The pro-p Iwahori Hecke algebra of a reductive p-adic group V (Parabolic induction) Stronger results or better redaction Corrections

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Abstract

We give basic properties of the parabolic induction and coinduction functors associated to R-algebras modelled on the pro-p-Iwahori-Hecke R-algebras $\mathcal{H}_R(G)$ and $\mathcal{H}_R(M)$ of a reductive p-adic group G and of a Levi subgroup M when R is a commutative ring. We show that the parabolic induction and coinduction functors are faithful, have left and right adjoints that we determine, respect finitely generated R-modules, and that the induction is a twisted coinduction.

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1 Introduction

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When R is an algebraically closed field of characteristic p, Abe [Abe, Section 4] proved that the induction is a twisted coinduction, when he classified the simple $\mathcal{H}_R(G)$-modules in term of supersingular simple $\mathcal{H}_R(M)$-modules. In two forthcoming articles [OV] and [AHHV2], we will use this paper to compute the images of an irreducible admissible R-representation of G by the basic functors: invariants by a pro-p-Iwahori subgroup, left or right adjoint of the parabolic induction.

Let R be a commutative ring and let $\mathcal{H}$ be a pro-p Iwahori Hecke R-algebra, associated to a pro-p Iwahori Weyl group $W(1)$ and parameter maps $\mathfrak{S} \xrightarrow{\mathfrak{q}} R,\ \mathfrak{S}(1) \xrightarrow{\mathfrak{c}} R[Z_k]$ [Vig1, §4.3], [Vig4].

For the reader unfamiliar with these definitions, we recall them briefly. The pro-p Iwahori Weyl group $W(1)$ is an extension of an Iwahori Weyl group W by a finite commutative group Z_k, $X(1)$ denotes the inverse image in $W(1)$ of a subset X of W, the Iwahori Weyl group contains a normal affine Weyl subgroup W^{aff}, $\mathfrak{S}$ is the set of all affine reflections of W^{aff}, $\mathfrak{q}$ is a W-equivariant map $\mathfrak{S} \to R$, W acting by conjugation on $\mathfrak{S}$ and trivially on R, $\mathfrak{c}$ is a $W(1) \times Z_k$-equivariant map $\mathfrak{S}(1) \to R[Z_k]$, $W(1)$ acting by conjugation and Z_k by multiplication on both sides.

The Iwahori Weyl group is a semidirect product $W = \Lambda \rtimes W_0$ where Λ is the (commutative finitely generated) subgroup of translations and W_0 is the finite Weyl subgroup of W^{aff}.

Let S^{aff} be a set of generators of W^{aff} such that (W^{aff}, S^{aff}) is an affine Coxeter system and $(W_0, S := S^{aff} \cap W_0)$ is a finite Coxeter system. The Iwahori Weyl group is also a semidirect product $W = W^{aff} \rtimes \Omega$ where Ω denotes the normalizer of S^{aff} in W. Let ℓ denote the length of (W^{aff}, S^{aff}) extended to W and then inflated to $W(1)$ such that $\Omega \subset W$ and $\Omega(1) \subset W(1)$ are the subsets of length 0 elements.

Let $\tilde{w} \in W(1)$ denote a fixed but arbitrary lift of $w \in W$.

The subset $\mathfrak{S} \subset W^{aff}$ of all affine reflections is the union of the W^{aff}-conjugates of S^{aff} and the map $\mathfrak{q}$ is determined by its values on S^{aff}, the map $\mathfrak{c}$ is determined by its values on any set $\tilde{S}^{aff} \subset S^{aff}(1)$ of lifts of S^{aff} in $W(1)$.

Definition 1.1. *The R-algebra $\mathcal{H}$ associated to $(W(1), \mathfrak{q}, \mathfrak{c})$ and S^{aff} is the free R-module of basis $(T_{\tilde{w}})_{\tilde{w} \in W(1)}$ and relations generated by the braid and quadratic relations:*

$$T_{\tilde{w}} T_{\tilde{w}'} = T_{\tilde{w}\tilde{w}'},\ T_{\tilde{s}}^2 = \mathfrak{q}(s)(\tilde{s})^2 + \mathfrak{c}(\tilde{s}) T_{\tilde{s}},$$

for all $\tilde{w}, \tilde{w}' \in W(1)$ with $\ell(w) + \ell(w') = \ell(ww')$ and all $\tilde{s} \in S^{aff}(1)$.

By the braid relations, the map $R[\Omega(1)] \to \mathcal{H}$ sending $\tilde{u} \in \Omega(1)$ to $T_{\tilde{u}}$ identifies $R[\Omega(1)]$ with a subring of $\mathcal{H}$ containing $R[Z_k]$. This identification is used in the quadratic relations. The isomorphism class of $\mathcal{H}$ in independent of the choice of S^{aff}.

Let S_M be a subset of S. We recall the definitions of the pro-p Iwahori Weyl group $W_M(1)$, the parameter maps $\mathfrak{S}_M \xrightarrow{\mathfrak{q}_M} R,\ \mathfrak{S}_M(1) \xrightarrow{\mathfrak{c}_M} R[Z_k]$ and S_M^{aff} given in [Vig4].

The set S_M generates a finite Weyl subgroup $W_{M,0}$ of W_0, $W_M := \Lambda \rtimes W_{M,0}$ is a subgroup of W, $W_M(1)$ is the inverse image of W_M in $W(1)$, $\mathfrak{S}_M(1) = \mathfrak{S}(1) \cap W_M(1)$, $\mathfrak{q}_M$ is the restriction of $\mathfrak{q}$ to $\mathfrak{S}_M$, and $\mathfrak{c}_M$ is the restriction of $\mathfrak{c}$ to $\mathfrak{S}_M(1)$. The subgroup $W_M^{aff} := W^{aff} \cap W_M \subset W_M$ is an affine Weyl group and S_M^{aff} denotes the set of generators of W_M^{aff} containing S_M such that (W_M^{aff}, S_M^{aff}) is an affine Coxeter system.

Definition 1.2. *For $S_M \subset S$, the R-algebra $\mathcal{H}_M$ associated to $(W_M(1), \mathfrak{q}_M, \mathfrak{c}_M)$ and S_M^{aff} is called a Levi algebra of $\mathcal{H}$.*

Let $(T_{\tilde{w}}^M)_{\tilde{w} \in W_M(1)}$ denote the basis of $\mathcal{H}_M$ associated to $(W_M(1), \mathfrak{q}_M, \mathfrak{c}_M)$ and S_M^{aff} and ℓ_M the length of $W_M(1)$ associated to S_M^{aff}.

Remark 1.3. When $S_M = S$, $\mathcal{H}_M = \mathcal{H}$. When $S_M = \emptyset$, $\mathcal{H}_M = R[\Lambda(1)]$.

In general when $S_M \neq S$, S_M^{aff} is not $W_M \cap S^{aff}$, and $\mathcal{H}_M$ is not a subalgebra of $\mathcal{H}$; it embeds in $\mathcal{H}$ only when the parameters $\mathfrak{q}(s) \in R$ for $s \in S^{aff}$ are invertible.

As in the theory of Hecke algebras associated to types, one introduces the subalgebra $\mathcal{H}_M^+ \subset \mathcal{H}_M$ of basis $(T_{\tilde{w}}^M)_{\tilde{w} \in W_{M^+}(1)}$ associated to the positive monoid $W_{M^+} := \{w \in W_M \mid w(\Sigma^+ - \Sigma_M^+) \subset \Sigma^{aff,+}\}$ where $\Sigma_M \subset \Sigma$ are the reduced root systems defining $W_M^{aff} \subset W^{aff}$, the upper index indicates the positive roots with respect to S^{aff}, S_M^{aff}, and Σ^{aff} is the set of affine roots of Σ. One chooses an element $\tilde{\mu}_M$ central in $W_M(1)$, in particular of length $\ell_M(\tilde{\mu}_M) = 0$, lifting a strictly positive element μ_M in $\Lambda_{M^+} := \Lambda \cap W_{M^+}$. The element $T_{\tilde{\mu}_M}^M$ of $\mathcal{H}_M$ is invertible of inverse $T_{\tilde{\mu}_M^{-1}}^M$ but in general $T_{\tilde{\mu}_M}$ is not invertible in $\mathcal{H}$.

Theorem 1.4. (i) *The R-submodule $\mathcal{H}_{M^+}$ of basis $(T_{\tilde{w}}^M)_{\tilde{w} \in W_{M^+}(1)}$ is a subring of $\mathcal{H}_M$, called the positive subalgebra of $\mathcal{H}_M$.*

(ii) *The R-algebra $\mathcal{H}_M = \mathcal{H}_{M^+}[(T_{\tilde{\mu}_M}^M)^{-1}]$ is a localization of $\mathcal{H}_{M^+}$ at $T_{\tilde{\mu}_M}^M$.*

(iii) *The injective linear map $\mathcal{H}_M \xrightarrow{\theta} \mathcal{H}$ sending $T_{\tilde{w}}^M$ to $T_{\tilde{w}}$ for $\tilde{w} \in W_M(1)$ restricted to $\mathcal{H}_{M^+}$ is a ring homomorphism.*

(iv) *As an $\theta(\mathcal{H}_{M^+})$-module, $\mathcal{H}$ is the almost localization of a left free $\theta(\mathcal{H}_{M^+})$-module $\mathcal{V}_{M^+}$ at $T_{\tilde{\mu}_M}$.*

The theorem was known in special cases. The part (iv) means that $\mathcal{H}$ is the union over $r \in \mathbb{N}$ of

$$ {}_r\mathcal{V}_{M^+} := \{x \in \mathcal{H} \mid T_{\tilde{\mu}_M}^r x \in \mathcal{V}_{M^+}\}, \quad \mathcal{V}_{M^+} = \oplus_{d \in {}^M W_0} \theta(\mathcal{H}_{M^+}) T_{\tilde{d}}. $$

Here ${}^M W_0$ is the set of elements of minimal lengths in the cosets $W_{M,0} \backslash W_0$ and $\tilde{d} \in W(1)$ is an arbitrary lift of d. The theorem admits a variant for the subalgebra $\mathcal{H}_{M^-} \subset \mathcal{H}_M$ associated the negative submonoid W_{M^-}, inverse of W_{M^+}, for the linear map $\mathcal{H}_M \xrightarrow{\theta^*} \mathcal{H}$ sending $(T_{\tilde{w}}^M)^*$ to $T_{\tilde{w}}^*$ for $\tilde{w} \in W_M(1)$ [Vig1, Prop. 4.14], and with *left* replaced by *right* in (iv): $\mathcal{H}_M = \mathcal{H}_{M^-}[T_{\tilde{\mu}_M}^M]$, θ^* restricted to $\mathcal{H}_{M^-}$ is a ring homomorphism, the right $\theta^*(\mathcal{H}_{M^-})$-module $\mathcal{H}$ is the almost localisation at $T_{\tilde{\mu}_M^{-1}}^*$ of a right free $\theta^*(\mathcal{H}_{M^-})$-module $\mathcal{V}_{M^-}^*$ of rank $|W_{M,0}|^{-1}|W_0|$, meaning that $\mathcal{H}$ is the union over $r \in \mathbb{N}$ of

$$ {}_r\mathcal{V}_{M^-}^* := \{x \in \mathcal{H} \mid x(T_{\tilde{\mu}_M^{-1}}^*)^r \in \mathcal{V}_{M^-}^*\}, \quad \mathcal{V}_{M^-}^* := \sum_{d \in W_0^M} T_{\tilde{d}}^* \, \theta^*(\mathcal{H}_{M^-}). $$

Here W_0^M is the inverse of ${}^M W_0$.

For a ring A, let Mod_A denote the category of right A-modules and ${}_A\mathrm{Mod}$ the category of left A-modules. Given two rings $A \subset B$, the induction $- \otimes_A B$ and the coinduction $\mathrm{Hom}_A(B, -)$ from Mod_A to Mod_B are the left and the right adjoint of the restriction Res_A^B. The ring B is considered as a left A-module for the induction, and as a right A-module for the coinduction.

The property (iv) and its variant describe $\mathcal{H}$ as a left $\theta(\mathcal{H}_{M^+})$-module and as a right $\theta^*(\mathcal{H}_{M^-})$-module. The linear maps θ and θ^* identify the subalgebras $\mathcal{H}_{M^+}, \mathcal{H}_{M^-}$ of $\mathcal{H}_M$ with the subalgebras $\theta(\mathcal{H}_{M^+}), \theta^*(\mathcal{H}_{M^-})$ of $\mathcal{H}$.

Definition 1.5. *The parabolic induction and coinduction from $\mathrm{Mod}_{\mathcal{H}_M}$ to $\mathrm{Mod}_{\mathcal{H}}$ are the functors $I_{\mathcal{H}_M}^{\mathcal{H}} = - \otimes_{\mathcal{H}_{M^+},\theta} \mathcal{H}$ and $\mathbb{I}_{\mathcal{H}_M}^{\mathcal{H}} = \mathrm{Hom}_{\mathcal{H}_{M^-},\theta^*}(\mathcal{H}, -)$.*

We show:

Theorem 1.6. *The parabolic induction $I_{\mathcal{H}_M}^{\mathcal{H}}$ is faithful, transitive, respects finitely generated R-modules, admits a right adjoint $\mathrm{Hom}_{\mathcal{H}_{M^+},\theta}(\mathcal{H}_M, -)$.*

If R is a field, the right adjoint functor respects finite dimension.

The transitivity of the parabolic induction means that for $S_M \subset S_{M'} \subset S$,

$$I_{\mathcal{H}_M}^{\mathcal{H}} = I_{\mathcal{H}_{M'}}^{\mathcal{H}} \circ I_{\mathcal{H}_M}^{\mathcal{H}_{M'}} \ : \ \mathrm{Mod}_{\mathcal{H}_M} \to \mathrm{Mod}_{\mathcal{H}_{M'}} \to \mathrm{Mod}_{\mathcal{H}}.$$

Let w_0 denote the longest element of W_0, $S_{w_0(M)}$ the subset $w_0 S_M w_0$ of S, $w_0^M := w_0 w_{M,0}$ where $w_{M,0}$ is the longest element of $W_{M,0}$. A lift $\tilde{w}_0^M \in W_0(1)$ of w_0^M defines an R-algebra isomorphism

$$\mathcal{H}_M \to \mathcal{H}_{w_0(M)}, \quad T_{\tilde{w}}^M \mapsto T^{w_0(M)}_{\tilde{w}_0^M \tilde{w} (\tilde{w}_0^M)^{-1}} \text{ for } \tilde{w} \in W_M(1), \tag{1}$$

inducing an equivalence of categories $\mathrm{Mod}_{\mathcal{H}_M} \xrightarrow{\tilde{\mathfrak{w}}_0^M} \mathrm{Mod}_{\mathcal{H}_{w_0(M)}}$, of inverse $\tilde{\mathfrak{w}}_0^{w_0(M)}$ defined by the lift $(\tilde{w}_0^M)^{-1} \in W_0(1)$ of $w_0^{w_0(M)} = (w_0^M)^{-1}$.

Definition 1.7. *The w_0-twisted parabolic induction and coinduction from $\mathrm{Mod}_{\mathcal{H}_M}$ to $\mathrm{Mod}_{\mathcal{H}}$ are the functors $I_{\mathcal{H}_{w_0(M)}}^{\mathcal{H}} \circ \tilde{\mathfrak{w}}_0^M$ and $\mathbb{I}_{\mathcal{H}_{w_0(M)}}^{\mathcal{H}} \circ \tilde{\mathfrak{w}}_0^M$.*

Modulo equivalence, these functors do not depend on the choice of the lift of w_0^M used for their construction.

Theorem 1.8. *The parabolic induction (resp. coinduction) is equivalent to the w_0-twisted parabolic coinduction (resp. induction):*

$$\mathbb{I}_{\mathcal{H}_M}^{\mathcal{H}} \simeq I_{\mathcal{H}_{w_0(M)}}^{\mathcal{H}} \circ \tilde{\mathfrak{w}}_0^M, \quad I_{\mathcal{H}_M}^{\mathcal{H}} \simeq \mathbb{I}_{\mathcal{H}_{w_0(M)}}^{\mathcal{H}} \circ \tilde{\mathfrak{w}}_0^M.$$

Using that the coinduction admits a left adjoint and that the induction is a twisted coinduction, one proves:

Theorem 1.9. *The parabolic induction $I_{\mathcal{H}_M}^{\mathcal{H}}$ admits a left adjoint equivalent to*

$$\tilde{\mathfrak{w}}_0^{w_0(M)} \circ (- \otimes_{\mathcal{H}_{w_0(M)^-}, \theta^*} \mathcal{H}_{w_0(M)}) : \mathrm{Mod}_{\mathcal{H}} \to \mathrm{Mod}_{\mathcal{H}_{w_0(M)}} \to \mathrm{Mod}_{\mathcal{H}_M}$$

When R is a field, the left adjoint functor respects finite dimension.

The coinduction satisfies the same properties as the induction:

Corollary 1.10. *The coinduction $\mathbb{I}_{\mathcal{H}_M}^{\mathcal{H}}$ is faithful, transitive, respects finitely generated R-modules, admits a left and a right adjoint. When R is a field, the left and right adjoint functors respect finite dimension.*

Note that the induction and the coinduction are exact functors, as they admit a left and a right adjoint. A localization functor is exact hence also the left adjoint of the induction and of the coinduction.

We prove Theorem 1.4 in chapter 2, Theorem 1.6 in chapter 3.2, Theorem 1.8, Theorem 1.9 in chapter 3.2.

Remark 1.11. One cannot replace $(\mathcal{H}, \mathcal{H}_M, \mathcal{H}_M^+)$ by $(\mathcal{H}, \mathcal{H}_M, \mathcal{H}_M^-)$ to define the induction $I_{\mathcal{H}_M}^{\mathcal{H}}$.

When no non-zero element of the ring R is infinitely p-divisible, is the parabolic induction functor $\mathrm{Mod}_{\mathcal{H}_M} \xrightarrow{I_{\mathcal{H}_M}^{\mathcal{H}}} \mathrm{Mod}_{\mathcal{H}}$ fully faithful ? The answer is yes for the parabolic induction functor $\mathrm{Mod}_R^\infty(M) \xrightarrow{\mathrm{Ind}_P^G} \mathrm{Mod}_R^\infty(G)$ when M is a Levi subgroup of a parabolic subgroup P of a reductive p-adic group G and $\mathrm{Mod}_R^\infty(G)$ the category of of smooth R-representations of G [Vig2, Theorem 5.3].

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2 Levi algebra

We prove Theorem 1.4 and its variant on the subalgebra $\mathfrak{H}_M^\epsilon \subset \mathfrak{H}_M$, its image in $\mathcal{H}$, on $\mathfrak{H}_M$ as a localisation of $\mathfrak{H}_M^\epsilon$ and on $\mathcal{H}$ as an almost left localisation of $\theta(\mathfrak{H}_M^+)$, and almost left localisation of $\theta^*(\mathfrak{H}_M^-)$.

2.1 Monoid W_{M^ϵ}

Let $S_M \subset S$ and $\epsilon \in \{+,-\}$. To S^{aff} is associated a submonoid $W_{M^\epsilon} \subset W_M$ defined as follows.

Let Σ denote the reduced root system of affine Weyl group W^{aff}, V the real vector space of dual generated by Σ, $\Sigma^{aff} = \Sigma + \mathbb{Z}$ the set of affine roots of Σ and $\mathfrak{H} = \{\mathrm{Ker}_V(\gamma) \mid \gamma \in \Sigma^{aff}\}$ the set of kernels of the affine roots in V. We fix a W_0-invariant scalar product on V. The affine Weyl group W^{aff} identifies with the group generated by the orthogonal reflections with respect to the affine hyperplanes of $\mathfrak{H}$.

Let $\mathfrak{A}$ denote the alcove of vertex 0 of $(V, \mathfrak{H})$ such that S^{aff} is the set of orthogonal reflections with respect to the walls of $\mathfrak{A}$ and S is the subset associated to the walls containing 0. An affine root which is positive on $\mathfrak{A}$ is called positive. Let $\Sigma^{aff,+}$ denote the set of positive affine roots, $\Sigma^+ := \Sigma \cap \Sigma^+_{aff}$, $\Sigma^{aff,-} := -\Sigma^{aff,-}, \Sigma^- := -\Sigma^+$.

