Remark on congruences between algebraic automorphic forms

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Abstract. The purpose of this paper is to present and to extend the method of **[9]** in order to obtain congruences between algebraic automorphic forms **[7]**. The congruences are "easy" to obtain; this is "soft" mathematics as Steve Rallis likes to say. However they are quite important; via the "hard" functorialities of Langlands, they allow one to get congruences between automorphic representations of $GL(n)$ or Galois representations, as for instance in **[5]**, **[6]**, or in the proof of the Langlands correspondence modulo ℓ **[9]**. It is a pleasure to dedicate this work to Steve Rallis, congruences are related to his work via the theory of L-functions where his contribution is so fundamental.

Let p be a prime number, F a number field, and G a reductive connected group over F such that $\{1\}$ is an arithmetic subgroup. We denote by $G' = \operatorname{Res}_{F/\mathbf{Q}} G$ the scalar restriction of G to $\mathbf{Q}$ **[8]** and by $G_A = G'_A = G_\infty \times G^\infty$ the adele group of G or of G'.

Theorem 0.1. *We consider*

- V, V' two absolutely irreducible algebraic representations of G',

- v a finite place of F and K_v an open compact subgroup of $G(F_v)$ of pro-order prime to p,

- $K^{v,p}$ an open compact subgroup of the component $G^{\infty,v,p}$ of G^∞ outside of v, p, and W^p a complex irreducible smooth representation of $K^p = K_v \times K^{v,p}$.

Then we have the following property:

Any irreducible automorphic representation π_A of G_A of algebraic type V which contains (K^p, W^p) is rational over a number field, its component $\pi^{\infty,p}$ outside of ∞, p is p-integral, and is congruent modulo p to an irreducible automorphic representation of G_A of algebraic type V' which contains (K^p, W^p) and has finite slope at p.

The term **finite slope** or **congruent modulo** p used in the theorem are defined as follows. An irreducible automorphic representation π_A of G_A has a **finite slope at** p, if the p-component π_p of π_A contains the trivial representation of a pro-p-Iwahori subgroup Iw_p of $G_p = G'(\mathbf{Q}_p)$.

Two irreducible automorphic representations π_A and π'_A of G_A are called **congruent modulo** p if there exists an isomorphism i_p from $\mathbf{C}$ to an algebraic closure $\overline{\mathbf{Q}}_p$ of $\mathbf{Q}_p$, a finite extension $E/\mathbf{Q}_p$ contained in $\overline{\mathbf{Q}}_p$ and a finite set S of places of F containing the places above ∞ and p, such that

- the components π^S and $\pi^{',S}$ of π_A and π'_A outside of S are unramified,

- the values of the characters $i_p\lambda^S, i_p\lambda^{',S} : \mathcal{H}^S \to \mathbf{C} \to \overline{\mathbf{Q}}_p$ of the commutative Satake-Hecke algebra $\mathcal{H}^S$ given by π^S and $\pi^{',S}$ belong to the ring of integers O_E of E and are congruent $i_p\lambda^S \equiv i_p\lambda^{',S}$ modulo the maximal ideal of O_E.

The theorem has a converse:

Let W_p be an irreducible complex representation of the pro-p-Iwahori group Iw_p. Any irreducible automorphic representation of G_A of algebraic type V', of finite slope at p, which contains (K^p, W^p), is congruent modulo p to an irreducible automorphic representation of G_A of algebraic type V, which contains $(Iw_p \times K^p, W_p \times W^p)$.

When $p \neq 2$, we obtain a slightly stronger form of the theorem and of its converse in §20.

One gets also congruences modulo $p^n, n > 1$, between automorphic forms instead of representations, in §17.

The method of the proof is a combination of arguments given in **[9]** only for automorphic forms trivial at ∞, and in the excellent introduction to algebraic automorphic forms **[7]**.

The beautiful recent results of Buzzard, Chenevier, Emerton and Kisin on p-adic interpolation of automorphic forms make this paper look out of date, but their results do not imply our theorem. The interest of Barry Mazur when I sent him these congruences in July 2002, motivated me to write their proof. It is a pleasure to thank him and Steve Rallis for many friendly discussions.

