, B, \varphi, p_F)$ of $\mathcal{W}$.*

The proof and definitions are given in section 3. The group $\mathfrak{U}$ is the maximal open normal pro-$p$-subgroup of $\mathfrak{B}$. The Hecke rings $\mathcal{H}_\mathbb{Z}(G, \mathfrak{B})$ and $\mathcal{H}_\mathbb{Z}(G, \mathfrak{U})$ are analogous to the Iwahori and unipotent Hecke rings of a reductive finite group. To $(\mathcal{W}, \mathfrak{q}, \mathfrak{c}, \iota)$ is associated a central subring $\mathcal{Z}_\mathbb{Z}(G, \mathfrak{U})^\iota$ of $\mathcal{H}_\mathbb{Z}(G, \mathfrak{U})$ (Definition 2.12).

**Example 2.16.** Let $\mathbf{H}$ be a reductive connected linear algebraic $F$-group which is anisotropic modulo the center (for example $\mathbf{Z}$). A maximal $F$-split torus $\mathbf{T_H}$ is central. The group $H$ has a unique parahoric subgroup $H_0$ and a unique pro-$p$ parahoric subgroup $H_1$ which is the pro-$p$ Sylow subgroup of $H_0$ and the quotient $H_k = H_0/H_1$ is the group of $k$-points of a $k$-torus. The Iwahori Hecke ring, resp. pro-$p$ Iwahori Hecke ring, is the group rings $\mathbb{Z}[H/H_0]$, resp. $\mathbb{Z}[H/H_1]$.

For the product $\mathbf{G} \times \mathbf{H}$ and the triple $(\mathbf{T} \times \mathbf{T_H}, \mathbf{B} \times \mathbf{H}, \varphi)$, the admissible datum $\mathcal{W}_{G\times H}$ has the same based root system than $\mathcal{W}$, the parameter map is $\mathfrak{S}(1) \times H_k \xrightarrow{\mathfrak{c}\otimes \mathrm{id}} \mathbb{Z}[Z_k] \otimes \mathbb{Z}[H_k]$, the Iwahori and pro-$p$ Iwahori Hecke rings are $\mathcal{H}_{\mathbb{Z}}(G, \mathfrak{B}) \otimes \mathbb{Z}[H/H_0]$ and $\mathcal{H}_{\mathbb{Z}}(G, \mathfrak{U}) \otimes \mathbb{Z}[H/H_1]$.

**Example 2.17.** Let $G'$ be the subgroup of $G$ generated the $G$-conjugates of $U$ [AHHV, **]. This is not in general the group of $F$-rational points of a connected reductive $F$-group. We have $G = ZG'$, the subgroup $G^{aff} := Z_0G' \subset G$ is generated by the parahoric subgroups of $G$, the subgroup $Z_1G' \subset G$ is generated by the pro-$p$ parahoric subgroups. Let denote $X' := G' \cap X$ for a subgroup $X \subset G$ and $(X/Y)' := X'/Y'$ for a normal subgroup $Y \subset X$. We have

$$\Lambda^{aff} = \Lambda',\ W^{aff} = W'$$

and $Z'_k \subset Z_k$ (it is often different, for instance if $G = GL(2, F)$ where $G' = SL(2, F)$ ). Set

$$\mathcal{W}' := \mathcal{W}(G', T', B', \varphi) := (\Sigma, \Delta, \Lambda', \nu|_{\Lambda'}, W', Z'_k, W'_1). \tag{5}$$

This is an affine admissible datum, the only difference with $\mathcal{W}^{aff} = \mathcal{W}^{aff}(G, T, B, \varphi)$ is $Z'_k \subset Z_k$ and $W'_1 \subset W_1^{aff}$. The Hecke rings $\mathcal{H}_{\mathbb{Z}}(G', \mathfrak{B}')$ and $\mathcal{H}_{\mathbb{Z}}(G', \mathfrak{U}')$ are naturally subrings of $\mathcal{H}_{\mathbb{Z}}(G, \mathfrak{B})$ and $\mathcal{H}_{\mathbb{Z}}(G, \mathfrak{U})$ respectively, and (Example 3.3):

$$\mathcal{H}_{\mathbb{Z}}(G', \mathfrak{B}') \simeq \mathcal{H}_{\mathbb{Z}}(\mathcal{W}', \mathfrak{q}, \mathfrak{q} - 1),\ \mathcal{H}_{\mathbb{Z}}(G', \mathfrak{U}') \simeq \mathcal{H}_{\mathbb{Z}}(\mathcal{W}', \mathfrak{q}, \mathfrak{c}). \tag{6}$$

for the parameter map $\mathfrak{c} = \mathfrak{c}(G, T, B, \varphi), \mathfrak{q} = \mathfrak{q}(G, T, B, \varphi)$ restricted to $\mathcal{W}'$. As in (2), (3), we have isomorphisms

$$\mathcal{H}_{\mathbb{Z}}(G, \mathfrak{B}) \simeq \mathcal{H}_{\mathbb{Z}}(G', \mathfrak{B}') \rtimes_{\mathbb{Z}} \mathbb{Z}[\Omega], \quad \mathcal{H}_{\mathbb{Z}}(G, \mathfrak{U}) \simeq \mathcal{H}_{\mathbb{Z}}(G', \mathfrak{U}') \rtimes_{\mathbb{Z}[Z'_k]} \mathbb{Z}[\Omega_1]. \tag{7}$$

The splitting $\iota = \iota(G, T, B, \varphi, p_F)$ gives a splitting of $\mathcal{W}'$. When $R$ is a commutative ring, the $R$-algebras $\mathcal{H}_R(G, \mathfrak{U}) = R \otimes_{\mathbb{Z}} \mathcal{H}_{\mathbb{Z}}(G, \mathfrak{U})$ and $\mathcal{Z}_R(G, \mathfrak{U})^\flat_* = R \otimes_{\mathbb{Z}} \mathcal{Z}_{\mathbb{Z}}(G, \mathfrak{U})^\flat_*$ (where $*$ stands for $\ell = 0$ or $\ell > 0$), satisfy the same properties.

## 2.3 Levi datum

In section 4, we return to a general admissible datum $\mathcal{W} = (\Sigma, \Delta, \Omega, \Lambda, \nu, W, Z_k, W_1)$ (Definition **??**) and we introduce the Levi data of $\mathcal{W}$.

**Let $\Delta_M$ be a subset of $\Delta$.**

**Definition 2.18.** *The Levi datum $\mathcal{W}_M$ of $\mathcal{W}$ associated to $\Delta_M$ is*

$$\mathcal{W}_M = (\Sigma_M, \Delta_M, \Omega_M, \Lambda, \nu_M, W_M, Z_k, W_{M,1})$$

*where*

(i) $\Sigma_M \subset \Sigma$ *is the reduced root subsystem generated by* $\Delta_M$. The objects associated as in Definition 2.1 to the based root system $(\Sigma_M, \Delta_M)$ are indicated with a lower index $M$. We have the surjective linear map $V \xrightarrow{p_M} V_M$ defined by $\langle \alpha, v\rangle = \langle \alpha, p_M(v)\rangle$ for $v \in V, \alpha \in \Sigma_M$.

(ii) $W_M = \Lambda \rtimes W_{M,0} \subset W$ *and* $W_{M,1}$ *is the inverse image of* $W_M$ *in* $W_1$.

(iii) $\nu_M = p_M \circ \nu$.

(iv) $\Omega_M$ *is the* $W_M$*-stabilizer of* $\mathfrak{C}_M$ (see lemma 4.1).

We note that $\Lambda, Z_k$ and the $W_0$-equivariant extension $1 \to Z_k \to \Lambda_1 \to \Lambda \to 1$ is the same for $\mathcal{W}$ and $\mathcal{W}_M$, which have therefore the same splittings.

Given a commutative ring $R$ and a parameter map $\mathfrak{S}(1) \xrightarrow{\mathfrak{c}} R[Z_k]$, let $\mathfrak{c}_M$ be the restriction of $\mathfrak{c}$ to $\mathfrak{S}(1) \cap W_{M,1}$.

**Proposition 2.19.** *The Levi datum* $\mathcal{W}_M$ *is admissible,* $p_M$ *is* $W_M$*-equivariant,* $W_M^{aff} = W^{aff} \cap W_M$, $\mathfrak{S}_M = \mathfrak{S} \cap W_M$, *and* $\mathfrak{c}_M$ *is a parameter map of* $(\mathcal{W}_M, R)$.

We note that $(\mathcal{W}^{Iw})_M = (\mathcal{W}_M)^{Iw}$ and $S_M^{aff} \subset \mathfrak{S}_M \subset \mathfrak{S}$. But in general $\Omega_M \not\subset \Omega$, $S_M^{aff} \not\subset S^{aff}$, and the restriction of $\mathfrak{c}$ on $S_M^{aff}(1)$ is not easy to compute from the values of $\mathfrak{c}$ on $S^{aff}(1)$.

compare the Bruhat orders of $G$ and on $M$ for two elements of $M$

**Definition 2.20.** $\mathcal{H}_R(\mathcal{W}_M, \mathfrak{q}_M, \mathfrak{c}_M)$ *is called a Levi* $R$*-algebra of* $\mathcal{H}_R(\mathcal{W}, \mathfrak{q}, \mathfrak{c})$.

Naturally, the definition of a Levi datum and of a Levi algebra is motivated by a Levi subgroup of a reductive connected $p$-adic group. As well known, a Levi algebra is generally not isomorphic to a subalgebra [Vig5].

Withe the notations $F, \mathbf{G}, \mathbf{T}, \mathbf{B}, \varphi, p_F$ introduced earlier, let $\mathbf{M}$ be a Levi subgroup of $\mathbf{G}$ centralizing a $F$-split subtorus of $\mathbf{T}$; we set $\mathbf{B_M} = \mathbf{B} \cap \mathbf{M}$ and let $\varphi_M$ be the restriction of $\varphi$ to the root datum of $M$ with respect to $T$. To $(M, T, B_M, \varphi_M)$ is associated an admissible datum, a splitting, a parameter map, an Iwahori subgroup and a pro-$p$ Iwahori subgroup by Theorem 2.15.

To $M$ is associated a subset $\Pi_M$ of the basis $\Pi$ of the root system $\Phi$ of $\mathbf{T}$ in $\mathbf{G}$ relative to $\mathbf{B}$, and the natural bijection between $\Pi$ and the basis $\Delta$ of the reduced root system $\Sigma$ (Theorem 2.15) sends $\Pi_M$ onto a subset $\Delta_M \subset \Delta$. With the notations of Theorem 2.15 and of Proposition 2.19, we have:

**Theorem 2.21.** *The Levi subdatum* $\mathcal{W}_M$ *and the map* $\mathfrak{c}_M$ *associated to* $\mathcal{W}(G, T, B, \varphi)$, $\Delta_M \subset \Delta$ *and the parameter map* $\mathfrak{c}(G, T, B, \varphi)$, *are the admissible datum and the parameter map associated to* $(M, T, B_M, \varphi_M)$.

*The splittings* $\iota(M, T, B_M, \varphi_M) = \iota(G, T, B, \varphi), p_F)$ *are equal.*

$\mathfrak{B}(M, T, B_M, \varphi_M) = M \cap \mathfrak{B}(G, T, B, \varphi)$ *and* $\mathfrak{U}(M, T, B_M, \varphi_M) = M \cap \mathfrak{U}(G, T, B, \varphi)$.

$\mathcal{W}_M'^{aff} = (\mathcal{W}')_M$

We arrive now to the core of this article which is the comparison of the pro-$p$ Iwahori Hecke rings of central extensions of connected reductive $p$-adic groups, done in section 5. We introduce:

**Definition 2.22.** *A morphism* $\mathcal{W}_H \xrightarrow{i} \mathcal{W}$ *between admissible data (notation and definition 2.1) with the same based reduced root system* $(\Sigma, \Delta)$, *is a set of compatible group homomorphims, all denoted by* $i$,

$$(\Omega_H, \Lambda_H, W_H, Z_{H,k}, W_{H,1}) \xrightarrow{i} (\Omega, \Lambda, W, Z_k, W_1),$$

*such that* $W_H \xrightarrow{i} W$ *is the identity on* $W^{aff}$, *and* $\nu_H = \nu \circ i : \Lambda_H \xrightarrow{i} \Lambda \xrightarrow{\nu} V$.

The morphism $\mathcal{W}_H \xrightarrow{i} \mathcal{W}$ induces morphisms between the affine and Iwahori data $\mathcal{W}_H^{aff} \xrightarrow{i} \mathcal{W}^{aff}$ and $\mathcal{W}_H^{Iw} \xrightarrow{i} \mathcal{W}^{Iw}$. The homomorphism $W_{H,1} \xrightarrow{i} W_1$ respects the length, the kernel of $W_{H,1} \xrightarrow{i} W_1$, of $\Omega_{H,1} \xrightarrow{i} \Omega_1$ and of $\Lambda_{H,1} \xrightarrow{i} \Lambda_1$ are equal. We denote $X_{f=1}$ the kernel of a group homomorphism $X \xrightarrow{f} Y$ and $A_{f=0}$ the kernel of a ring homomorphism $A \xrightarrow{f} B$.

The image $i(\mathcal{W}_H) = (\Sigma, \Delta, i(\Omega_H), i(\Lambda_H), \nu|_{i(\Lambda_H)}, i(W_H), i(Z_{H,k}), i(W_{H,1}))$ of $\mathcal{W}_H$ is an admissible datum. The subgroup $i(W_H) = W^{aff} \rtimes i(\Omega_H)$ of $W = W^{aff} \rtimes \Omega$ is normal of quotient $\Omega/i(\Omega_H)$ and $W_1 = i(W_{H,1})\Omega_1$. We have $\mathfrak{S}_H = \mathfrak{S}$ and $i(\mathfrak{S}_H(1)) \subset \mathfrak{S}(1)$. The restriction to $i(\mathfrak{S}_H(1))$ of a parameter map $\mathfrak{c}$ of $(\mathcal{W}, R)$ is a parameter map of $(i(\mathcal{W}_H), R)$, still denoted by $\mathfrak{c}$.

**Definition 2.23.** *Let $\mathcal{W}_H \xrightarrow{i} \mathcal{W}$ be a morphism between admissible data with the same based reduced root system. Parameter maps $(\mathfrak{c}_H, \mathfrak{c})$ of $(\mathcal{W}_H, R), (\mathcal{W}, R)$ and splittings $(\iota_H, \iota)$ of $(\mathcal{W}_H, \mathcal{W})$ are called $i$-compatible when the following diagrams are commutative:*

$$\begin{array}{ccc} \mathfrak{S}_H(1) & \xrightarrow{i} & \mathfrak{S}(1) \\ {\scriptstyle \mathfrak{c}_H}\downarrow & & \downarrow{\scriptstyle \mathfrak{c}} \\ R[Z_{H,k}] & \xrightarrow[i]{} & R[Z_k] \end{array} \qquad \begin{array}{ccc} \Lambda^\flat_{H,1} & \xrightarrow{i} & \Lambda^\flat_1 \\ \uparrow{\scriptstyle \iota_H} & & \uparrow{\scriptstyle \iota} \\ \Lambda^\flat_H & \xrightarrow[i]{} & \Lambda^\flat \end{array}$$

*Splittings $(\iota', \iota)$ of $\mathcal{W}$ compatible for the identity map $\mathcal{W} \xrightarrow{\mathrm{id}} \mathcal{W}$ are called compatible.*

Let $\Lambda^\flat_H \xrightarrow{\iota_H} \Lambda^\flat_{H,1}$ be a splitting of $\mathcal{W}_H$. The subgroup $i(\Lambda^\flat_H) \subset \Lambda$ is $W_0$-stable. If $\iota_H$ is compatible with a splitting of $\mathcal{W}$, then $i \circ \iota_H(\Lambda^\flat_H)$ is central in $\Lambda_1$. If this is true and if $i(\Lambda^\flat_H)$ has a finite index in $\Lambda$, the unique splitting $i(\Lambda^\flat_H) \xrightarrow{\iota} i(\Lambda^\flat_{H,1})$ on $i(\Lambda^\flat_H)$ compatible with $i$ is called the image of $\iota_H$ by $i$.

Let $\mathcal{W}_H \xrightarrow{i} \mathcal{W}$ be a morphism between admissible data with the same based reduced root system, let $(\mathfrak{c}_H, \mathfrak{c})$, be $i$-compatible parameter maps of $(\mathcal{W}_H, R), (\mathcal{W}, R)$ and let $(\mathfrak{q}_H, \mathfrak{q})$ be $i$-compatible parameter maps of $(\mathcal{W}^{Iw}_H, R), (\mathcal{W}^{Iw}, R)$. Let

$$\mathcal{H}_R(\mathcal{W}_H, \mathfrak{q}_H, \mathfrak{c}_H) \xrightarrow{i} \mathcal{H}_R(\mathcal{W}, \mathfrak{q}, \mathfrak{c}), \tag{8}$$

denote the linear map sending $T^H_{\tilde{w}_H}$ to $T_{\tilde{w}}$ for $\tilde{w} = i(\tilde{w}_H), \tilde{w}_H \in W_{H,1}$ (the upper index $H$ indicates that the element is relative to $\mathcal{W}_H$).

**Proposition 2.24.** *The map $i$ (8) is an algebra homomorphism respecting the alcove walk elements*

$$i(E^H_o(\tilde{w}_H)) = E_o(i(\tilde{w}_H)) \quad (\tilde{w}_H \in W_{H,1},\ o \text{ an orientation of } (V, \mathfrak{H})),$$

*of kernel $R[(\Omega_{H,1})_{i=1}]_{\epsilon=0}$. Therefore, we have the exact sequence*

$$0 \to R[(\Omega_{H,1})_{i=1}]_{\epsilon=0} \to \mathcal{H}_R(\mathcal{W}_H, \mathfrak{q}_H, \mathfrak{c}_H) \xrightarrow{i} \mathcal{H}_R(i(\mathcal{W}_H), \mathfrak{q}, \mathfrak{c}) \to 0,$$

*and the twisted tensor products*

$$\mathcal{H}_R(\mathcal{W}, \mathfrak{q}, \mathfrak{c}) \simeq \mathcal{H}_R(i(\mathcal{W}_H), \mathfrak{q}, \mathfrak{c})) \rtimes_{R[i(\Omega_{H,1})]} R[\Omega_1],$$
$$\mathcal{H}_R(i(\mathcal{W}_H), \mathfrak{q}, \mathfrak{c})) \simeq \mathcal{H}_R(\mathcal{W}^{aff}, \mathfrak{q}, \mathfrak{c})) \rtimes_{R[i(Z_{H,k})]} R[i(\Omega_{H,1})].$$

We return to $F, \mathbf{G}, \mathbf{T}, \mathbf{B}, \varphi, p_F$ introduced before Theorem 2.15. The inclusion $G' \subset G$ induces a morphism $\mathcal{W}' \to \mathcal{W}$ between the admissible data (Theorem 2.15, (5)) with the same root system.

### 2.4 Central extension

Let $\mathbf{H} \xrightarrow{\mathbf{i}} \mathbf{G}$ be a central $F$-extension of connected reductive $F$-groups [Borel, 22.3]. An isogeny is a surjective homomorphism with finite kernel; every separable isogeny is central; two groups are strictly isogenous when there is a group and central isogenies from this group to the two groups (this relation is transitive) [T0, 1.2.1].

There is a profusion of examples: a $z$-extension $\tilde{\mathbf{G}} \xrightarrow{\tilde{\mathbf{i}}} \mathbf{G}$ of $\mathbf{G}$, the multiplication map $\mathbf{C^0} \times \mathbf{G_{der}} \xrightarrow{\mathbf{j}} \mathbf{G}$ where $\mathbf{C^0}$ is the connected component of the center of $\mathbf{G}$ and $\mathbf{G}_{der}$ the derived group of $\mathbf{G}$, the simply connected cover $\mathbf{G}_{sc} \xrightarrow{i_{sc}^{der}} \mathbf{G}_{der}$ of $\mathbf{G}_{der}$, the natural morphism $\mathbf{C^0} \times \mathbf{G_{sc}} \xrightarrow{j \circ (\mathrm{id} \times i_{sc}^{der})} \mathbf{G}$, a separable isogeny. When the characteristic of $F$ is 2, the standard isogeny $\mathbf{SL_2} \to \mathbf{PGL_2}$ is not separable but is central while the isogeny $\mathbf{PGL_2} \to \mathbf{SL_2}$ is not central. references

The kernel $\boldsymbol{\mu}$ of $\mathbf{H} \xrightarrow{\mathbf{i}} \mathbf{G}$ is a central algebraic $F$-subgroup of $\mathbf{H}$. The subgroup $i(H) \subset G$ is the kernel of the natural homomorphism from $G$ to the first cohomology group $H^1(F, \boldsymbol{\mu})$. When the algebraic group $\boldsymbol{\mu}$ is affine, the group $H^1(F, \boldsymbol{\mu})$ is finite [PR, Theorem 6.14] hence $G/i(H)$ is finite, but there are examples where $G/i(H)$ is infinite, hence also $H^1(F, \boldsymbol{\mu})$, [Spr, 16.3.9. Exercise (1) (b)]. For the $F$-isogeny $\mathbf{SL(2)} \to \mathbf{PGL(2)}$, the group $H^1(F, \boldsymbol{\mu}) \simeq PGL(2,F)/PSL(2,F) \simeq F^*/(F^*)^2$ is finite if and only if the characteristic of $F$ is not 2.

The group $\mathbf{T}_H = i^{-1}(\mathbf{T})$ is a maximal $F$-split subtorus of $\mathbf{H}$ such that $i(\mathbf{T}_H) = \mathbf{T}$, the group $\mathbf{B}_H = i^{-1}(\mathbf{B})$ is a minimal $F$-parabolic sugroup of $H$ such that $i(\mathbf{B}_H) = \mathbf{B}$, $\mathbf{U_H} \xrightarrow{\mathbf{i}} \mathbf{U}$ is an isomorphism, $\mathbf{Z}_H = i^{-1}(\mathbf{Z})$ is the $\mathbf{H}$-centralizer of $\mathbf{T_H}$ and $i(\mathbf{Z}_H) = \mathbf{Z}$, $\mathbf{N}_H = i^{-1}(\mathbf{N})$ is the $\mathbf{H}$-normalizer of $\mathbf{T_H}$ and $i(\mathbf{N}_H) = \mathbf{N}$ [Borel, Theorem 22.6].

The special discrete valuation $\varphi$ compatible with $\omega$ of the root datum $(Z, (U_\alpha)_{\alpha\in\Phi})$ generating $G$ is also a special discrete valuation $\varphi_H$ compatible with $\omega$ of the root datum $(Z_H, (U_{H,\alpha})_{\alpha\in\Phi_H})$ generating $H$. By Theorem 2.15, we have the admissible data $\mathcal{W}_H = \mathcal{W}(H, T_H, B_H, \varphi)$ and $\mathcal{W} = \mathcal{W}(G,T,B,\varphi)$, the parameter maps $\mathfrak{c}_H = \mathfrak{c}(H, T_H, B_H, \varphi)$ and $\mathfrak{c} = \mathfrak{c}(G,T,B,\varphi)$, the splittings $\iota_H = \iota(H, T_H, B_H, \varphi, p_F)$ and $\iota = \iota(G,T,B,\varphi,p_F)$.

**Theorem 2.25.** *Let $\mathbf{H} \xrightarrow{\mathbf{i}} \mathbf{G}$ be a central $F$-extension of connected reductive $F$-groups.*

(i) *The homomorphism $H \xrightarrow{i} G$ induces an homomorphism $\mathcal{W}_H \xrightarrow{i} \mathcal{W}$ between the admissible data $\mathcal{W}_H$ and $\mathcal{W} = \mathcal{W}(G,T,B,\varphi)$ which have the same based reduced root system. The parameter maps $\mathfrak{c}_H$ and $\mathfrak{c}$ are $i$-compatible. The splitting $\iota$ is the image by $i$ of the splitting $\iota_H$. Proposition 2.24 applies to the pro-p Iwahori rings.*

(ii) *The homomorphism $H \xrightarrow{i} G$ sends the (pro-p) parahoric subgroup of $H$ fixing a facet of $(V, \mathfrak{H})$ into the (pro-p) parahoric subgroup of $G$ fixing the same facet. We have $i(H') = G'$ and the semidirect product $i(H)Z^1$ has a finite index in $G$.*

(iii) *The homomorphism $\mathcal{H}_{\mathbb{Z}}(H, \mathcal{U}_H) \xrightarrow{i} \mathcal{H}_{\mathbb{Z}}(G, \mathcal{U})$ between the pro-p Iwahori Hecke rings respects the central elements:*

$$i(E^H(C_{H,1}(\mu_H))) = E(C_1(i\circ\mu_H)) \quad (\mu_H \in X_*(\mathbf{T_H}),$$

*induces an isomorphism $\mathcal{Z}_{\mathbb{Z}}(H,\mathcal{U}_H)^\flat_{\ell>0} \xrightarrow{i} \mathcal{Z}_{\mathbb{Z}}(G,\mathcal{U})^\flat_{\ell>0}$, and $i(\mathcal{Z}_{\mathbb{Z}}(H,\mathcal{U}_H)^\flat_{\ell=0}) = \mathcal{Z}_{\mathbb{Z}}(G,\mathcal{U})^\flat_{\ell=0}$. The homomorphism $\mathcal{Z}_{\mathbb{Z}}(H,\mathcal{U}_H)^\flat \xrightarrow{i} \mathcal{Z}_{\mathbb{Z}}(G,\mathcal{U})$ is surjective.*

(iv) *The kernel of $W_{H,1} \xrightarrow{i} W_1$ is $i^{-1}(Z_1)/Z_{H,1}$. When it is finite, the homomorphism $\mathcal{Z}_{\mathbb{Z}}(H,\mathcal{U}_H)^\flat \xrightarrow{i} \mathcal{Z}_{\mathbb{Z}}(G,\mathcal{U})$ is injective.*

We assume now that $R$ is a field and we consider $R$-representations. For an $R$-representation $\pi$ of $G$, we denote by $\pi_H$ the inflation to $H$ of $\pi|_{i(H)}$, by $\pi^{\mathfrak{U}}$ the right

$\mathcal{H}_R(G,\mathfrak{U})$-module of $\mathfrak{U}$-invariants of $\pi$, and by $\pi_H^{\mathfrak{U}_H}$ the right $\mathcal{H}_R(H,\mathfrak{U}_H)$-module of $\mathfrak{U}_H$-invariants of $\pi_H$. A supercuspidal $R$-representation of $G$ is an irreducible admissible $R$-representation of $G$ which is not the quotient of a parabolically induced representation from an irreducible admissible $R$-representation of a proper Levi subgroup [AHHV, I.3].