Let Δ_M denote the set of positive roots $\alpha \in \Sigma^+$ such that $\mathrm{Ker}\,\alpha$ is a wall of $\mathfrak{A}$ and the orthogonal reflection s_α of V with respect to $\mathrm{Ker}\,\alpha$ belongs to S_M, $\Sigma_M \subset \Sigma$ the reduced root system generated by Δ_M, $\Sigma_M^\epsilon := \Sigma_M \cap \Sigma^\epsilon_{aff}$.

Definition 2.1. *The positive monoid $W_{M^+} \subset W_M$ is $\{w \in W_M \mid w(\Sigma^+ - \Sigma_M^+) \subset \Sigma^{aff,+}\}$.*

The negative monoid $W_{M^-} := \{w \in W_M \mid w^{-1} \in W_{M^+}\}$ is the inverse monoid.

It is well known that the finite Weyl group $W_{M,0}$ is the W_0-stabilizer of $\Sigma^\epsilon - \Sigma_M^\epsilon$. This implies

$$W_{M^\epsilon} = \Lambda_{M^\epsilon} \rtimes W_{M,0} \quad \text{where} \quad \Lambda_{M^\epsilon} := \Lambda \cap W_{M^\epsilon}.$$

Let $\Lambda \xrightarrow{\nu} V$ denote the homomorphism such that $\lambda \in \Lambda$ acts on V by translation by $\nu(\lambda)$.

Lemma 2.2. $\Lambda_{M^\epsilon} = \{\lambda \in \Lambda \mid -(\gamma \circ \nu)(\lambda) \geq 0$ *for all* $\gamma \in \Sigma^\epsilon - \Sigma_M^\epsilon\ \}$.

Proof. Let $\lambda \in \Lambda$. By definition, $\lambda \in \Lambda_{M^+}$ if and only if $\lambda(\gamma)$ is positive for all $\gamma \in \Sigma^+ - \Sigma_M^+$. We have $\lambda(\gamma) = \gamma - \nu(\lambda)$. The minimum of the values of γ on $\mathfrak{A}$ is 0[Vig1, (35)]. So $\gamma(v - \nu(\lambda)) \geq 0$ for $\gamma \in \Sigma^+ - \Sigma_M^+$ and $v \in \mathfrak{A}$ is equivalent to $-(\gamma \circ \nu)(\lambda) \geq 0$ for all $\gamma \in \Sigma^+ - \Sigma_M^+$. □

When $S_M \subset S_{M'} \subset S$, we have the inclusion $\Sigma_M^\epsilon \subset \Sigma_{M'}^\epsilon$, the inverse inclusion $\Sigma^\epsilon - \Sigma_M^\epsilon \subset \Sigma^\epsilon - \Sigma_{M'}^\epsilon$, and the inclusions $W_M \subset W_{M'}$ and $W_{M^\epsilon} \subset W_{M'}^\epsilon$.

Remark 2.3. Set $\mathcal{D}^\epsilon := \{v \in V \mid \gamma(v) \geq 0 \text{ for } \gamma \in \Sigma^\epsilon\}$ and $\Lambda^\epsilon := (-\nu)^{-1}(\mathcal{D}^\epsilon)$. The antidominant Weyl chamber of V is $\mathcal{D}^-$ and the dominant Weyl chamber is $\mathcal{D}^+$. Careful: [Vig3, §1.2 (v)] uses a different notation: $\Lambda^\epsilon = (\nu)^{-1}(\mathcal{D}^\epsilon)$.

The Bruhat order $\leq$ of the affine Coxeter system (W^{aff}, S^{aff}) extends to W: for $w_1, w_2 \in W^{aff}, u_1, u_2 \in \Omega$, $w_1u_1 \leq w_2u_2$ if $u_1 = u_2$ and $w_1 \leq w_2$ [VigRT, Appendice]. We write $w < w'$ if $w \leq w'$ and $w \neq w'$ for $w, w' \in W$. Careful: the Bruhat order $\leq_M$ on W_M associated to (W_M^{aff}, S_M^{aff}) is not the restriction of $\leq$ when S_M^{aff} is not contained in S^{aff} [Vig4].

Remark 2.4. The basic properties of (W^{aff}, S^{aff}) extend to W:

(i) If $x \leq y$ for $x, y \in W$ and $s \in S^{aff}$,

$$sx \leq (y \text{ or } sy), \quad xs \leq (y \text{ or } ys), \quad (x \text{ or } sx) \leq sy, \quad (x \text{ or } xs) \leq ys$$

[Vig3, Lemma 3.1, Remark 3.2].

(ii) $W = \sqcup_{\lambda \in \Lambda^\epsilon} W_0 \lambda W_0$ [HV1, 6.3 Lemma].

(iii) For $\lambda \in \Lambda^+$, $W_0\lambda W_0$ admits a unique element of maximal length $w_\lambda = w_0\lambda$ where w_0 is the unique element of maximal length in W_0, and $\ell(w_\lambda) = \ell(w_0) + \ell(\lambda)$ [Vig3, Lemma 3.5].

(iv) For $\lambda \in \Lambda^+$, $\{w \in W \,|w \leq w_\lambda\} \supset \sqcup_{\mu \in \Lambda^+, \mu \leq \lambda} W_0 \mu W_0$ [Vig3, Lemma 3.5].

Remark 2.5. $\{w \in W \,|w \leq w_\lambda\}$ is a union of (W_0, W_0)-classes only if $\lambda, \mu \in \Lambda^+$, $\mu \leq w_0 \lambda$ implies $\mu \leq \lambda$. I see no reason for this to be true.

Lemma 2.6. *The monoid W_{M^ϵ} is a lower subset of W_M for the Bruhat order $\leq_M$: for $w \in W_{M^\epsilon}$, any element $v \in W_M$ such that $v \leq_M w$ belongs to W_{M^ϵ}.*

Proof. [Abe, Lemma 4.1]. □

An element $w \in W$ admits a reduced decomposition in (W, S^{aff}), $w = s_1 \dots s_r u$ with $s_i \in S^{aff}, u \in \Omega$. As in [Vig1], we set for $w, w' \in W$,

$$q_w := \mathfrak{q}(s_1) \dots \mathfrak{q}(s_r), \quad q_{w,w'} := (q_w q_{w'} q_{ww'}^{-1})^{1/2}. \tag{2}$$

This is independent of the choice of the reduced decomposition. For $w, w' \in W_M$ and $s_i \in S_M^{aff}, u \in \Omega_M$, let $q_{M,w}, \ q_{M,w,w'}$ denote the similar elements. They may be different from $q_w, \ q_{w,w'}$.

Lemma 2.7. *We have $S_M^{aff} \cap W_{M^\epsilon} \subset S^{aff}$ and $q_{w,w'} = q_{M,w,w'}$ if $w, w' \in W_{M^\epsilon}$.*
In particular, $\ell_M(w) + \ell_M(w') - \ell_M(ww') = \ell(w) + \ell(w') - \ell(ww')$, if $w, w' \in W_{M^\epsilon}$.

Proof. [Abe, Lemma 4.4 and proof of lemma 4.5]. □

An element $\lambda \in \Lambda_{M^\epsilon}$ such that all the inequalities in (2.2) are strict is called strictly positive if $\epsilon = +$, and strictly negative if $\epsilon = +$. We choose

a central element $\tilde{\mu}_M$ of $W_M(1)$ lifting a stricty positive element μ_M of Λ.

We set $\tilde{\mu}_{M^+} := \tilde{\mu}_M$ and $\tilde{\mu}_{M^-} := \tilde{\mu}_M^{-1}$. The center of the pro-$p$ Iwahori Weyl group $W_M(1)$ is the set of elements in the center of $\Lambda(1)$ fixed by the finite Weyl group $W_{M,0}$ [Vig2]. Hence $\tilde{\mu}_{M^\epsilon}$ is an element of the center of $\Lambda(1)$ fixed by $W_{M,0}$ and $-\gamma \circ \nu(\mu_{M^\epsilon}) > 0$ for all $\gamma \in \Sigma^\epsilon - \Sigma_M^\epsilon$. We have $\gamma \circ \nu(\mu_{M^\epsilon}) = 0$ for $\gamma \in \Sigma_M$. The length of μ_{M^ϵ} is 0 in W_M, and is positive in W when $S_M \neq S$.

Let $\mathcal{H}_{M^\epsilon}$ denote the R-submodule of the Iwahori Hecke R-algebra $\mathcal{H}_M$ of M of basis $(T_{\tilde{w}}^M)_{\tilde{w} \in W_{M^\epsilon}(1)}$, and $\mathcal{H}_M \xrightarrow{\theta} \mathcal{H}$ (resp. $\mathcal{H}_M \xrightarrow{\theta^*} \mathcal{H}$) the linear map sending $T_{\tilde{w}}^M$ to $T_{\tilde{w}}$ (resp. $T_{\tilde{w}}^{M,*}$ to $T_{\tilde{w}}^*$) for $\tilde{w} \in W_M(1)$.

The proof of the properties (i), (ii), (iii) of Theorem 1.4 and its variant are as follows:

1. $\mathcal{H}_{M^\epsilon}$ is a subring of $\mathcal{H}_M$, because $T_{\tilde{w}}^M T_{\tilde{w}'}^M$ is a linear combination of elements $T_{\tilde{v}}$ such that $v \leq_M ww'$ [Vig1].

2. $\theta(T_{\tilde{w}_1}^M T_{\tilde{w}_2}^M) = T_{\tilde{w}_1} T_{\tilde{w}_2}$ and $\theta^*((T_{\tilde{w}_1}^M)^* (T_{\tilde{w}_2}^M)^*) = T_{\tilde{w}_1}^* T_{\tilde{w}_2}^*$ for $w_1, w_2 \in W_{M^\epsilon}$ for $w_1, w_2 \in W_{M^\epsilon}$. This follows from the braid relations if $\ell_M(w_1) + \ell_M(w_2) = \ell_M(w_1 w_2)$ because $\ell(w_1) + \ell(w_2) = \ell(w_1 w_2)$ (Lemma 2.7). If $w_2 = s \in S_M^{aff}$ with $\ell_M(w_1) - 1 = \ell_M(w_1 s)$ this follows from the quadratic relations

$$T_{\tilde{w}_1} T_{\tilde{s}} = T_{\tilde{w}_1 \tilde{s}^{-1}} (\mathfrak{q}(s)(\tilde{s})^2 + T_{\tilde{s}} \mathfrak{c}(\tilde{s})) = \mathfrak{q}(s) T_{\tilde{w}_1 \tilde{s}} + T_{\tilde{w}_1} \mathfrak{c}(\tilde{s}), \ T_{\tilde{w}_1}^* T_{\tilde{s}}^* = \mathfrak{q}(s) T_{\tilde{w}_1 \tilde{s}}^* - T_{\tilde{w}_1}^* \mathfrak{c}(\tilde{s}),$$

$s \in S^{aff}$, $\ell(w_1) - 1 = \ell(w_1 s)$ (Lemma 2.7) and $\mathfrak{q}(s) = \mathfrak{q}_M(s), \mathfrak{c}(\tilde{s}) = \mathfrak{c}_M(\tilde{s})$ [Vig4]. In general the formula is proved by induction on $\ell_M(w_2)$ [Abe, 4.1]. The proof of [Abe, Lemma 4.5] applies.

We have $\theta^*(T^M_{\tilde{w}}) = T^M_{\tilde{w}}$ for $w \in W_{M,0}$ because for $s \in S_M$,

$$\theta^*(T^M_{\tilde{s}}) = \theta^*(T^{M,*}_{\tilde{s}} + c^M_{\tilde{s}}) = T^*_{\tilde{s}} + c_{\tilde{s}} = T_{\tilde{s}}.$$

3. $\mathcal{H}_M = \mathcal{H}_{M^\epsilon}[(T^M_{\tilde{\mu}_{M^\epsilon}})^{-1}]$, because for $w \in W_M$ there exists $r \in \mathbb{N}$ such that $\mu_M^{\epsilon r} w \in W_{M^\epsilon}$.

Remark 2.8. If the parameters $\mathfrak{q}(s)$ are invertible in R, then $\mathcal{H}_{M^+} \xrightarrow{\theta} \mathcal{H}$ extends uniquely to an algebra homomorphism $\mathcal{H}_M \hookrightarrow \mathcal{H}$, sending $T^M_{\tilde{\mu}_M^{-\epsilon r}\tilde{w}}$ to $T^{-r}_{\tilde{\mu}_{M^\epsilon}} T_{\tilde{w}}$ for $\tilde{w} \in W_{M^+}(1), r \in \mathbb{N}$.

Remark 2.9. The trivial character $\chi_1 : \mathcal{H} \to R$ of $\mathcal{H}$ is defined by

$$\chi_1(T_{\tilde{w}}) = q_w \quad (\tilde{w} \in W(1)).$$

When $\mathcal{H}$ is the Hecke algebra of the pro-p-Iwahori subgroup of a reductive p-adic group G, $\mathcal{H}$ acts on the trivial representation of G by χ_1. Note that the restriction of the trivial character of $\mathcal{H}_M$ to $\theta(\mathcal{H}_{M^+})$ is not equal to $\chi_1 \circ \theta$ when $\ell_M(\mu_M) = 0, \ell(\mu_M) \neq 0$.

2.2 An anti-involution ζ

The R-linear bijective map

$$\mathcal{H} \xrightarrow{\zeta} \mathcal{H} \quad \text{such that} \quad \zeta(T_{\tilde{w}}) = T_{\tilde{w}^{-1}} \quad \text{for} \quad \tilde{w} \in W(1), \tag{3}$$

is an anti-involution when $\zeta(h_1 h_2) = \zeta(h_2)\zeta(h_1)$ for $h_1, h_2 \in \mathcal{H}$ because $\zeta \circ \zeta = \mathrm{id}$. For $S_M \subset S$, let $\mathcal{H} \xrightarrow{\zeta_M} \mathcal{H}_M$ denote the linear map such that $\zeta(T^M_{\tilde{w}}) = T^M_{\tilde{w}^{-1}}$ for $\tilde{w} \in W_M(1)$.

Lemma 2.10. *1. The following properties are equivalent:*
(i) ζ is an anti-involution,
(ii) $\zeta(\mathfrak{c}(\tilde{s})) = c_{(\tilde{s})^{-1}}$ for $\tilde{s} \in S^{aff}(1)$,
(iii) $\zeta \circ \mathfrak{c} = \mathfrak{c} \circ (-)^{-1}$ where $\mathfrak{S}(1) \xrightarrow{\mathfrak{c}} R[Z_k]$ is the parameter map.
2. If ζ is an anti-involution then ζ_M is an anti-involution.

Proof. Let $\tilde{w} = \tilde{s}_1 \ldots \tilde{s}_{\ell(w)}\tilde{u}$ be a reduced decomposition, $\tilde{s}_i \in S^{aff}(1), \tilde{u} \in W(1), \ell(\tilde{u}) = 0$ and let $\tilde{s} \in S^{aff}(1)$. Then,

$$\zeta(T_{\tilde{w}}) = T_{(\tilde{w})^{-1}} = T_{(\tilde{u})^{-1}} T_{\tilde{s}^{-1}_{\ell(w)}} \ldots T_{\tilde{s}_1^{-1}} = \zeta(T_{\tilde{u}})\zeta(T_{\tilde{s}_{\ell(w)}}) \ldots \zeta(T_{\tilde{s}_1}),$$
$$(\zeta(T_{\tilde{s}}))^2 = T^2_{\tilde{s}^{-1}} = \mathfrak{q}(s)\tilde{s}^{-2} + \mathfrak{c}(\tilde{s}^{-1})T_{\tilde{s}^{-1}}$$

Tthe map ζ is an anti-automorphism if and only if $\zeta(\mathfrak{c}(\tilde{s})) = \mathfrak{c}(\tilde{s}^{-1})$ for $\tilde{s} \in S^{aff}(1)$. This is equivalent to $\zeta \circ \mathfrak{c} = \mathfrak{c} \circ (-)^{-1}$ because $\mathfrak{S}(1)$ is the union of the $W(1)$-conjugates of $S^{aff}(1)$, $\mathfrak{c}$ is $W(1)$-equivariant and ζ commutes with the conjugation by $W(1)$.

If $\mathfrak{c}$ satisfies (iii), its restriction $\mathfrak{c}_M$ to $\mathfrak{S}_M(1)$ satisfies (iii). □

Lemma 2.11. *When $\mathcal{H} = \mathcal{H}(G)$ is the pro-p Iwahori Hecke R-algebra of a reductive p-adic group G, ζ is an anti-involution.*

Proof. Let $s \in \mathfrak{S}$, $\tilde{s}$ an admissible lift and $t \in Z_k$. Then $\mathfrak{c}(\tilde{s})$ is invariant by ζ [Vig1, Prop.4.4] If $u \in U^*_\gamma$ for $\gamma = \alpha + r \in \Phi^{aff}_{\mathrm{red}}$, then $u^{-1} \in U^*_\gamma$ and $m_\alpha(u)^{-1} = m_\alpha(u^{-1})$. Hence the set of admissible lifts of s is stable by the inverse map. As the group Z_k is commutative, we have

$$(\zeta \circ c)(t\tilde{s}) = \zeta(tc(s)) = t^{-1}c(s) = c(s)t^{-1} = c((t\tilde{s})^{-1}).$$

□

From now on, we suppose that ζ is an anti-involution. We recall the involutive automorphism [Vig1, Prop. 4.24]

$$\mathcal{H} \xrightarrow{\iota} \mathcal{H} \quad \text{such that} \quad \iota(T_{\tilde{w}}) = (-1)^{\ell(w)} T^*_{\tilde{w}} \quad \text{for} \quad \tilde{w} \in W(1),$$

and [Vig1, Prop. 4.13 2)]:

$$(4) \qquad T^*_{\tilde{s}} := T_{\tilde{s}} - \mathfrak{c}(\tilde{s}) \ \text{ for } \ \tilde{s} \in S^{aff}(1), \quad T^*_{\tilde{w}} := T^*_{\tilde{s}_1} \ldots T^*_{\tilde{s}_r} T_{\tilde{u}} \ \text{ for } \ \tilde{w} \in W(1)$$

of reduced decomposition $\tilde{w} = \tilde{s}_1 \ldots \tilde{s}_{\ell(w)} \tilde{u}$.

Remark 2.12. *We have $\zeta(T^*_{\tilde{w}}) = T^*_{(\tilde{w})^{-1}}$ for $\tilde{w} \in W(1)$, ζ and ι commute, $\zeta_M(\mathcal{H}_{M^\epsilon}) = \mathcal{H}^{-\epsilon}_M$, and $\theta \circ \zeta_M = \zeta \circ \theta$, $\theta^* \circ \zeta_M = \zeta \circ \theta^*$.*

2.3 ϵ-alcove walk basis

We define a basis of $\mathcal{H}$ associated to $\epsilon \in \{+,-\}$ and an orientation o of $(V, \mathfrak{H})$, that we call an ϵ-alcove walk basis associated to o.