Proof of the theorem (see §19)

1 *We consider a number field F and a reductive connected group G over F such that $G(F)$ is discrete in the group G^∞ of finite adeles.*

Equivalent properties are given in **[7]**. One of them is that the arithmetic subgroups are finite. This condition is very restrictive. There are only three possibilities:

- *$F = \mathbf{Q}$ and the split centre S of G is a maximal $\mathbf{R}$-split torus of G,*
- *F is an imaginary quadratic field and G is a F-split torus,*

- F is totally real and the infinite part G_∞ of G is compact (in particular the split centre of G is trivial).

The group G_∞ has a unique maximal compact subgroup K_∞. The connected components G_∞^o and K_∞^o are related to the split and anisotropic centres S and C of G' as follows. We have $K_\infty^o = (C(G', G'))_\infty$ and $G_\infty^o = S_\infty^o \times K_\infty^o$. We denote by S_1 the finite algebraic $\mathbf{Q}$-group of $x \in S$ with $x^2 = 1$. The groups G_∞, K_∞ are equal to $G_\infty = S_\infty K_\infty = S_{1,\infty} G_\infty^o$ and $K_\infty = S_{1,\infty} K_\infty^o$. They have the same group of components $\pi_{G,o} = G_\infty / G_\infty^o = K_\infty / K_\infty^o$, isomorphic to the 2-group $S_{1,\infty}/(S_{1,\infty} \cap G_\infty^o)$. We have $S_{1,\infty} = S_1(\mathbf{Q})$.

2 The representations are always left representations (unless right is specified).

An irreducible automorphic representation of G_A of dimension 1 is a Hecke character, i.e. a continuous morphism $G(F)\backslash G_A \to \mathbf{C}^*$ (for any reductive connected group over F). We denote by Ω the set of Hecke characters ω of $S_A = S(\mathbf{Q}) \times S_\infty^o \times \prod_p S(\mathbf{Z}_p)$.

An automorphic form on G_A is a finite sum $f = \sum_{\omega \in \Omega} f_\omega$ of functions $f_\omega : G(F)\backslash G_A \to \mathbf{C}$ which are smooth (continuous and K_∞-finite) on G_∞, locally constant on G^∞, and $f_\omega(sg) = \omega(s) f_\omega(g)$ for all $s \in S_A, g \in G_A$. The action of G_A by right translations on the space $\mathcal{S}_\mathbf{C} = \oplus_{\omega \in \Omega} \mathcal{S}_{\mathbf{C},\omega}$ of automorphic forms is semi-simple. The irreducible constituents are the irreducible automorphic representations of G_A.

A non-degenerate bilinear symmetric form $< f_{\omega,1}, f_{\omega,2} >$ on $\mathcal{S}_{\mathbf{C},\omega}$ is the integral of $f_{\omega,1} \overline{f}_{\omega,2} \eta^{-1}$ on the compact space $G(F) S_A \backslash G_A$, for an invariant Haar measure, where η is the positive Hecke character of G_A equal to $\overline{\omega}\omega$ on S_A.

3 Let (R, H, G, M, X) be any commutative ring R, a subgroup H of a group G, an R-module M with an action of H, and a space X with a right action of G. We denote by $C(X, R)$ the R-module of functions $f : X \to R$ with the natural action of G and by $\tilde{M}$ the R-module of linear forms on M with the contragredient action of H, such that $< \tilde{m}, m > = < h\tilde{m}, hm >$ for all $h \in H, m \in M, \tilde{m} \in \tilde{M}$. Then

$$\mathrm{Hom}_{RH}(M, C(X, R))$$

and the R-module $\mathcal{A}_H(X, M)$ of functions on X of type (H, M)

$$\phi : X \to \tilde{M}, \quad \phi(xh) = h^{-1}\phi(x) \text{ for all } h \in H,$$

are isomorphic by the map associating to a RH-morphism $m \to f_m$ the function ϕ such that $< \phi(x), m > = f_m(x)$. The relations $f_{hm}(x) = f_m(xh)$ and $< \phi(xh), m > = < \phi(x), hm > = < h^{-1}\phi(x), m >$ are equivalent.

When the groups and spaces have a topology, the representations and the functions ϕ are supposed continuous.