When $G/i(H)$ is finite, Clifford's theory can be used to obtain the irreducible admissible $R$-representations of $H$ knowing those of $G$ and vice versa.

**Proposition 2.26.** *We suppose that $G/i(H)$ is finite. Let $\pi$ be an irreducible admissible $R$-representation of $G$.*

(i) *The $R$-representation $\pi_H$ of $H$ is admissible semisimple of finite length.*
*$\pi$ is supercuspidal if and only if all the irreducible components of $\pi_H$ are supercuspidal if and only if some irreducible component of $\pi_H$ is supercuspidal.*

(ii) *Assume that the characteristic of the field $R$ is $p$.*
*$\pi^{\mathfrak{U}}$ contains a supersingular element if and only if $\pi_H^{\mathfrak{U}_H}$ contains a supersingular element.*
*$\pi^{\mathfrak{U}}$ is supersingular if and only if $\pi_H^{\mathfrak{U}_H}$ is supersingular.*

When $G/i(H)$ is finite and $R$ is an algebraically closed field of characteristic $p$, Theorem 2.27 describes $\pi_H$ using the classification of isomorphism classes of the irreducible admissible $R$-representations of $G$ have been classified [AHHV, Theorems 2 and 3].

The parabolic $F$-subgroups $\mathbf{P}$ of $\mathbf{G}$ containing $\mathbf{B}$, called standard, are in bijection with the subsets of simple roots of $\mathbf{T}$ in $\mathbf{B}$ hence with the subsets $\Delta_P$ of $\Delta$. A Levi decomposition $\mathbf{P} = \mathbf{MN}$ where the Levi subgroup $\mathbf{M}$ contains $\mathbf{Z}$ is called standard. We denote by $\mathbf{P_H} = \mathbf{M_H N_H}$ the standard decomposition of the parabolic subgroup of $\mathbf{H}$ with $\Delta_{P_H} = \Delta_P$. By restriction, we have the central extension $\mathbf{M_H} \xrightarrow{\mathbf{i}} \mathbf{M}$ of kernel $\boldsymbol{\mu}$. An element $\alpha \in \Delta$ corresponds to a minimal standard Levi subgroup $\mathbf{M}_\alpha$. An $R$-representation $\sigma$ of $M$ defines the standard parabolic subgroup $P(\sigma)$ with $\Delta_P \subset \Delta_{P(\sigma)}$ and $\alpha \in \Delta - \Delta_P$ lies in $\Delta_{P(\sigma)}$ if and only if $\sigma$ is trivial on $Z \cap M'_\alpha$ [AHHV, II.7 Proposition]. If $P, Q$ are two standard parabolic subgroups of $G$, $P \subset Q \subset P(\sigma)$, we denote by $\mathrm{Ind}_Q^G$ the smooth induction and $e_Q(\sigma)$ the representation of $Q$ trivial on $N$ extending $\sigma$. For $P \subset Q \subset Q' \subset P(\sigma)$, the representation $\mathrm{Ind}_{Q'}^G e_{Q'}(\sigma)$ identifies naturally with a subrepresentation of $\mathrm{Ind}_Q^G e_Q(\sigma)$.

If $\sigma$ is a supercuspidal representation of $M$, $(P,\sigma,Q)$ with $P \subset Q \subset P(\sigma)$ is called a supercuspidal standard triple of $G$ [AHHV, I.3]. For such a triple, the $R$-representation of $G$

$$I_G(P,\sigma,Q) = \frac{\mathrm{Ind}_Q^G e_Q(\sigma)}{\sum_{Q \subsetneq Q' \subset P(\sigma)} \mathrm{Ind}_{Q'}^G e_{Q'}(\sigma)}$$

is irreducible admissible. Every irreducible admissible $R$-representation of $G$ is isomorphic to $I_G(P,\sigma,Q)$ for a unique supercuspidal standard triple $(P,\sigma,Q)$ of $G$.

Assume that $G/i(H)$ is finite. Then $M/i(M_H)$ is finite. Let $(P,\sigma,Q)$ be a supercuspidal standard triple of $G$. The restriction of $\sigma$ to $i(M_H)$ is a finite sum of irreducible representations $\sigma_j$. Let $\sigma_{j,M_H}$ denote the inflation of $\sigma_j$ to $M_H$ for all $j$, and $P_H = M_H N_H$ the standard Levi decomposition of the standard parabolic subgroup of $H$ with $\Delta_{P_H} = \Delta_P$.

**Theorem 2.27.** *Assume that $G/i(H)$ is finite. Then $(P_H, \sigma_{j,M_H}, Q_H)$ is a supercuspidal standard triple of $H$ for all $j$, and $(I_G(P,\sigma,Q))_H = \oplus_j I_H(P_H, \sigma_{j,M_H}, Q_H)$.*

We consider a variant of Theorem 2.25, Proposition 2.26 and Theorem 2.27, which applies to $\mathbf{G_{der}} \xrightarrow{\mathbf{i}} \mathbf{G}$, $\mathbf{G_{sc}} \xrightarrow{\mathbf{i} \circ \mathbf{i_{sc}}} \mathbf{G}$, which motivate this work. We recall that $\mathbf{C^0}$ is the connected center of $\mathbf{G}$.

**Theorem 2.28.** *Let $\mathbf{H} \xrightarrow{\mathbf{i}} \mathbf{G}$ be an $F$-homomorphism of reductive $F$-groups such that the map $\mathbf{H} \times \mathbf{C^0} \xrightarrow{\mathbf{j}} \mathbf{G}$ sending $(\mathbf{h}, \mathbf{c})$ to $\mathbf{i}(\mathbf{h})\mathbf{c}$ is a central $F$-extension of kernel $\boldsymbol{\mu}$.*

(i) *Theorem 2.25 remains valid except that in (iii) we have*
$\mathcal{Z}_{\mathbb{Z}}(G,\mathcal{U})^\flat_{\ell=0} = i(\mathcal{Z}_{\mathbb{Z}}(H,\mathfrak{U}_H)^\flat_{\ell=0})\mathbb{Z}[(C^0/C^0_0)^\flat_1]$,
$\mathcal{Z}_{\mathbb{Z}}(G,\mathcal{U})^\flat_{\ell>0} = i(\mathcal{Z}_{\mathbb{Z}}(H,\mathfrak{U}_H)^\flat_{\ell>0})\mathbb{Z}[(C^0/C^0_0)^\flat_1]$.

(ii) *Proposition 2.26 remains valid when $\pi$ has a central character.*

(iii) *Theorem 2.27 remains valid.*

In section 6, we reformulate our results for the homomorphisms $\mathbf{G}_{\mathbf{sc},1} \xrightarrow{\mathbf{i_{sc}}} \mathbf{G}_{\mathbf{der},1} \xrightarrow{\mathbf{i}} \mathbf{G}$ in Proposition 6.11 and Theorem 6.12, after Lemma 6.5 where we compare the pro-$p$ parahoric subgroups $Z_{sc,1} \xrightarrow{i_{sc}} Z_{der,1} \xrightarrow{i} Z_1$ of the mininal Levi subgroups.

As an application, we give Theorem 2.29 motivated by a forthcoming article [OV]. We suppose that $R$ is a field of characteristic $p$. We consider the two properties of $G$ (where $\pi$ is any irreducible admissible $R$-representation $\pi$ of $G$ with a central character):

(i) $\pi$ is supercuspidal if and only if $\pi^{\mathfrak{U}}$ is supersingular,

(ii) $\pi^{\mathfrak{U}}$ is supersingular if and only if $\pi^{\mathfrak{U}}$ contains a supersingular element.

**Theorem 2.29.** *If (i), resp. (ii), is satisfied for all simply connected, $F$-simple and $F$-isotropic $F$-groups $\mathbf{G}$, then (i), resp. (ii), is satisfied for all connected reductive $F$-groups $\mathbf{G}$ such that $G/i_{sc}(G_{sc})C^0$ is finite.*

When $R$ is an algebraically closed field of characteristic $p$, it is proved in [OV] that (i) and (ii) are satisfied for all simply connected, $F$-simple and $F$-isotropic $F$-groups $\mathbf{G}$.

# 3 Reductive $F$-group

## 3.1 Elementary lemmas

We start with elementary lemmas which are useful throughout this paper. Let $K$ be a profinite group having an open pro-$p$ subgroup. By [HV1, 3.6], the group $K$ has a largest open normal pro-$p$ subgroup $K_1$, called the pro-$p$ radical. Any normal pro-$p$ subgroup $H \subset K$ is contained in $K_1$ because $HK_1 \subset K$ is a normal open pro-$p$ subgroup.

A closed subgroup $H \subset K$ is profinite with an open pro-$p$ subgroup $H \cap K_1$. If $H$ is normal, the quotient $K/H$ with the quotient topology is profinite with an open pro-$p$ subgroup.

If the order of $K/K_1$ is prime to $p$ , then $K_1$ is an open pro-$p$ Sylow subgroup of $K$; as $K_1$ is normal, $K_1 \subset K$ is the unique pro-$p$ Sylow subgroup.

**Lemma 3.1.** *Let $K \xrightarrow{f} K'$ be a continuous homomorphism between profinite groups having open pro-p radicals $K_1$ and $K'_1$, and let $H$ be a closed normal subgroup of $K$.*

(i) *$H$ has an open pro-p radical $H_1$ and $H_1 = H \cap K_1$.*

(ii) *The subgroup $f(K) \subset K'$ is closed, has an open pro-p radical $f(K)_1$ and $f(K_1) \subset f(K)_1$.*

(iii) *If the orders of $K/K_1$ and of $K'/K'_1$ are prime to p, then $f(K_1) = f(K)_1 = f(K) \cap K'_1$ and $f$ induces an exact sequence*

$$0 \to \operatorname{Ker} f/(\operatorname{Ker} f)_1 \to K/K_1 \xrightarrow{\overline{f}} f(K)/f(K)_1 \to 0.$$

*Proof.* (i) The pro-$p$ subgroup $H \cap K_1 \subset H$ is normal hence $H \cap K_1 \subset H_1$. We prove the reverse inclusion: for $k \in K$, the pro-$p$ subgroup $kH_1k^{-1} \subset H$ is normal as for $h \in H$, $hkH_1k^{-1}h^{-1} = k(k^{-1}hk)H_1(k^{-1}h^{-1}k)k^{-1} \subset kH_1k^{-1}$. Hence $kH_1k^{-1} \subset H_1$ implying that $H_1$ is normalized by $K$ and that $H_1K_1 \subset K$ is a normal open pro-$p$-subgroup containing $K_1$, hence $H_1K_1 = K_1$. Therefore $H \cap K_1 \supset H_1$.

(ii) The subgroup $f(K) \subset K'$ is closed (a profinite subgroup is compact and Hausdorff) hence profinite. The pro-$p$ subgroup $f(K_1) \subset f(K)$ is normal hence $f(K_1) \subset f(K)_1$.

(iii) The order of $K/K_1$ is prime to $p$, and the same is true its quotient $f(K)/f(K_1)$ and for the subgroup $f(K)_1/f(K_1) \subset f(K)/f(K_1)$. As $f(K)_1$ is a pro-$p$ groups, it must be equal to $f(K_1)$. The order of $K'/K'_1$ is prime to $p$, and the same is true for its subgroup $f(K)/f(K) \cap K'_1$. The pro-$p$ subgroup $f(K) \cap K'_1 \subset f(K)$ is normal hence $f(K) \cap K'_1 \subset f(K)_1$. As the index is prime to $p$, we have $f(K) \cap K'_1 = f(K)_1$. This implies the existence of $K/K_1 \xrightarrow{\overline{f}} K'/K'_1$ and the values of the kernel and of the image of this homomorphism. ☐

**Lemma 3.2.** *Let $H \subset G$ be a closed normal subgroup of a topological group $G$ and let $K \subset G$ be an open subgroup such that for any $g \in G$, the double coset $KgK$ is the union of finite cosets $Kg'$, and also of finite cosets $g''K$. Then the inclusions $H \subset HK \subset G$ induce respectively an isomorphism and an inclusion of Hecke rings*

$$\mathcal{H}_{\mathbb{Z}}(H, K \cap H) \xrightarrow{\simeq} \mathcal{H}_{\mathbb{Z}}(HK, K) \hookrightarrow \mathcal{H}_{\mathbb{Z}}(G, K).$$

The finiteness of left and right $K$-cosets in a double coset $KgK$ for any $g \in G$ allows to form the Hecke ring $\mathcal{H}_{\mathbb{Z}}(G, K)$.

*Proof.* As the subgroup $H \subset G$ is normal, $HK \subset G$ is a subgroup and the Hecke ring $\mathcal{H}_{\mathbb{Z}}(HK, K)$ is naturally isomorphic to the subring of elements in $\mathcal{H}_{\mathbb{Z}}(G, K)$ with support in $HK$. We write $C = K \cap H$. The inclusion $H \subset HK$ induces a bijection of cosets $C \backslash H \to K \backslash KH$, and also of double cosets $C \backslash H/C \to K \backslash HK/K$. The bijection between the cosets respects the convolution product as

$$Kg_1K \cap Kg_2Kg = \sqcup_{g \in H(g_1,g_2)} Kg, \quad Cg_1C \cap Cg_2Cg = \sqcup_{g \in H(g_1,g_2)} Cg,$$

where $H(g_1, g_2)$ is a finite subset of $H$. We check these equalities. For $g_1, g_2 \in H$ the set $Kg_1K \cap Kg_2Kg$ is a disjoint union $\sqcup_{g \in H(g_1,g_2)} Kg$ for some finite subset $H(g_1, g_2) \subset H$, because $KHKHK \subset KHK$. The insersection with $H$ is $Kg_1K \cap Kg_2Kg \cap H = (\sqcup_{g \in H(g_1,g_2)} Kg) \cap H = \sqcup_{g \in H(g_1,g_2)} Cg$. As $g_1 \in Kg_2K$ implies $g_1 \in Cg_2C$ we have $Kg_1K \cap H = Cg_1C$ and $Kg_2Kg \cap H = Cg_2Cg$. ☐

**Example 3.3.** Recalling the notations of the introduction,

$$\mathcal{H}_{\mathbb{Z}}(G, \mathfrak{B}) \supset \mathcal{H}_{\mathbb{Z}}(Z_0G', \mathfrak{B}) = \mathcal{H}_{\mathbb{Z}}(G'\mathfrak{B}, \mathfrak{B}) \simeq \mathcal{H}_{\mathbb{Z}}(G', \mathfrak{B}'),$$
$$\mathcal{H}_{\mathbb{Z}}(G, \mathfrak{U}) \supset \mathcal{H}_{\mathbb{Z}}(Z_1G', \mathfrak{U}) = \mathcal{H}_{\mathbb{Z}}(G'\mathfrak{U}, \mathfrak{U}) \simeq \mathcal{H}_{\mathbb{Z}}(G', \mathfrak{U}').$$

We recall the Gordan's lemma on convex polytopes [HV1, 2.11 Lemma]:

**Lemma 3.4.** (Gordan's lemma) *If $\mathcal{L}$ is a finitely generated free abelian group and $\mathcal{T}$ a convex rational polyhedral closed cone in $\mathcal{L} \otimes \mathbb{R}$, then $\mathcal{L} \cap \mathcal{T}$ is a finitely generated monoid.*

We apply Gordan's lemma in the following context. Let $\mathcal{W}$ be an admissible datum and let $\Lambda^\flat$ be a $W_0$-stable finite index subgroup of $\Lambda$, $\Lambda^{\flat,+}$ the monoid of $\lambda \in \Lambda^\flat$ with $\nu(\lambda) \in \overline{\mathfrak{D}}$ and $(\Lambda^\flat)^{W_0} \subset \Lambda^\flat$ the subgroup of elements fixed by $W_0$ (Definitions 2.1, 2.11 and 2.12).

**Lemma 3.5.** *The abelian groups $\Lambda^\flat$, $(\Lambda^\flat)^{W_0}$ and the monoids $\Lambda^{\flat,+}$, $\Lambda^{\flat,+} - (\Lambda^\flat)^{W_0}$ are finitely generated.*

*Proof.* The monoid $\nu(\Lambda^{\flat,+}) = \nu(\Lambda^\flat) \cap \overline{\mathfrak{D}}$ is finitely generated by the Gordan's lemma. The submonoid $\nu(\Lambda^{\flat,+}) - \{0\}$ is also finitely generated. We have $\nu(\Lambda^\flat) = \cup_{w \in W_0} w(\nu(\Lambda^{\flat,+}))$

and the kernel of $\Lambda^\flat \xrightarrow{\nu} V$ is $\Lambda^\flat \cap \Omega = (\Lambda^\flat)^{W_0}$. The subgroups $\Lambda^\flat$, $(\Lambda^\flat)^{W_0}$ of the finitely generated abelian group $\Lambda$ are finitely generated. The exact sequence

$$1 \to (\Lambda^\flat)^{W_0} \to \Lambda^{\flat,+} \xrightarrow{\nu} \nu(\Lambda^{\flat,+}) \to 1$$

implies that the monoid $\Lambda^{\flat,+}$ is finitely generated. The inverse image $\Lambda^{\flat,+} - (\Lambda^\flat)^{W_0}$ of the finitely generated monoid $\nu(\Lambda^{\flat,+}) - \{0\}$ is also finitely generated. □

## 3.2 The admissible datum, the parameter map and the splitting of a reductive $p$-adic group

Let $\mathbf{G}$ be a reductive connected $F$-group and let $(\mathbf{T}, \mathbf{B}, \varphi, p_F)$ be a quadruple as in §2. We describe in this subsection the admissible datum $(\Sigma, \Delta, \Omega, \Lambda, \nu, W, Z_k, W_1)$, the Iwahori subgroup $\mathfrak{B}$ and the pro-$p$ subgroup $\mathfrak{U}$ of $G$ associated to the triple $(\mathbf{T}, \mathbf{B}, \varphi)$ and the splitting $\Lambda^\flat \xrightarrow{\iota} \Lambda_1^\flat$ associated to the triple $(\mathbf{T}, \mathbf{B}, p_F)$, following [Vig1, §3] and [Vig3, §1.3].

When $\mathbf{G}$ is anisotropic modulo its center, the maximal $F$-split subtorus $\mathbf{T}$ is central, $G$ contains a unique Iwahori subgroup $G_0$, and a unique pro-$p$ Iwahori subgroup $G_1$ equal to the unique pro-$p$-Sylow subgroup of $G_0$. The group $G_k = G_0/G_1$ is the group of $k$-points of a $k$-torus. The admissible datum is $\mathcal{W} = (G/G_0, G_k, G/G_1)$ with a trivial root system. An homomorphism $\mathbf{H} \xrightarrow{\mathbf{f}} \mathbf{G}$ between reductive connected $F$-groups which are anisotropic modulo its center, induces an homomorphism $H_0 \xrightarrow{f} G_0$ between the unique parahoric subgroups such that $f(H_1) = f(H) \cap G_1$ and induces an homomorphism $H_k \xrightarrow{f} G_k$ between the finite $k$-tori as Lemma 3.1 (iii). When $\mathbf{G}$ is a $F$-split torus, $G_0$ is the unique maximal compact subgroup of $G$.

We suppose now $\mathbf{G}$ general. The $\mathbf{G}$-centralizer $\mathbf{Z}$ of $\mathbf{T}$ is anisotropic modulo the center and we define $Z_0, Z_1, Z_k, \Lambda = Z/Z_0, \Lambda_1 = Z/Z_1$ as above. When $\mathbf{G}$ is semisimple and simply connected, $Z_0$ is the unique maximal compact subgroup of $Z$. Let $\mathfrak{N}$ be the $\mathbf{G}$-normalizer of $\mathbf{T}$. The finite, Iwahori, pro-$p$ Iwahori, Weyl groups of $G$ with respect to $T$ are respectively $W_0 = \mathfrak{N}/ = \mathfrak{N}/Z_0$, $W_1 = \mathfrak{N}/Z_1$. We denote by $\Phi$ the set of roots of $(\mathbf{T}, \mathbf{G})$ and by $\Phi^+ \subset \Phi$ the subset of roots of $(\mathbf{T}, \mathbf{B})$.

The group $\Lambda$ is abelian (it may have torsion when $\mathbf{G}$ is not $F$-split), finitely generated of rank the number of simple roots in $\Phi^+$; it is a normal subgroup of $W$ and $\Lambda_1$ is a normal subgroup of $W_1$. We denote by $Z \xrightarrow{\lambda} \Lambda$, $Z \xrightarrow{\lambda_1} \Lambda_1$ the quotient maps. Let $\Lambda^\flat = \lambda(T)$. The group $\Lambda^\flat$ is isomorphic to $T/T_0$. The group $\lambda_1(T)$ is central in $\Lambda_1$ and isomorphic to $T/T_1$. We denote by $X_*(\mathbf{T})$ the group of $F$-cocharacters of $\mathbf{T}$. Let $\Lambda_1^\flat = \{\lambda_1(\mu(p_F^{-1})) \mid \mu \in X_*(T)\}$; this is a subgroup of $\lambda_1(T)$. The uniformizer $p_F$ induces $W_0$-equivariant isomorphisms

$$X_*(T) \xrightarrow{\sim} \Lambda^\flat \xrightarrow{\sim} \Lambda_1^\flat, \quad \mu \mapsto \lambda(\mu(p_F^{-1})) \mapsto \lambda_1(\mu(p_F^{-1})). \tag{9}$$

The $W_0$-equivariance follows from $n\mu(p_F^{-1})n^{-1} = w(\mu)(p_F^{-1})$ for $n \in \mathfrak{N}$ of image $w \in W_0$. The second isomorphism from $\Lambda^\flat$ on to $\Lambda_1^\flat$ is a $W_0$-equivariant splitting $\iota$ of the quotient map $\Lambda_1^\flat \to \Lambda^\flat$.

For $\alpha \in \Phi$, let $U_\alpha \subset G$ denote the root group of $\alpha$ ($U_{2\alpha} \subset U_\alpha$ if $2\alpha \in \Phi$), $\varphi_\alpha : U_\alpha - \{1\} \to \mathbb{R}$ the map given by the valuation $\varphi$ of the root datum $(Z, (U_\alpha)_{\alpha \in \Phi})$ of type $\Phi$ generating $G$. A root $\alpha \in \Phi$ is called reduced if $\alpha/2 \notin \Phi$. There exist positive integers $(e_\alpha)_{\alpha \in \Phi}$ with $2e_{2\alpha} = e_\alpha$ if $\alpha, 2\alpha \in \Phi$, and $(f_\alpha)_{\alpha, 2\alpha \in \Phi}$ such that [Vig1, (39),(40)] the image of $\varphi_\alpha$ is

$$\Gamma_\alpha = \begin{cases} e_\alpha^{-1}\mathbb{Z} & \text{if } \alpha \text{ is reduced,} \\ e_{\alpha/2}^{-1} f_{\alpha/2}\mathbb{Z} & \text{otherwise.} \end{cases}$$

For $r \in \Gamma_\alpha$, $U_{\alpha+r} := \{1\} \cup \varphi_\alpha^{-1}(r + e_\alpha^{-1}\mathbb{N})$ is a subgroup of $U_\alpha$ [Vig1, §3.5]. The image $\Sigma$ of $\Phi$ by the map $\alpha \mapsto e(\alpha)\alpha$ is a reduced root system [Vig1, §3.4] of basis $\Delta$, image of the basis of $\Phi$ relative to $\mathbf{B}$. The Weyl groups of the root systems $\Phi$ and $\Sigma$ are isomorphic to $W_0$.