For $s \in S^{aff}$, let α_s denote the positive affine root such that s is the orthogonal reflection with respect to $\operatorname{Ker} \alpha_s$. For an orientation o of $(V, \mathfrak{H})$, let $\mathcal{D}_o$ denote the corresponding (open) Weyl chamber in $(V, \mathfrak{H})$, $\mathfrak{A}_o$ the (open) alcove of vertex 0 contained in $\mathcal{D}_o$, and $o.w$ the orientation of Weyl chamber $w^{-1}(\mathfrak{D}_o)$ for $w \in W$. We recall [Vig1]:

Definition 2.13. *The following three properties determine uniquely elements $E_o(\tilde{w}) \in \mathcal{H}$ for any orientation o of $(V, \mathfrak{H})$ and $\tilde{w} \in W(1)$. For $\tilde{w} \in W(1), \tilde{s} \in S^{aff}(1), \tilde{u} \in \Omega(1)$:*

$$(5) \qquad E_o(\tilde{s}) = \begin{cases} T_{\tilde{s}} & \text{if } \alpha_s \text{ is negative on } \mathfrak{A}_o, \\ T^*_{\tilde{s}} = T_{\tilde{s}} - \mathfrak{c}(\tilde{s}) & \text{if } \alpha_s \text{ is positive on } \mathfrak{A}_o, \end{cases}$$

$$(6) \qquad E_o(\tilde{u}) = T_{\tilde{u}},$$

$$(7) \qquad E_o(\tilde{s}) E_{o.s}(\tilde{w}) = q_{s,w} E_o(\tilde{s}\tilde{w}).$$

They imply, for $w' \in W, \lambda \in \Lambda$:

$$(8) \qquad E_o(\tilde{w}') E_{o.w'}(\tilde{w}) = q_{w',w} E_o(\tilde{w}'\tilde{w}), \quad E_o(\tilde{\lambda}) E_o(\tilde{w}) = q_{\lambda,w} E_o(\tilde{\lambda}\tilde{w}).$$

We recall that λ acts on V by translation by $\nu(\lambda)$. The Weyl chamber $\mathcal{D}_o$ of the orientation o is characterized by:

$$(9) \qquad E_o(\tilde{\lambda}) = T_{\tilde{\lambda}} \text{ when } \nu(\lambda) \text{ belongs to the closure of } \mathcal{D}_o.$$

The alcove walk basis of $\mathcal{H}$ associated to o is $(E_o(\tilde{w}))_{\tilde{w} \in W(1)}$ [Vig1]. The Bernstein basis $(E(\tilde{w}))_{\tilde{w} \in W(1)}$ is the alcove walk basis associated to the antidominant orientation of Weyl chamber $\mathcal{D}^-$ Remark 2.3. By (5) and (9), the Bernstein basis satisfies

$$E(\tilde{w}) = T_{\tilde{w}} \quad \text{for } \ w \in \Lambda^+ \cup W_0, \quad E(\tilde{w}) = T^*_{\tilde{w}} \quad \text{for } \ w \in \Lambda^-.$$

The alcove walk basis $(E_{o^+}(\tilde{w}))_{\tilde{w} \in W(1)}$associated to the dominant orientation of Weyl chamber $\mathcal{D}^+$ satisfies similar relations with $T^*_{\tilde{w}}$ permuted with $T_{\tilde{w}}$:

$$E_{o^+}(\tilde{w}) = T^*_{\tilde{w}} \quad \text{for } \ w \in \Lambda^+ \cup W_0, \quad E_{o^+}(\tilde{w}) = T_{\tilde{w}} \quad \text{for } \ w \in \Lambda^-.$$

Definition 2.14. *The ϵ-alcove walk basis $(E^\epsilon_o(\tilde{w}))_{\tilde{w} \in W(1)}$ of $\mathcal{H}$ associated to o is*

$$(10) \qquad E^\epsilon_o(\tilde{w}) := \begin{cases} E_o(\tilde{w}) & \text{if } \epsilon = +, \\ \zeta(E_o(\tilde{w}^{-1})) & \text{if } \epsilon = -. \end{cases}$$

Lemma 2.15. *The elements $E_o^-(\tilde{w})$ for any orientation o of $(V,\mathcal{H})$ and $\tilde{w} \in W(1)$ are determined by the following properties. For $\tilde{w} \in W(1), \tilde{s} \in S^{aff}(1), \tilde{u} \in \Omega(1)$:*

$$E_o^-(\tilde{s}) = E_o(\tilde{s}), \quad E_o^-(\tilde{u}) = E_o(\tilde{u}), \tag{11}$$

$$E_{o.s}^-(\tilde{w})E_o^-(\tilde{s}) = q_{w,s}E_o^-(\tilde{w}\tilde{s}). \tag{12}$$

They imply for $w' \in W, \lambda \in \Lambda$:

$$E_{o.w'^{-1}}^-(\tilde{w})E_o^-(\tilde{w}') = q_{w,w'}E_o^-(\tilde{w}\tilde{w}'), \quad E_o^-(\tilde{w})E_o^-(\tilde{\lambda}) = q_{w,\lambda}E_o^-(\tilde{w}\tilde{\lambda}). \tag{13}$$

Proof.

$$\begin{aligned}
E_o^-(\tilde{s}) &= \zeta(E_o((\tilde{s})^{-1})) = E_o(\tilde{s}),\\
E_o^-(\tilde{w}\tilde{u}) &= \zeta(E_o((\tilde{w}\tilde{u})^{-1})) = \zeta(E_o((\tilde{u})^{-1}(\tilde{w})^{-1})) = \zeta(T_{(\tilde{u})^{-1}}E_o((\tilde{w})^{-1}))\\
&= \zeta(E_o((\tilde{w})^{-1}))T_{\tilde{u}} = E_o^-(\tilde{w})T_{\tilde{u}},\\
E_{o.s}^-(\tilde{w})E_o^-(\tilde{s}) &= \zeta(E_{o.s}((\tilde{w})^{-1}))\zeta(E_o((\tilde{s})^{-1})) = \zeta(E_o((\tilde{s})^{-1})E_{o.s}((\tilde{w})^{-1}))\\
&= q_{s,w^{-1}}\zeta(E_o((\tilde{s})^{-1}(\tilde{w})^{-1})) = q_{w,s}\zeta(E_o((\tilde{w}\tilde{s})^{-1})) = q_{w,s}E_o^-(\tilde{w}\tilde{s}).
\end{aligned}$$

We used that $q_w = q_{w^{-1}}$ implies $q_{w_1^{-1},w_2^{-1}} = (q_{w_1^{-1}}q_{w_2^{-1}}q^{-1}_{w_1^{-1}w_2^{-1}})^{1/2} = (q_{w_1}q_{w_2}q^{-1}_{w_2w_1})^{1/2} = q_{w_2,w_1}$ for $w_1, w_2 \in W$. ☐

The ϵ-alcove walk bases satisfy the the triangular decomposition:

$$E_o^\epsilon(\tilde{w}) - T_{\tilde{w}} \in \sum_{\tilde{w}' \in W(1), \tilde{w}' < \tilde{w}} RT_{\tilde{w}'}. \tag{14}$$

Remark 2.16. We will denote $E_+(\tilde{w}) = E_{o^+}(\tilde{w})$ and $E_-(\tilde{w}) = E_{o^+}^-(\tilde{w})$ as in [Abe] and call $(E_\epsilon(\tilde{w}))_{\tilde{w}\in W(1)}$ the lower ϵ-Bernstein basis of $\mathcal{H}$ (the upper ϵ-Bernstein basis will be the usual Bernstein basis).

Similarly, we will denote by $(E_M^\epsilon(\tilde{w}))_{\tilde{w}\in W_M(1)}$ and $(E_\epsilon^M(\tilde{w}))_{\tilde{w}\in W_M(1)}$ the upper and lower ϵ-Bernstein bases associated to the dominant orientation for $(V_M, \mathfrak{H}_M)$; here V_M ise the real vector space of dual generated by Σ_M with a $W_{M,0}$-invariant scalar product and $\mathfrak{H}_M$ the corresponding set of affine hyperplanes.

Lemma 2.17. *For $\epsilon, \epsilon' \in \{+,-\}$ and any orientation o_M of $(V_M, \mathfrak{H}_M)$, $(E_{o_M}^{\epsilon'}(\tilde{w}))_{\tilde{w}\in W_{M^\epsilon}(1)}$ is a basis of $\mathcal{H}_{M^\epsilon}$.*

When $\mathfrak{q}(s) = 0$ [Abe, Lemma 4.2].

Proof. A basis of $\mathcal{H}_{M^\epsilon}$ is $(T_{\tilde{w}}^M)_{\tilde{w}\in W_{M^\epsilon}(1)}$. As $w <_M w'$ and $w' \in W_{M^\epsilon}$ implies $w \in W_{M^\epsilon}$ (Lemma 2.6), the triangular decomposition (14) implies that $(E_{o_M}^{\epsilon'}(\tilde{w}))_{\tilde{w}\in W_{M^\epsilon}(1)}$ is a basis of $\mathcal{H}_{M^\epsilon}$. ☐

Lemma 2.18. *The ϵ-Bernstein basis satisfies $E^\epsilon(\tilde{w}) = T_{\tilde{w}}$ if $w \in \Lambda^\epsilon \cup W_0$ and $E^\epsilon(\tilde{w}) = T_{\tilde{w}}^*$ if $w \in \Lambda^{-\epsilon}$. The basis $(E_\epsilon(\tilde{w}))$ satisfies similar relations with $T_{\tilde{w}}^*$ permuted with $T_{\tilde{w}}$: $E_\epsilon(\tilde{w}) = T_{\tilde{w}}^*$ if $w \in \Lambda^\epsilon \cup W_0$ and $E_-(\tilde{w}) = T_{\tilde{w}}$ if $w \in \Lambda^{-\epsilon}$.*

Proof. We described $E^+(\tilde{w})$ and $E_+(\tilde{w})$ for $w \in \Lambda^+ \cup \Lambda^- \cup W_0$ before Definition 2.14 and we have:

$$E^-(\tilde{w}) = \zeta(E(\tilde{w}^{-1})) = \begin{cases} \zeta(T_{\tilde{w}^{-1}}^*) = T_{\tilde{w}}^* & (w \in \Lambda^+) \\ \zeta(T_{\tilde{w}^{-1}}) = T_{\tilde{w}} & (w \in \Lambda^- \cup W_0) \end{cases}$$

$$E_-(\tilde{w}) = \zeta(E_{o^+}(\tilde{w}^{-1})) = \begin{cases} \zeta(T_{\tilde{w}^{-1}}^*) = T_{\tilde{w}}^* & (w \in \Lambda^- \cup W_0) \\ \zeta(T_{\tilde{w}^{-1}}) = T_{\tilde{w}} & (w \in \Lambda^+). \end{cases}$$

☐

The upper and lower ϵ-Bernstein bases are compatible with the linear embeddings θ and θ^* of $\mathcal{H}_M$ into $\mathcal{H}$:

Proposition 2.19. *We have* $\theta(E^\epsilon_M(\tilde{w})) = E^\epsilon(\tilde{w})$, $\theta^*(E^M_\epsilon(\tilde{w})) = E_\epsilon(\tilde{w})$ *for* $\tilde{w} \in W_{M^+}(1) \cup W_{M^-}(1)$.

This generalizes [Ollivier10, Prop. 4.7], [Ollivier14, Lemma 3.8], [Abe, Lemma 4.5].

Proof. It suffices to prove the proposition when the $\mathfrak{q}(s)$ are invertible. Let $\tilde{w} \in W(1)$. We write $\tilde{w} = \tilde{\lambda}\tilde{u} = \tilde{\lambda}_1(\tilde{\lambda}_2)^{-1}\tilde{u}$ with $u \in W_0$, and λ_1, λ_2 in Λ^ϵ. We have for any orientation o of $(V, \mathfrak{h})$

$$E_o(\tilde{\lambda}_1)E_o((\tilde{\lambda}_2)^{-1}) = q_{\lambda_1,\lambda_2^{-1}}E_o(\tilde{\lambda}), \quad E_o(\tilde{\lambda}_2)E_o((\tilde{\lambda}_2)^{-1}) = q_{\lambda_2,\lambda_2^{-1}} = q_{\lambda_2},$$
$$E_o(\tilde{\lambda}_1)E((\tilde{\lambda}_2)^{-1})E_o(\tilde{u}) = q_{\lambda_1,\lambda_2^{-1}}q_{\lambda,u}E_o(\tilde{w}).$$

Then, $E_o(\tilde{w}) = q_{\lambda_2}(q_{\lambda_1,\lambda_2^{-1}}q_{\lambda,u})^{-1}E_o(\tilde{\lambda}_1)E_o(\tilde{\lambda}_2)^{-1}E_o(\tilde{u})$. Applying Lemma 2.18 to the orientations o of Weyl chamber $\mathcal{D}^\pm$ we obtain:

$$E(\tilde{w}) = q_{\lambda_2}(q_{\lambda_1,\lambda_2^{-1}}q_{\lambda,u})^{-1}\begin{cases} T_{\tilde{\lambda}_1}T^{-1}_{\tilde{\lambda}_2}T_{\tilde{u}} & \text{if } \epsilon = + \\ T^*_{\tilde{\lambda}_1}(T^*_{\tilde{\lambda}_2})^{-1}T_{\tilde{u}} & \text{if } \epsilon = - \end{cases} \tag{15}$$

and similar formulas for $E_+(\tilde{w})$ with $T^*_{\tilde{w}}$ permuted with $T_{\tilde{w}}$. We suppose now $w \in W_{M^\epsilon}$, that is $\lambda \in \Lambda_{M^\epsilon}, u \in W_{M,0}$. Note $\Lambda^\epsilon \subset \Lambda_{M^\epsilon}$ and $q_{M,\lambda,u} = q_{\lambda,u}$ (Lemma 2.7).

Suppose $w \in W_{M^+}$. Then $E_M(\tilde{w}) = q_{M,\lambda_2}(q_{M,\lambda_1,\lambda_2^{-1}}q_{\lambda,u})^{-1}T^M_{\tilde{\lambda}_1}(T^M_{\tilde{\lambda}_2})^{-1}T^M_{\tilde{u}}$ and

$$\theta(E_M(\tilde{w})) = q_{M,\lambda_2}(q_{M,\lambda_1,\lambda_2^{-1}}q_{\lambda,u})^{-1}T_{\tilde{\lambda}_1}T^{-1}_{\tilde{\lambda}_2}T_{\tilde{u}}$$
$$= q_{M,\lambda_2}(q_{M,\lambda_1,\lambda_2^{-1}}q_{\lambda,u})^{-1}q^{-1}_{\lambda_2}q_{\lambda_1,\lambda_2^{-1}}q_{\lambda,u}E(\tilde{w}) = q_{M,\lambda_2}(q_{M,\lambda_1,\lambda_2^{-1}}q_{\lambda_2})^{-1}q_{\lambda_1,\lambda_2^{-1}}E(\tilde{w}).$$

The triangular decomposition of $E_M(\tilde{w})$ and $E(\tilde{w})$ implies $q_{M,\lambda_2}(q_{M,\lambda_1,\lambda_2^{-1}}q_{\lambda_2})^{-1}q_{\lambda_1,\lambda_2^{-1}} = 1$. Hence for $w \in W_{M^+}$ we have $\theta(E_M(\tilde{w})) = E(\tilde{w})$, and by the same arguments $\theta^*(E^M_+(\tilde{w})) = E_+(\tilde{w})$.

Suppose $w \in W_{M^-}$. We write $\tilde{w} = \tilde{\lambda}\tilde{w}_0$ with $\tilde{\lambda} \in \Lambda(1)$ M_1-negative and $s \in \tilde{w}_0 \in W_{M_1,0}$. We have $E(\tilde{w}) = q_{\lambda,w_0}T^*_{\tilde{\lambda}}T_{\tilde{w}_0}$ and $E_M(\tilde{w}) = q^M_{\lambda,w_0}T^{M,*}_{\tilde{\lambda}}T_{\tilde{w}_0}$ with $q_{\lambda,w_0} = q^M_{\lambda,w_0}$ (Lemma 2.7). Applying the homomorphism $\mathcal{H}_{M_1^-} \xrightarrow{\theta} \mathcal{H}$ we obtain $\theta(E_M(\tilde{w})) = E(\tilde{w})$. The same arguments show that $\theta^*(E^M_+(\tilde{w})) = E_+(\tilde{w})$.

Suppose $w \in W_{M^+} \cup W_{M^-}$. We proved that $\theta(E_M(\tilde{w})) = E(\tilde{w})$ and $\theta^*(E^M_+(\tilde{w})) = E_+(\tilde{w})$, i.e. that $E_o(\tilde{w})$ is the image of $E^M_o(\tilde{w})$ by θ and θ^* when o is the orientation of Weyl chamber dominant or anti-dominant. Using $E^-_o(\tilde{w}) = \zeta(E_o((\tilde{w})^{-1}))$ and that $\zeta \circ \theta = \theta \circ \zeta_M, \zeta \circ \theta^* = \theta^* \circ \zeta_M$ (Remark 2.12), this implies that $E^-_o(\tilde{w})$ is the image of $E^-_{M,o}(\tilde{w})$ by θ and θ^*, as $E^-_o(\tilde{w}) = (\zeta \circ \theta)(E_{M,o}((\tilde{w})^{-1})) = (\theta \circ \zeta_M)(E_{M,o}((\tilde{w})^{-1})) = \theta(E^-_{M,o}(\tilde{w}))$. ☐

2.4 w_0-twist

Let $S_M \subset S$, w_0 denote the longest element of W_0 and $S_{w_0(M)} = w_0 S_M w_0 \subset w_0 S w_0 = S$. The longest element $w_{M,0}$ of $W_{M,0}$ satisfies $w_{M,0}(\Sigma^\epsilon_M) = \Sigma^{-\epsilon}_M$, and $w_{M,0}(\Sigma^\epsilon - \Sigma^\epsilon_M) = \Sigma^\epsilon - \Sigma^\epsilon_M$. The longest element $w_{w_0(M),0}$ of $W_{w_0(M),0}$ is $w_0 w_{M,0} w_0$.

Let $w^M_0 := w_0 w_{M,0}$. Its inverse ${}^M w_0 := w_{M,0} w_0$ is $w_0^{w_0(M)}$ and $w^M_0(\Sigma^\epsilon_M) = \Sigma^\epsilon_{w_0(M)}$. This implies that $w^M_0(\Sigma^{aff,\epsilon}_M) = \Sigma^{aff,\epsilon}_{w_0(M)}$. Indeed the image by w^M_0 of the simple roots of Σ_M is the set of simple roots of $\Sigma_{w_0(M)}$, and this remains true for the simple affine roots which are not roots. Note that the irreducible components $\Sigma_{M,i}$ of Σ_M have a unique highest root $a_{M,i}$, and that the $-a_{M,i} + 1$ are the simple affine roots of Σ which are not roots. We have $w_0^M(-a_{M,i}+1) = w_0 w_{M,0}(-a_{M,i}+1) = w_0(a_{M,i})+1$. The irreducible components of $\Sigma_{w_0(M)}$ are the $w_0(\Sigma_{M,i})$ and $-w_0(a_{M,i})$ is the highest root of $w_0(\Sigma_{M,i})$.

We deduce:

$$w_0^M S_M^{aff}(w_0^M)^{-1} = S_{w_0(M)}^{aff},\ w_0^M W_{M,0}^{aff}(w_0^M)^{-1} = W_{w_0(M,)0}^{aff},\ w_0^M W_{M,0}(w_0^M)^{-1} = W_{w_0(M,)0}.$$

We have $\Lambda = w_0^M \Lambda (w_0^M)^{-1}$ and $w_0^M \Lambda_M^{\epsilon}(w_0^M)^{-1} = \Lambda_{w_0(M)}^{-\epsilon}$. Recalling $W_M = \Lambda \rtimes W_{M,0}$, $W_{M^\epsilon} = \Lambda_{M^\epsilon} \rtimes W_{M,0}$ and the group Ω_M of elements which stabilize $\mathfrak{A}_M$, we deduce:

(16)
$$w_0^M W_M (w_0^M)^{-1} = W_{w_0(M)},\ w_0^M \Omega_M (w_0^M)^{-1} = \Omega_{w_0(M)},\ w_0^M W_{M^\epsilon}(w_0^M)^{-1} = W_{w_0(M)}^{-\epsilon}.$$

Let ν_M denote the action of W_M on V_M and $\mathfrak{A}_M$ the dominant alcove of $(V_M, \mathfrak{H}_M)$. The linear isomorphism

$$V_M \xrightarrow{w_0^M} V_{w_0(M)}, \quad \langle \alpha, x\rangle = \langle w_0^M(\alpha), w_0^M(x)\rangle \ \text{ for } \ \alpha \in \Sigma_M,$$

satisfies

$$w_0^M \circ \nu_M(w) = \nu_{w_0(M)}(w_0^M w (w_0^M)^{-1}) \circ w_0^M \quad \text{for } w \in W_M.$$

It induces a bijection $\mathfrak{H}_M \to \mathfrak{H}_{w_0(M)}$ sending $\mathfrak{A}_M$ to $\mathfrak{A}_{w_0(M)}$, a bijection $\mathfrak{D}_M \mapsto w_0^M(\mathfrak{D}_M)$ between the Weyl chambers, a bijection $o_M \mapsto w_0^M(o_M)$ between the orientations such that $\mathfrak{D}_{w_0^M(o_M)} = w_0^M(\mathfrak{D}_{o_M})$.

