REPRESENTATIONS OF $GL_n(D)$ NEAR THE IDENTITY

GUY HENNIART AND MARIE-FRANCE VIGNÉRAS

Abstract. Let p be a prime number, F a finite extension of $\mathbb{Q}_p$ or of $\mathbb{F}_p((t))$. We consider the group $G = GL_n(D)$ for a positive integer n and a central finite dimensional division F-algebra D of F-dimension d^2. For an irreducible smooth complex representation π of G, inspired by work of R. Howe when $D = F$, we establish the existence and uniqueness of integers $c_\pi(\lambda)$, for partitions λ of n, such that for any small enough compact open subgroup K of G the restriction of π to K is the same as that of the virtual representation $\sum c_\pi(\lambda)\,\mathrm{Ind}_{P_\lambda}^G 1$, where the sum is over partitions λ of n and P_λ is a parabolic subgroup of G in the associate class determined by λ. When P_λ is minimal such that $c_\pi(\lambda) \neq 0$ we prove that $c_\pi(\lambda)$ is positive, equal to the dimension of a generalized Whittaker model of π. We elucidate the behaviour of c_π under the Jacquet-Langlands correspondence LJ of Badulescu from $GL_{dn}(F)$ to G. We extend the above result on π near identity to a representation of G over a field R with characteristic not p. For any Moy-Prasad pro-p subgroup K of G, we determine from the integers $c_\pi(\lambda)$ a polynomial $P_{\pi,K}$ with integral coefficients and degree $d(\pi)$ independent on K, such that, for large enough integers j, the dimension of fixed points in π under the j-th congruence subgroup K_j of K is $P_{\pi,K}(q^{dj})$ where q is the cardinality of the residue field of F.

Mathematics Subject Classification 2020: Primary 22E50; Secondary 11F70

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Date: October 9, 2024.

1. Introduction

Let p be prime number, F a finite extension of $\mathbb{Q}_p$ or of $\mathbb{F}_p((T))$. Let $\underline{G}$ be a reductive connected group over F, and put $G = \underline{G}(F)$. Let R be a field, and π a smooth admissible representation of G on an R-vector space V.

Our first motivation was in the following question, when π is of finite length: Let x be a point in the Bruhat-Tits building of G and r a positive real number. For any integer $j \geq 0$, let $d(j)$ be the dimension of the space of fixed points in V under the Moy-Prasad subgroup $G_{x,r+j}$ of G.

Question 1.1. *Is there a polynomial P with integer coefficients such that $d(j) = P(p^j)$ for large enough j ? If so, what can we say about its degree and its leading coefficient ?*

When the characteristic char_R of R is p, precise knowledge of those dimensions for irreducible π is available only for $G = GL(2, \mathbb{Q}_p)$ (S. Morra, see §12.5). Apart for groups G of relative rank one those dimensions seem unknown.

Our paper studies the case where $\mathrm{char}_R \neq p$. Then a smooth finite length R-representation π of G is automatically admissible, and its restriction to a pro-p subgroup K of G is semisimple, with finite multiplicities. We write $[\pi]_K$ for the image of that restriction in the Grothendieck group of admissible R-representations of K. We ask a more ambitious question:

Question 1.2. *Is there an open pro-p subgroup K of G where we can control $[\pi]_K$?*

In the case of $GL_2(F)$ an answer to that question was offered by Casselman [Casselman73]. In this paper we consider the case where $G = GL_n(D)$ for a central division algebra D over F, with finite degree d^2 over F. For a partition $\lambda = (\lambda_1, \dots, \lambda_r)$ of n, we let P_λ be the upper block triangular subgroup of G with blocks of size $\lambda_1, \dots, \lambda_r$ down the diagonal, and put $d_\lambda = \sum_{i<j} \lambda_i \lambda_j$. We have $d_\lambda \geq d_\mu$ if $\lambda \leq \mu$ for the classical partial order $\leq$ on partitions. We let π_λ be the representation of G non-normalized parabolically induced from the trivial representation of P_λ; it has finite length.