The center $\mathbf{C}$ of $\mathbf{G}$ is the intersection of the kernels of the roots of $\mathbf{G}$ relative to a maximal subtorus of $\mathbf{G}$ [Spr, 8.1.8]. We choose on the $\mathbb{R}$-vector space

$$V = (X_*(\mathbf{T}) \otimes \mathbb{R})/(X_*(\mathbf{C}) \otimes \mathbb{R})$$

a $W_0$-invariant scalar product. The group $\mathfrak{N}$ acts on $V$ by affine automorphisms respecting the set $\mathfrak{H} \subset V$ of kernels of the affine roots of $\Sigma$ [Vig1, §3.3]. We denote by $\mathfrak{C}$ the alcove of $(V, \mathfrak{H})$ with vertex $0 \in V$ contained in the open Weyl chamber $\mathfrak{D} = \{v \in V \mid \langle \alpha, v \rangle \geq 0\}$ for $\alpha \in \Phi^+$. For $\alpha \in \Phi$ and $u \in U_\alpha - \{1\}$, the unique element $m(u)$ in $\mathfrak{N} \cap U_{-\alpha} u U_{-\alpha}$ acts by orthogonal reflection with respect to the affine hyperplane $\operatorname{Ker}(\alpha + \varphi_\alpha(u)) \in \mathfrak{H}$. The group $\mathfrak{N}$ is generated by $Z$ and the $m(u)$ for $\alpha \in \Phi$ and $u \in U_\alpha - \{1\}$. An element $z \in Z$ acts on $V$ by translation by the element $\nu(z) \in V$ determined by

$$(\alpha \circ \nu)(z) = -n^{-1}(\omega \circ \alpha)(z^n x) \quad (\alpha \in \Phi), \tag{10}$$

for any positive integer $n$ and $x \in Z_0$ such that $z^n x \in T$. The group $Z_0$ is contained in the kernel of $\nu$. We still denote by $\Lambda \xrightarrow{\nu} V$ or $\Lambda_1 \xrightarrow{\nu} V$ the induced homomorphisms. The action of $\mathfrak{N}$, denoted also by $\nu$, being trivial in $Z_0$ gives an action $\nu$ of $W_1$ and of $W$, on $(V, \mathfrak{H})$. The elements $\lambda \in \Lambda$ acts by translations by $\nu(\lambda)$.

The normal subgroup $W^{aff} \subset W$ generated by the images of $m(u)$ for $\alpha \in \Phi, u \in U_\alpha - \{1\}$, is isomorphic by $\nu$ to the affine Weyl group of $\Sigma$. Let $S^{aff} \subset W^{aff}$ corresponding to the orthogonal reflections with respect to the walls of the alcove $\mathfrak{C}$ and $S$ corresponding to the walls containing $0 \in V$. The subgroup of $W^{aff}$ generated by $S$ is isomorphic to the finite Weyl group $W_0$. The $W$-normalizer $\Omega$ of $S^{aff}$ is an abelian finitely generated group, isomorphic to the image of the Kottwitz homomorphism $\kappa_G$ [Ko, 7.1-4], [Vig1, §3.9] as noticed by Haines, Rapoport and Richartz. The kernel $\operatorname{Ker} \kappa_G$ of $\kappa_G$ is the subgroup of $G$ generated by the parahoric subgroups of $G$. In particular, $Z_0 = \operatorname{Ker} \kappa_Z$. We have the

For $x \in V$, let $\mathfrak{N}_x$ denote the $\mathfrak{N}$-stabilizer of $x$ and $U_x$ the subgroup of $G$ generated by $\cup_{\alpha \in \Phi} U_{\alpha + r_x(\alpha)}$ and $r_x(\alpha) \in \Gamma_\alpha$ the smallest element such that $\alpha(x) + r_x(\alpha) \geq 0$ [Vig1, (44)]. We have the subgroup $\mathfrak{P}_x := \mathfrak{N}_x U_x \subset G$. The semisimple Bruhat-Tits building $\mathfrak{BT}(G)$ is the quotient of $G \times V$ by the equivalence relation $(g, x) \sim (g', x') \Leftrightarrow$ there exists $n \in \mathfrak{N}$ such that $x' = \nu(n)(x)$ and $g^{-1} g' n \in \mathfrak{P}_x$, with the natural action of $G$ [Vig1, Definition 3.12].

The parahoric subgroups of $G$ are the $G$-conjugates of the $\operatorname{Ker} \kappa_G$-stabilisers $\mathfrak{K}_{\mathfrak{F}}$ of the facets $\mathfrak{F}$ of $(V, \mathfrak{H})$. The pro-$p$ parahoric subgroups of $G$ are the $G$-conjugates of the largest open normal pro-$p$-sugroups $\mathfrak{K}_{\mathfrak{F},1}$ of $\mathfrak{K}_{\mathfrak{F}}$ (§3.1, [HV1, 3.6]). The quotient $\mathfrak{K}_{\mathfrak{F},k} = \mathfrak{K}_{\mathfrak{F}}/\mathfrak{K}_{\mathfrak{F},1}$ is group of $k$-points of a connected reductive $k$-group. The parahoric subgroup $\mathfrak{K}_{\mathfrak{F}}$ and the pro-$p$-parahoric subgroup $\mathfrak{K}_{\mathfrak{F},1}$ are generated by their intersections $\mathfrak{K}_{\mathfrak{F}} \cap U_\alpha = \mathfrak{K}_{\mathfrak{F},1} \cap U_\alpha$ with the root groups $U_\alpha$ for the reduced roots $\alpha \in \Phi$, and by their intersections $\mathfrak{K}_{\mathfrak{F}} \cap Z = Z_0, \mathfrak{K}_{\mathfrak{F},1} \cap Z = Z_1$, with $Z$. We have

$$\mathfrak{K}_{\mathfrak{F},1} = (\mathfrak{K}_{\mathfrak{F},1} \cap U^-) Z_1 (\mathfrak{K}_{\mathfrak{F},1} \cap U)$$

with any order.

The Iwahori subgroup and the pro-$p$ Iwahori subgroup of $G$ determined by $(G, T, B, \varphi)$ are the parahoric and pro-$p$ parahoric groups $\mathfrak{B} = \mathfrak{K}_{\mathfrak{C}}, \mathfrak{U} = \mathfrak{K}_{\mathfrak{F},1}$ fixing the alcove $\mathfrak{C}$. The natural maps from $\mathfrak{N}$ to $B\backslash G/B$, $\mathfrak{B}\backslash G/\mathfrak{B}$, $\mathfrak{U}\backslash G/\mathfrak{U}$, induce bijections $W_0 \simeq B\backslash G/B$, $W \simeq \mathfrak{B}\backslash G/\mathfrak{B}$, $W_1 \simeq \mathfrak{U}\backslash G/\mathfrak{U}$.

## 3.3 The parameter map of a reductive $p$-adic group

We describe the parameter map $\mathfrak{c} : \mathfrak{S}(1) \to \mathbb{Z}[Z_k]$ associate to the triple $(\mathbf{T}, \mathbf{B}, \varphi)$. The value of $\mathfrak{c}$ is given first on the set of admissible elements $\tilde{s} \in \mathfrak{S}(1)$, defined as follows.

**Definition 3.6.** (i) *Let $\alpha \in \Phi$ and $u \in U_\alpha - \{1\}$. The pair $(\alpha, u)$ is called admissible when $\alpha$ is either*

*reduced and not multipliable,*

*or multipliable and $U_{\alpha+\varphi_\alpha(u)} \neq U_{\alpha+\varphi_\alpha(u)+e_\alpha^{-1}} U_{2\alpha+\varphi_{2\alpha}(u)}$,*

*or not reduced and $U_{\alpha/2+\varphi_{\alpha/2}(u)} = U_{\alpha/2+\varphi_{\alpha/2}(u)+e_{\alpha/2}^{-1}} U_{\alpha+\varphi_\alpha(u)}$.*

(ii) *An element $\tilde{s} \in \mathfrak{S}(1)$ lifting $s \in \mathfrak{S}$ is called admissible if there exists an admissible pair $(\alpha, u)$ such that $\tilde{s}$ is the image of $m(u) \in \mathfrak{N}$ in $W_1$. The triple $(\alpha, u, \tilde{s})$ is called admissible.*

The definition of an admissible pair comes from [Vig1, §4.2]. An admissible pair $(\alpha, u)$ determines an admissible triple $(\alpha, u, \tilde{s})$, where the affine hyperplane $H_s \subset V$ fixed by $s$ is $\mathrm{Ker}(\alpha + \varphi_\alpha(u))$. The admissible pair $(\alpha, u)$ such that $H_s = \mathrm{Ker}(\alpha + \varphi_\alpha(u))$ is not determined by $s$. If $r = \varphi_\alpha(u)$, all the other admissible pairs are

$$\{(\alpha, y) \ |y \in \varphi_\alpha^{-1}(r)\} \cup \{(-\alpha, z) \ |z \in \varphi_{-\alpha}^{-1}(-r)\}. \tag{11}$$

Let $(\alpha, u, \tilde{s})$ be an admissible triple. We define a subgroup $Z_{s,k} \subset Z_k$ and an element $c(\alpha, u) \in \mathbb{N}[Z_{s,k}]$ which will be $\mathfrak{c}(\tilde{s})$ [Vig1, §4.2]. For this, we choose an alcove of $(V, \mathfrak{H})$ having a face $\mathfrak{F}_s$ fixed by $s$. The parahoric subgroup $\mathfrak{K}_{\mathfrak{F}_s} \subset G$ fixing $\mathfrak{F}_s$ contains the groups $Z_0 U_{\alpha+\varphi_\alpha(u)}$ and $G_{\alpha,\varphi_\alpha(u)}$ generated by $U_{\alpha+\varphi_\alpha(u)} \cup U_{-\alpha-\varphi_\alpha(u)}$. The finite reductive quotient $\mathfrak{K}_{s,k}$ of $\mathfrak{K}_{\mathfrak{F}_s}$ does not depend on the choice of $\mathfrak{F}_s$. The image of $Z_0 U_{\alpha+\varphi_\alpha(u)}$ in $\mathfrak{K}_{s,k}$ is a Borel subgroup of Levi decomposition $Z_k U_{s,k}$ where $U_{s,k} \simeq U_{\alpha+\varphi_\alpha(u)}/U_{\alpha+\varphi_\alpha(u)+e_\alpha^{-1}}$. The unipotent group $U_{s,k}^{op}$ opposite to $U_{s,k}$ is isomorphic to $U_{-\alpha-\varphi_\alpha(u)}/U_{-\alpha-\varphi_\alpha(u)+e_\alpha^{-1}}$ (as $e_\alpha = e_{-\alpha}$). The image of $G_{\alpha,\varphi_\alpha(u)}$ in $\mathfrak{K}_{s,k}$ is the subgroup $G_{s,k}$ generated by $U_{s,k} \cup U_{s,k}^{op}$. The image of $Z_0 \cap G_{\alpha,\varphi_\alpha(u)}$ is $Z_{s,k} = Z_k \cap G_{s,k}$. These groups, in particular $Z_{s,k}$, are determined by $s$. The image $u_k \in U_{s,k}$ of $u$ is not trivial. Let $m(u_k)$ denote the unique element of $U_{s,k}^{op} u_k U_{s,k}^{op}$ normalizing $Z_{s,k}$. We consider the map uniquely defined by [Vig1, Step 2 of proof of Proposition 4.4], [CE, Proof of Proposition 6.8(iii)]:

$$x_k \mapsto z(x_k) : U_{s,k} - \{1\} \to Z_{s,k}, \quad m(u_k) x_k^{-1} m(u_k) \in U_{s,k} m(u_k) z(x_k) U_{s,k}. \tag{12}$$

The element $c(\alpha, u)$ is the sum of $z(x_k)$ for all $x_k \in U_{s,k} - \{1\}$,

$$c(\alpha, u) = \sum_{x_k \in U_{s,k} - \{1\}} z(x_k). \tag{13}$$

We note the properties

$$\epsilon(c(\alpha, u)) = q_s - 1, \quad t c(\alpha, u) = c(\alpha, u) s(t), \quad s(c(\alpha, u)) = c(\alpha, u), \tag{14}$$

where $\mathbb{Z}[Z_k] \xrightarrow{\epsilon} \mathbb{Z}$ is the augmentation morphism, $q_s$ is the order of $U_{s,k}$ (a power of the order $q$ of the residual field $k$ of $F$), $t \in Z_k$, $s(t) \in Z_k$ such that $t m(u_k) = m(u_k) s(t)$. We have $t c(\alpha, u) = c(\alpha, u) s(t)$ because $z(t x_k^{-1} t^{-1}) = t s(t^{-1}) z(x_k)$ as $s(t) m(u_k) x_k^{-1} m(u_k) s(t)^{-1} = m(u_k) t x_k^{-1} t^{-1} m(u_k)$ lies in $s(t) U_{s,k} m(u_k) z(x_k) U_{s,k} s(t)^{-1} = U_{s,k} m(u_k) t z(x_k) s(t)^{-1} U_{s,k}$. We have $s(c(\alpha, u)) = c(\alpha, u)$ by the quadratic relation $T_{m(u_k)}^2 = q_s T_{m(u_k)^2} + T_{m(u_k)} c(\alpha, u)$ in the finite Hecke complex algebra $\mathcal{H}_R(G_{s,k}, U_{s,k})$ [CE, Proof of Proposition 6.8(iii) where $T_{m(u_k)}$ is denoted $a_{m(u_k)}$]. When $p$ is invertible in $R$, we multiply the quadratic relation on the right or left by $T_{m(u_k)}^{-1}$ to get $T_{m(u_k)} = q_s T_{m(u_k)} + c(\alpha, u) = q_s T_{m(u_k)} + T_{m(u_k)} c(\alpha, u) T_{m(u_k)}^{-1} = q_s T_{m(u_k)} + s(c(\alpha, u))$ by the braid relations.

**Theorem 3.7.** *There exists a unique map $\mathfrak{S}(1) \xrightarrow{\mathfrak{c}} \mathbb{Z}[Z_k]$ satisfying*

$$\mathfrak{c}(\tilde{s}) := c(\alpha, u), \quad \mathfrak{c}(t\tilde{s}) := t\mathfrak{c}(\tilde{s}),$$

*for all admissible triples* $(\alpha, u, \tilde{s})$ *and* $t \in Z_k$. *The map* $\mathfrak{c}$ *is* $W_1 \times Z_k$*-equivariant:*

$$\mathfrak{c}(\tilde{w}\,\tilde{s}\,\tilde{w}^{-1}) = \tilde{w}\mathfrak{c}(\tilde{s})\tilde{w}^{-1}, \quad \mathfrak{c}(t\,\tilde{s}) = \mathfrak{c}(\tilde{s}\,t) = t\mathfrak{c}(\tilde{s}),$$

*for* $\tilde{w} \in W_1, t \in Z_k,\ \tilde{s} \in \mathfrak{S}(1)$.

The theorem follows from [Vig1, Proposition 4.4, Theorem 4.7, Remark 4.8] where we prove the formula $\mathfrak{c}(t\tilde{w}\,\tilde{s}\,\tilde{w}^{-1}) = t\tilde{w}\mathfrak{c}(\tilde{s})\tilde{w}^{-1}$ when $\tilde{s}$ and $\tilde{w}\,\tilde{s}\,\tilde{w}^{-1}$ belong to $S^{aff}(1)$. We give here a simpler proof.

*Proof.* An element $s \in \mathfrak{S}$ admits always an admissible lift $\tilde{s}$. The lifts of $s \in \mathfrak{S}$ are $t\tilde{s}$ for $t \in Z_k$. If its exists, the map $\mathfrak{c}$ is unique. The map $\mathfrak{c}$ exists if and only if $c(\alpha, u) = tc(\beta, v)$ for the admissible triples $(\alpha, u, \tilde{s})$ and $(\beta, v, t\tilde{s})$ with $t \in Z_k$. Note that $\mathfrak{c}$ will be left and right $Z_k$-equivariant by (14) because $t\tilde{s} = \tilde{s}s(t)$ and (14).

We need a lemma before the proof the existence of $\mathfrak{c}$.

For $u \in U_\alpha - \{1\}$, there exist unique elements $v, v' \in U_{-\alpha} - \{1\}$ such that $u = vm(u)v'$ [BT1, 6.1.2 (2) ]. If $u \in \varphi_\alpha^{-1}(r)$ we have $v, v' \in \varphi_{-\alpha}^{-1}(-r)$ by [BT1, property (V5)]. Let $G_{\alpha,r} \subset G$ denote the compact subgroup generated by $U_{\alpha,r} \cup U_{-\alpha-r}$.

**Lemma 3.8.** *We have* $m(v) = m(v') = m(u^{-1}) = m(u)^{-1}$. *The elements* $m(u)^{-1}m(u')$, $m(u')m(u)^{-1}$ *lie in* $Z_0 \cap G_{\alpha,r}$.

*Proof.* We have $m(v) = m(v') = m(u^{-1})$ because $v = um(u)^{-1}m(u)v'^{-1}m(u)^{-1}$ and similarly for $v'$. We have $m(u^{-1}) = m(u)^{-1}$ by inverting $u = vm(u)v'$. For the second assertion we can cite [Vig1, Lemma 4.5] or give the following arguments. For a facet $\mathfrak{F}$ of $(\mathfrak{A}, \mathfrak{H})$ contained in $\mathrm{Ker}(\alpha + r)$, the parahoric subgroup $K_{\mathfrak{F}} \subset G$ fixing $\mathfrak{F}$ contains $G_{\alpha,r}$[Vig1, (44)] and $Z \cap K_{\mathfrak{F}} = Z_0$. Obviously $m(u)^{-1}m(u'), m(u')m(u)^{-1}$ lie in $G_{\alpha,r} \cap \mathfrak{N}$. They lie in $Z$ because their image in $W_0$ is trivial. □

We start the proof of the existence of $\mathfrak{c}$. Let $s \in \mathfrak{S}$ and let $(\alpha, u)$ be an admissible pair such that $\mathrm{Ker}(\alpha + \varphi_\alpha(u))$ is the affine hyperplane of $V$ fixed by $s$. The other admissible pairs with this property are given in (11). There exists $t_y \in Z_{s,k}$ such that $m(y_k) = t_y m(u_k) = m(u_k)s(t_y)$ by Lemma 3.8 and the paragraph above (12). The image of $m(y)$ in $W_1$ is $t_y\tilde{s} = \tilde{s}s(t_y)$. Let $v, v' \in U_{-\alpha}$ be the elements such that $u = vm(u)v'$. By Lemma 3.8, $(-\alpha, v, \tilde{s}^{-1})$ is an admissible triple. To show the existence of $\mathfrak{c}$, it suffices to show

$$c(\alpha, y) = c(\alpha, u)s(t_y), \quad c(-\alpha, v) = \tilde{s}^{-2}c(\alpha, u).$$

The equality $c(\alpha, y) = c(\alpha, u)s(t_y)$ follows from (12) which implies $m(y_k)x_k^{-1}m(y_k) = t_y m(u_k)x_k^{-1}m(u_k)s(t_y) \in t_y U_{s,k}m(u_k)z(x_k)U_{s,k}s(t_y) = U_{s,k}m(y_k)z(x_k)s(t_y)U_{s,k}$.

We show now the second equality. By Lemma 3.8, $m(v_k) = m(u_k)^{-1}$ and $m(u_k)^2 = \tilde{s}^2$. When $x_k^{op}$ runs through $U_{s,k}^{op} - \{1\}$, then $x_k := m(u_k)^{-1}x_k^{op}m(u_k)$ runs through $U_{s,k} - \{1\}$. Let $z(x_k) \in Z_{s,k}$ such that $x_k^{-1} \in m(v_k)U_{s,k}m(u_k)z(x_k)U_{s,k}m(v_k) = U_{s,k}^{op}z(x_k)m(v_k)U_{s,k}^{op}$. Then $m(v_k)(x_k^{op})^{-1}m(v_k) = m(v_k)^2x_k^{-1}$ lies in the set

$U_{s,k}^{op}m(v_k)^2z(x_k)m(v_k)U_{s,k}^{op} = U_{s,k}^{op}m(v_k)^3m(v_k)^{-1}z(x_k)m(v_k)U_{s,k}^{op}$.

Recalling (14), we obtain the second equality:

$c(-\alpha, v) = m(v_k)c(\alpha, u)m(v_k) = \tilde{s}^{-2}m(v_k)^{-1}c(\alpha, u)m(v_k) = \tilde{s}^{-2}s(c(\alpha, u)) = \tilde{s}^{-2}c(\alpha, u)$.

It remains only to prove that $\mathfrak{c}$ is $W_1$-equivariant. Let $\tilde{s} \in S(1)$. We note that $\tilde{w}\mathfrak{c}(\tilde{s})\tilde{w}^{-1} = \mathfrak{c}(\tilde{w}\tilde{s}\tilde{w}^{-1})$ for all $\tilde{w} \in W_1$, implies $\mathfrak{c}(\tilde{w}\,t\tilde{s}\,\tilde{w}^{-1}) = \tilde{w}\mathfrak{c}(t\tilde{s})\tilde{w}^{-1}$ for all $\tilde{w} \in W_1$ and all $t \in Z_k$, because the left side is $\tilde{w}t\tilde{w}^{-1}\mathfrak{c}(\tilde{w}\,\tilde{s}\,\tilde{w}^{-1})$ and the right side is $\tilde{w}t\tilde{w}^{-1}\tilde{w}\mathfrak{c}(\tilde{s})\tilde{w}^{-1}$ by $Z_k$-equivariance of of $\mathfrak{c}$.

So, we are reduced to $\mathfrak{c}(\tilde{s}) = c(\alpha, u)$ for an admissible triple $(\alpha, u, \tilde{s})$. Let $n \in \mathfrak{N}$ lifting $\tilde{w} \in W_1$. The root $w(\alpha)$ is reduced if and only if $\alpha$ is reduced. We have $U_{w(\alpha)} = nU_\alpha n^{-1}$ and $m(nun^{-1}) = nm(u)n^{-1}$. The triple $(w(\alpha), nun^{-1}, \tilde{w}\tilde{s}(\tilde{w})^{-1})$ is admissible and $\mathfrak{c}(\tilde{w}\tilde{s}(\tilde{w})^{-1}) = c(w(\alpha), nun^{-1})$. We have to prove $\tilde{w}c(\alpha, u)\tilde{w}^{-1} = c(w(\alpha), nun^{-1})$

The image by $n$ of an alcove of $(V,\mathfrak{H})$ having a face $\mathfrak{F}_s$ fixed by $s$ is an alcove having a face $\mathfrak{F}_{wsw^{-1}}$ fixed by $wsw^{-1}$. The conjugation by $n$ induces an isomorphism between the (pro-$p$) parahoric subgroups of $G$ fixing $\mathfrak{F}_s$ and $\mathfrak{F}_{wsw^{-1}}$, hence an isomorphism $j_k$ between their reductive finite quotients. We have $j_k(Z_{s,k}U_{s,k}) = Z_{wsw^{-1},k}U_{wsw^{-1},k}$. For $z \in Z_0$ of image $t \in Z_k$, the image of $nzn^{-1} \in Z_0$ in $Z_k$ is $j_k(t) = \tilde{w}t\tilde{w}^{-1}$. Hence $j_k(c(\alpha,u)) = \tilde{w}c(\alpha,u)\tilde{w}^{-1}$. The image of $nun^{-1}$ in $G_{wsw^{-1},k}$ is $j_k(m(u_k))$. For $x_k \in U_{s,k} - \{1\}$ we have $j_k(m(u_k)x_k^{-1}m(u_k)) \in j_k(U_{s,k}m(u_k)z(x_k)U_{s,k}) = U_{wsw^{-1},k}j_k(m(u_k))j_k(z(x_k))U_{wsw^{-1},k}$. By (13), $j_k(c(\alpha,u)) = c(w(\alpha), nun^{-1})$. This ends the proof of Theorem 3.7. □

The Hecke rings

$$\mathcal{H}_{\mathbb{Z}}(G,\mathfrak{B}) \simeq \mathcal{H}_{\mathbb{Z}}(\mathcal{W}^{Iw},\mathfrak{q},\mathfrak{q}-1), \quad \mathcal{H}_{\mathbb{Z}}(G,\mathfrak{U}) \simeq \mathcal{H}_{\mathbb{Z}}(\mathcal{W},\mathfrak{q},\mathfrak{c}), \quad \mathfrak{q} = \epsilon \circ \mathfrak{c} + 1.$$

Isomorphism Hecke ring . Reflechir s'il ne faut pas mettre la suite de cette section dans le cadre general

The two isomorphisms of (9) induce bijective maps between the $W_0$-conjugacy class of $\mu$, the $W$-conjugacy class $C(\mu)$ of $\lambda(\mu(p_F^{-1}))$ and the $W_1$-conjugacy class $C_1(\mu)$ of $\lambda_1(\mu(p_F^{-1}))$. The monoid $X_*(T)^+$ of dominant cocharacters $\mu$ such that $\alpha \circ \mu(p_F) \in O_F$ for $\alpha \in \Phi^+$, is isomorphic to $\Lambda^{\flat,+}$ by the first isomorphism; the subgroup of invertible elements in $X_*(T)^+$ equal to the group $(X_*(T))^{W_0}$ of cocharacters $\mu \in X_*(T)$ fixed by $W_0$, is isomorphic to $(\Lambda^\flat)^{W_0}$; $X_*(T)^+$ is a system of representatives of the $W_0$-conjugacy classes of $X_*(T)$. We denote by $\mathcal{Z}_{\mathbb{Z}}(G,\mathfrak{U})^\flat \subset \mathcal{H}_{\mathbb{Z}}(G,\mathfrak{U})$ the central subalgebra of basis $(E(C_1(\mu))_{\mu\in X_*(T)^+}$, and by $\mathcal{Z}_{\mathbb{Z}}(G,\mathfrak{U})^\flat_{\ell=0}$, respectively $\mathcal{Z}_{\mathbb{Z}}(G,\mathfrak{U})^\flat_{\ell>0}$ the subrings of basis $E(C_1(\mu))$ for $\mu$ running in $(X_*(T))^{W_0}$, respectively $X_*(T)^+ - (X_*(T))^{W_0}$.