Proposition 2.20. *Let $\tilde{w}_0^M \in W_0(1)$ be a lift of w_0^M. The R-linear map*

$$\mathcal{H}_M \xrightarrow{j} \mathcal{H}_{w_0(M)}, \quad T_{\tilde{w}}^M \mapsto T^{w_0(M)}_{\tilde{w}_0^M \tilde{w} (\tilde{w}_0^M)^{-1}} \ \textit{for} \ \ \tilde{w} \in W_M(1),$$

is a R-algebra isomorphism sending $\mathcal{H}_{M^\epsilon}$ onto $\mathcal{H}_{w_0(M)^{-\epsilon}}$ and respecting the ϵ'-alcove walk basis

$$j(E_{o_M}^{\epsilon'}(\tilde{w})) = E^{\epsilon'}_{w_0^M(o_M)}(\tilde{w}_0^M \tilde{w} (\tilde{w}_0^M)^{-1}) \ \textit{for} \ \ \tilde{w} \in W_M(1),$$

for any orientation o_M of $(V_M, \mathfrak{H}_M)$ and $\epsilon, \epsilon' \in \{+,-\}$.

Proof. The proof is formal using the properties given above the proposition and the characterization of the elements in the ϵ'-alcove walks bases given by (5), (6), (7) if $\epsilon' = +$ and (11), (12) if $\epsilon' = -$. □

We study now the transitivity of the w_0-twist. Let $S_M \subset S_{M'} \subset S$. We have the subset $w_{M',0} S_M w_{M',0} = S_{w_{M',0}(M)}$ of S and we associate to the conjugation by a lift $\tilde{w}_{M',0}$ of $w_{M',0}$ in $W(1)$ an isomorphism $\mathcal{H}_M \xrightarrow{j'} \mathcal{H}_{w_{M',0}(M)}$ similar to $\mathcal{H}_M \xrightarrow{j} \mathcal{H}_{w_0(M)}$ in Proposition 2.20. We will show that j factorizes by j'.

We have $w_0^M = w_0^{M'} w_{M'}^M$, where $w_{M'}^M := w_{M',0} w_{M,0}$ (equal to w_0^M if $S = S_{M'}$),

$$W_{w_{M',0}(M)} = w_{M'}^M W_M (w_{M'}^M)^{-1}, \quad W_{w_0(M)} = w_0^{M'} W_{w_{M',0}(M)} (w_0^{M'})^{-1} = w_0^M W_M (w_0^M)^{-1}.$$

For $S_{M_1} \subset S_{M'}$, let $W_{M_1^{\epsilon, M'}} \subset W_{M_1}$ denote the submonoid associated to $S_{M'}^{aff}$ as in Definition 2.1 (the pair $(\Sigma^+ - \Sigma^+_{M_1}, \Sigma^{aff,+})$ is replaced by the pair $(\Sigma^+_{M'} - \Sigma^+_{M_1}, \Sigma^{aff,+}_{M'})$). We note that:

$$\begin{aligned} &W_{w_{M',0}(M)^{-\epsilon, M'}} = w_{M'}^M W_{M^\epsilon} (w_{M'}^M)^{-1}, \\ &W_{w_0(M)^{-\epsilon}} = w_0^{M'} W_{w_{M',0}(M)^{-\epsilon, M'}} (w_0^{M'})^{-1} = w_0^M W_{M^\epsilon} (w_0^M)^{-1}. \end{aligned}$$

Let $\tilde{w}_0^M, \tilde{w}_0^{M'}, \tilde{w}_{M'}^M$ in $W_0(1)$ lifting $w_0^M, w_0^{M'}, w_{M'}^M$ and satisfying $\tilde{w}_0^M = \tilde{w}_0^{M'} \tilde{w}_{M'}^M$. The algebra isomorphisms

$$\mathcal{H}_M \xrightarrow{j'} \mathcal{H}_{w_{M',0}(M)}, \quad \mathcal{H}_{M'} \xrightarrow{j''} \mathcal{H}_{w_0(M')}, \quad \mathcal{H}_M \xrightarrow{j} \mathcal{H}_{w_0(M)}$$

defined by $\tilde{w}_{M'}^M, \tilde{w}_0^{M'}, \tilde{w}_0^M$ respectively, as in Proposition 2.20, send the ϵ-subalgebra to the $-\epsilon$-subalgebra and are compatible with the ϵ'-Bernstein bases. We cannot compose j' with the map j'' defined by $\tilde{w}_0^{M'}$, but we can compose j' with the bijective R-linear map defined by the conjugation by $\tilde{w}_0^{M'}$ in $W(1)$:

$$\mathcal{H}_{w_{M',0}(M)} \xrightarrow{k''} \mathcal{H}_{w_0(M)}, \quad T_{\tilde{w}}^{w_{M',0}(M)} \mapsto T^{w_0(M)}_{\tilde{w}_0^{M'} \tilde{w} (\tilde{w}_0^{M'})^{-1}} \quad \text{for } \tilde{w} \in W_{w_{M',0}(M)}(1).$$

Proposition 2.21. *$j = k'' \circ j'$ and k'' is an R-algebra isomorphism respecting the ϵ-subalgebras and the ϵ-Bernstein bases: $k''(\mathcal{H}_{w_{M',0}(M)^\epsilon}) = \mathcal{H}_{w_0(M)^\epsilon}$ and $k''(E^\epsilon_{w_{M',0}(M)}(\tilde{w})) = E^\epsilon_{w_0(M)}(\tilde{w}_0^{M'} \tilde{w} (\tilde{w}_0^{M'})^{-1})$ for $\epsilon \in \{+,-\}$, $w \in W_{w_{M',0}(M)}$.*

Proof. The relations between the groups W_* and W_{*^ϵ} imply obviously that $j = k'' \circ j'$ and that k'' respects the ϵ-subalgebras.

k'' is an algebra isomorphism respecting the ϵ'-Bernstein bases because j, j' are algebra isomorphisms respecting the ϵ'-Bernstein bases and $k'' = j \circ (j')^{-1}$. □

2.5 Distinguished representatives of W_0 modulo $W_{M,0}$

The classical set ${}^M W_0$ of representatives on $W_{M,0} \backslash W_0$ is equal to ${}_M D_1 = {}_M D_2$ where [Carter, 2.3.3]

$$(17) \qquad {}_M D_1 := \{d \in W_0 \mid d^{-1}(\Sigma_M^+) \in \Sigma^+\},$$

$$(18) \qquad {}_M D_2 := \{d \in W_0 \mid \ell(wd) = \ell(w) + \ell(d) \text{ for all } w \in W_{M,0}\}.$$

The properties of ${}^M W_0$ used in this article that we are going to prove are probably well known. Note that the classical set of representatives of $W_0 \backslash W$ is studied in [Vig3], that $+$ can be replaced by $\epsilon \in \{+,-\}$ in the definition of ${}_M D_1$, that ${}^M w_0 = w_{M,0} w_0 \in {}^M W_0$ and that ${}^M W_0 \cap S = S - S_M$.

Taking inverses, we get the classical set W_0^M of representatives on $W_0/W_{M,0}$ equal to $D_{M,1} = D_{M,2}$, where

$$(19) \qquad D_{M,1} := \{d \in W_0 \mid d(\Sigma_M^+) \subset \Sigma^+\},$$

$$(20) \qquad D_{M,2} := \{d \in W_0 \mid \ell(dw) = \ell(d) + \ell(w) \text{ for all } w \in W_{M,0}\}.$$

The length of an element of W is equal to the length of its inverse, and [Vig1, Cor. 5.10]: for $\lambda \in \Lambda, w \in W_0$,

$$(21) \qquad \ell(\lambda w) = \sum_{\beta \in \Sigma^+ \cap w(\Sigma^+)} |\beta \circ \nu(\lambda)| + \sum_{\beta \in \Phi_w} |-\beta \circ \nu(\lambda) + 1|.$$

where $\Phi_w := \Sigma^+ \cap w(\Sigma^-)$. If $w = s_1 \ldots s_{\ell(w)}$ is a reduced decomposition in (W_0, S), $\Phi_w = \{\alpha_{s_1}\} \cup s_1(\Phi_{s_1 w})$ and $\ell(w)$ is the order of Φ_w. If $w \in W_{M,0}$, $\Phi_w \subset \Sigma_M^+$. Let $\ell_\beta(\lambda w)$ denote the contribution of $\beta \in \Sigma^+$ to the right side of (21).

We show now that $W_{M,0}$ can be replaced by W_{M^+} in (18) and by W_{M^-} in (20) (taking the inverses). It is also a variant of the equivalence $\ell(\lambda w) < \ell(\lambda) + \ell(w) \Leftrightarrow \beta \circ \nu(\lambda) > 0$ for some $\beta \in \Phi_w$ for λ, w as in (21).

Lemma 2.22. (i) *$\ell(wd) = \ell(w) + \ell(d)$ for $w \in W_{M^+}$ and $d \in {}^M W_0$.*

$\ell(dw) = \ell(d) + \ell(w)$ for $w \in W_{M^-}$ and $d \in W_0^M$.

(ii) *For* $\lambda \in \Lambda, w \in W_{M,0}, d \in {}^M W_0$*, then* $\ell(\lambda w d) < \ell(\lambda w) + \ell(d)$ *is equivalent to*

$$w(\beta) \circ \nu(\lambda) > 0 \quad \text{and } d^{-1}(\beta) \in \Sigma^- \text{ for some } \beta \in \Sigma^+ - \Sigma_M^+.$$

Proof. [Ollivier10, Lemma 2.3], [Abe, Lemma 4.8].

Let $\lambda \in \Lambda, w \in W_{M,0}, d \in {}^M W_0$ and $\beta \in \Sigma^+$.

Suppose $\beta \in \Sigma_M^+$. Then $\ell_\beta(d) = 0, \Phi_d = \emptyset$ because $d^{-1}(\Sigma_M^\epsilon) \subset \Sigma^\epsilon$ (17), and $\ell_\beta(\lambda w d) = \ell_\beta(\lambda w)$ because $w^{-1}(\beta) \in \Sigma^\epsilon \Leftrightarrow w^{-1}(\beta) \in \Sigma_M^\epsilon \Rightarrow d^{-1}w^{-1}(\beta) \in \Sigma^\epsilon$ (17).

Suppose $\beta \in \Sigma^+ - \Sigma_M^+$. Then $w^{-1}(\beta) \in \Sigma^+ - \Sigma_M^+$ and $\ell_\beta(\lambda w) = |\beta \circ \nu(\lambda)|$.

The number $\ell(d)$ of $\beta \in \Sigma^+ - \Sigma_M^+$ such that $d^{-1}(\beta) \in \Sigma^-$ is equal to the number of $\beta \in \Sigma^+ - \Sigma_M^+$ such that $(wd)^{-1}(\beta) \in \Sigma^-$.

When $\lambda \in \Lambda_{M^+}$ and $(wd)^{-1}(\beta) \in \Sigma^-$, then $\beta \circ \nu(\lambda) \leq 0$ and $\ell_\beta(\lambda w d) = |\beta \circ \nu(\lambda)| + 1$. Therefore $\ell(\lambda w d) = \ell(\lambda w) + \ell(d)$, which gives (i).

When $\lambda \notin \Lambda - \Lambda_{M^+}$, $\ell(\lambda w d) < \ell(\lambda w) + \ell(d)$ if and only if there exists $\beta \in \Sigma^+ - \Sigma_M^+$ such that $\beta \circ \nu(\lambda) > 0$ and $d^{-1}w^{-1}(\beta) \in \Sigma^-$. This gives (ii) because $\beta \mapsto w^{-1}(\beta)$ is a permutation map of $\Sigma^+ - \Sigma_M^+$. □

Lemma 2.23. (i) *For* $\lambda \in \Lambda, w \in W_0$*, we have* $q_\lambda = q_{w\lambda w^{-1}}$, $q_w = q_{w_0 w w_0}$*, and* $\ell(w_0) = \ell(w) + \ell(w^{-1}w_0) = \ell(w_0 w^{-1}) + \ell(w)$.

(ii) *For* $w \in W_{M,0}$*, we have* $q_w = q_{w_0^M w (w_0^M)^{-1}}$.

Proof. (i) [Vig1, Prop. 5.13]. The length on W_0 is invariant by inverse and by conjugation by w_0 because $w_0 S w_0 = S$ and [Bki, VI §1 Cor. 3].

(ii) $q_w = q_{w_{M,0} w w_{M,0}^{-1}} = q_{w_0^M w (w_0^M)^{-1}}$ for $w \in W_{M,0}$. □

Lemma 2.24. $W_0^M = W_0^{w_0(M)} w_0^M = w_0 W_0^M w_{M,0}$.

Proof. By (19), $d \in W_0^M \Leftrightarrow d(\Sigma_M^+) \subset \Sigma^+ \Leftrightarrow d(w_0^M)^{-1}(\Sigma_{w_0(M)}^+) \subset \Sigma^+ \Leftrightarrow d(w_0^M)^{-1} \in W_0^{w_0(M)}$. This proves the equality $W_0^M = W_0^{w_0(M)} w_0^M$. The equality $W_0^M = w_0 W_0^M w_{M,0}$, follows from $d(w_0^M)^{-1}(\Sigma_{w_0(M)}^+) \subset \Sigma^+ \Leftrightarrow w_0 d w_{M,0} w_0(\Sigma_{w_0(M)}^+) \subset \Sigma^- \Leftrightarrow w_0 d w_{M,0}(\Sigma_M^-) \subset \Sigma^- \Leftrightarrow w_0 d w_{M,0} \in W_0^M$. □

Remark 2.25. $W_M = \Lambda \rtimes W_{M,0}$ but $q_{\lambda w} = q_{w_0^M \lambda w (w_0^M)^{-1}}$ could be false for $\lambda \in \Lambda, w \in W_{M,0}$ such that $\ell(\lambda w) < \ell(\lambda) + \ell(w)$.

Lemma 2.26. $\ell(w_0^M) = \ell(w_0^M d^{-1}) + \ell(d)$ *for any* $d \in W_0^M$.

Proof. For $d \in W_0^M$ we have $\ell(d w_{M,0}) = \ell(d) + \ell(w_{M,0})$ by (20) and $w = w_0^M d^{-1}$ satisfies $w_0 = w d w_{M,0}$ and $\ell(w_0) = \ell(w) + \ell(d w_{M,0})$. We have $w_0^M = w_0 w_{M,0} = wd$ and $\ell(w_0^M) = \ell(w_0) - \ell(w_{M,0}) = \ell(w) + \ell(d)$. □

The Bruhat order $x \leq x'$ in W_0 is defined by the following equivalent two conditions:

(i) There exists a reduced decomposition of x' such that by omitting some terms one obtains a reduced decomposition of x.

(ii) For any reduced decomposition of x', by omitting some terms one obtains a reduced decomposition of x.

A reduced decomposition of $w \in W_0$ followed by a reduced decomposition of $w' \in W_0$ is a reduced decomposition of ww' if and only $\ell(ww') = \ell(w) + \ell(w')$. A reduced decomposition of $d \in W_0^M$ cannot end by a non trivial element $w \in W_{M,0}$.

Lemma 2.27. *For* $w, w' \in W_{M,0}, d, d' \in W_0^M$*, we have* $dw \leq d'w'$ *if and only if there exists a factorisation* $w = w_1 w_2$ *such that* $\ell(w) = \ell(w_1) + \ell(w_2)$, $dw_1 \leq d'$ *and* $w_2 \leq w'$.

Proof. We prove the direction "only if" (the direction "if" is obvious). If $dw \leq d'w'$, a reduced decomposition of dw is obtained by omitting some terms of the product of a reduced decomposition of d' and of a reduced decomposition of w'. We have $dw = d_1 w_2$ with $d_1 \leq d', w_2 \leq w'$ and $\ell(d_1 w_2) = \ell(d_1) + \ell(w_2)$. We have $d_1 = dw_1, w_1 := ww_2^{-1}$. As $w, w_2 \in w_{M,0}$ and $d \in W_0^M$ we have $\ell(dw_1) = \ell(d) + \ell(w_1)$ and $\ell(dw) = \ell(d) + \ell(w)$. Hence $\ell(w_1) + \ell(w_2) = \ell(w)$. □

Lemma 2.28. *Let $d' \in {}^{w_0(M)}W_0, d \in W_0^M$.*

(i) If there exists $u \in W_{M,0}, u' \in W_0^M$ such that $v = w_0^M u \leq w = du'$, then $d = w_0^M$.

(ii) $d'd \in w_0^M W_{M,0}$ if and only if $d'd = w_0^M$.

Proof. (i) As $\ell(w) = \ell(d) + \ell(u')$, we have $u = u_1 u_2$ with $w_0^M u_1 \leq d, u_2 \leq u'$ and $u_1, u_2 \in W_{M,0}$ (Lemma 2.27). We have $\ell(w_0^M u_1) = \ell(w_0^M) + \ell(u_1) = \ell(w_0^M d^{-1}) + \ell(d) + \ell(u_1)$ (Lemma 2.26). Hence $d = w_0^M, u_1 = 1$.

(ii) If there exists $u \in W_{M,0}$ such that $d = d'^{-1} w_0^M u$ we have $d = d'^{-1} w_0^M$ because $d'^{-1} w_0^M \in W_0^M$ (Lemma 2.24). □

2.6 $\mathcal{H}$ as a left $\theta(\mathcal{H}_{M^+})$-module and a right $\theta^*(\mathcal{H}_{M^-})$-module

We prove Theorem 1.4 (iv) on the structure of the left $\theta(\mathcal{H}_{M^+})$-module $\mathcal{H}$ and its variant for the right $\theta^*(\mathcal{H}_{M^-})$-module $\mathcal{H}$. We suppose $S_M \neq S$.

Recalling the properties (i), (ii), (iii) of Theorem 1.4, $\mathcal{H}_M = \mathcal{H}_{M^+}[(T^M_{\tilde{\mu}_M})^{-1}]$ is the localisation of the subalgebra $\mathcal{H}_{M^+}$ at the central element $T^M_{\tilde{\mu}_M}$. The algebra $\mathcal{H}_{M^+}$ embeds in $\mathcal{H}$ by θ. Recalling (17), (18) we choose a lift $\tilde{d} \in W(1)$ for any element d in the classical set of representatives ${}^M W_0$ of $W_{M,0}\backslash W_0$. We define

$$\mathcal{V}_{M^+} = \sum_{d \in {}^M W_0} \theta(\mathcal{H}_{M^+}) T_{\tilde{d}}. \tag{22}$$

Proposition 2.29. (i) *$\mathcal{V}_{M^+}$ is a free left $\theta(\mathcal{H}_{M^+})$-module of basis $(T_{\tilde{d}})_{d \in {}^M W_0}$*

(ii) *For any $h \in \mathcal{H}$, there exists $r \in \mathbb{N}$ such that $T^r_{\tilde{\mu}_M} h \in \mathcal{V}_{M^+}$.*

(iii) *If $\mathfrak{q} = 0$, $T_{\tilde{\mu}_M}$ is a left and right zero divisor in $\mathcal{H}$.*

For $GL(n, F)$, (ii) is proved in [Ollivier10, Prop. 4.7] for $(\mathfrak{q}(s)) = (0)$. When the $\mathfrak{q}(s)$ are invertible, $T_{\tilde{w}}$ is invertible in $\mathcal{H}$ for $\tilde{w} \in W(1)$.

Proof. (i) As ${}^M W_0$ is a set of representatives of $W_{M^+}\backslash W$, a set of representatives of $W_{M^+}(1)\backslash W(1)$ is the set $\{\tilde{d} \mid d \in {}^M W_0\}$ of lifts of ${}^M W_0$ in $W(1)$. The canonical bases of $\mathcal{H}_{M^+}$ and of $\mathcal{H}$ are respectively $(T_{\tilde{w}})_{(\tilde{w}) \in W_{M^+}(1)}$ and $(T_{\tilde{w}\tilde{d}})_{(\tilde{w}, d) \in W_{M^+}(1) \times {}^M W_0}$, and $T_{\tilde{w}\tilde{d}} = T_{\tilde{w}} T_{\tilde{d}}$ by the additivity of lengths (Lemma 2.22).

(ii) We can suppose that h runs over in a basis of $\mathcal{H}$. We cannot take the Iwahori-Matsumoto basis $(T_{\tilde{w}})_{\tilde{w} \in W(1)}$ and we explain why. For $\tilde{w} = \tilde{w}_M \tilde{d}$ with $\tilde{w}_M \in W_{M^+}(1), d \in {}^M W_0$ we choose $r \in \mathbb{N}$ such that $\tilde{\mu}^r_M \tilde{w}_M \in W_{M^+}(1)$. By the length additivity (Lemma 2.22) $T_{\tilde{\mu}^r_M \tilde{w}} = T_{\tilde{\mu}^r_M \tilde{w}_M} T_{\tilde{d}}$ lies in $\theta(\mathcal{H}_{M^+}) T_{\tilde{d}}$, but we cannot deduce that $T_{\tilde{\mu}^r_M} T_{\tilde{w}}$ lies in $\theta(\mathcal{H}_{M^+}) T_{\tilde{d}}$.