4 We decompose $\mathcal{S}_{\mathbf{C}}$ according to the set $\hat{G}^o_\infty$ of isomorphism classes of irreducible representations of G^o_∞,

$$\mathcal{S}_{\mathbf{C}} \simeq \oplus_{\pi^o_\infty \in \hat{G}^o_\infty} \pi^o_\infty \otimes \rho_{\pi^o_\infty}.$$

For each $(\pi^o_\infty, V_\infty) \in \hat{G}^o_\infty$*, the complex representation* $\rho_{\pi^o_\infty}$ *of the locally profinite group* $G^o_\infty \backslash G_A$ *is smooth, semisimple and admissible.*

The semisimplicity results from §2 because G_∞ / G^o_∞ is finite, the admissibility results from the finiteness of $G(F)G_\infty \backslash G_A / K$ for any open compact subgroup K of G^∞, for any reductive connected group over F **[2]** 5.1.

The model $\mathcal{A}_{G^o_\infty}(G(F)\backslash G_A, V_\infty)$ of $\rho_{\pi^o_\infty} \simeq \mathrm{Hom}_{\mathbf{C}G^o_\infty}(V_\infty, \mathcal{S}_{\mathbf{C}})$ obtained by applying §3, is the complex space of functions

$$\phi : G(F)\backslash G_A \to \tilde{V}_\infty, \;\; \phi(g g_\infty) = g_\infty^{-1} \phi(g) \;\; \text{for all } g_\infty \in G^o_\infty,$$

locally constant on G^∞, with $G^o_\infty \backslash G_A$ acting by right translations.

We can extend the representation π^o_∞ of G^o_∞ to G_∞ because $\pi_{G,o}$ is a finite 2-group. We choose an extension π_∞. Then we set

$$\phi(g) = g_\infty^{-1} \psi(g),$$

where g_∞ is the infinite component of $g \in G_A$. Now ψ is invariant by right translation by G^o_∞ and we have, for $\gamma \in G(F)$,

$$\psi(\gamma g) = \gamma g_\infty \phi(\gamma g) = \gamma \psi(g).$$

The natural action of $G^o_\infty \backslash G_A$ by right translations on the space $\mathcal{A}(V_\infty)$ of locally constant functions

$$\psi : G^o_\infty \backslash G_A \to \tilde{V}_\infty, \quad \psi(\gamma g) = \gamma \psi(g) \quad \text{for all} \quad \gamma \in G(F),$$

is a new model of $\rho_{\pi^o_\infty}$, depending on the choice of π_∞. We could replace $G^o_\infty \backslash G_A$ by $\pi_{G,o} \times G^\infty$ of course.

5 When π^o_∞ is trivial, we deduce that $\rho_{\pi^o_\infty}$ is isomorphic to the natural representation of $G^o_\infty \backslash G_A$ on the space of locally constant complex functions on $G(F)G^o_\infty \backslash G_A$. The functions with values in $\mathbf{Z}$ form a $G^o_\infty \backslash G_A$-stable $\mathbf{Z}$-lattice. An irreducible automorphic representation of G_A trivial on G^o_∞ is **defined over the ring of integers of a number field [9]**.

6 *We consider*
- *a number field* L,

- *a place* $\lambda_\infty \in \mathrm{Hom}(L, \mathbf{C})$,
- *a finite dimensional* L*-vector space* V *with an absolutely irreducible algebraic action of* G'.

In §4, we fix π_∞, resp. π^o_∞, to be the representation of G_∞, resp. G^o_∞, equal to the restriction of the algebraic action of $G'(\mathbf{C})$ on $V_\infty = V \otimes_{L,\lambda_\infty} \mathbf{C}$.

The representation $\pi_\infty \otimes \varepsilon$ *of* G_∞*, twist of* π_∞ *by a character* ε *of the component group* $\pi_{G,o}$*, is called of* **algebraic type** (V, ε) *or* V*. An irreducible automorphic representation of* G_A *of infinite component* $\pi_\infty \otimes \varepsilon$ *is also called of algebraic type* (V, ε) *or* V.

The irreducible complex representations of G_∞ of algebraic type are characterized by the property that their restriction to the component G^o_∞ is algebraic.
Any continuous irreducible complex representation to K^o_∞ is algebraic, and any continuous irreducible complex representation of G^o_∞ is the twist by a character of an algebraic representation because $G^o_\infty = S^o_\infty \times K^o_\infty$ (see §1).