Let π be a finite length smooth representation of G on an R-vector space V.

Theorem 1.3. *There is a unique function c_π from partitions of n to $\mathbb{Z}$ and an open pro-p subgroup $K = K_\pi$ of G such that $[\pi]_K = \sum_\lambda c_\pi(\lambda)[\pi_\lambda]_K$.*

If λ is minimal in the support of c_π, then $c_\pi(\lambda)$ is positive.

Theorem 1.3 has consequences to our first question. We let q be the cardinality of the residue field of F, so the residue field of D has cardinality q^d. Let x a point in the Bruhat-Tits building of G and r a positive real number.

Theorem 1.4. *Let $P = P_{\pi,G_{x,r}}$ be the polynomial*

$$P_{\pi,G_{x,r}}(X) = \sum_{\lambda} |P_\lambda \backslash G/G_{x,r}|\, c_\pi(\lambda)\, X^{d_\lambda}. \tag{1.1}$$

Then $\dim_R V^{G_{x,r+j}} = P(q^{dj})$ for large enough integers j. The degree of P is $d(\pi) = \max(d_\lambda)$ where the maximum is over partitions λ in the support of c_π. The leading coefficient is

$$a_{\pi,G_{x,r}} = \sum_{\lambda, d_\lambda = d(\pi)} |P_\lambda \backslash G/G_{x,r}|\, c_\pi(\lambda). \tag{1.2}$$

The function c_π has good properties with respect to natural operations, apart from being additive on exact sequences, hence factoring to a function on the Grothendieck group of finite length smooth representations of G. If χ is a character of G, $c_{\chi\pi} = c_\pi$. If π' the base change of π to an extension R' of R, then $c_{\pi'} = c_\pi$; in particular c_π is invariant under automorphisms of R. When $p \neq 2, G = GL_n(F)$ and $\mathrm{char}_F = 0$ the support of π contains a single partition λ with $d_\lambda = d(\pi)$ [Moeglin-Waldspurger87]. This may be true for any p, F and D.

Parabolic induction Let P be an upper block triangular subgroup of G, with block diagonal Levi subgroup M a product $GL_{n_1}(D) \times \ldots \times GL_{n_r}(D)$. For $i = 1, \ldots, r$ let σ_i be a finite length representation of $GL_{n_i}(D)$, and put $\sigma = \sigma_1 \otimes \ldots \otimes \sigma_r$ a finite length representation of M. Given a partition λ_i of n_i for $i = 1, \ldots, r$, we have the induced partition λ of n obtained by gathering all the parts of the λ_i's and putting them in decreasing order.

Theorem 1.5. *Let $\pi = \mathrm{ind}_P^G(\sigma)$. For each partition λ of n, $c_\pi(\lambda) = \sum \prod_{i=1,\ldots,r} c_{\sigma_i}(\lambda_i)$, where the sum is over r-tuples of partitions $(\lambda_1, \ldots, \lambda_r)$ inducing to λ.*

Whittaker models Assume that R contains all the roots of unity of p-power order. We have the notion of Whittaker models, possibly degenerate. Let U be the upper triangular subgroup of G, and θ a character of U. We let V_θ be the maximal quotient of the space V of π on which U acts via θ. Its dimension is finite and depends on θ only up to conjugation by the diagonal subgroup T of G. The orbits of T on the characters of U are parametrized by the compositions of n. To each composition λ of n is attached a partition $\lambda^\dagger$ obtained by gathering the parts of λ in decreasing order. The Whittaker support of π is the set of partitions of n of the form $\lambda^\dagger$ where λ is a composition of n such that $V_\theta \neq 0$ for θ corresponding to the composition λ.