We take the Bernstein basis (2.18) and we suppose that $\mathfrak{q}(s) = \mathbf{q}_s$ is indeterminate (but not invertible) with the same arguments as in [Ollivier10, Prop. 4.8]. Then $E(\tilde{d}) = T_{\tilde{d}}$ for $d \in {}^M W_0$. If we prove that $E(\tilde{\mu}^r_M \tilde{w})$ lies in $\theta(\mathcal{H}_{M^+}) T_{\tilde{d}}$ then $E(\tilde{\mu}_M)^r E_o(\tilde{w}) = \mathbf{q}_{\mu^r_M, w} E(\tilde{\mu}^r_M \tilde{w})$ lies also in $\theta(\mathcal{H}_{M^+}) T_{\tilde{d}}$. This implies $T^r_{\tilde{\mu}_M} E_o(\tilde{w}) \in \theta(\mathcal{H}_{M^+}) T_{\tilde{d}}$.

Now we prove $E(\tilde\mu_M^r\tilde w)\in\theta(\mathcal{H}_{M^+})T_{\tilde d}$. We write $\tilde w_M=\tilde\lambda\tilde w_{M,0},\tilde\lambda\in\Lambda(1),\tilde w_{M,0}\in W_{M,0}(1)$. Recalling $E(*)=T_*$ for $*\in W_0(1)$ and the additivity of the length (Lemma 2.22),

$$\begin{aligned}\mathbf{q}_{\mu_M^r\lambda,w_{M,0}d}E(\tilde\mu_M^r\tilde w)&=E(\tilde\mu_M^r\tilde\lambda)E(\tilde w_{M,0}\tilde d)=E(\tilde\mu_M^r\tilde\lambda)T_{\tilde w_{M,0}\tilde d}=E(\tilde\mu_M^r\tilde\lambda)T_{\tilde w_{M,0}}T_{\tilde d},\\&=\mathbf{q}_{\mu_M^r\lambda,w_{M,0}}E(\tilde\mu_M^r\tilde w_M)T_{\tilde d}\end{aligned}$$

The monoid W_{M^ϵ} is a lower subset of $(W_M,\leq_M)$ (Lemma 2.6). The triangular decomposition (14) implies $E_M(\tilde\mu_M^r\tilde w_M)\in\mathcal{H}_{M^+}$. By Proposition 2.19 $E(\tilde\mu_M^r\tilde w_M)\in\theta(\mathcal{H}_{M^+})$ and by the additivity of the length (Lemma 2.22),

$$\mathbf{q}_{w_{M,0}d}=\mathbf{q}_{w_{M,0}}\mathbf{q}_d,\quad \mathbf{q}_{\mu_M^r\lambda w_{M,0}d}=\mathbf{q}_{\mu_M^r\lambda w_{M,0}}\mathbf{q}_d,$$

implying $\mathbf{q}_{\mu_M^r\lambda}\mathbf{q}_{w_{M,0}d}\mathbf{q}^{-1}_{\mu_M^r\lambda w_{M,0}d}=\mathbf{q}_{\mu_M^r\lambda}\mathbf{q}_{w_{M,0}}\mathbf{q}^{-1}_{\mu_M^r\lambda w_{M,0}}$ hence $\mathbf{q}_{\mu_M^r\lambda,w_{M,0}d}=\mathbf{q}_{\mu_M^r\lambda,w_{M,0}}$.

(iii) We have $\ell(\mu_M)\neq 0$ and equivalently, $\nu(\mu_M)\neq 0$ in V. We choose $w\in W_0$ with $w(\nu(\mu_M)\neq\nu(\mu_M)$. Then $\nu(w\mu_Mw^{-1})=w(\nu(\mu_M))$ and $\nu(\mu_M)$ belong to different Weyl chambers. The alcove walk basis $(E_o(\tilde w))_{\tilde w\in W(1)}$ of $\mathcal{H}$ associated to an orientation o of V of Weyl chamber containing $\nu(\mu_M)$ satisfies

$$E_o(\tilde\mu_M)=T_{\tilde\mu_M},\quad E_o(\tilde\mu_M)E_o(\tilde w\tilde\mu_M\tilde w^{-1})=E_o(\tilde w\tilde\mu_M\tilde w^{-1})E_o(\tilde\mu_M)=0. \tag{23}$$

□

The properties of the left $\theta(\mathcal{H}_{M^+})$-module $\mathcal{H}$ transfer to properties of the right $\theta^*(\mathcal{H}_{M^-})$-module $\mathcal{H}$, with the involutive anti-automorphism $\zeta\circ\iota$ of $\mathcal{H}$ (Remark 2.12) exchanging $T_{\tilde w}$ and $(-1)^{\ell(w)}T^*_{(\tilde w)^{-1}}$ for $\tilde w\in W(1)$, $\theta(\mathcal{H}_{M^+})$ and $\theta^*(\mathcal{H}_{M^-})$, $\mathcal{V}_{M^+}$ and

$$\mathcal{V}^*_{M^-}:=\sum_{d\in W_0^M}T^*_{\tilde d}\theta^*(\mathcal{H}_{M^-}), \tag{24}$$

where $W_0^M=\{d'^{-1}\mid d'\in {}^MW_0\}$ is the set of classical representatives of $W_0/W_{M,0}$ (19), and $\tilde d=(\tilde{d'})^{-1}$ if $d=d'^{-1}$.

Corollary 2.30. (i) *$\mathcal{V}^*_{M^-}$ is a free right $\theta^*(\mathcal{H}_{M^-})$-module of basis $(T^*_{\tilde d})_{d\in W_0^M}$.*

(ii) *For any $h\in\mathcal{H}$, there exists $r\in\mathbb{N}$ such that $h(T^*_{(\tilde\mu_M)^{-1}})^r\in\mathcal{V}^*_{M^-}$.*

(iii) *If $\mathfrak{q}=0$, $T^*_{\tilde\mu_M^{-1}}$ is a left and right zero divisor in $\mathcal{H}$.*

3 Induction and coinduction

3.1 Almost localisation of a free module

In this chapter, all rings have unit elements.

Definition 3.1. *Let A be a ring, and $a\in A$ a central non-zero divisor. We say that a left A-module B is an almost a-localisation of a left A-module $B_D\subset B$ of basis D when :*

(i) *D is a finite subset of B, and the map $\oplus_{d\in D}A\to B,(x_d)\to\sum x_dd$ is injective,*

(ii) *for any $b\in B$, there exists $r\in\mathbb{N}$ such that a^rb lies in $B_D:=\sum_{d\in D}Ad$.*

Example 3.2. *Our basic example is $(A,a,B,D)=(\mathcal{H}_{M^+},T_{\mu_M},\mathcal{H},(T_{\tilde d})_{d\in{}^MW_0})$ (Thm. 2.29).*

As a is central and not a zero divisor in A, the a-localisation of A is ${}_aA = A_a = \cup_{n\in\mathbb{N}} Aa^{-n}$. The left multiplication by a in A is an injective A-linear endomorphism $A \to A, x \mapsto ax$, and the left multiplication by a in B is a A-linear endomorphism $a_B : x \mapsto ax$ of B which may be not injective hence B may be not a flat A-module. The ring B is the union for $r \in \mathbb{N}$, of the A-submodules

$$ {}_rB_D := \{b \in B \mid a^r b \in B_D\}, $$

and looks like a localisation of B_D at a.

Definition 3.3. *Let A be a ring and $a \in A$ a central non-zero divisor. We say that a right A-module B is an almost a-localisation of a right A-module ${}_DB$ of basis D if :*

(i) *D is a finite subset of B, and the map $\oplus_{d\in D} A \to B, (x_d) \to \sum d\, x_d$ is injective,*

(ii) *for any $b \in B$, there exists $r \in \mathbb{N}$ such that $ba^r \in {}_DB := \sum_{d\in D} dA$.*

The ring B is the union for $r \in \mathbb{N}$ of the A-submodules

$$ {}_DB_r = \{b \in B \mid ba^r \in {}_DB\}. $$

Example 3.4. *Our basic example is $(A, a, B, D) = (\mathcal{H}_{M^-}, T_{\mu_M^{-1}}, \mathcal{H}, (T_{\tilde{d}})_{d\in W_0^M})$ (Theorem 2.30).*

We note that $(A_a, B) = (\mathcal{H}_M, \mathcal{H})$ in Example 3.2 and in Example 3.4.

3.2 Induction and coinduction

3.2.1

For a ring A, let Mod_A denote the category of right A-modules, and ${}_A\mathrm{Mod}$ the category of left A-modules. The A-duality $X \mapsto X^* := \mathrm{Hom}_A(X, A)$ exchanges left and right A-modules.

A functor from Mod_A to a category admits a left adjoint if and only if it is left exact and commutes with small direct products (small projective limits); it admits a right adjoint if and only if it is right exact and commutes with small direct sums (small injective limits) [Vigadjoint, Prop. 2.10].

For two rings $A \subset B$, are defined two functors:

$$ \text{the induction } I_A^B := - \otimes_A B \text{ and the coinduction } \mathbb{I}_A^B := \mathrm{Hom}_A(B, -) : \mathrm{Mod}_A \to \mathrm{Mod}_B, $$

where B is seen as a (A, B)-module for the induction, and as a (B, A)-module for the coinduction. For $\mathcal{M} \in \mathrm{Mod}_A$, we have $(m \otimes x)b = m \otimes xb, (fb)(x) = f(bx)$ if $x, b \in B$ and $m \in \mathcal{M}, f \in \mathrm{Hom}_A(B, \mathcal{M})$.

The restriction $\mathrm{Res}_A^B : \mathrm{Mod}_B \to \mathrm{Mod}_A$ is equal to $\mathrm{Hom}_B(B, -) = - \otimes_B B$ where B is seen first as a (A, B)-module and then as a (B, A)-module. The induction and the coinduction are the left and right adjoints of the restriction [Benson, 2.8.2].

For two rings A and B and an (A, B)-module $\mathcal{J}$, the functor

$$ - \otimes_A \mathcal{J} : \mathrm{Mod}_A \to \mathrm{Mod}_B \text{ is left adjoint to } \mathrm{Hom}_B(\mathcal{J}, -) : \mathrm{Mod}_B \to \mathrm{Mod}_A. $$

Let $\mathcal{M} \in \mathrm{Mod}_A,\ \mathcal{N} \in \mathrm{Mod}_B$. The adjunction is given by the functorial isomorphism

$$ \mathrm{Hom}_B(\mathcal{M} \otimes_A \mathcal{J}, \mathcal{N}) \xrightarrow{\alpha} \mathrm{Hom}_A(\mathcal{M}, \mathrm{Hom}_B(\mathcal{J}, \mathcal{N})), \quad f(m \otimes x) = \alpha(f)(m)(x), $$

for $f \in \mathrm{Hom}_B(\mathcal{M} \otimes_A \mathcal{J}, \mathcal{N}), m \in \mathcal{M}, x \in \mathcal{J}$ [Benson, Lemma 2.8.2].

For three rings $A \subset B, A \subset C$, the isomorphism α applied to $\mathcal{M} = C, \mathcal{J} = B$ gives an isomorphism:

$$ \mathrm{Hom}_B(C \otimes_A B, -) \simeq \mathrm{Hom}_A(C, -) : \mathrm{Mod}_B \to \mathrm{Mod}_C\,. $$

3.2.2

Let $A \subset B$ be two rings and $a \in A$ a central non-zero divisor. Let $A_a = A[a^{-1}]$ denote the localisation of A at a. There is a natural inclusion $A \subset A_a$. The restriction $\mathrm{Mod}_{A_a} \to \mathrm{Mod}_A$ identifies Mod_{A_a} with the A-modules where the action of a is invertible. For $\mathcal{M}, \mathcal{M}'$ in Mod_{A_a}, we have

$$\mathrm{Hom}_{A_a}(\mathcal{M}, \mathcal{M}') = \mathrm{Hom}_A(\mathcal{M}, \mathcal{M}'), \quad \mathcal{M} \otimes_{A_a} \mathcal{M}' = \mathcal{M} \otimes_A \mathcal{M}'. \tag{25}$$

For $f \in \mathrm{Hom}_A(\mathcal{M}, \mathcal{M}'), m \in \mathcal{M}, m' \in \mathcal{M}'$, we have $f(aa^{-1}m) = af(a^{-1}m) \Rightarrow a^{-1}f(m) = f(a^{-1}m)$, and $m \otimes a^{-1}m' = ma^{-1}a \otimes a^{-1}m' = ma^{-1} \otimes m'$ in $\mathcal{M} \otimes_A \mathcal{M}'$. We view Mod_{A_a} as a full subcategory of Mod_A.

The restriction followed by the induction, resp. the coinduction, $\mathrm{Mod}_A \to \mathrm{Mod}_B$ defines an induction, resp. coinduction,

$$I_{A_a}^B = I_A^B \circ \mathrm{Res}_A^{A_a} = - \otimes_A B, \quad \mathbb{I}_{A_a}^B = \mathbb{I}_A^B \circ \mathrm{Res}_A^{A_a} = \mathrm{Hom}_A(B, -) \ : \ \mathrm{Mod}_{A_a} \to \mathrm{Mod}_B,$$

even when A_a is not contained in B. The induction $I_{A_a}^B$ admits a right adjoint

$$\mathbb{I}_A^{A_a} \circ \mathrm{Res}_A^B = \mathrm{Hom}_A(A_a, -) \ : \ \mathrm{Mod}_B \to \mathrm{Mod}_{A_a},$$

because the restriction $\mathrm{Res}_A^{A_a}$ and the induction I_A^B admit a right adjoint: the coinduction $\mathbb{I}_A^{A_a}$ and the restriction Res_A^B. The coinduction $\mathbb{I}_{A_a}^B$ admits a left adjoint

$$I_A^{A_a} \circ \mathrm{Res}_A^B = - \otimes_A A_a \ : \ \mathrm{Mod}_B \to \mathrm{Mod}_{A_a},$$

because the restriction $\mathrm{Res}_A^{A_a}$ and the coinduction $\mathbb{I}_A^B$ admit a left adjoint: the induction $I_A^{A_a}$ and the restriction Res_A^B.

When a is invertible in B we have $A_a \subset B$ and they coincide with the induction and coinduction from A_a to B.

The induction and the coinduction of A_a seen as a right A_a-module, are the (A_a, B)-modules

$$I_{A_a}^B(A_a) = A_a \otimes_A B, \quad \mathbb{I}_{A_a}^B(A_a) = \mathrm{Hom}_A(B, A_a). \tag{26}$$

Lemma 3.5. *Let $\mathcal{M} \in \mathrm{Mod}_{A_a}$. Then $I_{A_a}^B(\mathcal{M}) = \mathcal{M} \otimes_{A_a} I_{A_a}^B(A_a)$ in Mod_B.*

Proof. $\mathcal{M} \otimes_A B = (\mathcal{M} \otimes_{A_a} A_a) \otimes_A B = \mathcal{M} \otimes_{A_a} (A_a \otimes_A B)$. □

3.2.3

Let (A, a, B, D) satisfying Definition 3.1. Let $\mathcal{M} \in \mathrm{Mod}_{A_a}$. As R-modules,

$$I_{A_a}^B(\mathcal{M}) = \mathcal{M} \otimes_A B_D \tag{27}$$

because the action of a on $\mathcal{M}$ is invertible hence $\mathcal{M} \otimes_A {}_rB_D = \mathcal{M} \otimes_A B_D$ for $r \in \mathbb{N}$. In particular:

Lemma 3.6. *The left A_a-module $I_{A_a}^B(A_a)$ is free of basis $(1 \otimes d)_{d \in D}$.*

Remark 3.7. The A-dual $(B_D)^*$ of the left A-module B_D is the right A-module $\oplus_{d \in D} d^* A$ of basis the dual basis $D^* = \{d^* \mid d \in D\}$ of D. Let $\mathcal{M} \in \mathrm{Mod}_{A_a}$. We have canonical isomorphisms of R-modules:

$$\oplus_{d \in D} \mathcal{M} \xrightarrow{\simeq} \mathcal{M} \otimes_A B_D \xrightarrow{\simeq} \mathrm{Hom}_A((B_D)^*, \mathcal{M})$$
$$(x_d) \mapsto \sum_{d \in D} x_d \otimes d \mapsto (d^* \mapsto x_d)_{d \in D}.$$

The tensor product over A by a free A-module is exact and faithful hence the induction is exact and faithful.

Let $R \subset A$ be a subring central in B. The ring R is automatically commutative and a central subring of the localisation A_a of A. The modules over A_a or B are naturally R-modules.

Let $\mathcal{M} \in \mathrm{Mod}_{A_a}$ be a finitely generated R-module. The R-module $\mathcal{M} \otimes_{A_a} I_{A_a}^B(A_a)$ is finitely generated.

Let $\mathcal{N} \in \mathrm{Mod}_B$ be a finitely generated R-module. The R-module $\mathrm{Hom}_A(A_a, \mathcal{N})$ is finitely generated if R is a field by the Fitting's lemma applied to the action of a on $\mathcal{N}$. There exists a positive integer n such that $\mathcal{N}$ is a direct sum $\mathcal{N} = \mathcal{N}_a \oplus \mathcal{N}'_a$ where a^n acts on $\mathcal{N}_a$ as an automorphism and a^n is 0 on $\mathcal{N}'_a$. Then, $\mathrm{Hom}_A(A_a, \mathcal{N}) \simeq \mathcal{N}_a$ is finite dimensional.

We obtain:

Proposition 3.8. *Let (A, a, B, D) satisfying Definition 3.1. The induction functor*

$$I_{A_a}^B = - \otimes_A B : \mathrm{Mod}_{A_a} \to \mathrm{Mod}_B$$

is exact, faithful and admits a right adjoint $R_{A_a}^B := \mathrm{Hom}_A(A_a, -)$.

Let $R \subset A$ be a subring central in B. Then $I_{A_a}^B$ respects finitely generated R-modules. If R is a field, $R_{A_a}^B$ respects finite dimension over R.

3.2.4

Let (A, a, B, D) satisfying Definition 3.3.

For $\mathcal{M} \in \mathrm{Mod}_A$, the set $\mathcal{M}_d$ of $f \in \mathrm{Hom}_A({}_D B, \mathcal{M})$ vanishing on $D - \{d\}$ is isomorphic to $\mathcal{M}$ by the value at d. The A-dual $({}_D B)^*$ of ${}_D B$ is a free left A-module of basis D^*. We have

$$\mathrm{Hom}_A({}_D B, \mathcal{M}) = \oplus_{d \in D} \mathcal{M}_d \simeq \oplus_{d^* \in D^*} \mathcal{M} \otimes d^* = \mathcal{M} \otimes_A ({}_D B)^*. \quad (28)$$

The A-modules $\mathcal{M}_d$ and $\mathcal{M} \otimes d^*$ are isomorphic by $f \mapsto f(d) \otimes d^*$.

For $\mathcal{M} \in \mathrm{Mod}_{A_a}$, we have linear isomorphisms

$$\mathbb{I}_{A_a}^B(\mathcal{M}) = \mathrm{Hom}_A(B, \mathcal{M}) \simeq \mathrm{Hom}_A({}_D B, \mathcal{M}), \quad \mathcal{M} \otimes_A ({}_D B)^* = \mathcal{M} \otimes_A A_a \otimes_A ({}_D B)^*.$$

For $d \in D$, let $f_d \in \mathrm{Hom}_A(B, A_a)$ equal to 1 on d and 0 on $D - \{d\}$. We deduce from these arguments:

Lemma 3.9. *Let (A, a, B, D) satisfying Definition 3.3. The left A_a-module $\mathbb{I}_{A_a}^B(A_a)$ is free of basis $(f_d)_{d \in D}$ and $\mathbb{I}_{A_a}^B(\mathcal{M}) \simeq \mathcal{M} \otimes_{A_a} \mathbb{I}_A^B(A_a)$.*

Let $R \subset A$ be a subring central in B. Let $\mathcal{M} \in \mathrm{Mod}_{A_a}$ be a finitely generated R-module. The R-module $\mathcal{M} \otimes_{A_a} \mathbb{I}_{A_a}^B(A_a)$ is finitely generated. If R is a field, and the dimension of $\mathcal{N} \in \mathrm{Mod}_B$ is finite over R, then $\mathcal{N} \otimes_A A_a = \mathcal{N}_a \otimes_A A_a \simeq \mathcal{N}_a$ has finite dimension over R by the Fitting's lemma, as in the proof of Proposition 3.8. We obtain:

Proposition 3.10. *Let (A, a, B, D) satisfying Definition 3.3. The coinduction*

$$\mathbb{I}_{A_a}^B = \mathrm{Hom}_A(B, -) : \mathrm{Mod}_{A_a} \to \mathrm{Mod}_B$$

is exact, faithful, and admits a left adjoint $L_{A_a}^B = - \otimes_A A_a$.