The L-vector space $\mathcal{A}(V)$ is a $G^o_\infty \backslash G_A$-stable L-structure of the semi-simple admissible representation $\mathcal{A}(V_\infty)$. The finite part of an irreducible automorphic representation of G_A of algebraic type is **defined over a number field.**

7 In order to obtain an integral structure when V is not trivial (§5), we will work p-adically. We choose

- *an isomorphism* $i_p : \mathbf{C} \to \overline{\mathbf{Q}}_p$.

The place λ_∞ and i_p give a p-adic place $\lambda_p = i_p \lambda_\infty \in \mathrm{Hom}(L, \overline{\mathbf{Q}}_p)$ which allows to identify the complex space V_∞ defined in §6 with the $\overline{\mathbf{Q}}_p$-vector space $V_p = V \otimes_{L,\lambda_p} \overline{\mathbf{Q}}_p$.
The interest is that there is a natural action of the p-adic group $G_p = G'(\mathbf{Q}_p) \subset G'(\overline{\mathbf{Q}}_p)$ on V_p, such that the actions of $G(F)$ on V_p as a subgroup of G_p and on V_∞ as a subgroup of G_∞ coincide via the isomorphism $V_\infty \to V_p$. One identifies $\mathcal{A}_{G^o_\infty}(G(F)\backslash G_A, V_\infty)$ and $\mathcal{A}_{G^o_\infty}(G(F)\backslash G_A, V_p)$ and the trick is to set

$$f(g) = g_p^{-1} \psi(g) = (g_\infty, g_p^{-1}) \phi(g)$$

where $\phi \in \mathcal{A}_{G^o_\infty}(G(F)\backslash G_A, V_\infty)$ and (g_∞, g_p) is the (∞, p)-component of $g \in G_A$. There exists an open compact subgroup $K = K_p \times K^p$ of $G^\infty = G_p \times G^{\infty,p}$ (depending on ϕ) such that ϕ, ψ are right invariant by K. The relation $\psi(\gamma g k) = \gamma \psi(g)$ for $(\gamma, k) \in G(F) \times K$ is equivalent to

$$f(\gamma g k) = (\gamma g_p k_p)^{-1} \psi(\gamma g k) = (\gamma g_p k_p)^{-1} \gamma \psi(g) = k_p^{-1} f(g).$$

These are the functions studied in §3 when

$$(R, H, G, M, X) = (\overline{\mathbf{Q}}_p, K, G^\infty, V_p, G(F)G^o_\infty\backslash G_A).$$

the action of K on V_p is the inflation of the action of K_p (trivial on K^p).

8 *The $\overline{\mathbf{Q}}_p$-vector space $\mathcal{A}_K(G(F)G^o_\infty\backslash G_A, V_p)$ of p-adic automorphic forms on G_A of locally algebraic type (K, V_p) is*

$$\{f : G(F)G^o_\infty\backslash G_A \to \tilde{V}_p, \quad f(gk) = k_p^{-1} f(g) \text{ for all } k \in K\},$$

with $G^o_\infty\backslash G^p_A$ acting by right translations.

The inductive limit

$$\mathcal{A}(G(F)G^o_\infty\backslash G_A, V_p) = \lim\nolimits_K \mathcal{A}_K(G(F)G^o_\infty\backslash G_A, V_p)$$

over all open compact subgroups $K = K_p \times K^p$ of G^∞, is the space of p-adic automorphic forms on G_A of locally algebraic type V_p. The complex and p-adic spaces

$$\mathcal{A}_{G^o_\infty}(G(F)\backslash G_A, V_\infty), \quad \mathcal{A}(G(F)G^o_\infty\backslash G_A, V_p)$$

are isomorphic via i_p. The isomorphism is $\pi_{G,o} \times G^{\infty,p}$-equivariant only. The action of G_p is locally algebraic on $\mathcal{A}(G(F)G^o_\infty\backslash G_A, V_p)$ and is smooth on $\mathcal{A}_{G^o_\infty}(G(F)\backslash G_A, V_\infty)$.