Theorem 1.6. *The minimal elements in the support of c_π and in the Whittaker support of π are the same. If μ is such a minimal partition, λ is a composition of n with $\lambda^\dagger = \mu$ and θ a character of U corresponding to λ, then $c_\pi(\mu) = \dim_R V_\theta$.*

Jacquet-Langlands correspondence I.Badulescu has extended the classical Jacquet-Langlands correspondence to a morphism $LJ_{\mathbb{C}}$ from the Grothendieck group of smooth finite length complex representations of $GL_{dn}(F)$ to that of G. Let ℓ be a prime number different from p. For an algebraic closure $\mathbb{Q}_\ell^{ac}$ of $\mathbb{Q}_\ell$, with a chosen square root of q, A.Minguez and V.Sécherre have transported $LJ_{\mathbb{C}}$ to $\mathbb{Q}_\ell^{ac}$-representations, and showed that it descends to a map $LJ_{\mathbb{F}_\ell^{ac}}$ of $\mathbb{F}_\ell^{ac}$-representations, where $\mathbb{F}_\ell^{ac}$ is the residue field of $\mathbb{Q}_\ell^{ac}$. We define LJ_R for our field R, provided it be algebraically closed, and get:

Theorem 1.7. *Assume R to be algebraically closed. Let τ be a finite length smooth R-representation of $GL_{dn}(F)$ and $\pi = LJ_R(\tau)$. For any partition λ of n, we have $(-1)^n c_\pi(\lambda) = (-1)^{dn} c_\tau(d\lambda)$.*

For $R = \mathbb{C}$ and a discrete series π, the result is due to D.Prasad [Prasad00].

We show in §11 how to get Theorem 1.4 from Theorem 1.3; this amounts to computing the dimensions of fixed points for the π_λ's. Our method establishes the other results first for $R = \mathbb{C}$, and then extends them to R. Let us hasten to mention that when $R = \mathbb{C}$ part of the results were known. Indeed when $D = F$ and $\mathrm{char}_F = 0$, the first part of Theorem 1.3 is due to [Howe74]. We actually adapt Howe's arguments to our setting. Similarly when $\mathrm{char}_F = 0$ one can obtain Theorems 1.5, 1.6 (and the second part of Theorem 1.3) from the much more general results of [Moeglin-Waldspurger87], and we get inspiration from their proofs. [1]

We now give more detail on our method of proof. First we take $R = \mathbb{C}$. In that case, knowing $[\pi]_K$ for an open compact subgroup K of G is equivalent to knowing the character $\mathrm{trace}(\pi)$ on smooth functions on G supported in K. An expression of $\mathrm{trace}(\pi)$ on small enough K as a linear combination of finitely many easier distributions is usually called a germ expansion for π. When $\mathrm{char}_F = 0$, the theory of germ expansions has a long history. For a reductive group G and π irreducible Harish-Chandra established a germ expansion of $\mathrm{trace}(\pi)$ as a linear combination of Fourier transforms of nilpotent orbital integrals on the Lie algebra $\mathfrak{g}$ of G, with coefficients a priori only complex numbers [Harish-Chandra70]. To get from functions on G to functions on $\mathfrak{g}$, he used the exponential map, which is not available to us when $\mathrm{char}_F > 0$. The interest of our group $G = GL_n(D)$ is that $\mathfrak{g} = M_n(D)$, so that nilpotent orbits of G in $\mathfrak{g}$ are parametrized by partitions of n, and that one can use the map $e : X \mapsto 1 + X$ from $\mathfrak{g}$ to G as a substitute for the exponential. When $D = F$, Howe proved using e that the Fourier transform of the nilpotent orbital integral corresponding to a partition λ is proportional to $\mathrm{trace}(\pi_\lambda)$, and got a germ expansion $\mathrm{trace}(\pi) = \sum_\lambda c_\pi(\lambda)\,\mathrm{trace}(\pi_\lambda)$ on the i-th congruence subgroup K_i for i large enough. He showed that the $c_\pi(\lambda)$ are integers by constructing for any $i > 0$ a character ξ_λ of K_i which appears with multiplicity 1 in π_λ and multiplicity 0 in π_μ unless $\lambda \geq \mu$ [Howe74]. We show the existence of such characters for D in Lemma 6.2. For our group G and π irreducible, B.Lemaire proved the local integrability of the distribution $\mathrm{trace}(\pi)$ (that was new when $\mathrm{char}_F = p$) and adapted Howe's arguments to get a germ expansion as a linear combination of Fourier transforms of nilpotent integrals [Lemaire04], which by our Proposition 5.5 translates into a germ expansion as in Theorem 1.3. Our characters ξ_λ then yields the integrality statement and the positivity statement.