Let $R \subset A$ be a subring central in B. Then $\mathbb{I}_{A_a}^B$ respects finitely generated R-modules. If R is a field, $L_{A_a}^B$ respects finite dimension over R.

4 Parabolic induction and coinduction from $\mathcal{H}_M$ to $\mathcal{H}$

We prove Theorems 1.6, 1.8 and 1.9 giving the properties of the parabolic induction from $\mathcal{H}_M$ to $\mathcal{H}$.

4.1 Basic properties of the parabolic induction and coinduction

The example 3.2 satisfies Definition 3.1 and the example 3.4 satisfies Definition 3.3. In these two examples $(A_a, B) = (\mathcal{H}_M, \mathcal{H})$. The first one

$$(A, a, D) = (\theta(\mathcal{H}_{M^+}), T_{\tilde{\mu}_M}, (T_{\tilde{d}})_{d \in {}^M W_0}),$$

where we identify $\mathcal{H}_{M^+}$ with $\theta(\mathcal{H}_{M^+})$, defines the parabolic induction $I_{\mathcal{H}_M}^{\mathcal{H}} = - \otimes_{\mathcal{H}_{M^+},\theta} \mathcal{H} : \mathrm{Mod}_{\mathcal{H}_M} \to \mathrm{Mod}_{\mathcal{H}}$. The second one

$$(A, a, D) = (\theta^*(\mathcal{H}_{M^-}), T^*_{(\tilde{\mu}_M)^{-1}}, (T^*_{\tilde{d}})_{d \in W_0^M}),$$

where we identify $\mathcal{H}_{M^-}$ with $\theta^*(\mathcal{H}_{M^-})$, defines the parabolic coinduction $\mathbb{I}_{\mathcal{H}_M}^{\mathcal{H}} = \mathrm{Hom}_{\mathcal{H}_{M^-},\theta^*}(\mathcal{H}, -) : \mathrm{Mod}_{\mathcal{H}_M} \to \mathrm{Mod}_{\mathcal{H}}$. Propositions 3.8 and 3.10 imply:

Proposition 4.1. *The parabolic induction $I_{\mathcal{H}_M}^{\mathcal{H}}$ and the coinduction $\mathbb{I}_{\mathcal{H}_M}^{\mathcal{H}}$ are exact, faithful and respect finitely generated R-modules. The parabolic induction admits a right adjoint*

$$R_{\mathcal{H}_M}^{\mathcal{H}} = \mathrm{Hom}_{\mathcal{H}_{M^+},\theta}(\mathcal{H}_M, -) : \mathrm{Mod}_{\mathcal{H}} \to \mathrm{Mod}_{\mathcal{H}_M}.$$

The parabolic coinduction admits a left adjoint

$$\mathbb{L}_{\mathcal{H}_M}^{\mathcal{H}} := - \otimes_{\mathcal{H}_{M^-},\theta^*} \mathcal{H}_M : \mathrm{Mod}_{\mathcal{H}} \to \mathrm{Mod}_{\mathcal{H}_M}.$$

If R is a field, the adjoint functors $R_{\mathcal{H}_M}^{\mathcal{H}}$ and $\mathbb{L}_{\mathcal{H}_M}^{\mathcal{H}}$ respect finite dimension over R.

4.2 Transitivity

Let $S_M \subset S_{M'} \subset S$. Let $W_{M^\epsilon,M'} = \Lambda_{M^\epsilon,M'} \rtimes W_{M,0}$ denote the submonoid of W_M associated to $S_{M'}^{aff}$ as in Definition 2.1 (see before Proposition 2.21), and

$$\Lambda_{M^\epsilon,M'} = \Lambda \cap W_{M^\epsilon,M'} = \{\lambda \in \Lambda \mid -(\gamma \circ \nu)(\lambda) \geq 0 \text{ for all } \gamma \in \Sigma^\epsilon_{M'} - \Sigma^\epsilon_M \},$$

By the property (i), (ii), (iii) of Theorem 1.4, the R-submodule $\mathcal{H}_{M^\epsilon,M'}$ of $\mathcal{H}_M$ of basis $(T^M_{\tilde{w}})_{\tilde{w} \in W_{M^\epsilon,M'}(1)}$, is a subring of $\mathcal{H}_M$, the restriction to $\mathcal{H}_{M^\epsilon,M'}$ of the injective linear map

$$\mathcal{H}_M \xrightarrow{\theta'} \mathcal{H}_{M'}, \quad T^M_{\tilde{w}} \mapsto T^{M'}_{\tilde{w}} \quad \text{for } \tilde{w} \in W_M(1),$$

respects the product, and $\mathcal{H}_M = \mathcal{H}_{M^\epsilon,M'}[(T^M_{\tilde{\mu}_{M^\epsilon}})^{-1}]$. Obviously, the map $\mathcal{H}_M \xrightarrow{\theta} \mathcal{H}$ satisfies $\theta = \theta_{M'} \circ \theta'$ for the linear map $\mathcal{H}_{M'} \xrightarrow{\theta_{M'}} \mathcal{H}$, $T^{M'}_{\tilde{w}} \mapsto T_{\tilde{w}}$ for $\tilde{w} \in W_{M'}(1)$.

Lemma 4.2. *We have*

(i) $W_M \subset W_{M'}$, $W_{M^\epsilon} = W_{M^\epsilon,M'} \cap W_{M'^\epsilon}$, $\theta'(\mathcal{H}_{M^\epsilon}) = \theta'(\mathcal{H}_{M^\epsilon,M'}) \cap \mathcal{H}_{M'^\epsilon}$.

(ii) $\tilde{\mu}_{M^\epsilon}\tilde{\mu}_{M'^\epsilon}$ *is central in* $W_M(1)$, *satisfies* $-(\gamma \circ \nu)(\mu_{M^\epsilon}\mu_{M'^\epsilon}) > 0$ *for all* $\gamma \in \Sigma^\epsilon - \Sigma^\epsilon_M$, *and the additivity of the lengths* $\ell(\mu_{M^\epsilon}\mu_{M'^\epsilon}) = \ell(\mu_{M^\epsilon}) + \ell(\mu_{M'^\epsilon})$.

(iii) ${}^M W_0 = {}^M W_{M',0} \, {}^{M'} W_0$.

Proof. (i) We have $W_{M,0} \subset W_{M',0}$ and $\Lambda_{M^\epsilon} = \Lambda'_{M^\epsilon} \cap \Lambda_{M'^\epsilon}$. Therefore $W_M = \Lambda \rtimes W_{M,0} \subset \Lambda \rtimes W_{M',0} = W_{M'}$, and $W_{M^\epsilon, M'} \cap W^\epsilon_{M'} = (\Lambda'_{M^\epsilon} \rtimes W_{M,0}) \cap (\Lambda'_{M'^\epsilon} \rtimes W_{M',0}) = (\Lambda'_{M^\epsilon} \cap \Lambda_{M'^\epsilon}) \rtimes W_{M,0} = \Lambda_{M^\epsilon} \rtimes W_{M,0} = W_{M^\epsilon}$.

(ii) $\tilde{\mu}_{M'^\epsilon}$ is central in $W_{M'}(1)$ which contains $W_M(1)$, $\tilde{\mu}_{M^\epsilon}$ is central in $W_M(1)$, hence $\tilde{\mu}_{M^\epsilon}\tilde{\mu}_{M'^\epsilon}$ is central in $W_M(1)$. We have

$-(\gamma \circ \nu)(\ \mu_{M'^\epsilon}) > 0$ for all $\gamma \in \Sigma^\epsilon - \Sigma^\epsilon_{M'}$, $-(\gamma \circ \nu)(\ \mu_{M'^\epsilon}) = 0$ for all $\gamma \in \Sigma_{M'}$,

$-(\gamma \circ \nu)(\ \mu_{M^\epsilon}) > 0$ for all $\gamma \in \Sigma^\epsilon - \Sigma^\epsilon_{M}$, $-(\gamma \circ \nu)(\ \mu_{M^\epsilon}) = 0$ for all $\gamma \in \Sigma_{M}$.

Hence $-(\gamma \circ \nu)(\mu'_{M^\epsilon}\mu_{M'^\epsilon}) > 0$ for all $\gamma \in \Sigma^\epsilon - \Sigma^\epsilon_M$ and $\ell(\mu_{M^\epsilon}\mu_{M'^\epsilon}) = \ell(\mu_{M^\epsilon}) + \ell(\mu_{M'^\epsilon})$.

(iii) Let $u \in {}^M W_{M',0}, v \in {}^{M'} W_0$ and let $w \in W_{M,0}$. We have $\ell(wuv) = \ell(wu) + \ell(v) = \ell(w) + \ell(u) + \ell(v) = \ell(w) + \ell(uv)$ hence $uv \in {}^M W_0$. The injective map $(u, v) \mapsto uv : {}^M W_{M',0} \times {}^{M'} W_0 \to {}^M W_0$ is bijective because

$|{}^M W_0| = |W_{M,0} \backslash W_0| = |W_{M,0} \backslash W_{M',0}|\, |W_{M',0} \backslash W_0| = |{}^M W_{M',0}|\, |{}^{M'} W_0|$,

where $|X|$ denotes the number of elements of a finite set X.

□

Proposition 4.3. *The induction is transitive:*

$$I^{\mathcal{H}}_{\mathcal{H}_M} = I^{\mathcal{H}}_{\mathcal{H}_{M'}} \circ I^{\mathcal{H}_{M'}}_{\mathcal{H}_M} \ :\ \mathrm{Mod}_{\mathcal{H}_M} \to \mathrm{Mod}_{\mathcal{H}_{M'}} \to \mathrm{Mod}_{\mathcal{H}}\,.$$

The coinduction is also transitive. This is proved at the end of this paper.

Proof. By lemma 3.5, the proposition is equivalent to

$$\mathcal{H}_M \otimes_{\mathcal{H}_{M^+}} \mathcal{H} \simeq \mathcal{H}_M \otimes_{\mathcal{H}_{M^+,M'}} \mathcal{H}_{M'} \otimes_{\mathcal{H}_{M'^+}} \mathcal{H}$$

in $\mathrm{Mod}_{\mathcal{H}}$. As $\mathcal{H}_{M'} = \mathcal{H}_{M'^+}[(T^{M'}_{\tilde{\mu}_{M'^+}})^{-1}]$ is the localisation of the ring $\mathcal{H}_{M'^+}$ at the central element $T^{M'}_{\tilde{\mu}_{M'^+}} \in \mathcal{H}_{M'^+}$, the right $\mathcal{H}$-module $\mathcal{H}_{M'} \otimes_{\mathcal{H}_{M'^+}} \mathcal{H}$ is the inductive limit of $(T^{M'}_{\tilde{\mu}_{M'^+}})^{-r} \otimes \mathcal{H}$ for $r \in \mathbb{N}$ with the transition maps

$$(T^{M'}_{\tilde{\mu}_{M'^+}})^{-r} \otimes x \mapsto (T^{M'}_{\tilde{\mu}_{M'^+}})^{-r-1} \otimes T_{\tilde{\mu}_{M'^+}} x, \quad \text{for } x \in \mathcal{H}.$$

As $\mathcal{H}_M = \mathcal{H}_{M^+,M'}[(T^M_{\tilde{\mu}_{M^+}})^{-1}]$ is the localisation of the ring $\mathcal{H}_{M^+,M'}$ at the central element $T^M_{\tilde{\mu}_{M^+}} \in \mathcal{H}_{M^+,M'}$, the right $\mathcal{H}$-module $\mathcal{H}_M \otimes_{\mathcal{H}_{M^+,M'}} \mathcal{H}_{M'} \otimes_{\mathcal{H}_{M'^+}} \mathcal{H}$ is the inductive limit of $(T^M_{\tilde{\mu}_{M^+}})^{-s} \otimes \mathcal{H}_{M'} \otimes_{\mathcal{H}_{M'^+}} \mathcal{H}$ for $s \in \mathbb{N}$ with the transition maps

$$(T^M_{\tilde{\mu}_{M^+}})^{-s} \otimes y \mapsto (T^M_{\tilde{\mu}_{M^+}})^{-s-1} \otimes T^{M'}_{\tilde{\mu}_{M^+}} y, \quad \text{for } y \in \mathcal{H}_{M'} \otimes_{\mathcal{H}_{M'^+}} \mathcal{H}.$$

Using that $T^{M'}_{\tilde{\mu}_{M'^+}}$ is central in $\mathcal{H}_{M'}$ and $T^{M'}_{\tilde{\mu}_{M^+}} \in \mathcal{H}_{M'^+}$, we have for $y = (T^{M'}_{\tilde{\mu}_{M'^+}})^{-r} \otimes x$:

$$T^{M'}_{\tilde{\mu}_{M^+}} y = T^{M'}_{\tilde{\mu}_{M^+}} (T^{M'}_{\tilde{\mu}_{M'^+}})^{-r} \otimes x = (T^{M'}_{\tilde{\mu}_{M'^+}})^{-r} T^{M'}_{\tilde{\mu}_{M^+}} \otimes x = (T^{M'}_{\tilde{\mu}_{M'^+}})^{-r} \otimes T_{\tilde{\mu}_{M^+}} x.$$

Alltogether, the right $\mathcal{H}$-module $\mathcal{H}_M \otimes_{\mathcal{H}_{M^+,M'}} \mathcal{H}_{M'} \otimes_{\mathcal{H}_{M'^+}} \mathcal{H}$ is the inductive limit of $(T^M_{\tilde{\mu}_{M^+}})^{-s} \otimes (T^{M'}_{\tilde{\mu}_{M'^+}})^{-r} \otimes \mathcal{H}$ for $r, s \in \mathbb{N}$ with the transition maps

$$(T^M_{\tilde{\mu}_{M^+}})^{-s} \otimes (T^{M'}_{\tilde{\mu}_{M'^+}})^{-r} \otimes x \mapsto (T^M_{\tilde{\mu}_{M^+}})^{-s-1} \otimes (T^{M'}_{\tilde{\mu}_{M'^+}})^{-r} \otimes T_{\tilde{\mu}_{M^+}} x,$$
$$(T^M_{\tilde{\mu}_{M^+}})^{-s} \otimes (T^{M'}_{\tilde{\mu}_{M'^+}})^{-r} \otimes x \mapsto (T^M_{\tilde{\mu}_{M^+}})^{-s} \otimes (T^{M'}_{\tilde{\mu}_{M'^+}})^{-r-1} \otimes T_{\tilde{\mu}_{M'^+}} x.$$

The right $\mathcal{H}$-module $\mathcal{H}_M \otimes_{\mathcal{H}_{M^+,M'}} \mathcal{H}_{M'} \otimes_{\mathcal{H}_{M'^+}} \mathcal{H}$ is also the inductive limit of the modules $(T^M_{\tilde{\mu}_{M^+}})^{-r} \otimes (T^{M'}_{\tilde{\mu}_{M'^+}})^{-r} \otimes \mathcal{H}$ for $r \in \mathbb{N}$ with the transition maps

$$(T^M_{\tilde{\mu}_{M^+}})^{-r} \otimes (T^{M'}_{\tilde{\mu}_{M'^+}})^{-r} \otimes x \mapsto (T^M_{\tilde{\mu}_{M^+}})^{-r-1} \otimes (T^{M'}_{\tilde{\mu}_{M'^+}})^{-r-1} \otimes T_{\tilde{\mu}_{M^+}} T_{\tilde{\mu}_{M'^+}} x.$$

By Lemma 4.2 (ii), $T_{\tilde{\mu}_{M^+}} T_{\tilde{\mu}_{M'^+}} = T_{\tilde{\mu}_{M^+}\tilde{\mu}_{M'^+}}$. Hence, we have in $\mathrm{Mod}_{\mathcal{H}}$

$$\mathcal{H}_M \otimes_{\mathcal{H}_{M^+,M'}} \mathcal{H}_{M'} \otimes_{\mathcal{H}_{M'^+}} \mathcal{H} \simeq \varinjlim_{x \mapsto T_{\tilde{\mu}_{M^+}\tilde{\mu}_{M'^+}} x} \mathcal{H}.$$

On the other hand, $\mathcal{H}_M = \mathcal{H}_{M^+}[(T^M_{\tilde{\mu}_{M^+}\tilde{\mu}_{M'^+}})^{-1}]$ is the localisation of $\mathcal{H}_{M^+}$ at $T^M_{\tilde{\mu}_{M^+}\tilde{\mu}_{M'^+}}$ (Lemma 4.2), hence $\mathcal{H}_M \otimes_{\mathcal{H}_{M^+}} \mathcal{H}$ is the inductive limit of $(T^M_{\tilde{\mu}_{M^+}\tilde{\mu}_{M'^+}})^{-r} \otimes \mathcal{H}$ for $r \in \mathbb{N}$ with the transition maps

$$(T^M_{\tilde{\mu}_{M^+}\tilde{\mu}_{M'^+}})^{-r} \otimes x \mapsto (T^M_{\tilde{\mu}_{M^+}\tilde{\mu}_{M'^+}})^{-r-1} \otimes T_{\tilde{\mu}_{M^+}\tilde{\mu}_{M'^+}} x.$$

We deduce that

$$\mathcal{H}_M \otimes_{\mathcal{H}_{M^+}} \mathcal{H} \simeq \varinjlim_{x \mapsto T_{\tilde{\mu}_{M^+}\tilde{\mu}_{M'^+}} x} \mathcal{H}$$

is isomorphic to $\mathcal{H}_M \otimes_{\mathcal{H}_{M^+,M'}} \mathcal{H}_{M'} \otimes_{\mathcal{H}_{M'^+}} \mathcal{H}$ in $\mathrm{Mod}_{\mathcal{H}}$. □

4.3 w_0-twisted induction = coinduction

We prove Theorem 1.8. When $\mathcal{H} = \mathcal{H}_R(G)$ is the pro-p Iwahori Hecke algebra of a reductive p-adic group G over an algebraically closed field R of characteristic p, Theorem 1.8 is proved by Abe [Abe, Prop. 4.14]. We will extend his arguments to the general algebra $\mathcal{H}$.

Let $\tilde{w}_0^M \in W_0(1)$ lifting w_0^M. The algebra isomorphism $\mathcal{H}_M \simeq \mathcal{H}_{w_0(M)}$ defined by $\tilde{w}_0^M$ (Proposition 2.20) induces an equivalence of categories :

$$\mathrm{Mod}_{\mathcal{H}_M} \xrightarrow{\tilde{\mathfrak{w}}_0^M} \mathrm{Mod}_{\mathcal{H}_{w_0(M)}} \tag{29}$$

called a w_0-twist. Let $\mathcal{M}$ be a right $\mathcal{H}_M$-module. The underlying R-module of $\tilde{\mathfrak{w}}_0^M(\mathcal{M})$ and of $\mathcal{M}$ is the same; the right action of $T^M_{\tilde{w}}$ on $\mathcal{M}$ is equal to the right action of $T^{w_0(M)}_{\tilde{w}_0^M \tilde{w} (\tilde{w}_0^M)^{-1}}$ on $\tilde{\mathfrak{w}}_0^M(\mathcal{M})$, for $\tilde{w} \in W_M(1)$. The inverse of $\tilde{\mathfrak{w}}_0^M$ is the algebra isomorphism induced by $(\tilde{w}_0^M)^{-1}$ lifthing ${}^M w_0 := (w_0^M)^{-1} = w_{M,0} w_0 = w_0 w_0 w_{M,0} w_0 = w_0^{w_0(M)}$.