9 We denote by $E = L_{\lambda_p}$, O_E the ring of integers, p_E an uniformizer, k_E the residual field. We choose:

- *an open compact subgroup K^o_p of G_p,*
- *an O_E-structure $\mathcal{L}(V_p)$ of V_p stable by K^o_p.*

The linear O_E-dual $\tilde{\mathcal{L}}(V_p) = \mathcal{L}(\tilde{V}_p)$ is an O_E-structure of the contragredient $\tilde{V}_p$. In §8, we can suppose that the groups K_p are contained in K^o_p.

The O_E-module $\mathcal{A}(G(F)G^o_\infty\backslash G^p_A, \mathcal{L}(V_p))$ is free, $G^o_\infty\backslash G_A$-stable, and generates the $\overline{\mathbf{Q}}_p$-vector space $\mathcal{A}(G(F)G^o_\infty\backslash G_A, V_p)$. The finite part outside of p of an irreducible automorphic representation of G_A of algebraic type is **defined over the ring of integers of a finite extension of $\mathbf{Q}_p$**.

We could not do this in the complex realization of §4, because the representation V_∞ of G_∞, when it is not trivial, has no integral structure stable by $G(F)$, by density.

10 We return to the generalities of §3. We suppose that G is a locally profinite group acting continuously on a locally profinite space X, and that $H = K$ is an open subgroup of G acting smoothly on an R-module M. We consider the compact induction from the R-representations of K to the R-representations of G

$$\operatorname{ind}_K^G M = \{f : G \to M,\ f(kg) = kf(g),\ (k,g) \in K \times G,$$

$$f \text{ with compact support}\},$$

the restriction res_K^G in the other direction, and the Hecke R-algebra

$$\mathcal{H}(G, K, M) = \operatorname{End}_{RG} \operatorname{ind}_K^G M.$$

These functors respect smoothness. In the abelian category of smooth R-representations, the induction from K to G is left adjoint to the restriction from G to K [**10**],

$$\operatorname{Hom}_{RG}(\operatorname{ind}_K^G M, V) \simeq \operatorname{Hom}_{RK}(M, \operatorname{res}_K^G V).$$

Hence $\operatorname{Hom}_{RK}(M, \operatorname{res}_K^G \cdot)$ simply denoted by $\operatorname{Hom}_{RK}(M, \cdot)$ is a functor from the smooth R-representations of K to the right $\mathcal{H}(G, K, M)$-modules.

The sum V^{sm} of the smooth subrepresentations of an R-representation V of G is a smooth subrepresentation of V and for any smooth representation M of K

$$Hom_{RK}(M, V) = \operatorname{Hom}_{RK}(M, V^{sm})$$

because K is open. One then deduces that $\mathcal{A}_K(X, M)$ *is naturally a right* $\mathcal{H}(G, K, M)$*-module.*

This remains true when the action of K on M is locally algebraic. The proof follows the same pattern.

11 We can add to the machine:

- a finite dimensional L-vector space W with an absolutely irreducible smooth action of an open compact subgroup K of G^∞.

The functor $V \to \operatorname{Hom}_{RK}(W, V)$ (see §10) gives a bijection from the isomorphism classes of the irreducible smooth representations of G^∞ which contain (K, W) onto the isomorphism classes of the simple right modules of the Hecke algebra $\mathcal{H}(G^\infty, K, W)$ (consider the idempotent of the global Hecke algebra of G^∞ associated to (K, W)).

The representation $\rho_{\pi_\infty^o}$ of $\pi_{G,o} \times G^\infty$ is admissible and semisimple. The isomorphism class of the subrepresentation of $\rho_{\pi_\infty^o}$ generated by the irreducible subrepresentations which contain (K, W), is identified with (see §3, §4):

The right $\mathcal{H}(G^\infty, K, W)$-module $\mathcal{A}_{G_\infty^o \times K}(G(F)\backslash G_A, V_\infty \otimes_{L,\lambda_\infty} W)$ of automorphic forms on G^∞ of algebraic type V and smooth type (K, W) .