[1] While we were writing our results, the preprint [Suzuki22] reached us. When $R = \mathbb{C}, D = F$ and $\mathrm{char}_F = 0$, Suzuki establishes Theorem 1.4 for the congruence subgroups $K_j = 1 + M_n(P_F^j)$ of $GL_n(F)$. He also gets a result equivalent to Theorem 1.5 and Theorem 1.7 for square integrable τ. His methods are similar to ours.

Theorem 1.5 follows from the known behaviour of traces with respect to parabolic induction. In §7, we give a treatment valid whatever char_F is.

As already said, when $\mathrm{char}_F = 0$, Theorem 4 can be obtained from results of C.Moeglin and J.-L.Waldspurger for a reductive group G and π irreducible. They attach to a nilpotent orbit $\mathfrak{O}$ of G in $\mathfrak{g}$ a number of generalized Whittaker spaces of π. They consider the Harish-Chandra germ expansion of π as a linear combination $\sum c_\pi(\mathfrak{O})D_{\mathfrak{O}}$ over the nilpotent orbits $\mathfrak{O}$, where $D_{\mathfrak{O}}$ is the Fourier transform of the orbital integral along $\mathfrak{O}$. They show that if $\mathfrak{O}$ is maximal in the support of c_π then the dimension of any Whittaker space attached to $\mathfrak{O}$ is $c_\pi(\mathfrak{O})$. The nilpotent orbits with that maximality property go by the name of wave front set of π and there is a large literature on that subject. In our more restricted setting, but allowing $\mathrm{char}_F = p$, we get Theorem 1.6 by adapting arguments of [Rodier74][2] and [Moeglin-Waldspurger87].

Still with $R = \mathbb{C}$, to prove Theorem 1.7 in §9 we use that the Jacquet-Langlands correspondence LJ is expressed by character identities, where the characters are considered as locally L^1 functions on regular semisimple elements (by the result of B.Lemaire alluded to above).

In §10 we pass from $R = \mathbb{C}$ to the general case. To transfer the results from a field R to an isomorphic field R' we use that the theory of smooth representations is essentially algebraic. That gives the case of $\mathbb{Q}_\ell^{ac}$ which is isomorphic to $\mathbb{C}$. We then get the case of $\mathbb{F}_\ell^{ac}$ by reduction, using the results of [Minguez-Sécherre14]. To transfer the results from an algebraically closed field R to an algebraically closed extension R', we use the fact that for a cuspidal R'-representation π of G, there is a character χ of G into R'^* such that $\chi\pi$ comes by base change from an R-cuspidal representation of G. To get the result for any R we show that Theorem 1.3 over an algebraically closed extension R^{ac} of R implies Theorem 1.3 over R essentially because base change preserves finite length.

When $n = 2$ and $D = F$, we compute in §12 the two coefficients $c_\pi(\lambda)$ for all irreducible π. When $n = 3$ or 4, $D = F, \mathrm{char}_F = 0, R = \mathbb{C}$, F.Murnaghan computes the three coefficients $c_\pi(\lambda)$ for cuspidal representations π of G induced from $F^*GL_n(O_F)$[Murnaghan91]. For any split reductive group G over F, R.Meyer and M.Solleveld using the Bruhat-Tits building of G, give an upper bound on $\dim_R V^{C_r}$ for some special cases C_r, of Moy-Prasad subgroups ([Meyer-Solleveld12]Theorem 8.5). Their result is far less precise than ours.