Remark 4.4. The lifts of w_0^M are $t\tilde{w}_0^M = \tilde{w}_0^M t'$ with $t, t' \in Z_k$, the elements $T^M_{t'} \in \mathcal{H}_M, T^{w_0(M)}_t \in \mathcal{H}_{w_0(M)}$ are invertible, and the conjugation by T_t in $\mathcal{H}_M$, by $T^{w_0(M)}_t$ in $\mathcal{H}_{w_0(M)}$ induce equivalence of categories

$$\mathrm{Mod}_{\mathcal{H}_M} \xrightarrow{\mathfrak{t}'} \mathrm{Mod}_{\mathcal{H}_M}, \quad \mathrm{Mod}_{\mathcal{H}_{w_0(M)}} \xrightarrow{\mathfrak{t}} \mathrm{Mod}_{\mathcal{H}_{w_0(M)}}$$

such that $\mathfrak{t}\tilde{\mathfrak{w}}_0^M = \mathfrak{t} \circ \tilde{\mathfrak{w}}_0^M = \tilde{\mathfrak{w}}_0^M \circ \mathfrak{t}' = \tilde{\mathfrak{w}}_0^M \mathfrak{t}'$.

Remark 4.5. The trivial characters of $\mathcal{H}_M$ and $\mathcal{H}_{w_0(M)}$ correspond by $\tilde{\mathfrak{w}}_0^M$.

We will prove that, for all $S_M \subset S$, the coinduction $\mathrm{Mod}_{\mathcal{H}_M} \xrightarrow{\mathbb{I}^{\mathcal{H}}_{\mathcal{H}_M}} \mathrm{Mod}_{\mathcal{H}}$ is equivalent to the w_0-twist induction

$$\mathrm{Mod}_{\mathcal{H}_M} \xrightarrow{\tilde{\mathfrak{w}}_0^M} \mathrm{Mod}_{\mathcal{H}_{w_0(M)}} \xrightarrow{I^{\mathcal{H}}_{\mathcal{H}_{w_0(M)}}} \mathrm{Mod}_{\mathcal{H}} .$$

This proves Theorem 1.8 because

$$\mathbb{I}^{\mathcal{H}}_{\mathcal{H}_M} \simeq I^{\mathcal{H}}_{\mathcal{H}_{w_0(M)}} \circ \tilde{\mathfrak{w}}_0^M \Leftrightarrow I^{\mathcal{H}}_{\mathcal{H}_M} \simeq \mathbb{I}^{\mathcal{H}}_{\mathcal{H}_{w_0(M)}} \circ \tilde{\mathfrak{w}}_0^M. \tag{30}$$

Indeed, if the left hand side is true for all $S_M \subset S$, permuting M and $w_0(M)$ we have $\mathbb{I}_{\mathcal{H}_{w_0(M)}} \simeq I^{\mathcal{H}}_{\mathcal{H}_M} \circ \tilde{\mathfrak{w}}_0^{w_0(M)}$, and composing with $(\tilde{\mathfrak{w}}_0^{w_0(M)})^{-1}$, we get $I^{\mathcal{H}}_{\mathcal{H}_M} \simeq \mathbb{I}^{\mathcal{H}}_{\mathcal{H}_{w_0(M)}} \circ (\tilde{\mathfrak{w}}_0^{w_0(M)})^{-1} \simeq \mathbb{I}^{\mathcal{H}}_{\mathcal{H}_{w_0(M)}} \circ \tilde{\mathfrak{w}}_0^{M}$ as $w_0^{w_0(M)} = (w_0^M)^{-1}$ The arguments can be reversed to get the equivalence.

Let $\mathcal{M} \in \mathrm{Mod}_{\mathcal{H}_M}$. We will construct an explicit functorial isomorphism in $\mathrm{Mod}_{\mathcal{H}}$:

$$(I^{\mathcal{H}}_{\mathcal{H}_{w_0(M)}} \circ \tilde{\mathfrak{w}}_0^M)(\mathcal{M}) \xrightarrow{\mathfrak{b}} \mathbb{I}^{\mathcal{H}}_{\mathcal{H}_M}(\mathcal{M}). \tag{31}$$

From Lemmas 3.5, 3.6, 3.9 and Examples 3.2, 3.4, we get:

(i) $I^{\mathcal{H}}_{\mathcal{H}_{w_0(M)}}(\mathcal{H}_{w_0(M)}) = \mathcal{H}_{w_0(M)} \otimes_{\mathcal{H}_{w_0(M)^+},\theta} \mathcal{H}$ is a left free $\mathcal{H}_{w_0(M)}$-module of basis $1 \otimes T_{\tilde{d}'}$ for $d' \in {}^{w_0(M)}W_0$, and

$$(I^{\mathcal{H}}_{\mathcal{H}_{w_0(M)}} \circ \tilde{\mathfrak{w}}_0^M)(\mathcal{M}) = \tilde{\mathfrak{w}}_0^M(\mathcal{M}) \otimes_{\mathcal{H}_{w_0(M)}} I^{\mathcal{H}}_{\mathcal{H}_{w_0(M)}}(\mathcal{H}_{w_0(M)}).$$

(ii) $\mathbb{I}^{\mathcal{H}}_{\mathcal{H}_M}(\mathcal{H}_M) = \mathrm{Hom}_{\mathcal{H}_{M^-},\theta^*}(\mathcal{H}, \mathcal{H}_M)$ where $\mathcal{H}$ is seen as a right $\theta^*(\mathcal{H}_{M^-})$-module, is a left free $\mathcal{H}_M$-module of basis $(f^*_{\tilde{d}})_{d \in W_0^M}$, where $f^*_{\tilde{d}}(T^*_{\tilde{d}}) = 1$ and $f^*_{\tilde{d}}(T^*_{\tilde{x}}) = 0$ for $x \in W_0^M - \{d\}$, and

$$\mathbb{I}^{\mathcal{H}}_{\mathcal{H}_M}(\mathcal{M}) = \mathcal{M} \otimes_{\mathcal{H}_M} \mathbb{I}^{\mathcal{H}}_{\mathcal{H}_M}(\mathcal{H}_M).$$

It is an exercise to prove that the left $\mathcal{H}_M$-module $\mathbb{I}^{\mathcal{H}}_{\mathcal{H}_M}(\mathcal{H}_M)$ admits also the basis $(f_{\tilde{d}})_{d \in W_0^M}$, where $f_{\tilde{d}}(T_{\tilde{d}}) = 1$ and $f_{\tilde{d}}(T_{\tilde{x}}) = 0$ for $x \in W_0^M - \{d\}$. We will prove that the linear map

$$m \otimes T_{\tilde{d}'} \mapsto m \otimes f_{\tilde{w}_0^M} T_{\tilde{d}'} : \oplus_{d' \in {}^{w_0(M)}W_0} \tilde{\mathfrak{w}}_0^M(\mathcal{M}) \otimes T_{\tilde{d}'} \xrightarrow{\mathfrak{b}} \oplus_{d \in W_0^M} \mathcal{M} \otimes f_{\tilde{d}} \tag{32}$$

is a functorial isomorphism in $\mathrm{Mod}_{\mathcal{H}}$. The bijectivity follows from the bijectivity of the map $d' \mapsto d'^{-1} w_0^M : {}^{w_0(M)}W_0 \to W_0^M$ (Lemma 2.24) and:

Lemma 4.6.

$$f_{\tilde{w}_0^M} T_{\tilde{d}'} - f_{\widetilde{(d'^{-1} w_0^M)}} \quad \textit{lies in} \quad \oplus_{x \in W_0^M, x < d'^{-1} w_0^M} \mathcal{M} \otimes f_{\tilde{x}}.$$

Proof. For $d \in W_0^M$ we have $(f_{\tilde{w}_0^M} T_{\tilde{d}'})(T_{\tilde{d}}) = f_{\tilde{w}_0^M}(T_{\tilde{d}'} T_{\tilde{d}}) = f_{\tilde{w}_0^M}(T_{\tilde{d}'\tilde{d}}) + x$ where $x \in \sum R f_{\tilde{w}_0^M}(T_{\tilde{w}})$ the sum over the $\tilde{w} \in W_0(1)$ with $w < d'd$ and $w \in w_0^M W_{M,0}$. If $d'd \notin w_0^M W_{M,0}$, there is no $w \in w_0^M W_{M,0}$ with $w < d'd$ (Lemma 2.26). We have $d'd \in w_0^M W_{M,0}$ if and only if $d = d'^{-1} w_0^M$ (part (ii) of Lemma 2.28). □

The restriction $\mathrm{Res}^{\mathcal{H}}_{\mathcal{H}_{w_0(M)^+},\theta} : \mathrm{Mod}_{\mathcal{H}} \to \mathrm{Mod}_{\mathcal{H}_{w_0(M)^+}}$ is left adjoint to $- \otimes_{\mathcal{H}_{w_0(M)^+},\theta} \mathcal{H}$ and the $\mathcal{H}_{w_0(M)^+}$-equivariance of the linear map

$$m \mapsto m \otimes f_{\tilde{w}_0^M} : \tilde{\mathfrak{w}}_0^M(\mathcal{M}) \to \mathbb{I}^{\mathcal{H}}_{\mathcal{H}_M}(\mathcal{M}) \tag{33}$$

implies the $\mathcal{H}$-equivariance of (31), i.e. of (32). Let $\mathcal{H}_M \xrightarrow{j} \mathcal{H}_{w_0(M)}$ denote the isomorphism induced by $\tilde{w}_0^M$ (Proposition 2.20), and θ_M the linear map $\mathcal{H}_M \xrightarrow{\theta} \mathcal{H}$. The $\mathcal{H}_{w_0(M)^+}$-invariance of the map $m \mapsto m \otimes f_{\tilde{w}_0^M}$ is equivalent to:

$$f_{\tilde{w}_0^M} \theta_{w_0(M)}(h) = j^{-1}(h) f_{\tilde{w}_0^M} \quad \text{for } h \in \mathcal{H}_{w_0(M)^+}, \tag{34}$$

We can suppose that h lies in the Bernstein basis of $\mathcal{H}_{w_0(M)^+}$. Let $\tilde{w} \in W_{w_0(M)^+}(1)$ and $h = E_{w_0(M)}(\tilde{w})$. As $\theta_{w_0(M)}(E_{w_0(M)}(\tilde{w})) = E(\tilde{w})$, and $j^{-1}(E_{w_0(M)}(\tilde{w}))$ is equal to $E_M((\tilde{w}_0^M)^{-1} \tilde{w} \tilde{w}_0^M)$, (34) is equivalent to:

Proposition 4.7. $f_{\tilde{w}_0^M} E(\tilde{w}) = E_M((\tilde{w}_0^M)^{-1}\tilde{w}\tilde{w}_0^M) f_{\tilde{w}_0^M}$ *for* $w \in W_{w_0(M)^+}$.

Proof. By the usual reduction arguments, we suppose that the $\mathfrak{q}(s)$ are invertible in R. Using $W_{w_0(M)^+} = \Lambda_{w_0(M)^+} \rtimes W_{w_0(M),0}$, the product formula (8) and Lemma 2.23 we reduce to $w \in \Lambda_{w_0(M)^+} \cup W_{w_0(M),0}$. By induction on the length in $W_{w_0(M),0}$ with respect to $S_{w_0(M)}$, we reduce to $w \in \Lambda_{w_0(M)^+} \cup S_{w_0(M)}$.

Let $d \in W_0^M$. We have $(f_{\tilde{w}_0^M} E(\tilde{w}))(T_{\tilde{d}}) = f_{\tilde{w}_0^M}(E(\tilde{w})T_{\tilde{d}})$ in $\mathcal{H}_M$. We have to prove

$$(35)\qquad f_{\tilde{w}_0^M}(E(\tilde{w})T_{\tilde{d}}) = \begin{cases} 0 & \text{if } d \neq w_0^M, \\ E_M((\tilde{w}_0^M)^{-1}\tilde{w}\tilde{w}_0^M) & \text{if } \tilde{d} = \tilde{w}_0^M. \end{cases}$$

for $w \in \Lambda_{w_0(M)^+} \cup S_{w_0(M)}$.

(i) $w = \lambda \in \Lambda_{w_0(M)^+}$. Let $\mathcal{A}$ denote the subalgebra of $\mathcal{H}$ of basis $(E(\tilde{x}))_{\tilde{x}\in\Lambda(1)}$ [Vig1, Cor. 2.8]. By the Bernstein relations [Vig1, Thm. 2.9], we have

$E(\tilde{\lambda})T_{\tilde{d}} = T_{\tilde{d}} E((\tilde{d})^{-1}\tilde{\lambda}\tilde{d}) + \sum T_{\tilde{w}} a_{\tilde{w}}$,

where $a_{\tilde{w}} \in \mathcal{A}$ and the sum is over $\tilde{w} \in W_0(1), w < d$. If $d \neq w_0^M$, the image by $f_{\tilde{w}_0^M}$ of the right hand side vanishes because $w \in w_0^M W_{M,0}, w \leq d$ implies $w = d = w_0^M$; hence $f_{\tilde{w}_0^M}(E(\tilde{\lambda})T_{\tilde{d}}) = 0$ as we want. For $\tilde{d} = \tilde{w}_0^M$, using $(w_0^M)^{-1}\lambda\tilde{w}_0^M \in W_{w_0(M)^-}$, we have $f_{\tilde{w}_0^M}(E(\tilde{\lambda})T_{\tilde{w}_0^M}) = f_{\tilde{w}_0^M}(T_{\tilde{w}_0^M} E((\tilde{w}_0^M)^{-1}\tilde{\lambda}\tilde{w}_0^M) = \theta^*(E((\tilde{w}_0^M)^{-1}\tilde{\lambda}\tilde{w}_0^M)) = E_M((\tilde{w}_0^M)^{-1}\tilde{\lambda}\tilde{w}_0^M)$.

(ii) $w = s \in S_{w_0(M)}$. We have $w_0 s w_0 \in S_M$, $w_0 s w_0 w_{M,0} < w_{M,0}$ and $s w_0^M = s w_0 w_{M,0} = w_0 w_0 s w_0 w_{M,0} > w_0 w_{M,0} = w_0^M$.

Assume $sd < d$. We deduce $d \neq w_0^M$. Assume $\tilde{d} = \tilde{s}(\tilde{sd})$. Then

$E(\tilde{s})T_{\tilde{d}} = T_{\tilde{s}}T_{\tilde{d}} = T_{\tilde{s}}^2 T_{(\tilde{sd})} = (\mathfrak{q}(s)(\tilde{s})^2 + \mathfrak{c}(\tilde{s})T_{\tilde{s}})T_{(\tilde{sd})} = \mathfrak{q}(s)(\tilde{s})^2 T_{(\tilde{sd})} + \mathfrak{c}(\tilde{s})T_{\tilde{d}}$. We deduce that $f_{\tilde{w}_0^M}(E(\tilde{s})T_{\tilde{d}}) = 0$.

Assume $sd > d$. We write $\tilde{s}\,\tilde{d} = \tilde{d}_1\tilde{u}$ with $d_1 \in W_0^M, u \in W_{M,0}$. Then $T_{\tilde{s}}T_{\tilde{d}} = T_{\tilde{s}\tilde{d}} = T_{\tilde{d}_1\tilde{u}}$. Therefore $f_{\tilde{w}_0^M}(E(\tilde{s})T_{\tilde{d}}) = f_{\tilde{w}_0^M}(T_{\tilde{d}_1\tilde{u}}) = 0$ if $d_1 \neq w_0^M$. We suppose now $d_1 = w_0^M$. We have $d \leq w_0^M \leq sd$ hence $w_0^M = d$ or $w_0^M = sd$. In the latter case, a reduced decomposition of w_0^M starts by s. But this is incompatible with $s \in S_{w_0(M)}$ because $w_0^M = {}^{w_0(M)}w_0$. We deduce that $d = w_0^M$. For $\tilde{d} = \tilde{w}_0^M$, we have $f_{\tilde{w}_0^M}(E(\tilde{s})T_{\tilde{w}_0^M}) = f_{\tilde{w}_0^M}(T_{\tilde{s}\,\tilde{w}_0^M}) = f_{\tilde{w}_0^M}(T_{\tilde{w}_0^M} T_{(w_0^M)^{-1}\tilde{s}\tilde{w}_0^M}) = f_{\tilde{w}_0^M}(T_{\tilde{w}_0^M} E_{(w_0^M)^{-1}\tilde{s}\tilde{w}_0^M}) = \theta^*(E_{(w_0^M)^{-1}\tilde{s}\tilde{w}_0^M})) = E_M((\tilde{w}_0^M)^{-1}\tilde{s}\tilde{w}_0^M)$. This ends the proof of Proposition 4.7 hence of Theorem 1.8. ☐

Corollary 4.8. *The right $\mathcal{H}$-modules $\mathcal{H}_M \otimes_{\mathcal{H}_{M^+},\theta} \mathcal{H}$ and $\mathrm{Hom}_{\mathcal{H}_{w_0(M)^-},\theta^*}(\mathcal{H}, \mathcal{H}_{w_0(M)})$ are isomorphic.*

4.4 Transitivity of the coinduction

Let $S_M \subset S_{M'} \subset S$. By Proposition 2.21, the algebra isomorphisms

$$\mathcal{H}_M \xrightarrow{j} \mathcal{H}_{w_0(M)}, \quad \mathcal{H}_M \xrightarrow{j'} \mathcal{H}_{w_{M',0}(M)} \xrightarrow{k''} \mathcal{H}_{w_0(M)}$$

corresponding to $\tilde{w}_0^M, \tilde{w}_{M'}^M, \tilde{w}_0^{M'}$, $\tilde{w}_0^M = \tilde{w}_0^{M'}\tilde{w}_{M'}^M$, satisfy $j = k'' \circ j'$. The associated equivalences of categories, denoted by

$$(36)\qquad \mathcal{M}_{\mathcal{H}_M} \xrightarrow{\tilde{\mathfrak{w}}_0^M} \mathcal{M}_{\mathcal{H}_{w_0(M)}}, \quad \mathcal{M}_{\mathcal{H}_M} \xrightarrow{\tilde{\mathfrak{w}}_{M'}^M} \mathcal{M}_{\mathcal{H}_{w_{M',0}(M)}} \xrightarrow{\tilde{\mathfrak{w}}_{0,k}^{M'}} \mathcal{M}_{\mathcal{H}_{w_0(M)}},$$

satisfy $\tilde{\mathfrak{w}}_0^M = \tilde{\mathfrak{w}}_{0,k}^{M'} \circ \tilde{\mathfrak{w}}_{M'}^M$. We refer to this as the transitivity of the w_0-twisting.

Lemma 4.9. *The functors $\tilde{\mathfrak{w}}_0^{M'} \circ I_{\mathcal{H}_{w_{M',0}(M)}}^{\mathcal{H}_{M'}}$ and $I_{\mathcal{H}_{w_0(M)}}^{\mathcal{H}_{w_0(M')}} \circ \tilde{\mathfrak{w}}_{0,k}^{M'}$ from $\mathrm{Mod}_{\mathcal{H}_{w_{M',0}(M)}}$ to $\mathrm{Mod}_{\mathcal{H}_{w_0(M')}}$ are isomorphic.*

The proof gives an explicit isomorphism.

Proof. Let $\mathcal{M} \in \mathrm{Mod}_{\mathcal{H}_{w_{M',0}(M)}}$. The R-module $\mathcal{M} \otimes_{\mathcal{H}_{w_{M',0}(M)^+},\theta} \mathcal{H}_{M'}$ with the right action of $\mathcal{H}_{w_0(M')}$ defined by $(x \otimes T^{M'}_{\tilde{u}}) T^{w_0(M')}_{\tilde{w}_o^{M'} \tilde{v} (\tilde{w}_o^{M'})^{-1}} = x \otimes T^{M'}_{\tilde{u}} T^{M'}_{\tilde{v}}$ for $x \in \mathcal{M}, u, v \in W_{M'}$, is $\tilde{\mathfrak{w}}_0^{M'} \circ I^{\mathcal{H}_{M'}}_{\mathcal{H}_{w_{M',0}(M)}}(\mathcal{M})$.