An element $\lambda(\mu(p_F^{-1})) \in \Lambda^\flat \cap \Omega$ if and only if it is fixed by $W_0$ if and only if $\lambda_1(\mu(p_F^{-1}))$ is fixed by $W_1$ if and only if $E(C_1(\mu)) = T_{\lambda_1(\mu(p_F^{-1}))}$. The linear map

$$\mu \mapsto T_{\lambda_1(\mu(p_F^{-1}))} : \mathbb{Z}[X_*(T)^{W_0}] \xrightarrow{\simeq} \mathcal{Z}_{\mathbb{Z}}(G,\mathfrak{U})^\flat_{\ell=0}$$

is a ring isomorphism. By Lemma 3.5, the ring $\mathcal{Z}_{\mathbb{Z}}(G,\mathcal{U})^\flat_{\ell=0}$ is finitely generated.

**Lemma 3.9.** *Assume that $R$ is a commutative ring of characteristic $p$.*

*The linear map $\mu \mapsto E(C_1(\mu)) : R[X_*(T)^+] \to \mathcal{Z}_R(G,\mathfrak{U})^\flat$ is an $R$-algebra isomorphism. The $R$-algebras $\mathcal{Z}_R(G,\mathcal{U})^\flat$, $\mathcal{Z}_R(G,\mathfrak{U})^\flat_{\ell>0}$ are finitely generated.*

*Proof.* When $G$ is split [OComp, Proposition 2.10]. The proof is valid in general, and is as follows. We have $E(C_1(\mu)) = \sum_{\mu'\in W_0(\mu)} E_o(\lambda_1(\mu'))$ where $o$ is an orientation of $(V,\mathfrak{H})$. When the characteristic of the ring $R$ is $p$, for $\mu_1,\mu_2 \in X_*(T)$, the product $E_o(\lambda_1(\mu_1))E_o(\lambda_1(\mu_2))$ is equal to $E_o(\lambda_1(\mu_1\mu_2))$ if $\mu_1,\mu_2 \in w(X_*(T)^+)$ for some $w \in W_0$, and is 0 otherwise. For $\mu_1,\mu_2 \in X_*(T)^+$, the map $(\mu_1',\mu_2') \mapsto \mu_1'\mu_2'$ yields a bijection from the set of $(\mu_1',\mu_2') \in W_0(\mu_1) \times W_0(\mu_2)$ with $\mu_1',\mu_2' \in w(X_*(T)^+)$ for some $w \in W_0$, onto $W_0(\mu_1\mu_2)$.

Then, Lemma 3.5 follows for the second assertion. □

# 4 Levi subgroup

Let $\mathcal{W}$ be an admissible datum of based reduced root system $(\Sigma,\Delta)$ and let $\Delta_M \subset \Delta$. In Definition (2.18), we defined a Levi datum $\mathcal{W}_M$ of based reduced root system $(\Sigma_M,\Delta_M)$ and a linear map $V \xrightarrow{p_M} V_M$ the linear map such that $\langle\alpha,v\rangle = \langle\alpha,p_M(v)\rangle$ for $v \in V, \alpha \in \Delta_M$. We have the set $\mathfrak{H}$ of affine hyperplanes $\mathrm{Ker}_V(\alpha+r)$ in $V$ for $(\alpha,r) \in \Sigma\times\mathbb{Z}$, and the set $\mathfrak{H}_M$ of affine hyperplanes $\mathrm{Ker}_{V_M}(\alpha+r)$ in $V_M$ for $(\alpha,r) \in \Sigma_M\times\mathbb{Z}$. Before proving that $\mathcal{W}_M$ is admissible, we examine the compatibility of $p_M$ with $\mathfrak{H}$ and $\mathfrak{H}_M$.

**Lemma 4.1.** *(i) For $(\alpha, r) \in \Sigma_M \times \mathbb{Z}$, the inverse image $p_M^{-1}(H_M)$ of the affine hyperplane $H_M = \mathrm{Ker}_{V_M}(\alpha + r) \in \mathfrak{H}_M$ is the affine hyperplane $H = \mathrm{Ker}_V(\alpha + r) \in \mathfrak{H}$, and $p_M(H) = H_M$.*

*(ii) The image $p_M(\mathfrak{F})$ of a facet $\mathfrak{F}$ of $(V, \mathfrak{H})$ is contained in a facet of $(V_M, \mathfrak{H}_M)$, that we denote by $\mathfrak{p}_M(\mathfrak{F})$.*

*(iii) For any facet $\mathfrak{F}_M$ of $(V_M, \mathfrak{H}_M)$, there exists a facet $\mathfrak{F}$ of $(V, \mathfrak{H})$ such that $p_M(\mathfrak{F}) = \mathfrak{F}_M$*

*Proof.* (i) is obvious.

Let $\mathfrak{F}$ be a facet of $(V, \mathfrak{H})$. For $x, y$ in $\mathfrak{F}$, $\alpha \in \Sigma_M, r \in \mathbb{Z}$, the real numbers $\langle \alpha + r, x \rangle = \langle \alpha + r, p_M(x) \rangle$ and $\langle \alpha + r, y \rangle = \langle \alpha + r, p_M(y) \rangle$ are both zero, positive or negative. Hence $p_M(\mathfrak{F})$ is contained in a facet of $(V_M, \mathfrak{H}_M)$. The image of the dominant alcove $\mathfrak{C}$ of $(V, \mathfrak{H})$ associated to $\Delta$ is contained in the dominant alcove $\mathfrak{C}_M$ of $(V_M, \mathfrak{H}_M)$ associated to $\Delta_M$, $p_M(\mathfrak{C}) \subset \mathfrak{C}_M$.

□

A point $x$ in $V$ is **$\mathfrak{H}$-special** if for any $\alpha \in \Sigma$, there exists $r \in \mathbb{Z}$ such that $\alpha(x) + r = 0$ [BT1, (1.3.7)]. It suffices to suppose $\alpha \in \Delta$. The origin of $V$ is $\mathfrak{H}$-special.

**Lemma 4.2.** *(i) The image $y = p_M(x)$ of a $\mathfrak{H}$-special point $x \in V$ is $\mathfrak{H}_M$-special.*

*(ii) A $\mathfrak{H}_M$-special point $y \in V_M$ is the image $y = \mathfrak{p}_M(x)$ of a $\mathfrak{H}$-special point $x \in V$.*

*Proof.* (i) is obvious.

(ii) $\Delta$ is a basis of the dual of $V$. There exists $x \in V$ with $\alpha(x) = 0$ for $\alpha \in \Delta \setminus \Delta_M$, and $\langle \alpha, x \rangle = \langle \alpha, y \rangle$ for $v \in V, \alpha \in \Delta_M$. Then $x$ is special and $p_M(x) = y$. □

**Lemma 4.3.** *The group $W_M = \Lambda \rtimes W_{M,0}$ acts on $(V_M, \mathfrak{H}_M)$ and is a semidirect product $W_M = W_M^{aff} \rtimes \Omega_M$. The surjective map $V \xrightarrow{p_M} V_M$ is $W_M$-equivariant.*

*Proof.* The subgroup $W_M = \Lambda \rtimes W_{0,M} \subset W$ acts on $(V, \mathfrak{H})$ and on $(V_M, \mathfrak{H}_M)$: $\Lambda$ by translation by $\nu$ on $(V, \mathfrak{H})$ and by $\nu_M = p_M \circ \nu$ on $(V_M, \mathfrak{H}_M)$, and $W_{0,M}$ by its natural action: for $w \in W_{0,M}, v \in V, v_M \in V_M, \alpha \in \Sigma, \alpha_M \in \Sigma_M$, we have $\langle \alpha, w(v) \rangle = \langle w^{-1}(\alpha), v \rangle$ and $\langle \alpha_M, w(v_M) \rangle = \langle w^{-1}(\alpha_M), v_M \rangle$ The map $p_M$ is cleary $\Lambda$-equivariant; it is $W_{0,M}$-equivariant because $\langle \alpha_M, w(v) \rangle = \langle w^{-1}(\alpha_M), v \rangle = \langle w^{-1}(\alpha_M), p_M(v) \rangle = \langle \alpha_M, w(p_M(v)) \rangle$. Therefore $p_M$ is $W_M$-equivariant. □

We prove Proposition 2.19. We choose, as we can, the scalar products such that $V \xrightarrow{p_M} V_M$ such that

$$p_M \circ s_{\alpha+r} = s_{\alpha+r,M} \circ p_M : V \to V_M,$$

for $\alpha \in \Sigma_M, r \in \mathbb{Z}$, if $s_{\alpha+r}$ denote the orthogonal reflection of $V$ with respect to $\mathrm{Ker}_V(\alpha+r)$ and $s_{\alpha+r,M}$ the orthogonal reflection of $V_M$ with respect to $\mathrm{Ker}_{V_M}(\alpha + r)$.

The map $s_{\alpha+r,M} \mapsto s_{\alpha+r}$ for $\alpha \in \Sigma_M, r \in \mathbb{Z}$ injects $\mathfrak{S}_M$ into $\mathfrak{S}$ and induces an injective homomorphism $W_M^{aff} \to W^{aff}$ of image $W^{aff} \cap W_M$. We identify $W_M^{aff}$ with $W^{aff} \cap W_M$, hence $\mathfrak{S}_M$ with $\mathfrak{S} \cap W_M$. We have $W_M = W_M^{aff} \rtimes \Omega_M$ because $W_M$ acts on $(V_M, \mathfrak{H}_M)$. Although the group $\Omega_M$ is not contained in $\Omega$, it is isomorphic to a subgroup of $\Omega$, hence is abelian and finitely generated, because $\Omega_M \simeq W_M / W_M^{aff} \simeq W_M / W^{aff} \cap W_M$ embeds in $W/W^{aff} \simeq \Omega$.

As $W_{M,1}$ is the inverse image of $W_M \subset W$ in $W_1$, we have $\mathfrak{S}_M(1) \subset \mathfrak{S}(1)$ and the inclusion is $W_{M,1} \times Z_k$-equivariant. Hence the restriction $\mathfrak{c}_M$ to $\mathfrak{S}_M(1)$ of a parameter map $\mathfrak{c}$ of $(\mathcal{W}, R)$ is a parameter map of $(\mathcal{W}_M, R)$. This ends the proof of Proposition 2.19.

Let $\mathbf{M}$ be a Levi subgroup of $\mathbf{G}$. We recall the natural surjective linear map $V \xrightarrow{p_M} V_M$, and for a facet $\mathfrak{F}$ of $(V, \mathfrak{H})$, the facet $\mathfrak{p}_M(\mathfrak{F})$ of $(V_M, \mathfrak{H}_M)$ containing $p_M(\mathfrak{F})$ (Lemma 4.1). Let $K_{\mathfrak{F}}, K_{\mathfrak{p}_M(\mathfrak{F})}$ denote the parahoric subgroup of $G, M$ fixing $\mathfrak{F}, \mathfrak{p}_M(\mathfrak{F})$, and $K_{\mathfrak{F},1}, K_{\mathfrak{p}_M(\mathfrak{F}),1}$

denote their pro-$p$ radicals. We have $\mathfrak{p}_M(\mathfrak{C}) = \mathfrak{C}_M$ and $K_\mathfrak{C} = \mathfrak{B}, K_{\mathfrak{C}_M} = \mathfrak{B}_M, K_{\mathfrak{C},1} = \mathfrak{U}, K_{\mathfrak{C}_M,1} = \mathfrak{U}_M$.

The map $\mathfrak{F} \mapsto \mathfrak{p}_M(\mathfrak{F})$ from the set of facets of $(V, \mathfrak{H})$ to the set of facets of $(V_M, \mathfrak{H}_M)$ is surjective because the map $V \xrightarrow{p_M} V_M$ is surjective.

**Proposition 4.4.** *Let $\mathfrak{F}$ be a facet of $(V, \mathfrak{H})$ and $H_M \in \mathfrak{H}_M$. Then,*

(i) *$\mathfrak{p}_M(\mathfrak{F}) \subset H_M$ if and only if $p_M(\mathfrak{F}) \subset H_M$.*

(ii) *$K_{\mathfrak{p}_M(\mathfrak{F})} = M \cap K_\mathfrak{F}$ and $K_{\mathfrak{p}_M(\mathfrak{F}),1} = M \cap K_{\mathfrak{F},1}$.*

*Proof.* (i) is obvious.

(ii) The equality $K_{\mathfrak{p}_M(\mathfrak{F})} = M \cap K_\mathfrak{F}$ is proved in [Morris, Lemma 1.13] using the extended buildings (where the apartment attached to $T$ is the same for $G$ and for $M$), and in [HRo, Lemma 4.1.1].

We prove $K_{\mathfrak{p}_M(\mathfrak{F}),1} = M \cap K_{\mathfrak{F},1}$. A (pro-$p$) parahoric subgroup of $G$ or of $M$ is generated by its intersections $U_\alpha$ for $\alpha$ in $\Phi$ or $\Phi_M$ and by the (pro-$p$) parahoric subgroup of $Z$. We check that for $\alpha \in \Phi_M$, $U_\alpha \cap K_{\mathfrak{p}_M(\mathfrak{F})} = U_\alpha \cap K_\mathfrak{F}$ and $U_\alpha \cap K_{\mathfrak{p}_M(\mathfrak{F}),1} = U_\alpha \cap K_{\mathfrak{F},1}$ using [Vig1, (43), (51), (52)].

The smallest element $r_\mathfrak{F}(\alpha) \in \Gamma_\alpha$ denote such that $\alpha(x) + r_\mathfrak{F}(\alpha) \geq 0$ for $x \in \mathfrak{F}$ is equal to $r_{\mathfrak{p}_M(\mathfrak{F})}(\alpha)$, hence $U_\alpha \cap K_{\mathfrak{p}_M(\mathfrak{F})} = U_{\alpha + r_{\mathfrak{p}_M(\mathfrak{F})}(\alpha)} = U_{\alpha + r_\mathfrak{F}(\alpha)} = U_\alpha \cap K_\mathfrak{F}$.

We have $\mathfrak{F} \subset \mathrm{Ker}_V(\alpha + r_\mathfrak{F}(\alpha))$ if and only if $\mathfrak{p}_M(\mathfrak{F}) \subset \mathrm{Ker}_{V_M}(\alpha + r_\mathfrak{F}(\alpha))$ by (i), the element $r^*_\mathfrak{F}(\alpha) = r_\mathfrak{F}(\alpha)$ if $\mathfrak{F} \subset \mathrm{Ker}(\alpha + r_\mathfrak{F}(\alpha))$, $r^*_\mathfrak{F}(\alpha) = r_\mathfrak{F}(\alpha) + e_\alpha^{-1}$ otherwise, is equal to $r^*_{\mathfrak{p}_M(\mathfrak{F})}(\alpha)$, hence $U_\alpha \cap K_{\mathfrak{p}_M(\mathfrak{F}),1} = U_{\alpha + r^*_{\mathfrak{p}_M(\mathfrak{F})}(\alpha)} = U_{\alpha + r^*_\mathfrak{F}(\alpha)} = U_\alpha \cap K_{\mathfrak{F},1}$.

We can only deduce $K_{\mathfrak{p}_M(\mathfrak{F})} \subset M \cap K_\mathfrak{F}$, but the Iwahori decomposition of $K_{\mathfrak{F},1}$ [Vig1, Proposition 3.19] implies $K_{\mathfrak{p}_M(\mathfrak{F}),1} = M \cap K_{\mathfrak{F},1}$. ☐

We prove Theorem 2.21.

Proposition 4.4 implies that the (pro-$p$) Iwahori subgroup of $(M, T, B_M, \varphi_M)$ is the intersection with $M$ of the (pro-$p$) Iwahori subgroup of $(G, T, B, \varphi)$.

We check that the datum $\mathcal{W}_M$ of $(M, T, B_M, \varphi_M)$ is equal to the datum (2.18) associated to the datum $\mathcal{W}$ of $(G, T, B, \varphi)$ and $S_M$. The $\mathbf{M}$-centralizer of $\mathbf{T}$ is $\mathbf{Z}$, hence $\mathcal{W}_M, \mathcal{W}$ have the same $\Lambda, Z_k$. Recalling from section 3 the relation between $\Phi$ and the reduced root system $\Sigma$ and the definition of the basis $\Delta$, the reduced root system $\Sigma_M$ for $M$ is $\{e_\alpha \alpha \mid \alpha \in \Phi_M\}$ because $\varphi_{M,\alpha} = \varphi_\alpha$ for $\alpha \in \Phi_M$ and the basis $\Delta_M$ of $\Sigma_M$ corresponding to $\mathbf{B_M} = \mathbf{B} \cap \mathbf{M}$ is $\Delta \cap \Sigma_M$. The property (ii) of (2.18) is clear. The property (iii) also because the $\mathbf{M}$-normalizer of $\mathbf{T}$ is $\mathfrak{N}_M = \mathfrak{N} \cap \mathbf{M}$.

We check that the parameter map $\mathfrak{c}_M$ of $(M, T, B_M, \varphi_M)$ and the parameter map $\mathfrak{c}$ of $(G, T, B, \varphi)$ are equal on $\mathfrak{S}_M(1)$. Let $\alpha \in \Phi_M, u \in U_\alpha - \{1\}$ and $\tilde{s} \in \mathfrak{S}_M(1)$. The definition of the admissibility of the pair $(\alpha, u)$ or of the triple $(\alpha, u, \tilde{s})$ (Definition 3.6) is the same for $M$ and $G$. The parameter maps are $Z_k$-equivariant hence it suffices to check that $\mathfrak{c}_M$ and $\mathfrak{c}$ are equal on admissible elements of $\mathfrak{S}_M(1)$. Let $(\alpha, u, \tilde{s})$ be an admissible triple. We have to show that $c(\alpha, u)$ (13) is the same for $M$ and $G$. Let $H_s \in \mathfrak{H}$ and $H_{M,s} \in \mathfrak{H}_M$ fixed by $s$. We have $H_s = p_M^{-1}(H_{M,s})$. Let $\mathfrak{A}_s$ be an alcove of $(V, \mathfrak{H})$ with a face $\mathfrak{F}_s \subset H_s$. The unique facet of $(V_M, \mathfrak{H}_M)$ containing $p_M(\mathfrak{A}_s)$ is an alcove $\mathfrak{A}_{M,s}$ with a face $\mathfrak{F}_{M,s} \subset H_{M,s}$ containing $p_M(\mathfrak{F}_s)$. Let $\mathfrak{K}_{M,s}, \mathfrak{K}_s$ denote the parahoric subgroups of $M, G$ fixing $\mathfrak{F}_{M,s}, \mathfrak{F}_s$, $\mathfrak{K}_{M,s,1}, \mathfrak{K}_{s,1}$ their pro-$p$ radicals, $\mathfrak{K}_{M,s,k}, \mathfrak{K}_{s,k}$ their finite reductive quotients.

**Lemma 4.5.** $\mathfrak{K}_{M,s,k} = \mathfrak{K}_{s,k}$.

*Proof.* By proposition, $\mathfrak{K}_{M,s} = M \cap \mathfrak{K}_s$, $\mathfrak{K}_{M,s,1} = M \cap \mathfrak{K}_{s,1}$. This implies $\mathfrak{K}_{M,s,k} \subset \mathfrak{K}_{s,k}$. Both groups generated by $Z_k, U_{s,k} = U_{\alpha+r}/U_{\alpha+r+e_\alpha^{-1}}$ hence they are equal. ☐

The lemma implies that $c(\alpha, u)$ is the same for $M$ and $G$. This ends the proof of Theorem 2.21.

# 5 Central extension

## 5.1 Morphism of admissible data with the same based reduced root system

Let $\mathcal{W}_H \xrightarrow{i} \mathcal{W}$ be a morphism of admissible data with the same based reduced root system, and let $(\mathfrak{q}_H, \mathfrak{q})$ and $(\mathfrak{c}_H, \mathfrak{c})$ be $i$-compatible parameter maps of $(\mathcal{W}_H^{Iw}, R), (\mathcal{W}^{Iw}, R)$ and $(\mathcal{W}_H, R), (\mathcal{W}, R)$. We prove Proposition 2.24.

The linear map $\mathcal{H}(\mathcal{W}_H, \mathfrak{q}_H, \mathfrak{c}_H) \xrightarrow{i} \mathcal{H}(\mathcal{W}, \mathfrak{q}, \mathfrak{c})$ respects the product, because it respects the braid relations as $W_{H,1} \xrightarrow{i} W_1$ respects the length, and the quadratic relations as the parameters are $i$-compatible. Obviously, its image is isomorphic to $\mathcal{H}(i(\mathcal{W})_H, \mathfrak{q}, \mathfrak{c})$ and its kernel is $R[(W_{H,1})_{i=1}]_{\epsilon=0}$. We prove that it respects the alcove walk elements. Let $o$ be an orientation of $(V, \mathfrak{H})$. We recall that $i$ is the identity on $W_H^{aff} = W^{aff}$. Let $s \in S_H^{aff} = S^{aff}$, $w \in W^{aff}$ such that $\ell(ws) = \ell(w) + 1$, and $\tilde{s}_H \in S_H^{aff}(1)$ lifting $s$ in $W_{H,1}$, the definition (4) implies:

$$(15) \qquad i(T_{\tilde{s}_H}^{H,\epsilon_o(w,s)}) = T_{i(\tilde{s}_H)}^{\epsilon_o(w,s)}$$

where $i(\tilde{s}_H) \in S(1)$ lifts $s$ in $W_1$. Let $\tilde{w}_H \in W_{H,1}$ of reduced decomposition $\tilde{w}_H = \tilde{s}_{H,1} \ldots \tilde{s}_{H,r} \tilde{u}_H, r = \ell(w_H), \tilde{s}_{H,i} \in S_H^{aff}(1), \tilde{u}_H \in \Omega_{H,1}$. A reduced decomposition of $i(\tilde{w}_H)$ is $i(\tilde{w}_H) = \tilde{s}_1 \ldots \tilde{s}_r \tilde{u}, r = \ell(w), \tilde{s}_i = i(\tilde{s}_{H,i}) \in S^{aff}(1), \tilde{u} = i(\tilde{u}_H) \in \Omega_1$. As $i$ is an algebra homomomorphism, definition 2.10 and (15) imply $i(E_o^H(\tilde{w}_H)) = E_o(i(\tilde{w}_H))$.

We have $i(\Omega_H) \subset \Omega$ and $i(W_H)$ is the subgroup $W^{aff} \rtimes i(\Omega_H)$ of $W = W^{aff} \rtimes \Omega$. The exact sequence

$$1 \to i(Z_{H,k}) \to i(W_{H,1}) \to i(W_H) \to 1$$

is contained in the exact sequence $1 \to Z_k \to W_1 \to W \to 1$. We have

$$W_1 = i(W_{H,1})\Omega_1 = \Omega_1 i(W_{H,1}), \quad i(\Omega_{H,1}) = \Omega_1 \cap i(W_{H,1}).$$

We deduce from (2) that the algebra $\mathcal{H}_R(\mathcal{W}, \mathfrak{q}, \mathfrak{c})$ is isomorphic to

$$i(\mathcal{H}_R(\mathcal{W}_H, \mathfrak{q}_H, \mathfrak{c}_H)) \otimes_{R[i(\Omega_{H,1})]} R[\Omega_1] \simeq R[\Omega_1] \otimes_{R[i(\Omega_{H,1})]} i(\mathcal{H}_R(\mathcal{W}_H, \mathfrak{q}_H, \mathfrak{c}_H)).$$

This ends the proof of Proposition 2.24.

**Remark 5.1.** The homomorphism $\mathcal{W}_{H,1}^{aff} \xrightarrow{i} \mathcal{W}_1^{aff}$ is surjective (injective) if only if the homomorphism $Z_{H,k} \xrightarrow{i} Z_k$ is surjective (injective).

## 5.2 Pro-$p$ Iwahori Hecke algebras of central extensions

Let $\mathbf{H} \xrightarrow{\mathbf{i}} \mathbf{G}$ be a central extension of connected $F$-reductive groups. We indicate with a lower or upper index $H$ an object relative to $H$. as in §2, we associate to a triple $(T, B, \varphi)$ of $G$ a triple $(T_H, B_H, \varphi)$ of $H$. The homomorphism $\mathbf{i}$ induces a bijection $\alpha \mapsto \alpha \circ i$ from the root system $\Phi$ of $(\mathbf{G}, \mathbf{T})$ onto the root system $\Phi_H$ of $(\mathbf{H}, \mathbf{T_H})$ respecting the positive roots relative to $\mathbf{B}$ and $\mathbf{B_H}$, and an $F$-isomorphism $\mathbf{U}_{\mathbf{H},\alpha\circ\mathbf{i}} \xrightarrow{\mathbf{i}} \mathbf{U}_\alpha$ between the root group of $\alpha \circ i$ in $\mathbf{H}$ and of $\alpha$ in $\mathbf{G}$ for all $\alpha \in \Phi$. Let $\mathcal{W}, \mathfrak{c}$ be the admissible datum and the parameter map associated to $(G, T, B, \varphi)$ as in section 3 and Theorem 2.15. The special discrete valuation $\varphi = (\varphi_\alpha)_{\alpha \in \Phi}$ compatible with $\omega$ of the root datum $(Z, (U_\alpha)_{\alpha \in \Phi})$ generating $G$ is also a special discrete valuation compatible with $\omega$ of the root datum $(Z_H, (U_{H,\alpha})_{\alpha \in \Phi_H})$ generating $H$.

We prove Theorem 2.25.