Acknowledgments. We thank I.Badulescu, D.Bernardi, P.Broussous, B.Lemaire, G.McNinch, A.Minguez, S.Morra, V.Sécherre for a number of conversations about the topic of the paper. The second author has talked about this work at numerous conferences in 2022 (Stockholm, Singapore, Grenoble, Heidelberg, Bangalore), and thanks the organizers for their invitation. A part of this work was done at the Institute for Mathematical Sciences of the National University of Singapore. The final version of that paper was written when the first-named author was enjoying the hospitability of the Graduate School of Mathematical Sciences of the University of Tokyo.

[2]Rodier assumed $\mathrm{char}_F = 0, G$ split and the support of c_π contains the maximal nilpotent orbit

2. Notations

Let p be a prime number, and F a local non archimedean field of residual characteristic p. We denote by O_F the ring of integers of F, P_F the maximal ideal of O_F, p_F a generator of P_F, $k_F = O_F/P_F$ the residue field of order $q = p^f$ where $f = [k_F : \mathbb{F}_p]$ is the residual degree, and F^{ac} an algebraic closure of F. Let $|\ |$ denote the absolute value of F^{ac} such that for $x \in F^{ac}$ non-zero, and $N_{E/F}$ the norm of a finite extension E of F containing x, we have $|x|^{[E:F]} = |N_{E/F}(x)| = |O_F/N_{E/F}(x)O_F|$ ([Cassels67] 10.Theorem). In particular $|p_F| = q^{-1}$.

Let D be a central division F-algebra of finite dimension d^2. We denote by O_D the maximal order of D, P_D the maximal ideal of O_D, p_D a generator of P_D, $k_D = O_D/P_D$ the residue field of cardinal q^d; we have $p_F O_D = P_D^d$ [Reiner75].

Let n be a positive integer and $G = GL_n(D)$. Put $K_0 = GL_n(O_D)$ and $K_i = 1 + M_n(P_D^i)$ for a positive integer i. Let $Z \simeq F^*$ denote the center and $\mathfrak{g} = M_n(D)$ the Lie algebra of G. Let $\mathrm{trd}, \mathrm{nrd} : M_n(D) \to F$ be the reduced trace, the reduced norm. The symmetric G-invariant bilinear form $(X, Y) \mapsto \mathrm{trd}(XY) : M_n(D) \times M_n(D) \to F$ is not degenerate and $G = \{Z \in M_n(D) \mid \mathrm{nrd}(Z) \neq 0\}$.

The letter P will denote a parabolic subgroup of G, its unipotent radical is usually written N, and M is used for a Levi subgroup so that $P = MN$. We write $\mathfrak{p}, \mathfrak{m}, \mathfrak{n}$ for their Lie algebras.

A composition $\lambda = (\lambda_i)$ of $n = \lambda_1 + \ldots + \lambda_r,\ \lambda_i \in \mathbb{N}_{>0}$, is called a partition of n when the sequence (λ_i) is decreasing. To a composition λ of n is associated a parabolic subgroup P_λ of $G = GL_n(D)$ with Levi subgroup M_λ block-diagonal with blocks of size $\lambda_1, \ldots, \lambda_r$ down the diagonal, and unipotent radical N_λ contained in the upper triangular subgroup B. We let $P_\lambda^- = M_\lambda N_\lambda^-$ the parabolic subgroup opposite to P_λ with respect to M_λ. We have $G = P_{(n)}$ and $P_{(1,\ldots,1)} = B$. We denote by T and U the group $M_{(1,\ldots,1)}$ of diagonal matrices with entries in D^* and the strictly upper triangular group $N_{(1,\ldots,1)}$. A parabolic subgroup P of G is conjugate to P_λ for a unique composition λ of n and is associated to $P_{\lambda^\dagger}$ for the unique partition $\lambda^\dagger$ of n deduced from λ by re-ordering its elements. Let $\mathfrak{P}(n)$ denote the set of partitions of n. For $\lambda = (\lambda_1, \ldots, \lambda_r) \in \mathfrak{P}(n)$, $d\lambda = (d\lambda_1, \ldots, d\lambda_r) \in \mathfrak{P}(dn)$.