This semi-simple module is isomorphic to

$$\mathcal{A}(V_\infty \otimes_{L,\lambda_\infty} W) = \{\psi : G^o_\infty \backslash G_A \to \tilde{V}_\infty \otimes_{L,\lambda_\infty} \tilde{W},$$
$$\psi(\gamma g k) = (\gamma, k^{-1})\psi(g), \text{ for all } (\gamma, k) \in G(F) \times K\}.$$

The finite set $G(F)G^o_\infty \backslash G_A/K$ has a system of representatives $g_1, \ldots, g_n$ in G^∞ because $G(F)$ is dense in G_∞ (for any reductive connected group) **[2]**. The values $\psi(g_i)$ are invariant by the finite arithmetic group $\Gamma_i = G(F) \cap (G^o_\infty \times g_i K g_i^{-1})$. As a complex vector space,

$$\mathcal{A}(V_\infty \otimes_{L,\lambda_\infty} W) \simeq \oplus_{i=1}^n (\tilde{V}_\infty \otimes_{L,\lambda_\infty} \tilde{W})^{\Gamma_i}, \quad \psi \mapsto (\psi(g_i))_{1 \le i \le n}.$$

It is easy to choose K such that $G(F) \cap K$ is trivial, or even such that all the finite groups Γ_i are trivial, noting that if $K = \prod_v K_v$ is factorizable, then the order of Γ_i divides the pro-order of K_v for any finite place v of F.

12 One deduces from §11 a formula for the finite dimension of the space of automorphic representations with given algebraic and smooth types. In particular:

For any algebraic absolutely irreducible representation V of G', there exists an irreducible automorphic representation of algebraic type V. If K is an open compact subgroup K of G^∞ such that $K \cap G(F) = \{1\}$, for any irreducible complex representation W of K, there exists an irreducible automorphic representation of algebraic type V which contains (K, W).

13 There exists a finite set S of places of F which contains the infinite places, such that $K = K_S \times K^S$ (see §11) where $K_S \subset G_S$ and outside of S, K^S is a good maximal open compact subgroup of G^S, and W^S is trivial.

The Hecke **Z**-algebra $\mathcal{H}^S = \mathcal{H}(G^S, K^S)$ is commutative and acts semi-simply on $\mathcal{A}(V_\infty \otimes_{L,\lambda_\infty} W)$ with eigenvalues λ_{π^S} where λ_{π^S} is the character of $\mathcal{H}^S$ acting on the one dimensional space of K^S-invariants of π^S, for all irreducible automorphic representations $\pi_A = \pi_S \otimes \pi^S$ of G_A of algebraic type V which contain (K, W).

14 We suppose in §11 that $K = K_p \times K^p$ with $K_p \subset K^o_p$ (see §9) and $W = W_p \otimes_L W^p$ has a factorization compatible with the factorization of $G^\infty = G_p \times G^{\infty,p}$. The group K acts on $V_p \otimes_{L,\lambda_p} W$ by the diagonal locally algebraic irreducible action of K_p and by the smooth action of K^p natural on W and trivial on V_p. We choose

- a *K-stable O_L-structure $\mathcal{L}(W) = \mathcal{L}(W_p) \otimes_{O_L} \mathcal{L}(W^p)$ of W over the ring of integers O_L of L* (see §11).

The linear O_L-dual $\tilde{\mathcal{L}}(W) = \mathcal{L}(\tilde{W})$ is an O_L-structure of the contragredient $\tilde{W}$. As in §8,

15 *The $\overline{\mathbf{Q}}_p$-space of p-adic automorphic forms of locally algebraic type $(K, V_p \otimes_{L,\lambda_p} W)$ is*

$$\mathcal{A}_K(G(F)G_\infty^o \backslash G_A, V_p \otimes_{L,\lambda_p} W) = \{f : G(F)G_\infty^o \backslash G_A \to \tilde{V}_p \otimes_L \tilde{W},$$
$$f(gk) = k^{-1} f(g) \ \text{ for all } k \in K\}.$$

This is a right $\pi_{G,o} \times \mathcal{H}(G^\infty, K, V_p \otimes_{L,\lambda_p} W)$-module as in §10, but the proof of the theorem uses only its $\pi_{G,o} \times \mathcal{H}(G^{\infty,p}, K^p, W^p)$-module structure. The natural isomorphism

$$\mathcal{A}_{G_\infty^o \times K}(G(F)\backslash G_A, V_\infty \otimes_{L,\lambda_\infty} W) \to \mathcal{A}_K(G(F)G_\infty^o \backslash G_A, V_p \otimes_{L,\lambda_p} W)$$

is $\pi_{G,o} \times \mathcal{H}(G^{\infty,p}, K^p, W^p)$-equivariant.