(i) We sometimes identify $\alpha$ and $\alpha \circ i$, hence $\Phi_H$ and $\Phi$, $V_H$ and $V$. The action of $\mathfrak{N}_H$ and of $\mathfrak{N}$ on the semisimple apartment $(V, \mathfrak{H})$ associated to $\Phi$ and $\varphi$ are compatible with the homomorphism $\mathfrak{N}_H \xrightarrow{i} \mathfrak{N}$. The based reduced root systems of the admissible datum $\mathcal{W}_H$ of $(H, T_H, B_H, \varphi)$ and of the admissible datum $\mathcal{W}$ of $(G, T, B, \varphi)$ are the same. The functoriality of the Kottwitz homomorphism applied to $\mathbf{Z_H} \xrightarrow{\mathbf{i}} \mathbf{Z}$ implies that $i(Z_{H,0}) \subset Z_0$. Lemma 3.1 (ii), (iii) applied to $Z_{H,0} \xrightarrow{i} Z_0$ implies $i(Z_{H,1}) \subset (i(Z_{H,0}))_1 = Z_{H,0} \cap Z_1 \subset Z_1$. We deduce that the homomomorphism $\mathfrak{N}_H \xrightarrow{i} \mathfrak{N}$ induces compatible homomorphisms

$$(\Lambda_H, W_H, Z_{H,k}, W_{H,1}) \xrightarrow{i} (\Lambda, W, Z_k, W_1)$$

equal to the identify on $W^{aff}$, and $\nu_H = \nu \circ i$. Hence $H \xrightarrow{i} G$ induces an homomorphism $\mathcal{W}_H \xrightarrow{i} \mathcal{W}$ between the admissible data with the same reduced root system.

(ii) The homomorphism between the $F$-rational points does no remain surjective in general. The subgroup $i(H) \subset G$ is normal because it is the kernel of the natural homomorphism $G \to H^1(F, \mu)$. The group $G/i(H)$ may be infinite ($PGL(2, F)/PSL(2, F)$ is infinite when the characteristic of $F$ is 2). But we note the finiteness property:

**Lemma 5.2.** *$\Lambda/i(\Lambda_H)$ is finite.*

*Proof.* The kernel $\mu$ is central and $\Phi_H \simeq \Phi$ have the same number $r$ of simple roots. The groups $\Lambda$ and $\Lambda_H$ are finitely generated of rank $r$. ☐

For later use, let $P = MN, P_H = M_H N_H$ be standard parabolic subgroups of $G, H$ corresponding to the same subset of $\Delta$, with their standard Levi decomposition.

**Lemma 5.3.** *$i(P_H) = i(M_H)\, N = P \cap i(H)$ and $P\, i(H) = G$.*

*Proof.* The isomorphism $\mathbf{N_H} \xrightarrow{\mathbf{i}} \mathbf{N}$ implies $i(P_H) = i(M_H)N$ and $(M \cap i(H))N = P \cap i(H)$. We recall that $G = ZG'$ where $G'$ is generated by the root subgroups $U_\alpha$ for $\alpha$ in the root system $\Phi$ of $\mathbf{T}$ in $\mathbf{G}$ and $G' = i(H')$. We have $M = ZM' = Zi((M_H)')$ and $Z \cap i(H) = i(Z_H)$. Hence $M \cap i(H) = i(Z_H)i((M_H)') = i(M_H)$ and $G = ZG' = Z\, i(H) = P\, i(H)$. ☐

The homomorphism $\mathfrak{N}_H/Z_{H,1} \xrightarrow{i} \mathfrak{N}/Z_1$ has kernel $i^{-1}(Z_1)/Z_{H,1}$ and image $i(\mathfrak{N}_H)Z_1/Z_1$.
We deduce that $i(H) = G \Leftrightarrow i(Z_H) = Z \Leftrightarrow i(\mathfrak{N}_H) = \mathfrak{N}$. The latter equivalence follows from the isomorphism $W_{H,0} = \mathfrak{N}_H/Z_H \xrightarrow{i} \mathfrak{N}/Z = W_0$.

(iii) The map $(h, x) \mapsto (i(h), x) : H \times V \to G \times V$ induces a map $\mathfrak{BT}_H \xrightarrow{i} \mathfrak{BT}$ between the semisimple Bruhat-Tits buildings of $H$ and $G$ (the definition and notation is recalled in section 3). Indeed, for $x \in V$, we have the isomorphism $U_{H,x+r_x(\alpha\circ i)} \xrightarrow{i} U_{x+r_x(\alpha)}$ for $\alpha \in \Phi$, homomorphisms $\mathfrak{N}_{H,x} \xrightarrow{i} \mathfrak{N}_x$ between the $\mathfrak{N}_H$ and $\mathfrak{N}$ stabilizers of $x$, and $\mathfrak{P}_{H,x} = \mathfrak{N}_{H,x}U_{H,x} \xrightarrow{i} \mathfrak{P}_x = \mathfrak{N}_x U_x$. Let $\mathfrak{F}$ be a facet of $(V, \mathfrak{H})$. We denote by $\mathfrak{K}_{H,\mathfrak{F}} \subset H$ the parahoric subgroup fixing $\mathfrak{F}$, by $\mathfrak{K}_{H,\mathfrak{F},1}$ and by $\mathfrak{K}_{H,\mathfrak{F},k}$ the finite reductive quotient.

**Lemma 5.4.** *$i(\mathfrak{K}_{H,\mathfrak{F}})$, $i(\mathfrak{K}_{H,\mathfrak{F},1})$ are open normal subgroups of $\mathfrak{K}_{\mathfrak{F}}, \mathfrak{K}_{\mathfrak{F},1}$ and $i$ induces an homomorphism $\mathfrak{K}_{H,\mathfrak{F},k} \xrightarrow{i} \mathfrak{K}_{\mathfrak{F},k}$.*

*Proof.* For a reduced root $\alpha \in \Phi$, we have $i(K_{H,\mathfrak{F}} \cap U_{H,\alpha\circ i}) = K_{\mathfrak{F}} \cap U_\alpha$ and $i(K_{H,\mathfrak{F},1} \cap U_{H,\alpha\circ i}) = K_{\mathfrak{F},1} \cap U_\alpha$. The group $\mathfrak{K}_{H,\mathfrak{F}}$ is generated by $Z_{H,0}$ and $K_{H,\mathfrak{F}} \cap U_{H,\alpha\circ i}$ for all reduced root $\alpha \in \Phi$, the group $\mathfrak{K}_{H,\mathfrak{F},1}$ is generated by $Z_{H,1}$ and $K_{H,\mathfrak{F},1} \cap U_{H,\alpha\circ i}$. We deduce that $i(\mathfrak{K}_{H,\mathfrak{F}})$, $i(\mathfrak{K}_{H,\mathfrak{F},1})$ are open subgroups of $\mathfrak{K}_{\mathfrak{F}}, \mathfrak{K}_{\mathfrak{F},1}$. ☐

(iv) We prove that the parameter maps $\mathfrak{c}_H$ of $(H, T_H, B_H, \varphi)$ and $\mathfrak{c}$ of $(G, T, B, \varphi)$ are $i$-compatible. Let $(\alpha \circ i, u_H, \tilde{s}_T)$ be an admissible pair for $(H, T_H, B_H, \varphi)$ and $t_H \in Z_{H,k}$. Write $(u, \tilde{s}, t) = i(u_H, \tilde{s}_H, t_H)$. Then $(\alpha, u, \tilde{s})$ is an admissible pair for $(G, T, B, \varphi)$ and $t \in Z_k$. By Theorem 3.7,

$$\mathfrak{c}_H(\tilde{s}_H t_H) = \sum_{x_{H,k} \in U_{H,s_H,k} - \{1\}} z_H(x_{H,k}) t_H, \quad \mathfrak{c}(\tilde{s}t) = \sum_{x_k \in U_{s,k} - \{1\}} z(x_k) t.$$

Let $\mathfrak{F}_{s_H} = \mathfrak{F}_s$ be a face fixed by $s_H$ hence by $s$ of an alcove of $(V, \mathfrak{H})$. By Lemma 5.4, the homomorphism $G \xrightarrow{i} H$ induces an homomorphism $\mathfrak{K}_{H,s_H,k} \xrightarrow{i} \mathfrak{K}_{s,k}$ between the finite reductive quotients of the parahoric subgroups fixing this face. This homomorphism restricts to an isomorphism $U_{H,s_H,k} \simeq U_{s,k}$, $i(\mathfrak{N}_{H,s_H,k}) \subset \mathfrak{N}_{s,k}$, $i(G_{H,s_H,k}) \subset G_{s,k}$, $i(Z_{H,s_H,k}) \subset Z_{s,k}$. As (12), the element $z_H(x_{H,k}) \in Z_{H,s_H,k}$ for $x_{H,k} \in U_{H,s_H,k} - \{1\}$, is defined by

$$m_H(u_{H,k}) x_{H,k}^{-1} m_H(u_{H,k}) \in U_{H,s_H,k} m_H(u_{H,k}) z_H(x_{H,k}) U_{H,s_H,k},$$

where $u_{H,k} \in U_{H,s_H,k} - \{1\}$ is the image of $u_H$, $\{m_H(u_{H,k})\} = \mathfrak{N}_{H,s_H,k} \cap U^{op}_{H,s_H,k} u_{H,k} U^{op}_{H,s_H,k}$. We have $i(m_H(u_{H,k})) = m(u_k)$ where $u_k$ is the image of $u$ and $i(z_H(x_{H,k})) = z(x_k)$ where $i(x_{H,k}) = x_k$. We deduce that $i(\mathfrak{c}_H(\tilde{s}_H t_H)) = \mathfrak{c}(\tilde{s}t)$. Hence the the parameter maps $\mathfrak{c}_H$ and $\mathfrak{c}$ are $i$-compatible.

The augmentation maps satisfy $\mathbb{Z}[Z_{H,k}] \xrightarrow{\epsilon_H} \mathbb{Z} = \mathbb{Z}[Z_{H,k}] \xrightarrow{i} \mathbb{Z}[Z_k] \xrightarrow{\epsilon} \mathbb{Z}$ hence the parameter maps $\mathfrak{q}_H = \epsilon_H \circ \mathfrak{c}_H, \mathfrak{q} = \epsilon \circ \mathfrak{c}$ of $\mathcal{W}_H^{Iw}, \mathcal{W}^{Iw}$ are $i$-compatible and we can apply Proposition 2.24 to the algebra homomorphism $\mathcal{H}_{\mathbb{Z}}(H, \mathfrak{U}_H) = \mathcal{H}_{\mathbb{Z}}(\mathcal{W}_H, \mathfrak{q}_H, \mathfrak{c}_H) \xrightarrow{i} \mathcal{H}_{\mathbb{Z}}(\mathcal{W}, \mathfrak{q}, \mathfrak{c}) = \mathcal{H}_{\mathbb{Z}}(G, \mathfrak{U})$ between the pro-$p$ Iwahori Hecke rings.

(v) The kernel $\mathbb{Z}[i^{-1}(Z_1)/Z_{H,1}]$ of $\mathcal{H}_{\mathbb{Z}}(H, \mathcal{U}_H) \xrightarrow{i} \mathcal{H}_{\mathbb{Z}}(G, \mathcal{U})$ (Proposition 2.24) is contained in $\mathbb{Z}[\Omega_{H,1}]$. Recalling the isomorphism (7), we have $\mathcal{H}_{\mathbb{Z}}(H, \mathfrak{U}_H) = \mathcal{H}_{\mathbb{Z}}(H', \mathfrak{U}'_H) \rtimes_{\mathbb{Z}[Z'_{H,k}]} \mathbb{Z}[\Omega_{H,1}]$. We have $i(\mathcal{H}_{\mathbb{Z}}(H', \mathfrak{U}'_H)) = \mathcal{H}_{\mathbb{Z}}(i(H')\mathfrak{U}, \mathfrak{U}) = \mathcal{H}_{\mathbb{Z}}(Z_1 G', \mathfrak{U}) \simeq \mathcal{H}_{\mathbb{Z}}(G', \mathfrak{U}')$ (Lemma 3.2), and

$$i(\mathcal{H}_{\mathbb{Z}}(H, \mathfrak{U}_H)) \simeq \mathcal{H}_{\mathbb{Z}}(G', \mathfrak{U}') \rtimes_{\mathbb{Z}[i(Z'_{H,k})]} \mathbb{Z}[i(\Omega_{H,1}].$$

**Remark 5.5.** *We have*

$$i(\mathcal{H}_{\mathbb{Z}}(H, \mathfrak{U}_H)) = \mathcal{H}_{\mathbb{Z}}(i(H)\mathfrak{U}, \mathfrak{U}) \simeq \mathcal{H}_{\mathbb{Z}}(i(H), (Z_1 \cap i(Z_H))\mathfrak{U}').$$

Indeed, $i(\mathcal{H}_{\mathbb{Z}}(H, \mathcal{U}_H)) = \mathcal{H}_{\mathbb{Z}}(i(H)\mathfrak{U}, \mathfrak{U}) \simeq \mathcal{H}_{\mathbb{Z}}(i(H), \mathfrak{U} \cap i(H))$ by Lemma 3.2 applied to the normal subgroup $i(H) \subset G$. By the Iwahori decomposition of a pro-$p$ Iwahori subgroup,

$$\mathfrak{U} = Z_1 \mathfrak{U}', \quad \mathfrak{U}' = \mathfrak{U} \cap G' = i(\mathfrak{U}'_H).$$

We have $i(H) = i(Z_H)G'$, $i(H)\mathfrak{U} = Z_1 i(Z_H) G'$, $\mathfrak{U} \cap i(H) = (Z_1 \cap i(Z_H))\mathfrak{U}'$.

(vi) The $F$-extension $\mathbf{T_H} \xrightarrow{\mathbf{i}} \mathbf{T}$ of $F$-split tori induces a surjective homomorphism $\mu_H \mapsto i \circ \mu_H : X_*(\mathbf{T_H}) \xrightarrow{i} X_*(\mathbf{T})$ and $i(\mu_H(p_F^{-1})) = (i \circ \mu_H)(p_F^{-1})$. This homomorphism is $W_0$-equivariant (we identify naturally $W_{H,0}$ and $W_0$).

The commutative diagram $Z_H \xrightarrow{\lambda} \Lambda_H \xrightarrow{i} \Lambda = Z_H \xrightarrow{i} Z \xrightarrow{\lambda} \Lambda$ and the inclusion $i(T_H) \subset T$ imply that $i(\Lambda_H^\flat) = (i \circ \lambda)(T_H) = (\lambda \circ i)(T_H) \subset \Lambda^\flat = \lambda(T)$. For $\mu_H \in X_*(\mathbf{T_H}), \mu = i \circ \mu_H$, we have $i(\lambda(\mu_H(p_F)^{-1})) = \lambda(\mu(p_F)^{-1})$.

The commutative diagram $Z_H \xrightarrow{\lambda_1} \Lambda_{H,1} \xrightarrow{i} \Lambda_1 = Z_H \xrightarrow{i} Z \xrightarrow{\lambda_1} \Lambda_1$ shows that $i(\lambda_1(\mu_H(p_F)^{-1})) = \lambda_1(\mu(p_F)^{-1})$.

The splitting $\Lambda_H^\flat \xrightarrow{\iota_H} \Lambda_{H,1}^\flat$ of $(H, T_H, B_H, \varphi, p_F)$ is defined by $\iota_H(\lambda(\mu_H(p_F)^{-1})) = \lambda_1(\mu_H(p_F)^{-1})$. It is $i$-compatible with the splitting $\Lambda^\flat \xrightarrow{\iota} \Lambda_1^\flat$ of $(G, T, B, \varphi, p_F)$ because $(i \circ \iota_H)(\lambda(\mu_H(p_F)^{-1})) = (i \circ \lambda_1)(\mu_H(p_F)^{-1}) = (\lambda_1 \circ i)(\mu_H(p_F)^{-1}) = (\iota \circ i)(\lambda(\mu_H(p_F)^{-1}))$.

All the homomorphisms $\lambda, \lambda_1, i$ are $W_0$-equivariant, and $\mathcal{H}_{\mathbb{Z}}(H,\mathfrak{U}_H) \xrightarrow{i} \mathcal{H}_{\mathbb{Z}}(G,\mathfrak{U})$ satisfying Proposition 2.24 respect the alcove walk elements. We deduce that the algebra homomorphism $i$ between the pro-$p$ Iwahori Hecke rings respects the central elements $i(E^H(C_{H,1}(\mu_H))) = E(C_1(i \circ \mu_H))$. Hence the homomorphism $\mathcal{Z}_{\mathbb{Z}}(H,\mathcal{U}_H)^\flat \xrightarrow{i} \mathcal{Z}_{\mathbb{Z}}(G,\mathcal{U})^\flat$ is surjective. Its kernel is $\mathcal{Z}_{\mathbb{Z}}(H,\mathcal{U}_H)^\flat \cap \mathbb{Z}[i^{-1}(Z_1)/Z_{H,1}]$. As $W_{H,1} \xrightarrow{i} W_1$ respects the length, $\mathcal{Z}_{\mathbb{Z}}(H,\mathcal{U}_H)^\flat \cap \mathbb{Z}[i^{-1}(Z_1)/Z_{H,1}] = \mathcal{Z}_{\mathbb{Z}}(H,\mathcal{U}_H)^\flat_{\ell=0} \cap \mathbb{Z}[i^{-1}(Z_1)/Z_{H,1}]$ by Remark **??**, and $\mathcal{Z}_{\mathbb{Z}}(H,\mathcal{U}_H)^\flat_{\ell>0} \xrightarrow{i} \mathcal{Z}_{\mathbb{Z}}(G,\mathcal{U})^\flat_{\ell>0}$ is an isomorphism.

As $(T/T_0)^{W_0} \simeq X_*(T)^{W_0} \simeq \mathcal{Z}_{\mathbb{Z}}(H,\mathcal{U}_H)^\flat_{\ell=0}$, contains no element of finite order, $\mathcal{Z}_{\mathbb{Z}}(H,\mathcal{U}_H)^\flat \cap \mathbb{Z}[i^{-1}(Z_1)/Z_{H,1}] = \{0\}$ if $i^{-1}(Z_1)/Z_{H,1}$ is finite.

This ends the proof of Theorem 2.25.

### 5.3 Supercuspidal representations and supersingular modules

Notations as in section 5.2. We denote by $\pi_H$ the inflation to $H$ of the restriction $\pi|_{i(H)}$ of a smooth $R$-representation $\pi$ of $G$. The functor $\pi \mapsto \pi_H$ from the $R$-representations of $G$ to those of $H$ is exact, of image the $R$-representations of $H$ where the kernel $L$ of $H \xrightarrow{i} G$ acts trivially. The $R$-submodules $\pi^K \subset \pi$ and $\pi_H^{K_H} \subset \pi_H$ fixed by open compact subgroups $K \subset G$ and $K_H \subset H$ with $i(K_H) \subset K$ satisfy

$$\pi^K \subset \pi_H^{K_H} = \pi^{i(K_H)}.$$

As the subgroup $i(H) \subset G$ is open, $i(K_H) \subset G$ is open (and compact), and an arbitrary open compact subgroup $K \subset G$ contains $i(K_H)$ for some open compact subgroup $K_H \subset H$. Therefore, the representation $\pi$ is smooth, or admissible if and only if the representation $\pi_H$ is smooth, or admissible. The $R$-module $\pi^K$ has a structure of right module over the Hecke $R$-algebra $\mathcal{H}_R(G,K)$, and the $R$-module $\pi_H^{K_H} = \pi^{i(K_H)}$ has a structure of right $\mathcal{H}_R(H,K_H)$-module and of right $\mathcal{H}_R(G,i(K_H))$-module. We note that $\mathcal{H}_R(i(H),i(K_H)) \subset \mathcal{H}_R(G,i(K_H))$.

Assume that $R$ is a field. By Clifford's theory, the restriction of the irreducible admissible $R$-representation $\pi$ of $G$ to the normal open subgroup $i(H) \subset G$ of finite index is a finite direct sum $\oplus_j \pi_j$ of $G$-conjugate irreducible admissible $R$-representations $\pi_j$ of $i(H)$ conjugate in $G$. The representations $\pi_j$ are $Z$-conjugate because $G = i(H)Z$. The induced representation $\rho_G(\pi) = \mathrm{Ind}_{i(H)}^G(\pi_j)$ of $G$ does not depend on the choice of $j$ modulo isomorphism. It is admissible of finite length and contains $\pi$ because the induction is the right adjoint of the restriction. The representation $\pi_H$ of $H$ is admissible semisimple of finite length, of irreducible components $\pi_{j,H}$ inflating $\pi_j$ for all $j$.

Let $\pi, \tau$ be irreducible admissible $R$-representations of $G, M$. We decompose $\pi|_{i(H)} = \oplus_j \pi_j$ and $\tau|_{i(M_H)} = \oplus_r \tau_r$ as a finite sum of irreducible admissible representations. We consider the parabolic induced representation $\mathrm{Ind}_P^G(\tau)$ (where $\tau$ is inflated to $P$).

**Lemma 5.6.** (i) *The restriction of* $\mathrm{Ind}_P^G(\tau)$ *to* $H$ *is equal to* $(\mathrm{Ind}_P^G(\tau))_H = \mathrm{Ind}_{P_H}^H(\tau_{M_H})$, *and it is also the inflation to* $H$ *of* $\mathrm{Ind}_{i(P_H)}^{i(H)}(\tau|_{i(M_H)})$.

(ii) *If* $\pi$ *is a subquotient of* $\mathrm{Ind}_P^G(\tau)$, *then* $\pi_H$ *is a subquotient of* $(\mathrm{Ind}_P^G(\tau))_H$.

(iii) *If* $\pi_{j,H}$ *is a subquotient of* $(\mathrm{Ind}_P^G(\tau))_H$ *for some* $j$, *then* $\rho_G(\pi)$ *is a subquotient of* $\mathrm{Ind}_P^G \rho_M(\tau)$.

*Proof.* (i) We have $G = Pi(H)$ and $P \cap i(H) = i(P_H)$ (Lemma 5.3). The restriction of $\mathrm{Ind}_P^G(\tau)$ to $i(H)$ is $\mathrm{Ind}_{i(P_H)}^{i(H)}(\tau|_{i(M_H)})$. The inflation of $\mathrm{Ind}_{i(P_H)}^{i(H)}(\tau|_{i(M_H)})$ to $H$ is $\mathrm{Ind}_{P_H}^H(\tau_{M_H})$ because the kernel of $H \xrightarrow{i} G$ is equal to the kernel of $M_H \xrightarrow{i} M$.

(ii) By exactitude of the inflation and of the restriction, if $\pi$ is a subquotient of $\mathrm{Ind}_P^G(\tau)$ then $\pi_H$ is a subquotient of $(\mathrm{Ind}_P^G(\tau))_H$.

(iii) We denote by $\mathrm{Ind}_{i(P_H)}^{i(H)}$ the functor from smooth representations of $i(M_H)$ to smooth representations of $i(H)$ similar to the parabolic induction: one induces smoothly the inflation to $i(P_H)$ of a smooth representation of $i(M_H)$. The functor $\mathrm{Ind}_{i(P_H)}^{i(H)}$ commutes with finite direct sums. Assume that $\pi_{j,H}$ is a subquotient of $(\mathrm{Ind}_P^G(\tau))_H$. Then $\pi_j$ is a subquotient of $\mathrm{Ind}_{i(P_H)}^{i(H)}(\tau|_{i(M_H)})$ by (i). There exists $r$ such that $\pi_j$ is a subquotient of $\mathrm{Ind}_{i(P_H)}^{i(H)}(\tau_r)$. By exactness of the induction, $\rho(\pi)$ is a subquotient of $\mathrm{Ind}_{i(H)}^G(\mathrm{Ind}_{i(P_H)}^{i(H)}(\tau_r))$. By transitivity of the induction $\mathrm{Ind}_{i(H)}^G(\mathrm{Ind}_{i(P_H)}^{i(H)}(\tau_r)) = \mathrm{Ind}_{i(P_H)}^G(\tau_r) = \mathrm{Ind}_P^G(\mathrm{Ind}_{i(P_H)}^P(\tau_r))$. As $i(P_H) = i(M_H)N, P = MN$, the representation $\mathrm{Ind}_{i(P_H)}^P(\tau_r)$ is the inflation to $P$ of $\rho_M(\tau) = \mathrm{Ind}_{i(M_H)}^M(\tau_r)$. Hence $\rho_G(\pi)$ is a subquotient of $\mathrm{Ind}_P^G \rho_M(\tau)$. □

We prove Proposition 2.26.

Let $R$ be a field and $\pi$ an irreducible admissible $R$-representation of $G$. We deduce from Lemma 5.6 (ii) that if $\pi$ is not supercuspidal then $\pi_{j,H}$ is not supercuspidal for all $j$, and from Lemma 5.6 (iii) that if $\pi_{j,H}$ is not supercuspidal for some $i$ that then $\pi$ is not supercuspidal. The part (i) of Proposition 2.26 is proved.

We denote by $\pi_H$ the inflation to $H$ of the restriction of $\pi$ to $i(H)$.

We consider the parabolic induction $\mathrm{Ind}_P^G$ from the smooth $R$-representations of $M$ to those of $G$ (the smooth induction from $P$ to $G$ of the inflation from $M$ to $P$), and similarly the parabolic induction $\mathrm{Ind}_{i(P_H)}^{i(H)}$ (from the smooth $R$-representations of $i(M_H)$ to those of $i(P_H)$ by inflation then to those of $G$ by smooth induction). The parabolic functors commute with finite direct sums.

Let $\tau$ be a smooth $R$-representation of $M$.