As $k''(\mathcal{H}_{w_{M',0}(M)^+}) = \mathcal{H}_{w_0(M)^+}$ (Proposition 2.21), the R-linear map $\mathcal{M} \otimes_R \mathcal{H}_{M'} \to \tilde{\mathfrak{w}}_{0,k}^{M'}(\mathcal{M}) \otimes_{\mathcal{H}_{w_0(M)^+},\theta} \mathcal{H}_{w_0(M')}$ defined by $x \otimes T^{M'}_{\tilde{u}} \to x \otimes T^{w_0(M')}_{\tilde{w}_0^{M'} \tilde{u} (\tilde{w}_0^{M'})^{-1}}$ is the composite of the quotient map $\mathcal{M} \otimes_R \mathcal{H}_{M'} \to \tilde{\mathfrak{w}}_0^{M'} \circ \mathcal{M} \otimes_{\mathcal{H}_{w_{M',0}(M)^+}} \mathcal{H}_{M'}$, and of the bijective linear map

$$\tilde{\mathfrak{w}}_0^{M'} \circ I^{\mathcal{H}_{M'}}_{\mathcal{H}_{w_{M',0}(M)}}(\mathcal{M}) \to \tilde{\mathfrak{w}}_{0,k}^{M'}(\mathcal{M}) \otimes_{\mathcal{H}_{w_0(M)^+},\theta} \mathcal{H}_{w_0(M')}.$$

The displayed map is clearly $\mathcal{H}_{w_0(M')}$-equivariant. □

Proposition 4.10. *The coinduction is transitive.*

Proof. By the transitivity of the w_0-twisting (36), Lemma 4.9, and the transitivity of the induction (Proposition 4.3), we have:
$\mathbb{I}^{\mathcal{H}}_{\mathcal{H}_{M'}} \circ \mathbb{I}^{\mathcal{H}_{M'}}_{\mathcal{H}_M} = I^{\mathcal{H}}_{\mathcal{H}_{w_0(M')}} \circ \tilde{\mathfrak{w}}_0^{M'} \circ I^{\mathcal{H}_{w_0(M')M'}}_{\mathcal{H}_{w_0(M)}} \circ \tilde{\mathfrak{w}}^M_{M'} = I^{\mathcal{H}}_{\mathcal{H}_{w_0(M')}} \circ I^{\mathcal{H}_{w_0(M')}}_{\mathcal{H}_{w_0(M)}} \circ \tilde{\mathfrak{w}}^{M'}_{0,k} \circ \tilde{\mathfrak{w}}^M_{M'} = I^{\mathcal{H}}_{\mathcal{H}_{w_0(M')}} \circ I^{\mathcal{H}_{w_0(M')}}_{\mathcal{H}_{w_0(M)}} \circ \tilde{\mathfrak{w}}_0^M = I^{\mathcal{H}}_{\mathcal{H}_{w_0(M)}} \circ \tilde{\mathfrak{w}}_0^M = \mathbb{I}^{\mathcal{H}}_{\mathcal{H}_M}$. □

Proof of Theorem 1.9. The induction $I^{\mathcal{H}}_{\mathcal{H}_M}$ is equivalent to $\mathbb{I}^{\mathcal{H}}_{\mathcal{H}_{w_0(M)}} \circ \tilde{\mathfrak{w}}_0^M$. The coinduction $\mathbb{I}^{\mathcal{H}}_{\mathcal{H}_M}$ is the composite of the restriction $\mathrm{Mod}_{\mathcal{H}_M} \to \mathrm{Mod}_{\mathcal{H}_{M^-}}$ and of $\mathrm{Hom}_{\mathcal{H}_{M^-},\theta^*}(\mathcal{H}, -) : \mathrm{Mod}_{\mathcal{H}_{M^-}} \to \mathrm{Mod}_{\mathcal{H}}$. These functors admit left adjoints, the restriction $\mathrm{Mod}_{\mathcal{H}} \to \mathrm{Mod}_{\mathcal{H}_{M^-}}$ for $\mathrm{Hom}_{\mathcal{H}_{M^-},\theta^*}(\mathcal{H}, -)$, the induction $- \otimes_{\mathcal{H}_{M^-}} \mathcal{H}_M : \mathrm{Mod}_{\mathcal{H}_{M^-}} \to \mathrm{Mod}_{\mathcal{H}_M}$ for the restriction $\mathrm{Mod}_{\mathcal{H}_M} \to \mathrm{Mod}_{\mathcal{H}_{M^-}}$, hence $- \otimes_{\mathcal{H}_{M^-},\theta^*} \mathcal{H}_M : \mathrm{Mod}_{\mathcal{H}} \to \mathrm{Mod}_{\mathcal{H}_M}$ for $\mathbb{I}^{\mathcal{H}}_{\mathcal{H}_M}$, and $(\tilde{\mathfrak{w}}_0^M)^{-1} \circ (- \otimes_{\mathcal{H}_{w_0(M)^-},\theta^*} \mathcal{H}_{w_0(M)}) \simeq \tilde{\mathfrak{w}}_0^{w_0(M)} \circ (- \otimes_{\mathcal{H}_{w_0(M)^-},\theta^*} \mathcal{H}_{w_0(M)})$ for $\mathbb{I}^{\mathcal{H}}_{\mathcal{H}_{w_0(M)}} \circ \tilde{\mathfrak{w}}_0^M$.

5

Let $\Delta = \Delta_1 \cup \Delta_2$ be an orthogonal decomposition, $\{i, j\} = \{1, 2\}$ and $\epsilon \in \{+, -\}$. In the notations, we will often replace a (lower or upper) index M_i by a (lower or upper) index i. The orthogonal decomposition of Δ corresponds to orthogonal decompositions $\Sigma = \Sigma_1 \cup \Sigma_2, S = S_2 \cup S_2, \Sigma^{aff} = \Sigma_1^{aff} \cup \Sigma_2^{aff}, S^{aff} = S_1^{aff} \cup S_2^{aff}$ and direct products $W^{aff} = W_1^{aff} \times W_2^{aff}, \Lambda^{aff} = \Lambda_1^{aff} \times \Lambda_2^{aff}, W_0 = W_{1,0} \times W_{2,0}$. We have the semidirect products $W_j^{aff} = \Lambda_j^{aff} \rtimes W_{j,0}, W^{aff} = \Lambda^{aff} \rtimes W_0, W_j = W_j^{aff} \rtimes \Omega_j = \Lambda \rtimes W_{j,0}$ analogous to $W = W^{aff} \rtimes \Omega = \Lambda \rtimes W_0$. The group W_j acts by the identity on Σ_i^{aff}. For $w \in W$ we have $w(\Sigma_i^{aff}) \subset \Sigma_i^{aff}$ and $\ell(w) = \ell_1(w) + \ell_2(w)$ where

$$\ell(w) = |\Sigma^{aff,+} \cap w(\Sigma^{aff,-})|, \quad \ell_i(w) = |\Sigma_i^{aff,+} \cap w(\Sigma_i^{aff,-})|. \tag{37}$$

The kernel of ℓ_i is $W_j^{aff}\Omega$ (hence Ω normalizes W_j^{aff}). For $(\lambda, w_0) \in \Lambda \times W_0$ we have:

$$\ell(\lambda w_0) = \sum_{\alpha \in \Sigma^+ \cap w_0(\Sigma^+)} |\langle \alpha, \nu(\lambda) \rangle| + \sum_{\alpha \in \Sigma^+ \cap w_0(\Sigma^-)} |\langle \alpha, \nu(\lambda) \rangle - 1|, \tag{38}$$

$$\ell_i(\lambda w_0) = \sum_{\alpha \in \Sigma_i^+ \cap w_0(\Sigma_i^+)} |\langle \alpha, \nu(\lambda) \rangle| + \sum_{\alpha \in \Sigma_i^+ \cap w_0(\Sigma_i^-)} |\langle \alpha, \nu(\lambda) \rangle - 1|. \tag{39}$$

For $\ell(\lambda w_0)$ see [Vig1, Cor. 5.10, Cor. 5.11]. For $\ell_i(\lambda w_0)$ *** Decomposing $\Sigma^+ = \Sigma_i^+ \sqcup \Sigma_j^+$ and recalling that $w_0 \in W_{0,i}$ fixes Σ_j, and that Σ_i vanishes on $\nu(\Lambda_j^{aff})$. The restriction of ℓ and of ℓ_i to W_i is the length associated to (W_i^{aff}, S_i^{aff}) and ℓ_i vanishes on W_j.

Lemma 5.1. *The group W normalizes Λ_i^{aff}. For $w \in W_i$ and $\mu \in \Lambda_j^{aff}$ we have $\ell(\mu w) = \ell(\mu) + \ell(w)$.*

Proof. The group Λ is commutative and contains Λ_i^{aff}, the group $W_{i,0}$ normalizes Λ_i^{aff}, and the elements of $W_{j,0}$ commute with those of Λ_i^{aff}. Hence the group $W = \Lambda \rtimes (W_{i,0} \times W_{j,0})$ normalizes Λ_i^{aff}.

Using $W_i = \Lambda \rtimes W_{0,i}$, we write $w = \lambda w_0$ where $(\lambda, w_0) \in \Lambda \times W_{0,i}$. We have $\Sigma^+ \cap w_0(\Sigma^+) = (\Sigma_i^+ \cap w_0(\Sigma_i^+)) \sqcup \Sigma_j^+$ and $\Sigma^+ \cap w_0(\Sigma^-) = \Sigma_i^- \cap w_0(\Sigma_i^-)$. We apply the formula (38) to $(\mu\lambda, w_0) \in \Lambda \times W_0$ to obtain the equality between the lengths:

$$\begin{aligned}\ell(\mu w) &= \sum_{\alpha \in \Sigma_i^+ \cap w_0(\Sigma_i^+)} |\langle \alpha, \nu(\mu\lambda)\rangle| + \sum_{\alpha \in \Sigma_j^+} |\langle \alpha, \nu(\mu\lambda)\rangle| + \sum_{\alpha \in \Sigma_i^+ \cap w_0(\Sigma_i^-)} |\langle \alpha, \nu(\mu\lambda)\rangle - 1| \\ &= \sum_{\alpha \in \Sigma_i^+ \cap w_0(\Sigma_i^+)} |\langle \alpha, \nu(\lambda)\rangle| + \sum_{\alpha \in \Sigma_j^+} |\langle \alpha, \nu(\mu)\rangle| + \sum_{\alpha \in \Sigma_i^+ \cap w_0(\Sigma_i^-)} |\langle \alpha, \nu(\lambda)\rangle - 1| \\ &= \ell(\mu) + \ell(w).\end{aligned}$$

□

Let ${}_1W^{aff} = {}_1W_1^{aff} \times {}_1W_2^{aff} \subset W^{aff}(1)$ be an extension of W^{aff}. We have $W(1) = {}_1W^{aff}\Omega(1)$. Let ${}_1W_{i,0}$ and ${}_1\Lambda_i^{aff}$ denote the inverse images in ${}_1W_i^{aff}$ of $W_{i,0}$ and Λ_i^{aff}.

Let $\mathcal{H}_i$ the Levi algebra of $\mathcal{H}$ of basis $(T^i(\tilde{w}))_{\tilde{w} \in W_i(1)}$ associated to Δ_i.

Lemma 5.2. *(i) The left ideal $\mathcal{J}_1 \subset \mathcal{H}_1$ generated by $T^1_{\tilde{\mu}} - 1$ for $\tilde{\mu} \in {}_1\Lambda_2^{aff}$ is equal to the right ideal generated by these elements, and also to the R-submodule generated by $E_1(\tilde{\mu}\tilde{w}) - E_1(\tilde{w})$ for $\tilde{\mu} \in {}_1\Lambda_2^{aff}, \tilde{w} \in W_1(1)$.*

(ii) The ideal $\mathcal{J} \subset \mathcal{H}$ generated by $T^*_{\tilde{w}} - 1$ for $\tilde{w} \in {}_1W_2^{aff}$ contains $E(\tilde{\mu}\tilde{w}) - E(\tilde{w})$ for $\tilde{\mu} \in {}_1\Lambda_2^{aff}, \tilde{w} \in W_1(1)$.

(iii) $\mathcal{J} = \oplus_{\tilde{v} \in {}_1W^{aff} \backslash W(1)} (\mathcal{J} \cap \sum_{\tilde{w} \in {}_1W^{aff}\tilde{v}} T_{\tilde{w}}) = \oplus_{\tilde{v} \in {}_1W^{aff} \backslash W(1)} (\mathcal{J} \cap \sum_{\tilde{w} \in {}_1W^{aff}\tilde{v}} E(\tilde{w}))$.

(iv) Let $w \in W_1(1)$ written as $w = ab, a \in {}_1W_2^{aff}, \ell_2(b) = 0$. Then $E(w) - T_b \in \sum_{c<b} \mathbb{Z}T_c + \mathcal{J}$.

(v) $\mathcal{J} \cap \sum_{b \in W(1), \ell_2(b)=0} \mathbb{Z}T_b$ is contained in the ideal of $\mathcal{H}$ generated by $T^1_{\tilde{\mu}} - 1$ for $\tilde{\mu} \in Z_k \cap {}_1W_2^{aff}$.

Proof. (i) Note that $\ell_1(\mu) = 0$, that W_1 normalizes Λ_2^{aff} (Lemma 5.1) and $W_1(1)$ normalizes ${}_1\Lambda_2^{aff}$ ***. This implies that $T^1_{\tilde{\mu}} = T^{1,*}_{\tilde{\mu}} = E_1(\tilde{\mu})$ and we have $E_1(\tilde{\mu})E_1(\tilde{w}) = E_1(\tilde{\mu}\tilde{w}) = E_1(\tilde{w}\tilde{\mu}') = E_1(\tilde{w})E_1(\tilde{\mu}')$ where $\tilde{w} \in W_1(1), \tilde{\mu}' = (\tilde{w})^{-1}\tilde{\mu}\tilde{w} \in {}_1\Lambda_2^{aff}$.

(ii) We have $\ell(\mu w) = \ell(\mu) + \ell(w)$ (Lemma 5.1), hence $E(\tilde{\mu}\tilde{w}) = E(\tilde{\mu})E(\tilde{w})$. If μ is dominant we have $E(\tilde{\mu}) = T^*_{\tilde{\mu}}$ and $E(\tilde{\mu}\tilde{w}) - E(\tilde{w}) \in \mathcal{J}$. For a general μ, choose $\mu_0 \in {}_1\Lambda_2^{aff}$ dominant such that $\mu_0\mu^{-1}$ is dominant and write $E(\tilde{\mu}\tilde{w}) - E(\tilde{w}) = E(\tilde{\mu}\tilde{w}) - E(\tilde{\mu}_0\tilde{\mu}^{-1}\tilde{\mu}\tilde{w}) + E(\mu_0\tilde{w}) - E(\tilde{w})$. We get $E(\tilde{\mu}\tilde{w}) - E(\tilde{w}) \in \mathcal{J}$. □

Proposition 5.3. *The homomorphism $\mathcal{H}_1^- \xrightarrow{\theta^*} \mathcal{H} \to \mathcal{H}/\mathcal{J}$ is surjective of kernel $\mathcal{H}_1^- \cap \mathcal{J}_1$.*

The proposition in the particular case of the pro-p Iwahori Hecke algebra of a reductive p-adic group over an algebraically closed field of characteristic p is proved in [Abe, Prop. 4.16].

Proof. (i) Surjectivity. Let $\tilde{w} \in W(1)$. We want to prove that $T^*_{\tilde{w}} \in \theta^*(\mathcal{H}_1^-) + \mathcal{J}$. Using the semidirect product $W = W^{aff} \rtimes \Omega$, we write $\tilde{w} = \tilde{w}_2 \tilde{w}_1 \tilde{u}$ with $\tilde{w}_i \in {}_1W'_i$ and $\tilde{u} \in \Omega(1)$. We suppose, as we can, $\tilde{w}_2$ not in $Z_k - \{1\}$. As seen above $\ell(\tilde{w}) = \ell(\tilde{w}_1) + \ell(\tilde{w}_2)$ hence $T^*_{\tilde{w}} = T^*_{\tilde{w}_2} T^*_{\tilde{w}_1} T^*_{\tilde{u}}$. If $\tilde{w}_2 \neq 1$ we have $T^*_{\tilde{w}} \in T^*_{\tilde{w}_1} T^*_{\tilde{u}} + \mathcal{J}$. Hence we can suppose $\tilde{w} = \tilde{w}_1 \tilde{u}$.

Suppose more generally $\ell_2(\tilde{w}) = 0$. As $T_{\tilde{w}} = E(\tilde{w}) + \sum_{\tilde{v} < \tilde{w}} E(\tilde{v})$ and $\tilde{v} < \tilde{w}$ imply $\ell_2(\tilde{v}) = 0$, to prove $T^*_{\tilde{w}} \in \theta^*(\mathcal{H}_1^-) + \mathcal{J}$, it suffices to prove $E(\tilde{w}) \in \theta^*(\mathcal{H}_1^-) + \mathcal{J}$.

Using the semidirect product $W = \Lambda \rtimes W_0$, we write $\tilde{w} = \tilde{\lambda} \tilde{w}_{2,0} \tilde{w}_{1,0}$ with $\tilde{\lambda} \in \Lambda(1), \tilde{w}_{i,0} \in {}_1W_{i,0}$. As $\ell_2(\tilde{w}) = 0$, we have $\alpha(\nu(\lambda) \in \{0,1\}$ for $\alpha \in \Sigma_2^+$ by *** hence $\tilde{\lambda} \tilde{w}_{1,0} \in W^-_{M_1}$. We have ***

$$E(\tilde{w}) T^*_{\tilde{w}_{2,0}^{-1}} = E(\tilde{\lambda} \tilde{w}_{1,0}).$$

This implies $E(\tilde{w}) \in E(\tilde{\lambda} \tilde{w}_{1,0}) + \mathcal{J} \in \theta^*(\mathcal{H}_1^-) + \mathcal{J}$. We proved that the homomorphism $\mathcal{H}_1^- \xrightarrow{\theta^*} \mathcal{H} \to \mathcal{H}/\mathcal{J}$ is surjective.

(ii) Kernel. Let $\sum_{\tilde{w} \in W_1(1)} c_{\tilde{w}} E_1(\tilde{w}) \in \mathcal{H}_1$ such that

By Lemma 5.2 (ii), the kernel $\mathrm{Ker}(\mathcal{H}_1^- \to \mathcal{H}/\mathcal{J})$ contains $\mathcal{H}_1^- \cap \mathcal{J}_1$. We prove the inverse inclusion: if $\sum_{\tilde{w} \in W_{1,-}(1)} c_{\tilde{w}} E(\tilde{w}) \in \mathcal{J}$ then $\sum_{\tilde{w} \in W_{1,-}(1)} c_{\tilde{w}} E_1(\tilde{w}) \in \mathcal{J}_1$.

Let $\tilde{v} \in W_{1,-}(1)$ and $\sum_{\tilde{w} \in {}_1W^{aff} \tilde{v}} c_{\tilde{w}} E(\tilde{w}) \in \mathcal{J}$.

Using $W_{1,-} = \Lambda_{1,-} W_{1,0}$ we write $\tilde{v} = \tilde{\lambda}' \tilde{w}'_0, \tilde{\lambda}' \in \Lambda_{1,-}(1), \tilde{w}'_0 \in W_{1,0}(1)$, Let $\tilde{\lambda} \in \Lambda(1), \tilde{w}_0 \in W_0(1)$ such that $\tilde{w} = \tilde{\lambda} \tilde{w}_0 \in {}_1W^{aff} \tilde{v}$. We have $\tilde{\lambda}' \tilde{\lambda}^{-1} \in \Lambda^{aff}$. Using ${}_1\Lambda^{aff} = {}_1\Lambda_1^{aff} \times {}_1\Lambda_2^{aff}$ we write $\tilde{\lambda}' \tilde{\lambda}^{-1} = \tilde{\lambda}_1 \tilde{\lambda}_2, \tilde{\lambda}_1 \in {}_1\Lambda_1^{aff}, \tilde{\lambda}_2 \in {}_1\Lambda_2^{aff}$. As $\ell_1(\lambda_2) = 0$ we have $E_1(\tilde{w}) - E_1(\tilde{\lambda}_2 \tilde{w}) = (1 - E_1(\tilde{\lambda}_2)) E_1(\tilde{w}) \in \mathcal{J}_1$.

As $\lambda' \in \Lambda_{1,-}$, $\lambda_2 \lambda \in \Lambda_{1,-}$

Using $W = (W_2^{aff} \times W_1^{aff}) \rtimes \Omega$ we write $\tilde{v} = \tilde{w}_2 \tilde{u}'_2, \tilde{w}_2 \in W_2^{aff}(1), u'_2 \in W_1^{aff}(1)\Omega(1)$. We have also $\tilde{w} = \tilde{w}_2 \tilde{u}_2, u'_2 \in W_1^{aff}(1)\Omega(1)$.

Put $r = \max \ell(\tilde{w}_2^{-1} \tilde{w}) \; | c_{\tilde{w}}) \neq 0$.

□

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