16 The integral structures chosen in §9 and in §14, give an O_E-structure $\mathcal{L} = \mathcal{L}(V_p) \otimes_{O_L} \mathcal{L}(W)$ of the locally algebraic representation of K on $V_p \otimes_L W$. As in §11, the O_E-module $\mathcal{A}_K(G(F)G_\infty^o \backslash G_A, \mathcal{L})$ is isomorphic to

$$\oplus_{i=1}^n \tilde{\mathcal{L}}^{\Gamma_i},$$

is O_E-free, stable by $\pi_{G,o} \times \mathcal{H}(G^{\infty,p}, K^p, \mathcal{L})$, and generates the $\overline{\mathbf{Q}}_p$-vector space $\mathcal{A}_K(G(F)G_\infty^o \backslash G_A, V_p \otimes_{L,\lambda_p} W)$.

17 We suppose that **the orders of the finite arithmetic groups Γ_i in §11, are prime to p.** The congruences between automorphic forms rely on the fact that the natural $\pi_{G,o} \times \mathcal{H}(G^{\infty,p}, K^p, \mathcal{L})$-equivariant map between

$$\mathcal{A}_K(G(F)G_\infty^o \backslash G_A, \mathcal{L}) \ / \ p_E^n \mathcal{A}_K(G(F)G_\infty^o \backslash G_A, \mathcal{L})$$

and

$$\mathcal{A}_K(G(F)G_\infty^o \backslash G_A, \mathcal{L}/p_E^n \mathcal{L})$$

is an isomorphism for any integer $n \geq 1$.

The congruences between automorphic representations when $n = 1$, rely on the Deligne-Serre lemma [**4**]. Let $\mathcal{X}$ be any commutative subalgebra of $\pi_{G,o} \times \mathcal{H}(G^{\infty,p}, K^p, \mathcal{L})$. For any eigenvalue μ of $\mathcal{X}$ on the reduction

$$\mathcal{A}_K(G(F)G_\infty^o \backslash G_A, \mathcal{L}/p_E \mathcal{L})$$

there exists a finite extension E'/E and an eigenvalue λ of $\mathcal{X}$ acting on

$$\mathcal{A}_K(G(F)G_\infty^o \backslash G_A, \mathcal{L} \otimes_{O_E} O_{E'}) = \mathcal{A}_K(G(F)G_\infty^o \backslash G_A, \mathcal{L}) \otimes_{O_E} O_{E'},$$

congruent modulo p to μ, i.e. $\lambda \equiv \mu \mod p_{E'}$.

18 The pro-p-radical Iw_p of an Iwahori subgroup Iw of G_p, called a **pro-p-Iwahori subgroup** of G_p, should be considered as the pro-p-Sylow of G_p

because it is a pro-p-group, uniquely defined modulo conjugation, which contains the pro-p-radical of any parahoric subgroup of G_p containing Iw. As Iw_p is a pro-p-group, it satisfies the following properties:

1) An irreducible locally algebraic representation of a pro-p-group over a field of characteristic p is trivial.

2) For any locally algebraic E-representation W of a pro-p-group I_p, there exists a finite extension E'/E and an $O_{E'}$-structure of $W \otimes_E E'$ with trivial reduction modulo $p_{E'}$ (for the action of I_p) **[3]**.

19 We are now ready to prove the theorem and its converse. Let π_A be an irreducible automorphic representation of G_A of algebraic type V and containing $(K^p = K_v \times K^{p,v}, W^p)$, let V' be as in the theorem, and let W_p be an irreducible smooth complex representation of Iw_p.