**Lemma 5.7.** (i) *The restriction of* $\mathrm{Ind}_P^G(\tau)$ *to* $i(H)$ *is equal to* $\mathrm{Ind}_{i(P_H)}^{i(H)}(\tau|_{i(M_H)})$. *The inflation ot* $H$ *of* $\mathrm{Ind}_P^G(\tau)|_{i(H)}$ *is equal to* $\mathrm{Ind}_{P_H}^H G(\tau_{M_H})$.

(ii) *If* $\pi$ *is a subquotient of* $\mathrm{Ind}_P^G(\tau)$, *then* $\pi_H$ *is a subquotient of* $(\mathrm{Ind}_P^G(\tau))_H$.

*Proof.* (i) We have $G = Pi(H)$ and $P \cap i(H) = i(P_H)$ (Lemma **??**). The restriction of $\mathrm{Ind}_P^G(\tau)$ to $i(H)$ is $\mathrm{Ind}_{i(P_H)}^{i(H)}(\tau|_{i(M_H)})$. The inflation of this latter representation to $H$ is $\mathrm{Ind}_{P_H}^H(\tau_{M_H})$ because the kernel of $H \xrightarrow{i} i(H)$ is equal to the kernel of $M_H \xrightarrow{i} i(M_H)$. **

(ii) Exactitude of the inflation and of the restriction. □

We assume from now on that $R$ is a field. Let $\pi$ be an irreducible admissible $R$-representation of $G$ and $\tau$ an irreducible admissible $R$-representation of $M$.

The subgroup $i(H) \subset G$ is normal open of finite index. By Clifford's theory, the restriction of $\pi$ to $i(H)$ is a finite direct sum $\oplus_j \pi_j$ of $G$-conjugate irreducible admissible $R$-representations $\pi_j$ of $i(H)$. The representation $\pi_H$ of $H$ is admissible semisimple of finite length, of irreducible components the representations $\pi_{j,H}$ of $H$ inflating $\pi_j$ for all $j$. The representations $\pi_j$ are $Z$-conjugate because $G = i(H)Z$. The representation $\rho_G(\pi)$ of $G$ induced from $\pi_j$ does not depend on the choice of $j$ modulo isomorphism. The representation $\rho_G(\pi)$ of $G$ is admissible of finite length and contains $\pi$ because the induction is the right adjoint of the restriction.

Similar considerations apply to the subgroup $i(M_H) \subset M$ and to the quotient map $M_H \to i(M_H)$. The restriction of $\tau$ to $i(M_H)$ is a finite direct sum $\oplus_r \tau_r$ of $Z$-conjugate irreducible admissible $R$-representations $\tau_r$ of $i(M_H)$ inflating to representations $\tau_{r,\times M_H}$ of $M_H$. The representation $\rho_M(\tau)$ of $M$ induced from $\tau_r$ of $M$ is admissible of finite length and contains $\tau$.

**Lemma 5.8.** *a enlever probablement*

*Assume that $R$ is a field and that $\pi, \tau$ are irreducible admissible $R$-representations of $G, M$. If $\pi_{j,H}$ is a subquotient of $(\mathrm{Ind}_P^G(\tau))_H$ for some $j$, then $\rho_G(\pi)$ (defined in ***) is a subquotient of $\mathrm{Ind}_P^G \rho_M(\tau)$.*

*Proof.* Assume that $\pi_{j,H}$ is a subquotient of $(\mathrm{Ind}_P^G(\tau))_H$. Then $\pi_j$ is a subquotient of the restriction of $\mathrm{Ind}_P^G(\tau)$ to $i(H)$, hence of $\mathrm{Ind}_{i(P_H)}^{i(H)}(\tau|_{i(M_H)})$ by Lemma 5.7 (i). As $\mathrm{Ind}_{i(P_H)}^{i(H)}(\tau|_{i(M_H)})$ is the finite direct sum of the representations $\mathrm{Ind}_{i(P_H)}^{i(H)}(\tau_r)$, there exists $r$ such that $\pi_j$ is a subquotient of $\mathrm{Ind}_{i(P_H)}^{i(H)}(\tau_r)$. By exactness of the induction, $\rho(\pi)$ is a subquotient of the representation of $G$ induced by $\mathrm{Ind}_{i(P_H)}^{i(H)}(\tau_r)$. The smooth induction from $i(P_H)$ to $i(H)$ followed by the induction from $i(H)$ to $G$ is the smooth induction from $i(P_H)$ to $G$. As $i(P_H) = i(M_H)N$ and $P = MN$, the smooth induction from $i(P_H)$ to $G$ is the smooth induction from $i(P_H)$ to $P$ to the smooth induction from $P$ to $G$, and the representation of $P$ smoothly induced from the the inflation of $\tau_r$ to $i(P_H)$ is equal to the inflation to $P$ of the induction $\rho_M(\tau)$ of $\tau_r$ to $M$. Hence $\rho_G(\pi)$ is a subquotient of $\mathrm{Ind}_P^G(\rho_M(\tau)$. □

**Lemma 5.9.** *Assume that $R$ is a field. An irreducible admissible $R$-representation of $H$ is the tensor product $\pi \otimes \pi_H$ of irreducible admissible representations $\pi, \pi_H$ of $, H$ which are unique modulo isomorphism.*

*Proof.* *** □

**Proposition 5.10.** *Assume that $R$ is a field. Let $\pi$ be an irreducible admissible $R$-representation of $G$, $\pi_j$ the irreducible components of the restriction of $\pi$ to $i(H)$ and $\pi_{j,H}$ the inflation of $\pi_j$ to $H$. Then, the representations $\pi_j, \pi_{j,H}$ of $i(H), H$ are irreducible admissible, and the following properties are equivalent:*

*$\pi$ is supercuspidal,*

*$\pi_{j,H}$ is supercuspidal for one $j$,*

*$\pi_{j,H}$ is supercuspidal for all $j$.*

*Proof.* If $\pi$ is a subquotient of $\mathrm{Ind}_P^G(\tau)$ for some $P, \tau$ as in Lemma 5.7, then the inflation $\pi_H = \oplus_j \pi_{j,H}$ to $H$ of the restriction of $\pi$ to $i(H)$ is a subquotient of

$$\mathrm{Ind}_{P_H}^H(\oplus_r \tau_{r,H}) = \oplus_r \mathrm{Ind}_{P_H}^H(\tau_{r,H})$$

** by Lemma 5.7 (ii). We deduce that for each $j$ there is $r$ such that $\pi_{j,H}$ is a subquotient of $\mathrm{Ind}_{P_H}^H(\tau_{r,H})$. We have $P \neq G$ if and only if $P_H \neq H$. Hence if $\pi$ is not supercuspidal, all $\pi_{j,H}$ are not supercuspidal.

Suppose that there exists $j$ such that $\pi_{j,H}$ is a subquotient of $\mathrm{Ind}_{P_H}^H(\sigma)$ for some $P_H, \sigma$ an irreducible admissible representation of $M_H$, then $\pi_{j,H}$ is a subquotient of $\mathrm{Ind}_{P_H}^H(\sigma)$, then $\sigma$ is trivial on the kernel of $M_H \to i(M_H)$ because this kernel is also the kernel of $H \to i(H)$, and this kernel acts trivially on $\pi_{j,H}$. Hence $\sigma$ is the inflation of a representation $\sigma_j$ of $i(H)$. The representation $\sigma_j$ of $i(H)$ is irreducible admissible because $\sigma$ is. The representation $\pi_j$ is a subquotient of $\mathrm{Ind}_{i(P_H)}^{i(H)}(\sigma_j)$. By adjunction, the representation $\pi$ is a subquotient of the representation of $G$ induced by $\mathrm{Ind}_{i(P_H)}^{i(H)}(\sigma_j)$. The smooth induction from $i(P_H)$ to $i(H)$ followed by the induction from $i(H)$ to $G$ is the smooth induction from $i(P_H)$ to $G$. As $i(P_H) = i(M_H)N$ and $P = MN$, the smooth induction from $i(P_H)$ to $G$ is the smooth induction from $i(P_H)$ to $P$ to the smooth induction from $P$ to $G$, and the representation of $P$ smoothly induced from the the inflation of $\sigma_j$ to $i(P_H)$ is equal to the inflation to $P$ of the induction $\sigma$ of $\sigma_j$ to $M$. Hence $\pi$ is a subquotient of $\mathrm{Ind}_P^G(\sigma)$. Hence if $\pi_{j,H}$ is not supercuspidal for one $j$, then $\pi$ is not supercuspidal. □

We assume now that $R$ is a field of characteristic $p$.

**Proposition 5.11.** *When $R$ is a field of characteristic $p$, a finite dimensional non-supersingular right $\mathcal{H}_R(G,\mathfrak{U})$-module contains a simple non-supersingular submodule.*

*Proof.* When $\mathbf{G}$ is $F$-split [OComp, §5.3]. The proof is valid for $\mathbf{G}$ general (this will be explained in [OV]). $\square$

By Lemma 3.2 we have natural isomorphisms

$$\mathcal{H}_{\mathbb{Z}}(i(H), i(\mathfrak{U}_H)) \simeq \mathcal{H}_{\mathbb{Z}}(i(H), i(\mathfrak{U}_H)),$$
$$\mathcal{H}_{\mathbb{Z}}(i(H), i(H)\cap\mathfrak{U}) \simeq \mathcal{H}_{\mathbb{Z}}(i(H)\mathfrak{U},\mathfrak{U}).$$

The inclusion $i(H)\subset G$ induces an homomorphism $\mathcal{H}_{\mathbb{Z}}(i(H), i(\mathfrak{U}_H)) \to \mathcal{H}_{\mathbb{Z}}(G,\mathfrak{U})$ of image the Hecke subring $\mathcal{H}_{\mathbb{Z}}(i(H)\mathfrak{U},\mathfrak{U})$. The homomorphism $H\to i(H)$ induces an homomorphism $\mathcal{H}_{\mathbb{Z}}(H,\mathfrak{U}_H)\to\mathcal{H}_{\mathbb{Z}}(i(H), i(\mathfrak{U}_H))$ which coincides, via the natural isomorphisms of Hecke rings, with the homomorphism $\mathcal{H}_{\mathbb{Z}}(H,\mathfrak{U}_H)\to\mathcal{H}_{\mathbb{Z}}(i(H), i(\mathfrak{U}_H))$ induced by $i$.

**Proposition 5.12.** *Assume that $R$ is a field of characteristic $p$. Let $\pi$ be a smooth $R$-representation of $G$, and $\pi_H$ the inflation to $H$ of $\pi|_{i(H)}$.*

(i) *The $\mathcal{H}_{\mathbb{Z}}(H,\mathfrak{U}_H)$-module $(\pi_H)^{\mathfrak{U}_H}$ contains a supersingular submodule if and only if the $\mathcal{H}_{\mathbb{Z}}(G,\mathfrak{U})$-module $\pi^{\mathfrak{U}}$ contains a supersingular submodule.*

(ii) *If $\pi$ is admissible, the $\mathcal{H}_{\mathbb{Z}}(H,\mathfrak{U}_H)$-module $(\pi_H)^{\mathfrak{U}_H}$ is supersingular if and only if the $\mathcal{H}_{\mathbb{Z}}(G,\mathfrak{U})$-module $\pi^{\mathfrak{U}}$ is supersingular.*

*Proof.* (i) The vector spaces $\pi_H^{\mathfrak{U}_H}$ and $\pi^{i(\mathfrak{U}_H)}$ are equal. We have $\mathfrak{U}=Z_1 i(\mathfrak{U}_H)$. A non-zero subspace of $\pi^{i(\mathfrak{U}_H)}$ fixed by $Z_1$ contains a non-zero element of $\pi^{\mathfrak{U}}$.

We recall that the map $\mathcal{Z}_R(H,\mathfrak{U}_H)^\flat_{\ell>0}\xrightarrow{i}\mathcal{Z}_R(G,\mathfrak{U})^\flat_{\ell>0}$ is an isomorphism (Theorem 2.25 (vi)). Hence $\mathcal{Z}_R(i(H), i(\mathfrak{U}_H))^\flat_{\ell>0}\simeq\mathcal{Z}_R(G,\mathfrak{U})^\flat_{\ell>0}$ ***. For a positive integer $n$, let

$X_{H,n}\subset\pi_H^{\mathfrak{U}_H}$ be the $\mathcal{H}_R(H,\mathfrak{U}_H)$-submodule killed by $(\mathcal{Z}_R(H,\mathfrak{U}_H)^\flat_{\ell>0})^n$,
$X'_n\subset\pi^{i(\mathfrak{U}_H)}$ be the $\mathcal{H}_R(i(H), i(\mathfrak{U}_H))$-submodule killed by $(\mathcal{Z}_R(G,\mathfrak{U})^\flat_{\ell>0})^n$,
$X_n\subset\pi^{\mathfrak{U}}$ be the $\mathcal{H}_R(G,\mathfrak{U})$-module killed by $(\mathcal{Z}_R(G,\mathfrak{U})^\flat_{\ell>0})^n$.

We have

$$X_{H,n}=X'_n,\quad X_n=\pi^{\mathfrak{U}}\cap X'_n.$$

Hence $X_n\neq 0$ implies $X'_n\neq 0$. But the space $X'_n$ is stable by $Z_1$. If $X'_n\neq 0$ is non-zero then it contains a non-zero element of $\pi^{\mathfrak{U}}$. We deduce

$$X'_n\neq 0\Leftrightarrow X_n\neq 0.$$

We have $X_n=\pi^{\mathfrak{U}}\cap X'_n$. Hence $X_n$ non-zero is equivalent to $X_{H,n}$ non-zero.

(ii) We set $X=\cup_{n>0}X_n$, $X'=\cup_{n>0}X'_n$, $X_H=\cup_{n>0}X_{H,n}$. The module $\pi^{\mathfrak{U}}$ is not supersingular if and only if $Y=\pi^{\mathfrak{U}}-X$ is non-zero. By (i)m $X_n=\pi^{\mathfrak{U}}\cap X'_n$, hence $X=\pi^{\mathfrak{U}}\cap X'$ and $Y=\pi^{\mathfrak{U}}\cap Y'$ where $Y'=\pi^{i(\mathfrak{U}_H)}-X'$. By (i), $Y'$ is equal to $Y_H=\pi_H^{\mathfrak{U}_H}-X_H$.

We saw that $\pi$ is admissible if and only if $\pi_H$ is admissible. As a pro-$p$ Iwahori subgroup is a pro-$p$ group and the characteristic of $R$ is $p$, this is also equivalent to $\pi^{\mathfrak{U}}$ is finite dimensional or to $\pi_H^{\mathfrak{U}_H}$ is finite dimensional.

We suppose that $\pi$ is admissible. The finite dimensional module $\pi_H^{\mathfrak{U}_H}$ is not supersingular if and only if $Y_H$ is non-zero, if and only if $\pi^{i(\mathfrak{U}_H)}$ contains a non-supersingular simple submodule by Proposition 5.11. By (i) there exists a non-zero element $v\in\pi^{\mathfrak{U}}$ in a simple submodule $\pi^{i(\mathfrak{U}_H)}$. If the submodule is not supersingular, then $v\in Y$. We have $Y'=Y_H$, $Y=\pi^{\mathfrak{U}}\cap Y'$ and $Y_H$ non-zero implies $Y$ non-zero. Hence $Y$ non-zero is equivalent to $Y_H$ non-zero. $\square$

Let $P = MN \subset G$ be a standard parabolic subgroup with its standard Levi decomposition, let $\sigma$ be an irreducible admissible representation of $M$, and let $Q = M_Q N_Q \subset G$ be a parabolic subgroup containing $P$ with its standard Levi decomposition.

The subgroup $i(M_H) \subset M$ is normal of finite index. As explained in the introduction, $\sigma|_{i(M_H)} = \oplus_j \sigma_j$ is a finite direct sum of $Z$-conjugate irreducible representations $\sigma_j$ of inflation $\sigma_{j,M_H}$ to $M_H$.

**Lemma 5.13.** *We have:*

(i) $(P(\sigma))_H = P_H(\sigma_{j,M_H})$ *for all* $j$.

(ii) $(P, \sigma, Q)$ *is a standard supercuspidal triple of* $G$*, if and only if* $(P_H, \sigma_{j,M_H}, Q_H)$ *is a standard supercuspidal triple of* $H$ *for one* $j$*, if and only if* $(P_H, \sigma_{j,M_H}, Q_H)$ *is a standard supercuspidal triple of* $H$ *for all* $j$.

(iii) *For* $P \subset Q \subset P(\sigma)$*, we have* $e_{Q_H}(\sigma_{M_H}) = \oplus_j e_{Q_H}(\sigma_{j,M_H})$.

*Proof.* (i) We recall that a simple root $\alpha \in \Delta - \Delta_P$ is contained in $P(\sigma)$ if and only if $\sigma$ is trivial on $M'_\alpha$. The group $M'_\alpha$ is contained in $i(H)$. Hence $\sigma$ is trivial on $M'_\alpha$ if and only if all $\sigma_j$ are trivial on $M'_\alpha$. But $\sigma_j$ is trivial on $M'_\alpha$ if and only if $\sigma_{j,M_H}$ is trivial on $M'_{H,\alpha}$ because $i(M'_{H,\alpha}) = M'_\alpha$. The group $Z$ normalizes $M'_\alpha$ and the $\sigma_j$ are $Z$-conjugate, hence if one $\sigma_j$ is trivial on $M'_\alpha$ then all $\sigma_j$ are trivial on $M'_\alpha$.

(ii) follows from (i) and Proposition 2.26 which says that $\sigma$ is supercuspidal if and only if $\sigma_{j,M_H}$ is supercuspidal for all $j$.

(iii) follows from (i). □

We prove now the equality $(I_G(P, \sigma, Q))_H = \oplus_j I_H(P_H, \sigma_{j,M_H}, Q_H)$ of Theorem 2.27. By exactness of the functor $\pi \mapsto \pi_H$ from the smooth representations of $G$ to those of $H$,

$$(I_G(P, \sigma, Q))_H = \frac{(\operatorname{Ind}_Q^G e_Q(\sigma))_H}{(\sum_{Q \subsetneq Q' \subset P(\sigma)} \operatorname{Ind}_{Q'}^G e_{Q'}(\sigma))_H}.$$

By Lemma 5.6 (i) we have for $P \subset Q \subset P(\sigma)$, $(\operatorname{Ind}_Q^G e_Q(\sigma))_H = \operatorname{Ind}_{Q_H}^H e_{Q_H}(\sigma_{M_H})$ and by Lemma 5.13 (i) we have $P_H(\sigma_{M_H}) = P(\sigma)$. Hence

$$(I_G(P, \sigma, Q))_H = \frac{\operatorname{Ind}_{Q_H}^H e_{Q_H}(\sigma_{M_H})}{\sum_{Q_H \subsetneq Q'_H \subset P_H(\sigma_{M_H})} \operatorname{Ind}_{Q'_H}^H e_{Q'_H}(\sigma_{M_H})}.$$

By 5.13 (i) we have $P_H(\sigma_{j,M_H}) = P_H(\sigma_H)$ for all $j$. This implies that for $P_H \subset Q_H \subset P_H(\sigma_{M_H})$, The parabolic induction commutes with finite direct sums, for $P \subset Q \subset P(\sigma)$, we have $e_{Q_H}(\sigma_{M_H}) = \oplus_j e_{Q_H}(\sigma_{j,M_H})$ and $P_H(\sigma_{j,M_H}) = P_H(\sigma_{M_H})$ for all $j$ by Lemma 5.13 (i), (iii) hence

$$(I_G(P, \sigma, Q))_H = \frac{\oplus_j \operatorname{Ind}_{Q_H}^H e_{Q_H}(\sigma_{j,M_H})}{\oplus_j \sum_{Q_H \subsetneq Q'_H \subset P_H(\sigma_{j,M_H})} \operatorname{Ind}_{Q'_H}^H e_{Q'_H}(\sigma_{j,M_H})} = \oplus_j I_H(P_H, \sigma_{j,M_H}, Q_H).$$

This ends the proof of Theorem 2.27.

## 5.4 Variant

Let $\mathbf{H} \xrightarrow{\mathbf{i}} \mathbf{G}$ be an $F$-homomorphism such that the map $\mathbf{H} \times \mathbf{C^0} \xrightarrow{\mathbf{j}} \mathbf{G}$ sending $(\mathbf{h}, \mathbf{c})$ to $\mathbf{i}(\mathbf{h})\mathbf{c}$ is a central $F$-extension (where $\mathbf{C^0}$ is the connected component of the center $\mathbf{C}$ of the reductive $F$-group $\mathbf{G}$). The kernel of $\mathbf{i}$ remains central in $\mathbf{H}$ but we have only $\mathbf{i}(\mathbf{H}) \subset \mathbf{G} = \mathbf{i}(\mathbf{H})\mathbf{C^0}$. Notation as in section 5.3 and **??**.

To prove Theorem 2.28, we review the proof of Theorem 2.25 for the central extension $\mathbf{H} \times \mathbf{C^0} \xrightarrow{\mathbf{j}} \mathbf{G}$ and we restrict the arguments to $\mathbf{H} \xrightarrow{\mathbf{i}} \mathbf{G}$.

The group $\mathbf{C^0}$ contains a unique maximal $F$-split torus $\mathbf{T_0}$ and defines an admissible datum with a trivial root system $\mathcal{W}_{C^0} = (C^0/C^0_0, C^0_k, C^0/C^0_1)$ with the notations after Definition 2.1 and Theorem 2.15. We have the groups $\mathbf{T_H}, \mathbf{B_H}, \mathbf{Z_H}, \mathfrak{N}_\mathbf{H}$ such that $\mathbf{T_H} \times \mathbf{T_0}, \mathbf{B_H} \times \mathbf{C^0}, \mathbf{Z_H} \times \mathbf{C^0}, \mathfrak{N}_\mathbf{H} \times \mathbf{C^0}$ satisfy the requirements given before Theorem 2.25 for the central $F$-extension $\mathbf{H} \times \mathbf{C^0} \xrightarrow{\mathbf{j}} \mathbf{G}$. The map $\alpha \mapsto \alpha \circ i$ identifies the root system $\Phi$ with the root system $\Phi_H$, respects the positivity defined by $\mathbf{B}, \mathbf{B_H}$ and the roots groups are isomorphic $\mathbf{U}_{\mathbf{H},\alpha\circ \mathrm{i}} \xrightarrow{\mathbf{i}} \mathbf{U}_\alpha$. The valuation $\varphi$ of $(Z, U_\alpha)_{\alpha\in\Phi}$ is also a valuation of $(Z_H, U_\alpha)_{\alpha\in\Phi_H}$.

The admissible root datum $\mathcal{W}_{H\times C^0} = \mathcal{W}_H \times \mathcal{W}_{C^0}$ (notation after Definition 2.1) has the same reduced root system than $\mathcal{W}_H$. The restriction $\mathfrak{N}_H \xrightarrow{i} \mathfrak{N}$ of $\mathfrak{N}_H \times C^0 \xrightarrow{j} \mathfrak{N}$ induces an homomorphism $\mathcal{W}_{H,1} \xrightarrow{i} \mathcal{W}_1$ which is the restriction of $\mathcal{W}_{H,1} \times C^0/C^0_1 \xrightarrow{j} \mathcal{W}_1$. The kernel of this last homomorphism is the image of $j^{-1}(Z_1) \subset \mathfrak{N}_H \times C^0$ in $\mathcal{W}_{H,1} \times C^0/C^0_1$. As $C^0_1 \subset Z_1$, the kernel of $\mathcal{W}_{H,1} \xrightarrow{i} \mathcal{W}_1$ is the image $i^{-1}(Z_1)/Z_{H,1}$ of $i^{-1}(Z_1) \subset \mathfrak{N}_H$ in $\mathcal{W}_{H,1}$.

The subgroups $j(H \times C^0) = i(H)C^0 \subset G, j(Z_H \times C^0) = i(Z_H)C^0 \subset Z, j(\mathfrak{N}_H \times C^0) = i((\mathfrak{N}_H)C^0 \subset \mathfrak{N}$ are normal open of finite index, and the subgroup $i(H) \subset i(H)C^0$ is normal. The subgroup $j(W_{H,1} \times C^0/C^0_1) = i(W_{H,1})C^0/C^0_1 \subset W_1$ is normal of finite index with cosets of representatives in $\Omega_1$ and the subgroup $i(W_{H,1}) \subset i(W_{H,1})C^0/C^0_1$ is normal. As $C^0/C^0_1 \subset \Omega_1$, the left and right cosets of the subgroup $i(W_{H,1}) \subset W_1$ admit representatives in $\Omega_1$.

The parahoric subgroups of $H \times C^0$ fixing a facet $\mathfrak{F}$ of $(V, \mathfrak{H})$ are $K_{H,\mathfrak{F}} \times C^0_0$ and $K_{H,\mathfrak{F}} \subset K_{H,\mathfrak{F}}C^0_0$ is contained in the parahoric subgroup of $G$ fixing $\mathfrak{F}$. The same proprery holds true for the pro-$p$ parahoric subgroups.

The parameter maps $\mathfrak{c}_{H\times C^0}$ and $\mathfrak{c}$ are $j$-compatible: $j \circ \mathfrak{c}_{H\times C^0} = \mathfrak{c} \circ j$ (Definition 2.23). We have $\mathfrak{S}_H(1) \times C_k = \mathfrak{S}_{H\times C^0}(1)$, and $\mathfrak{c}_{H\times C^0}(\tilde{s}, c) = \mathfrak{c}_H(\tilde{s})c$ for $(\tilde{s}, c) \in \mathfrak{S}_H(1) \times C_k$. We deduce that $\mathfrak{c}_H, \mathfrak{c}$ are $i$-compatible.

The pro-$p$ Iwahori Hecke ring of $H \times C^0$ is $\mathcal{H}_\mathbb{Z}(H, \mathfrak{H}) \otimes_\mathbb{Z} \mathbb{Z}[C^0/C^0_1]$. The homomorphism $\mathcal{H}_\mathbb{Z}(H \times C^0, \mathfrak{U}_H \times C^0_1) \xrightarrow{j} \mathcal{H}_\mathbb{Z}(G, \mathfrak{U})$ of image $i(\mathcal{H}_\mathbb{Z}(H, \mathfrak{U}_H)\mathbb{Z}[C^0/C^0_1]$ restricts to the homomorphism $\mathcal{H}_\mathbb{Z}(H, \mathfrak{U}_H) \xrightarrow{i} \mathcal{H}_\mathbb{Z}(G, \mathfrak{U})$. Recalling $C^0/C^0_1 \subset \Omega_1$, we have

$$\mathcal{H}_\mathbb{Z}(G, \mathfrak{U}) = i(\mathcal{H}_\mathbb{Z}(H, \mathfrak{U}_H))\mathbb{Z}[C^0/C^0_1]\mathbb{Z}[\Omega_1] = i(\mathcal{H}_\mathbb{Z}(H, \mathfrak{U}_H)) \otimes_{i(\Omega_{H,1})} \mathbb{Z}[\Omega_1].$$

The kernel of $\mathcal{H}_\mathbb{Z}(H, \mathfrak{U}_H) \xrightarrow{i} \mathcal{H}_\mathbb{Z}(G, \mathfrak{U})$ is $(\mathbb{Z}[i^{-1}(Z_1)/Z_{H,1}])_{\epsilon=0}$. The image is the subring $\mathcal{H}_\mathbb{Z}(\mathfrak{U}i(H)\mathfrak{U}, \mathfrak{U})$ of elements with support in $\mathfrak{U}i(H)\mathfrak{U}$.

We have $\mathbf{j}(\mathbf{T_H} \times \mathbf{T_{C^0}}) = \mathbf{T}$ and $j(X_*(T_{H\times C^0})) = j(X_*(\mathbf{T_H}) \times X_*(\mathbf{T_{C^0}}) = X_*(\mathbf{T})$, and the splitting $(\Lambda_H \times C^0/C^0_0)^\flat \xrightarrow{\iota_{H\times C^0}} (\Lambda_H \times C^0/C^0_0)^\flat_1$ satisfies $\iota \circ j = j \circ \iota_{H\times C^0}$. The splitting $\iota_{H\times C^0}$ is equal to $\Lambda^\flat_H \times (C^0/C^0_0)^\flat \xrightarrow{(\iota_H, \iota_{C^0})} \Lambda^\flat_{H,1} \times (C^0/C^0_0)^\flat_1$ hence $\iota_H \circ i = i \circ \iota_H$. The splittings $\iota_H, \iota$ are $i$-compatible. The homomorphism $\mathcal{H}_\mathbb{Z}(H \times C^0, \mathfrak{U}_H \times C^0_1) \xrightarrow{j} \mathcal{H}_\mathbb{Z}(G, \mathfrak{U})$ respects the central elements associated to $X_*(T_{H\times C^0})$. Clearly this is means that the homomorphism $\mathcal{H}_\mathbb{Z}(H, \mathfrak{U}_H) \xrightarrow{i} \mathcal{H}_\mathbb{Z}(G, \mathfrak{U})$ respects the central elements associated to $X_*(T_H)$.

We have $\mathcal{Z}_\mathbb{Z}(C^0, C^0_1)^\flat = \mathbb{Z}[(C^0/C^0_0)^\flat_1]$ and $\mathcal{Z}_\mathbb{Z}(G, \mathcal{U})^\flat$ is equal to

$$j(\mathcal{Z}_\mathbb{Z}(H \times C^0, \mathcal{U}_H \times C^0_1)^\flat) = j(\mathcal{Z}_\mathbb{Z}(H, \mathcal{U}_H)^\flat \otimes \mathbb{Z}[(C^0/C^0_0)^\flat_1]) = i(\mathcal{Z}_\mathbb{Z}(H, \mathfrak{U}_H)^\flat)\mathbb{Z}[(C^0/C^0_0)^\flat_1].$$

The length on $W_{H,1}$ is the restriction of the length of $W_{H\times C^0,1}$ and $j, i$ respects the length. We have

$$\mathcal{Z}_\mathbb{Z}(G, \mathcal{U})^\flat_{\ell=0} = i(\mathcal{Z}_\mathbb{Z}(H, \mathfrak{U}_H)^\flat_{\ell=0})\mathbb{Z}[(C^0/C^0_0)^\flat_1],\ \mathcal{Z}_\mathbb{Z}(G, \mathcal{U})^\flat_{\ell>0} = i(\mathcal{Z}_\mathbb{Z}(H, \mathfrak{U}_H)^\flat_{\ell>0})\mathbb{Z}[(C^0/C^0_0)^\flat_1].$$

The homomorphism $i$ is injective on $\mathcal{Z}_{\mathbb{Z}}(H,\mathcal{U}_H)^\flat_{\ell>0}$ because $j$ is injective on $\mathcal{Z}_{\mathbb{Z}}(H\times C^0,\mathcal{U}_H\times C^0_1)^\flat_{\ell>0}$. This ends the proof of (i) in Theorem 2.28.

We prove (ii) of Theorem 2.28. Let $R$ be a field and let $\pi$ be an irreducible admissible $R$-representation of $G$. We saw already that the representation $\pi|_{i(H)C^0}$ is a finite direct sum $\oplus_j\pi_j$ of irreducible admissible $R$-representations $\pi_j$ which are $Z$-conjugate, as $G=i(H)Z$.

We suppose that $C^0$ acts on $\pi$ by a character $\chi$ and we check that $\pi$ satisfies Proposition 2.26. Lemma 5.3, 5.6 and their proof remain valid in our new setting. Assume that $R$ is a field of characteristic $p$. Proposition 5.12 (ii) and (iii) remains valid for the following reason. We have $\mathcal{H}_R(H\times C^0,\mathfrak{U}_H\times C^0_1)=\mathcal{H}_R(H,\mathfrak{U}_H)\otimes_R R[C^0/C^0_1]$ and $\pi_{H\times C^0}^{\mathfrak{U}_H\times C^0_1}=\pi_H^{\mathfrak{U}_H}\otimes\chi$. The submodules of the $\mathcal{H}_R(H\times C^0,\mathfrak{U}_H\times C^0_1)$-submodule of $\pi_H^{\mathfrak{U}_H}\otimes\chi$ are the tensor product of the $\mathcal{H}_R(H,\mathfrak{U}_H)$-submodules of $\pi_H^{\mathfrak{U}_H}$ by $\chi$. A $\mathcal{H}_R(H,\mathfrak{U}_H)$-module is supersingular if and only if its product by $\chi$ is a supersingular $\mathcal{H}_R(H\times C^0,\mathfrak{U}_H\times C^0_1)$-module. Hence Proposition 5.12 (ii) and (iii) remains valid. Proposition 2.26 follows.

We prove (iii) of Theorem 2.28. Assume that $R$ is algebraically closed of characteristic $p$. Let $(P,\sigma,Q)$ be a standard supercuspidal triple of $G$, and let $\chi$ be the character of $C^0$ giving its action on $I_G(P,\sigma,Q)$. We have $P_{H\times C^0}=P_H\times C^0$. The representation $\sigma|_{i(M_H)C^0}=\oplus_j\sigma_j$ is a sum of irreducible admissible representations $\sigma_j$. The representations $\sigma_j|_{i(M_H)}$ and their inflations $\sigma_{j,M_H}$ to $M_H$ are irreducible admissible. The inflation of $\sigma|_{i(M_H)C^0}$ to $M_H\times C^0$ is $\sigma_{H\times C^0}=\oplus_j(\sigma_{j,M_H}\otimes\chi)$. We have

$$\begin{aligned}(I_G(P,\sigma,Q))_H\otimes\chi=(I_G(P,\sigma,Q))_{H\times C_0}&=\oplus_j I_{H\times C^0}(P_H\times C^0,\sigma_{j,M_H}\otimes\chi,Q_H\times C^0)\\&=\oplus_j I_H(P_H,\sigma_{j,M_H},Q_H)\otimes\chi\end{aligned}$$

The second equality follows from Theorem 2.27 applied to the central extension $\mathbf{H}\times\mathbf{C^0}\xrightarrow{\mathbf{j}}\mathbf{G}$. We deduce $(I_G(P,\sigma,Q))_H=\oplus_j I_H(P_H,\sigma_{j,M_H},Q_H)$.

# 6 Classical examples

## 6.1 $z$-extension

A $z$-extension $\tilde{\mathbf{G}}\xrightarrow{\tilde{\mathbf{i}}}\mathbf{G}$ of connected reductive $F$-groups is a central $F$-extension where the derived group of $\tilde{\mathbf{G}}$ is simply connected, $\tilde{\mathbf{G}}_{\mathbf{sc}}=\tilde{\mathbf{G}}_{\mathbf{der}}$, and the kernel of $\tilde{\mathbf{G}}\xrightarrow{\tilde{\mathbf{i}}}\mathbf{G}$ is a central $F$-induced torus $\mathbf{L}$. The homomorphism $\tilde{G}\xrightarrow{\tilde{i}}G$ between the rational $F$-points is surjective because $H^1(F,\mathbf{L})=0$ [Spr, 11.3.4, 12.4.7]. The torus $L$ has a unique parahoric subgroup $L_0$ and a unique pro-$p$ parahoric subgroup $L_1$. As in section 5, we associate to a triple $(\mathbf{T},\mathbf{B},\varphi)$ in $\mathbf{G}$ a similar triple in $\tilde{\mathbf{G}}$ and (pro-$p$) parahoric subgroups. We add an upper index $\tilde{}$ on an object relative to $\tilde{\mathbf{G}}$. By [HV1, 3.5], the parahoric groups form an exact sequence $1\to L_0\to\tilde{Z}_0\xrightarrow{\tilde{i}}Z_0\to 1$.

**Lemma 6.1.** *We have an exact sequence of pro-p parahoric subgroups*

$$1\to L_1\to\tilde{Z}_1\xrightarrow{\tilde{i}}Z_1\to 1,$$

*Proof.* $\tilde{i}(\tilde{Z}_1)=Z_1$ by Lemma 3.1 (iii) and $L_0\cap\tilde{Z}_1=L_1$ by Lemma 3.1 (i). ☐

**Remark 6.2.** Let $\mathfrak{F}$ be an arbitrary facet of $(V,\mathfrak{H})$. The (pro-$p$) parahoric subgroups fixing $\mathfrak{F}$ satisfy a similar exact sequence.

**Proposition 6.3.** *The pro-p Iwahori Hecke rings satisfy the exact sequence:*

$$0\to\mathbb{Z}[L/L_1]_{\epsilon=0}\to\mathcal{H}_{\mathbb{Z}}(\tilde{G},\tilde{\mathfrak{U}})\xrightarrow{\tilde{i}}\mathcal{H}_{\mathbb{Z}}(G,\mathfrak{U})\to 0.$$

*Proof.* Proposition 2.24 (i), Theorem 2.25 (v) and Lemma 6.1. □

Example: $\tilde{G} = GL(n, F) \xrightarrow{\tilde{i}} G = PGL(n, F)$. We have $\tilde{L}/L_1 = F^*/U_F$ where $U_F$ denotes the pro-$p$ Sylow subgroup of the group $O_F^*$ of units of $F$.

### 6.2 Simply connected cover of the derived group and adjoint group and scalar restriction

Let $\mathbf{G}$ be a connected reductive $F$-group, $\mathbf{G_{der}}$ its derived group, $\mathbf{C^0}$ the connected center of $\mathbf{G}$ (an $F$-torus [Spr, 8.1.8]). The multiplication map $\mathbf{G_{der}} \times \mathbf{C^0} \xrightarrow{j} \mathbf{G}$ is a central $F$-extension. The simply connected cover $\mathbf{G_{sc}} \xrightarrow{\mathbf{i_{sc}^{der}}} \mathbf{G_{der}}$ is a central $F$-extension. We have the $F$-central extension $\mathbf{G_{sc}} \times \mathbf{C^0} \xrightarrow{j\circ(i_{sc}^{der}\times \mathrm{id})} \mathbf{G}$. The groups $\mathbf{G}, \mathbf{G_{der}}, \mathbf{G_{sc}}$ are canonical $F$-central extensions of the adjoint group $\mathbf{G_{ad}}$ of $\mathbf{G_{der}}$, $\mathbf{G} \xrightarrow{\mathbf{i^{ad}}} \mathbf{G_{ad}}$, $\mathbf{G_{der}} \xrightarrow{\mathbf{i^{ad}_{der}}} \mathbf{G_{ad}}$, $\mathbf{G_{sc}} \xrightarrow{\mathbf{i^{ad}_{sc}} = \mathbf{i^{ad}_{der}} \circ \mathbf{i^{der}_{sc}}} \mathbf{G_{ad}}$. All the central extensions have a finite kernel.

The group $\mathbf{G_{sc}}$ is in a unique way a direct product of almost $F$-simple simply connected groups (a group is almost $F$-simple if it has no infinite normal $F$-subgroup). If $\mathbf{G_{sc}}$ is almost $F$-simple, there exist a separable finite field extension $F'/F$ and an (absolutely) almost simple simply connected $F'$-group $\mathbf{H}$ such that $\mathbf{G_{sc}}$ is $F$-isomorphic to the group $R_{F'/F}(\mathbf{H})$ obtained from $\mathbf{H}$ by restriction of the scalar field from $F'$ of $F$ [**?**, 6.21 (ii)]. We may everywhere replace "simply connected" by "adjoint", in which case, the "almost" can be dropped [T0, 3.1.2] [Borel, 14.10 Proposition, 22.10 Theorem].

We write $\mathbf{G_{sc}} = \mathbf{G_{sc}^{is}} \times \mathbf{G_{sc}^{anis}}$ where $\mathbf{G_{sc}^{is}} = \prod_{\mathbf{b} \in \mathbf{B_{sc}^{is}}} \mathbf{G_{sc,b}^{is}}$ denotes the product of the isotropic almost simple components $\mathbf{G_{sc,b}^{is}}$, and $\mathbf{G_{sc}^{anis}}$ the product of the anisotropic components. We write the same for the adjoint group. An object relative to $G'_*$ will be denote the same way with an upper index $'$ and lower index $*$. An object relative to $C^0$ with an index $C^0$.

As explained in section 5 for a general central extension, one associate to a triple $(\mathbf{T}, \mathbf{B}, \varphi)$ for $\mathbf{G}$, via $j$ and $i_{sc}$, a triple $(\mathbf{T}_{der} \times \mathbf{T}_{C^0}, \mathbf{B}_{der} \times C^0, \varphi)$ for $\mathbf{G_{der}} \times \mathbf{C^0}$ and a triple $(\mathbf{T}_{sc}, \mathbf{B}_{sc}, \varphi)$ for $\mathbf{G_{sc}}$ such that

$$j^{-1}(\mathbf{X}) = \mathbf{X}_{der} \times \mathbf{C^0}, i_{sc}^{-1}(\mathbf{X}_{der}) = \mathbf{X}_{sc} \text{ and } \mathbf{X} = \mathbf{X}_{der}\mathbf{C^0}, \mathbf{X}_{der} = i_{sc}(\mathbf{X}_{sc}) \text{ for } \mathbf{X} = \mathbf{Z}, \mathbf{B}, \mathbf{N},$$

and $\mathbf{U} = \mathbf{U}_{der}$ is homeomorphic to $\mathbf{U}_{sc}$ via $i_{sc}$. By factorization one gets triples for $\mathbf{G_{sc}^{is}}$ and $\mathbf{G_{sc,b}^{is}}$ for all $b$.

We consider the (pro-$p$) Iwahori subgroups, admissible data, parameter maps and splittings associated to these triples (we fixed an uniformizer $p_F$). The irreducible components of the based reduced root system $(\Sigma, \Delta)$ of $G, G_{der}, G_{sc}$ are the based reduced root systems $(\Sigma_b, \Delta_b)$ of $G_{sc,b}^{is}$ for all $b$.

As in the introduction, we denote by $G'$ the subgroup of $G$ generated by the set $U^G$ of conjugates of $U$ and we set $X' := X \cap G'$ and $(X/Y)' := X'/Y'$ for subgroups $Y \subset X \subset G$. The group $G'$ is also generated by $U$ and $U^{op}$

We consider first the product decomposition of the simply connected group $\mathbf{G_{sc}}$. As $G'_{sc} = G_{sc}^{is}$ [AHHV, II.4 Proposition] we have $Z'_{sc,k} = Z_{sc,k}^{is}$, $\mathfrak{U}'_{sc} = \mathfrak{U}_{sc}^{is}$, $\Omega_{sc}^{is} = \{1\}$ hence $\Omega_{sc,1}^{is} = Z_{sc,k}^{is}$. The factorisation $\mathbf{G_{sc}} = \mathbf{G_{sc}^{is}} \times \mathbf{G_{sc}^{anis}}$ transfers to a factorization of the pro-$p$ Iwahori subgroups $\mathfrak{U}_{sc} = \mathfrak{U}_{sc}^{is} \times \mathfrak{U}_{sc}^{anis}$ and of the pro-$p$ Iwahori Hecke rings and the central subrings.

**Lemma 6.4.** *We have*

$$\Omega_{sc,1} = Z^{is}_{sc,k} \times \Omega^{anis}_{sc,1}, \ \Omega^{anis}_{sc,1} = G^{anis}_{sc}/G^{anis}_{sc,1}$$
$$\mathcal{H}_{\mathbb{Z}}(G_{sc}, \mathfrak{U}_{sc}) = \mathcal{H}_{\mathbb{Z}}(G^{is}_{sc}, \mathfrak{U}^{is}_{sc}) \otimes_{\mathbb{Z}} \mathcal{H}_{\mathbb{Z}}(G^{anis}_{sc}, \mathfrak{U}^{anis}_{sc}),$$
$$\mathcal{Z}_{\mathbb{Z}}(G_{sc}, \mathfrak{U}_{sc})^{\flat}_{\ell=0} = \mathcal{Z}_{\mathbb{Z}}(G^{is}_{sc}, \mathfrak{U}^{is}_{sc})^{\flat}_{\ell=0} \otimes_{\mathbb{Z}} \mathcal{Z}_{\mathbb{Z}}(G^{anis}_{sc}, \mathfrak{U}^{anis}_{sc})^{\flat},$$
$$\mathcal{Z}_{\mathbb{Z}}(G_{sc}, \mathfrak{U}_{sc})^{\flat}_{\ell>0} = \mathcal{Z}_{\mathbb{Z}}(G^{is}_{sc}, \mathfrak{U}^{is}_{sc})^{\flat}_{\ell>0} \otimes_{\mathbb{Z}} \mathcal{Z}_{\mathbb{Z}}(G^{anis}_{sc}, \mathfrak{U}^{anis}_{sc})^{\flat},$$
$$\mathcal{H}_{\mathbb{Z}}(G^{anis}_{sc}, \mathfrak{U}^{anis}_{sc}) = \mathbb{Z}[G^{anis}_{sc}/(G^{anis}_{sc})_1], \ \mathcal{Z}_{\mathbb{Z}}(G^{anis}_{sc}, \mathfrak{U}^{anis}_{sc})^{\flat} \simeq \mathbb{Z}[T^{anis}_{sc}/(T^{anis}_{sc})_0].$$

The product decomposition of the adjoint group $\mathbf{G_{ad}}$ ***

We compare now $\mathbf{G_{sc}}$ and $\mathbf{G_{ad}}$ with $\mathbf{G_{der}}$ and $\mathbf{G}$. The differences between the (pro-$p$) Iwahori subgroups of $G_{sc}, G^{is}_{sc}, G_{der}, G$ or their images by $i_{sc}$ is seen by their intersections with the different groups $Z$. The Kottwitz's functoriality implies the inclusions

$$i_{sc}(Z_{sc,0}) \subset (i_{sc}(G_{sc}) \cap Z_0), \quad Z_{der,0} \subset (G_{der} \cap Z_0), \quad Z_{der,0}C_0 \subset (G_{der}C \cap Z_0).$$

Recalling $i_{sc}(G^{is}_{sc}) = G'$ we have also the inclusion $i_{sc}(Z^{is}_{sc,0}) \subset (G' \cap Z_0)$.

The Kottwitz homomorphism of $G_{sc}$ being trivial, the Iwahori subgroup $Z_{sc,0} \subset Z_{sc}$ is equal to the maximal compact subgroup $Z^{max}_{sc,0} \subset Z_{sc}$. As the kernel of $i_{sc}$ is finite, the images of the parahoric subgroups

$$i_{sc}(Z_{sc,0}) = i_{sc}(G_{sc}) \cap Z_0, \quad i_{sc}(Z^{is}_{sc,0}) = G' \cap Z_0 \tag{16}$$

are as big as possible because the inverse images by $i_{sc}$ of the compact groups on the right side of the equalities are compact subgroups of $Z_{sc}$ and $Z^{is}_{sc}$ hence equal to the maximal compact subgroups $Z_{sc,0}$ and $Z^{is}_{sc,0}$. The images of the unique pro-$p$ Sylow subgroups

$$i_{sc}(Z_{sc,1}) = i_{sc}(G_{sc}) \cap Z_1, \quad i_{sc}(Z^{is}_{sc,1}) = G' \cap Z_1 \tag{17}$$

are also as big as possible by Lemma 3.1 (iii). The $p$-part of the kernel of $G_{sc} \xrightarrow{i_{sc}} G$ is a central $p$-subgroup of $Z_{sc}$ hence is contained in the pro-$p$ Sylow subgroup of the maximal compact subgroup $Z_{sc,0} \subset Z_{sc}$. The inverse images by $i_{sc}$ of the groups on the right side of the above equalities are $\mu Z_{sc,1}$ and $(\mu \cap Z^{is}_{sc})Z^{is}_{sc,1}$ where $\mu$ is the prime to $p$ part of the kernel of $G_{sc} \xrightarrow{i_{sc}} G$. We deduce:

**Lemma 6.5.** *The finite group $\mu$ of order prime to $p$, and the group $\mu^{is} = \mu \cap Z^{is}_{sc}$ identify with the kernels of the surjective homomorphisms*

$$Z_{sc,k} \xrightarrow{i_{sc}} (Z_0 \cap i_{sc}(G_{sc}))/(Z_1 \cap i_{sc}(G_{sc})) \subset Z_k, \quad Z^{is}_{sc,k} \xrightarrow{i_{sc}} (G' \cap Z_0)/(G' \cap Z_1) = Z'_k \subset Z_k.$$

**Remark 6.6.** We have the inclusions $i_{sc}(Z_{sc,0}) \subset Z_{der,0} \subset Z_{der} \cap Z_0$. When the homomorphism $G_{sc} \xrightarrow{i_{sc}} G_{der}$ is surjective, $i_{sc}(Z_{sc,0}) = Z_{der,0}$ is the maximal compact subgroup $Z^{max}_{der,0} \subset Z_{der}$. Clearly $Z_{der,0} = Z^{max}_{der,0}$ implies $Z_{der,0} = Z_{der} \cap Z_0$ and $Z_{der,0} = Z_{der} \cap Z_0$ implies $Z_{der,1} = Z_{der} \cap Z_1$ by Lemma 3.1 (i).