In §11, we take $K = Iw_p \times K^p$. The orders of the arithmetic groups Γ_i in §11 are trivial because the pro-orders of K_v and of Iw_p are relatively prime. We choose the number field L and the place λ_∞ in §6, and the isomorphism i_p in §7, such that:

a) The algebraic representations V, V' of G' and the smooth representation $W_p \otimes_{\mathbf{C}} W^p$ of K are defined over L embedded in $\mathbf{C}$ via λ_∞.

b) The finite dimensional $\overline{\mathbf{Q}}_p$-vector spaces $U_p = V_p, V'_p, W_p \otimes_{\mathbf{C},i_p} \overline{\mathbf{Q}}_p$ have Iw_p-stable O_E-structures $\mathcal{L}(U_p)$ of trivial reduction modulo p_E (for the action of Iw_p), using §18 2).

c) The finite components prime to p in the finite set of irreducible automorphic representations of G_A

- of algebraic type V and containing $(K, W_p \otimes_{\mathbf{C}} W^p)$, or
- of algebraic type V' and containing $(K, \mathrm{id}_p \otimes_{\mathbf{C}} W^p)$ (trivial on Iw_p),

are defined over O_E.

In §11, the L-representation of K is either W' equal to a K^p-stable L-structure of W^p with K acting by inflation (trivial on Iw_p), or $W = W_{p,L} \otimes_L W'$ for a K_p-stable L-structure $W_{p,L}$ of W_p.

We choose a K^p-stable O_E-structure $\mathcal{L}(W^p)$ of $W^p \otimes_{\mathbf{C},i_p} \overline{\mathbf{Q}}_p$. With b), we obtain the O_E-structures in §9, §14 and in §16, $\mathcal{L}$ for $V_p \otimes_{L,\lambda_p} W$ and $\mathcal{L}'$ for $V'_p \otimes_{L,\lambda_p} W'$. We denote by d the complex dimension of W_p.

There is a K-isomorphism

$$\mathcal{L}/p_E\mathcal{L} \simeq \oplus^d \mathcal{L}'/p_E\mathcal{L}'.$$

The same is true for the contragredients. We deduce as in §17 that the reduction modulo p_E of

$$\mathcal{A}_K(G(F)G^o_\infty \backslash G_A, \mathcal{L}) \quad \text{and} \quad \oplus^d \mathcal{A}_K(G(F)G^o_\infty \backslash G_A, \mathcal{L}')$$

are $\pi_{G,o} \times \mathcal{H}(G^{\infty,p}, K^p, \mathcal{L}(W^p))$-isomorphic.

We suppose in §13 that S contains p. Then $\pi_{G,o} \times \mathcal{H}^S$ has the same eigenvalues on the reduction modulo p_E of

$$\mathcal{A}_K(G(F)G^o_\infty \backslash G_A, \mathcal{L}) \quad \text{and} \quad \oplus^d \mathcal{A}_K(G(F)G^o_\infty \backslash G_A, \mathcal{L}').$$

For $* = \mathcal{L}$ or $\mathcal{L}'$, we deduce from c) that for any finite extension E'/E, the eigenvalues of $\mathcal{H}^S$ on

$$\mathcal{A}_K(G(F)G^o_\infty \backslash G_A, *) \quad \text{or on} \quad \mathcal{A}_K(G(F)G^o_\infty \backslash G_A, * \otimes_{O_E} O_{E'})$$

are the same. Remembering from §1 that $\pi_{G,o}$ is a finite 2-group, we deduce as in §17:

When $p \neq 2$, the reduction modulo p_E of the eigenvalues of $\pi_{G,o} \times \mathcal{H}^S$ on

$$\mathcal{A}_K(G(F)G^o_\infty \backslash G_A, \mathcal{L}) \text{ and on } \mathcal{A}_K(G(F)G^o_\infty \backslash G_A, \mathcal{L}')$$

are the same. If $p = 2$, this remains true without $\pi_{G,o}$.

20 Using §13, we obtain a slightly stronger version of the theorem and of its converse.

With the notations of the theorem and of its converse, let ε be a character of $\pi_{G,o}$. If $p \neq 2$, any automorphic irreducible representation of G_A of algebraic type (V, ε), and containing $(Iw_p \times K^p, W_p \otimes W^p)$, is congruent to an automorphic irreducible representation of G_A of algebraic type (V', ε) and containing $(Iw_p \times K^p, \mathrm{id}_p \otimes W^p)$, and conversely.

If $p = 2$, this remains true without the character ε.

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