The kernel of $G_{sc} \xrightarrow{i_{sc}} G$ is a finite abelian subgroup $\mu_1\mu \subset Z_{sc,0}$ of $p$-part $\mu_1$ and prime to $p$-part $\mu$. The restriction $G^{is}_{sc} \xrightarrow{i^{is}_{sc}} G$ of $i_{sc}$ to $G^{is}_{sc} \subset G_{sc}$ has kernel $(\mu_1\mu)^{is} = \mu_1\mu \cap G^{is}_{sc}$ and image $G'_{der}$ as $G'_{sc} = G^{is}_{sc}$. By Remark 3.2 and (7)), the image of $\mathcal{H}_{\mathbb{Z}}(G^{is}_{sc}, \mathfrak{U}^{is}_{sc}) \xrightarrow{i^{is}_{sc}} \mathcal{H}_{\mathbb{Z}}(G, \mathfrak{U})$ is $\mathcal{H}_{\mathbb{Z}}(G'\mathfrak{U}, \mathfrak{U}) \simeq \mathcal{H}_{\mathbb{Z}}(G', \mathfrak{U}')$. We obtain:

**Lemma 6.7.** *We have an exact sequence*

$$0 \to \mathbb{Z}[\mu^{is}]_{\epsilon=0} \to \mathcal{H}_{\mathbb{Z}}(G^{is}_{sc}, \mathfrak{U}^{is}_{sc}) \xrightarrow{i^{is}_{sc}} \mathcal{H}_{\mathbb{Z}}(G', \mathfrak{U}') \to 0,$$

*inducing an isomorphism between the central subalgebras* $\mathcal{Z}_{\mathbb{Z}}(G^{is}_{sc}, \mathfrak{U}^{is}_{sc})^\flat \xrightarrow{\simeq} \mathcal{Z}_{\mathbb{Z}}(G', \mathfrak{U}')^\flat$ *respecting the decomposition by the length:*

$$\mathcal{Z}_{\mathbb{Z}}(G^{is}_{sc}, \mathfrak{U}^{is}_{sc})^\flat_{\ell=0} \xrightarrow{\simeq} \mathcal{Z}_{\mathbb{Z}}(G', \mathfrak{U}')^\flat_{\ell=0} \text{ and } \mathcal{Z}_{\mathbb{Z}}(G^{is}_{sc}, \mathfrak{U}^{is}_{sc})^\flat_{\ell>0} \xrightarrow{\simeq} \mathcal{Z}_{\mathbb{Z}}(G', \mathfrak{U}')^\flat_{\ell>0}.$$

*Proof.* It remains only to check the isomorphisms. The homomorphism $W^{is}_{sc} \xrightarrow{i_{sc}} W'$ respects the length hence the isomorphism $\mathcal{Z}_{\mathbb{Z}}(G^{is}_{sc}, \mathfrak{U}^{is}_{sc})^\flat \xrightarrow{\simeq} \mathcal{Z}_{\mathbb{Z}}(G', \mathfrak{U}')^\flat$ implies the two other ones. We have $(\mathbf{i} \circ \mathbf{i^{is}_{sc}})(\mathbf{T^{is}_{sc}} \times \mathbf{T^{anis}_{sc}}) \times \mathbf{T_{C^0}} = \mathbf{T}$. For $\mu^{is}_{sc} \in X_*(T^{is}_{sc})$ and $\mu \in X_*(T), \mu = (i \circ i^{is}_{sc}) \circ \mu$, we have $(i \circ i^{is}_{sc})(E^{is}_{sc}(\mu^{is}_{sc})) = E(\mu)$ and $\Lambda^{is,\flat}_{sc}$ ** ☐

**Theorem 6.8.** (i) *The homomorphisms* $G_{sc} \xrightarrow{i_{sc}} G_{der} \xrightarrow{i} G$ *induce homomorphisms* $\mathcal{W}_{sc} \xrightarrow{i_{sc}} \mathcal{W}_{der} \xrightarrow{i} \mathcal{W}$*, between the admissible data* $\mathcal{W}_{sc}, \mathcal{W}_{der}, \mathcal{W}$ *with the same based root system* $(\Sigma, \Delta)$*, compatible with the parameter maps and the splittings.*

(ii) $\mu$ *is the kernel of* $\Omega_{sc,1} \xrightarrow{i_{sc}} \Omega_{der,1}$ *and of* $\Omega_{sc,1} \xrightarrow{i \circ i_{sc}} \Omega_1$*,* $(Z_1 \cap Z_{der})/Z_{der,1}$ *is the kernel of* $\Omega_{der,1} \xrightarrow{i} \Omega_1$*. The subgroup* $i_{sc}(\Omega_{sc,1}) \subset \Omega_{der,1}$ *is normal of finite index, the subgroup* $i(\Omega_{der,1}) \subset \Omega_1$ *is normal.*

(iii) *The homomorphisms* $G_{sc} \xrightarrow{i_{sc}} G_{der} \xrightarrow{i} G$ *send the (pro-p) parahoric subgroup fixing a facet of* $(V, \mathfrak{H})$ *into the (pro-p) parahoric subgroup fixing the same facet.*

(iv) *The maps* $\mathcal{H}_{\mathbb{Z}}(G_{sc}, \mathcal{U}_{sc}) \xrightarrow{i_{sc}} \mathcal{H}_{\mathbb{Z}}(G_{der}, \mathcal{U}_{der}) \xrightarrow{i} \mathcal{H}_{\mathbb{Z}}(G, \mathcal{U})$ *between the pro-p Iwahori Hecke rings satisfy Proposition 2.24.*

(v) *The kernel of the homomorphism* $\mathcal{H}_{\mathbb{Z}}(G_{der}, \mathfrak{U}_{der}) \xrightarrow{i} \mathcal{H}_{\mathbb{Z}}(G, \mathfrak{U})$ *is* $\mathbb{Z}[(Z_1 \cap Z_{der})/Z_{der,1}]_{\epsilon=0}$*. The image of* $i$ *is*

$$\mathcal{H}_{\mathbb{Z}}(G', \mathcal{U}') \rtimes_{\mathbb{Z}[i(Z'_k)]} \mathbb{Z}[i(\Omega_{der,1})] = \mathcal{H}_{\mathbb{Z}}(G_{der}\mathfrak{U}, \mathfrak{U}) \simeq \mathcal{H}_{\mathbb{Z}}(G_{der}, (Z_1 \cap Z_{der})\mathfrak{U}'_{der}).$$

*In particular when* $Z_{der,0} = Z^{max}_{der,0}$*, the homomorphism* $\mathcal{H}_{\mathbb{Z}}(G_{der}, \mathfrak{U}_{der}) \xrightarrow{i} \mathcal{H}_{\mathbb{Z}}(G, \mathfrak{U})$ *is injective.*

*The kernels of* $\mathcal{H}_{\mathbb{Z}}(G_{sc}, \mathfrak{U}_{sc}) \xrightarrow{i_{sc}} \mathcal{H}_{\mathbb{Z}}(G_{der}, \mathfrak{U}_{der})$ *and of* $\mathcal{H}_{\mathbb{Z}}(G_{sc}, \mathfrak{U}_{sc}) \xrightarrow{i \circ i_{sc}} \mathcal{H}_{\mathbb{Z}}(G, \mathfrak{U})$ *are* $\mathbb{Z}[\mu]_{\epsilon=0}$*. The image of* $i_{sc}$ *is*

$$\mathcal{H}_{\mathbb{Z}}(G'_{der}, \mathcal{U}'_{der}) \rtimes_{\mathbb{Z}[Z'_{der,k}]} \mathbb{Z}[i_{sc}(\Omega_{sc,1})] = \mathcal{H}_{\mathbb{Z}}(i_{sc}(G_{sc})\mathfrak{U}_{der}, \mathfrak{U}_{der}) \simeq \mathcal{H}_{\mathbb{Z}}(i_{sc}(G_{sc}), i_{sc}(\mathfrak{U}_{sc})).$$

*The image of* $i \circ i_{sc}$ *is* $\mathcal{H}_{\mathbb{Z}}(G', \mathcal{U}') \rtimes_{\mathbb{Z}[Z'_k]} \mathbb{Z}[(i \circ i_{sc})(\Omega_{sc,1})]$.

(vi) *The homomorphisms* $i_{sc}$ *and* $i$ *between the pro-p Iwahori Hecke rings induce homomorphisms between the central subrings respecting the length*

*The homomorphism* $\mathcal{Z}_{\mathbb{Z}}(G_{sc}, \mathfrak{U}_{sc})^\flat_* \xrightarrow{i_{sc}} \mathcal{Z}_{\mathbb{Z}}(G_{der}, \mathfrak{U}_{der})^\flat_*$ *is an isomorphism*

*The homomorphism* $\mathcal{Z}_{\mathbb{Z}}(G_{der}, \mathcal{U}_{der})^\flat_* \xrightarrow{i} \mathcal{Z}_{\mathbb{Z}}(G, \mathcal{U})^\flat_*$ *is injective.*

*We have* $\mathcal{Z}_{\mathbb{Z}}(G, \mathcal{U})^\flat_{\ell=0} = i(\mathcal{Z}_{\mathbb{Z}}(G_{der}, \mathfrak{U}_{der})^\flat_{\ell=0})\mathbb{Z}[(C^0/C^0_0)^\flat_1]$,
$\mathcal{Z}_{\mathbb{Z}}(G, \mathcal{U})^\flat_{\ell>0} = i(\mathcal{Z}_{\mathbb{Z}}(G_{der}, \mathfrak{U}_{der})^\flat_{\ell>0})\mathbb{Z}[(C^0/C^0_0)^\flat_1]$.

*Proof.* Theorem 2.25 for $\mathbf{G_{sc}} \xrightarrow{\mathbf{i_{sc}}} \mathbf{G_{der}}$, Theorem 2.28 for $\mathbf{G_{der}} \xrightarrow{\mathbf{i}} \mathbf{G}$ and $\mathbf{G_{sc}} \xrightarrow{\mathbf{i \circ i_{sc}}} \mathbf{G}$, and Remark 5.5. Note that each subgroup $i_{sc}(G_{sc}) \subset G_{der} \subset G$ is normal in the next one, $\mu \simeq i_{sc}^{-1}(Z_{der,1})/Z_{sc,1} \simeq (i \circ i_{sc})^{-1}(Z_1)/Z_{sc,1}$, $(Z_{der,1} \cap i_{sc}(Z_{sc})) \subset i_{sc}(\mathfrak{U}_{sc})$, and if $Z_{der,0} = Z^{max}_{der,0}$ that $(Z_1 \cap Z_{der}) \subset \mathfrak{U}_{der}$ (Lemma 6.5 (i)). ☐

**Remark 6.9.** The Iwahori Hecke rings satisfy stronger results: the homomorphism $\mathcal{H}_{\mathbb{Z}}(G_{sc}, \mathfrak{B}_{sc}) \xrightarrow{i \circ i_{sc}} \mathcal{H}_{\mathbb{Z}}(G, \mathfrak{B})$ is injective, and the affine Iwahori Hecke rings are isomorphic to the Iwahori Hecke ring of $G^{is}_{sc}$:

$$\mathcal{H}_{\mathbb{Z}}(G^{is}_{sc}, \mathfrak{B}^{is}_{sc}) \simeq \mathcal{H}^{aff}_{\mathbb{Z}}(G_{sc}, \mathfrak{B}_{sc}) \xrightarrow{\sim} \mathcal{H}^{aff}_{\mathbb{Z}}(G_{der}, \mathfrak{B}_{der}) \xrightarrow{\sim} \mathcal{H}^{aff}_{\mathbb{Z}}(G, \mathfrak{B}) = \mathcal{H}_{\mathbb{Z}}(G', \mathfrak{B}').$$

**Remark 6.10.** The results are simpler when $\mathbf{G}$ is $F$-split. In this case,

$$G_{sc}^{is} = G_{sc},\ \Omega_{sc,1} = Z_{sc,k},\ Z_{der,0} = Z_{der,0}^{max},\ Z_0 = Z_0^{max},\ \Lambda = \Lambda^\flat,$$

the homomorphism $\mathcal{H}_\mathbb{Z}(G_{der}, \mathfrak{U}_{der}) \xrightarrow{i} \mathcal{H}_\mathbb{Z}(G, \mathfrak{U})$ is injective, and if $\mathbf{G_{der}}$ is simply connected, we have $\mathcal{H}_\mathbb{Z}(G, \mathcal{U}) \simeq \mathcal{H}_\mathbb{Z}(G_{sc}, \mathcal{U}_{sc}) \otimes_{\mathbb{Z}[Z_{sc,k}]} \mathbb{Z}[\Omega_1]$.

et dans le cas quasi-split

We consider now $R$-representations. For an $R$-representation $\pi$ of a subgroup of $G$ containing $i \circ i_{sc}(G_{sc})$, we denote by $\pi_{sc}$ the inflation to $G_{sc}$ of $\pi|_{i\circ i_{sc}(G_{sc})}$.

**Proposition 6.11.** *Let $\pi$ be an irreducible admissible $R$-representation of $G$.*

(i) *Assume that $R$ is a field. We have:*
$\pi|_{i\circ i_{sc}(G_{sc})} = \oplus_j \pi_j$ *and* $\pi_{sc} = \oplus_j \pi_{j,sc},\ \pi_{j,sc} = \pi_{j,sc}^{is} \otimes \pi_{j,sc}^{anis},\ \pi_{j,sc}^{is} = \oplus_j (\prod_i \pi_{j,sc,i}^{is})$ *where the sum is finite and* $\pi_j, \pi_{j,sc,i}^{is}, \pi_{j,sc}^{anis}$ *are irreducible admissible.*

*$\pi$ is supercuspidal if and only if $\pi_{j,sc}^{is}$ is supercuspidal for all $j$ if and only if $\pi_{j,sc}^{is}$ is supercuspidal for one $j$.*

*$\pi_{j,sc}^{is}$ is supercuspidal if and only if $\pi_{j,sc,i}^{is}$ is supercuspidal for all $i$.*

(ii) *Assume that that $R$ is a field of characteristic $p$. We have:*
*$\pi^{\mathfrak{U}}$ contains a supersingular module if and only if $(\pi_{j,sc})^{\mathfrak{U}_{sc}}$ contains a supersingular module for some $j$.*

*$\pi^{\mathfrak{U}}$ is supersingular if and only if all $(\pi_{j,sc})^{\mathfrak{U}_{sc}}$ is supersingular for all $j$.*

*$(\pi_{j,sc})^{\mathfrak{U}_{sc}}$ is supersingular if and only if $(\pi_{j,sc,i}^{is})^{\mathfrak{U}_{sc}^{is,i}}$ is supersingular for all $i$. We can replace "is supersingular" by "contains a supersingular module".*

*Proof.* Theorem 2.28 and Proposition 2.26 We apply applied to $\mathbf{G_{sc}} \xrightarrow{\mathbf{i\circ i_{sc}}} \mathbf{G}$. □

Let $P \subset G, P_{sc} \subset G_{sc}, P_{sc,i}^{is} \subset G_{sc,i}^{is}$ be standard parabolic subgroups with $\Delta_P = \Delta_{P_sc}, \Delta_P \cap \Delta_i = \Delta_{P_{sc,i}}$, and let $P = NM, P_{sc} = M_{sc}N_{sc}, P_{sc,i}^{is} = M_{sc,i}^{is} N_{sc,i}^{is}$ be the standard Levi decompositions. We have $P_{sc} = (\prod_i P_{sc,i}) \times G_{sc}^{anis}$.

Assume that $R$ is a field. Let $\sigma$ be a supercuspidal $R$-representation of $M$. Its restriction to $(i \circ i_{sc})(M)$ lifts to a semisimple finite length representation $\sigma_{M_{sc}} = \oplus_j \sigma_{j,M_{sc}} = \oplus_j (\prod_i \sigma_{j,M_{sc},i}^{is}) \otimes \sigma_{j,M_{sc}}^{anis}$ where $\sigma_{j,M_{sc},i}^{is}$ is supercuspidal for all $(j,i)$ by Proposition 6.11 (i).

**Theorem 6.12.** *Assume that $R$ is an algebraically closed field of characteristic $p$ and that $(P, \sigma, Q)$ is a supercuspidal standard triple of $G$.*

*Then $(P(\sigma))_{sc} = (\prod_i P(\sigma_{j,M_{sc},i})) \times G_{sc}^{anis}$, and*

$$(I_G(P, \sigma, Q))_{sc} = \oplus_j I_{G_{sc}}(P_{sc}, \sigma_{j,M_{sc}}, Q_{sc}),$$
$$I_{G_{sc}}(P_{sc}, \sigma_{j,M_{sc}}, Q_{sc}) = (\otimes_i I_{G_{sc,i}^{is}}(P_{sc,i}^{is}, \sigma_{j,M_{sc},i}^{is}, Q_{sc,i}^{is})) \otimes \sigma_{j,M_{sc}}^{anis}.$$

*Proof.* Theorem 2.28 and Theorem 2.27 applied to $\mathbf{G_{sc}} \xrightarrow{\mathbf{i\circ i_{sc}}} \mathbf{G}$. □

---

NEW

The relative local Dynkin diagram of $(\mathbf{G}, F)$ is the Dynkin diagram $\Delta = \Delta(\Phi_{af})$ of the affine root system $\Phi_{af}$ (or "échelonnage" [BT1, 1.4]) of $(\mathbf{G}, F)$. It is the Coxeter diagram of the affine reflection group $(W, S)$, where double and triple edges and possibly some fat ones are oriented, and some vertices (possibly none) are marked with a cross, such that for every vertex $\nu$ marked with a cross, all edges having $\nu$ as an extremity are double or fat and none of them is oriented away from $\nu$.

To each vertex $\nu$ of $\Delta$ is attached a positive integer $d(\nu)$ which depends not only on $\Phi_{af}$ and on $\nu$ but on $(\mathbf{G}, F)$ itself. If $\mathbf{G}$ is $F$-split, all $d(\nu)$ are equal to 1 [Tits, 1.8].

The index of $(\mathbf{G}, F)$ consists of

(a) The Dynkin diagram $\Delta_1 = \Delta(\Phi_{1af})$ of the affine root system $\Phi_{1af}$ (or "échelonnage" [BT1, 1.4]) of $(\mathbf{G}, F^{unr})$ where $F^{unr}$ is the maximal unramfied extension of $F$ (absolute local Dynkin diagram).

(b) The action of Gal$(F^{unr}/F)$ on $\Delta_1$.

(c) The Gal$(F^{unr}/F)$-invariant set of distinguished vertices of $\Delta_1$. When $\mathbf{G}$ is simple, all vertices are distinguished except for the unique anisotropic type ${}^dA_{d-1}$.

The index of $(\mathbf{G}, F)$ determines its relative local Dynkin diagram $\Delta$ and the integers $d(\nu)$ uniquely.

First of all, there is a canonical bijection $\nu \mapsto O(\nu)$ between the vertices of $\Delta$ and the Gal$(F^{unr}/F)$-orbits of distinguished vertices of $\Delta_1$. For every vertex $\nu$ of $\Delta$, $\Delta_{1,\nu}$ *** is the index of a semisimple group of relative rank 1 over the residue field $k$ of $F$, the integer $d(\nu)$ is half the total number of absolute roots of that group and $\nu$ is maked with a cross in $\Delta$ if and only if the relative root system of the group in question has type $BC_1$, that means that $\Delta_{1,\nu}$ is a disjoint union of diagrams of type $A_2$.

The type of the edge joining $\nu$ and $\nu'$ in $\Delta$ is determined by $\Delta_{1,\nu,\nu'} * **, O(\nu)$ and $O(\nu')$. This is an "empty edge" if and only if no connected component of $\Delta_{1,\nu,\nu'}$ meets both $O(\nu)$ and $O(\nu')$. Otherwise Gal$(F^{unr}/F)$ permutes the connected components of $\Delta_{1,\nu,\nu'}$ and the result can be described in terms of any one of them, say $\Delta^o_{1,\nu,\nu'}$. If the latter has only two vertices $\nu_1 \in O(\nu)$ and $\nu'_1 \in O(\nu')$, then $\nu$ and $\nu'$ are joined in $\Delta$ in the same way they are joined in $\Delta^o_{1,\nu,\nu'}$. When $\Delta^o_{1,\nu,\nu'}$ has at least three vertices, we refer to the tables which give $\Delta$. [Tits, 1.11]

The tables provide a list of all central isogeny classes of absolutely quasi-simple $F$-groups.

We say the $G$ is residually split if $G$ has the same rank over $F$ and over $F^{unr}$. A residually split group is quasi-split. The group is residually split if and only if

There is a smallest unramified extension $F'/F$ on which $G$ is residually split (the smallest splitting field of $T_1$), and $G$ being quasi-split over $F'$, has a smallest splitting field $F''$ over $F'$. The field $F''$ is the unique splitting field of $G$ over $F$ for which the degree $[F'' : F]$ and the ramification index $e(F''/F)$ are minimal for the lexicographic ordering.

A $F$-simple $F$-group $\mathbf{G}$ is the scalar restriction $\mathbf{G} = \mathbf{R_{F'/F}(G')}$ of a connected absolutely simple $F'$-group $\mathbf{G}'$ over a finite separable extension $F'/F$ [BorelTits, 6.21 (ii)]. The relative local Dynkin diagram, the integers $d(\nu)$, and the index of $(\mathbf{G}, F)$ can be deduced from those of $(\mathbf{G}', F')$ [Tits, 1.12]. We decompose $F'/F$ into its unramified and its totally ramified parts and handle the two cases separately.

If $F'/F$ is totally ramified, the local Dynkin diagram, the integers $d(\nu)$ and the index are the same for $(\mathbf{G}, F)$ as for $(\mathbf{G}', F')$.

If $F'/F$ is unramified of degree $f$, the index of $(\mathbf{G}, F)$ consists of $f$ copies of the index of $(\mathbf{G}', F')$ permuted transitively by Gal$(F^{sep}/F)$ whose action on the whole diagram us "induced up" from the action of Gal$(F^{sep}/F)$ on one copy, the relative local Dynkin diagram of $(\mathbf{G}, F)$ is the same as that of $(\mathbf{G}', F')$ and the integers $d(\nu)$ are $f$ times as big.

When $\mathbf{G}$ is semi-simple, the Iwahori-Hecke algebra of $(\mathbf{G}, F)$ is given by $(W, S)$, the integers $d(\nu)$, and a finite commutative subgroup $\Omega$ of the group $\mathbb{A}\mathrm{ut}\,\mathrm{Cox}(W, S)$ of automorphisms of the Coxeter diagram Cox$(W, S)$ of $(W, S)$.

## References


[Abde] Abdellatif R. *Autour des représenations modulo p des groupes réductifs p-adiques de rang* 1. Thesis, Université Paris-Sud XI, (2011).

[Abe] Abe N. *Modulo p parabolic induction of pro-p-Iwahori Hecke algebra.* Preprint 2014.

[AHHV] Abe N., Henniart G. Herzig F., Vignéras M.-F. *A classification of irreducible admissible mod p representations of p-adic reductive groups.* Preprint 2014.

[AHHV2] Abe N., Henniart G. Herzig F., Vignéras M.-F. *Parabolic induction, adjoints, and contragredients of mod p representations of p-adic reductive groups.* In preparation.

[Borel] Borel A. *Linear Algebraic groups.* 2nd edition, Springer 1991.

[BorelTits] Borel A. et Tits J. *Groupes réductifs.* Inst. Hautes Études Scient. Publications Mathématiques Vol. 27 (1965), pp. 55-150.

[BT1] Bruhat F. et Tits J. *Groupes réductifs sur un corps local. I. Données radicielles valuées.* Inst. Hautes Études Scient. Publications Mathématiques Vol. 41 (1972), pp. 5-252.

[BT2] Bruhat F. et Tits J. *Groupes réductifs sur un corps local. II Schémas en groupes. Existence d'une donnée radicielle valuées.* Inst. Hautes Études Scient. Publications Mathématiques Vol. 60 (1984), part II, pp. 197-376.

[CE] Cabanes M. and Enguehard M. *Representation theory of finite reductive groups.* Cambridge University Press 2004.

[HRa] Haines T., Rapoport M., *Appendix On parahoric subgroups.* Advances in Math. 219 (1), (2008), 188-198; appendix to G. Pappas, M. Rapoport, Twisted loop groups and their affine flag varieties, Advances in Math. 219 (1), (2008), 118-198.

[HRo] Haines T., Rostami S., *The Satake isomorphism for special maximal parahoric algebras* Representation Theory 14 (2010), 264-284.

[HV1] Henniart G., Vigneras M.-F. *A Satake isomorphism for representations modulo p of reductive groups over local fields* Journal fur die reine und angewandte Mathematik, 2015(701), pp. 33-75.

[Ko] Kottwitz R. *Isocrystals with additional structure II.* Compositio Math. Vol. 109 (1997) pp. 225-339.

[Koziol] Koziol K. *Pro-p-Iwahori invariants for $SL_2$ and L-packets of Hecke modules* Int. Math. Res. Notices. Published on line June 2015.

[KoziolF] Koziol K. *A classification of the irreducible mod-p representations of* $U(1,1)(\mathbb{Q}_{p^2}/\mathbb{Q}_p)$ to appear in Annales de l'Institut Fourier.

[Morris] Morris L. *Level zero G-types.* Compositio Math. 118, 135-157 (1999).

[OInv] Ollivier R. *An Inverse Satake isomorphism in characteristic p.* Preprint 2012.

[OComp] Ollivier R. *Compatibility between Satake and Bernstein isomorphisms in characteristic p.* To appear in ANT 2014.

[PR] Platonov V., Rapinchuk A. *Algebraic groups and number theory.* Translated from the 1991 Russian original by Rachel Rowen. Pure and Applied Mathematics, 139, Academic Press(1994).

[OV] Ollivier R., Vignéras M.-F. *The pro-p Iwahori Hecke algebra of a reductive p-adic group V (parabolic induction).* In progress.

[Spr] Springer T. A. *Linear algebraic groups.* Birkhauser PM 9 Second Edition (1998).

[T0] Tits J. *Reductive groups over local fields.* Proc. of Symposia in Pure Math. 9 (1979) part 1 29-69.

[Tits] Tits J. *Reductive groups over local fields.* Proc. of Symposia in Pure Math. 33 (1979) part 1 29-69.

[VigMA] Vigneras M.-F. *On a numerical Langlands correspondence modulo p with the pro–Iwahori Hecke ring.* Mathematische Annalen volume 331 No 3, 1432-1807, Erratum volume 333 No 3, 699-701 (2005).

[Vigsel] Vignéras M.-F. *Induced representations of reductive p-adic groups in characteristic $\ell \neq p$.* Selecta Mathematica 4 (1998) 549-623.

[Vig1] Vignéras M.-F. *The pro-p-Iwahori-Hecke algebra of a reductive p-adic group I.* Preprint 2013. To appear in Compositio Mathematica 2015.

[Vig2] Vigneras M.-F. *The pro-p-Iwahori-Hecke algebra of a reductive p-adic group II.* Volume in the honour of Peter Schneider. Münster J. of Math. 2014.

[Vig3] Vignéras M.-F. *The pro-p-Iwahori-Hecke algebra of a reductive p-adic group III (spherical Hecke algebras and supersingular modules).* Preprint 2014. Journal of the Institute of Mathematics of Jussieu / FirstView Article / August 2015, pp 1 - 38

[Vig5] Vignéras Marie-France *The pro-p-Iwahori-Hecke algebra of a reductive p-adic group V (parabolic induction).* Preprint 2015. To appear in a volume in the memory of R. Steinberg in Pacific Math. Journal.

Vignéras Marie-France
UMR 7586, Institut de Mathematiques de Jussieu, 4 place Jussieu, Paris 75005, France,
vigneras@math.jussieu.fr