Let R be a field. We denote by char_R the characteristic of R, and by $C_c^\infty(X; R)$ the R-module of locally constant functions on a locally profinite space X with compact support and values in R. The map

$$\varphi \mapsto f(1 + X) = \varphi(X) : C_c^\infty(M_n(P_D); R) \to C_c^\infty(K_1; R)$$

is a K_0-equivariant isomorphism. The extension by 0 embeds $C_c^\infty(M_n(P_D); R)$ in $C_c^\infty(\mathfrak{g}; R)$ and $C_c^\infty(K_1; R)$ in $C_c^\infty(G; R)$. An R-distribution on G or on $\mathfrak{g}$ is a linear form on $C_c^\infty(G; R)$ or $C_c^\infty(\mathfrak{g}; R)$. The group G acts on G and on $\mathfrak{g}$ by conjugation, and by functoriality on $C_c^\infty(G; R)$, $C_c^\infty(\mathfrak{g}; R)$ and on the distributions. A G-invariant distribution is called invariant.

For $R = \mathbb{C}$, dg will denote the Haar measure on G such that dg gives the volume 1 to K_0, et dZ the Haar measure on $\mathfrak{g}$ giving the volume $[K_0 : K_1]^{-1} = |GL_n(k_D)|^{-1}$ to $M_n(P_D)$, hence the volume $a = q^{dn^2}|GL_n(k_D)|^{-1}$ to $M_n(O_D)$. The Haar measures dZ and $dg = a|\operatorname{nrd} Z|_F^{-n} dZ$ ([Weil67] X, §1 Lemma 1) are compatible with the map $x \mapsto 1 + x : M_n(P_D) \to K_1$. The modulus of P is $\delta_P(p) = |\det(\operatorname{Ad} p)_{\mathfrak{n}}|_F$ ([Vigneras96] I.2.8). Let dk denote the restriction of dg to K_0, dp the left Haar measure on P such that $dg = \delta_P(p)dkdp$, dn^- the Haar measure on N^- such that $dn^- \, dp$ is the restriction of dg to $N^- P$ (open in G), dn the Haar measure on N giving the same volume to $N \cap K_0$ as the volume of $N^- \cap K_0$ for dn^-, and dm the Haar measure on M such that $dp = dm \, dn$. For each $f \in C_c^\infty(G; \mathbb{C})$,

$$\int_G f(g)dg = \int_{K_0 \times P} f(p^{-1}k) \, dk \, dp = \int_{K_0 \times P} f(kp) \, \delta_P(p) \, dk \, dp$$

$$= \int_{K_0 \times M \times N} f(kmn) \, \delta_P(m) \, dk \, dm \, dn.$$

Let dW, dY^-, dY be the Haar measures on $\mathfrak{h} = \mathfrak{p}, \mathfrak{n}^-, \mathfrak{n}$ such that dp and dW, dn^- and dY^-, dn and dY are compatible with the map $x \mapsto 1 + x$ for $x \in \mathfrak{h}(P_D) = \mathfrak{h} \cap M_n(P_D)$. We have $dZ = dW \, dY^-$.

Let π be a smooth representation of G on an R-vector space V. Each vector is fixed by some open compact subgroup K of G,

$$V = \cup_K V^K \quad \text{where } V^K = \{\text{vectors of } V \text{ fixed by } K\}. \tag{2.1}$$

π is called admissible when the dimension $\dim_R V^K$ of V^K is finite for any open compact subgroup K. The categories $\operatorname{Rep}_R^\infty(G)$ of smooth R-representations of G, $\operatorname{Rep}_R^{\infty,f}(G)$ of finite length smooth representations are abelian. When $\operatorname{char}_R \neq p$, the category of admissible R-representations of G is abelian and contains $\operatorname{Rep}_R^{\infty,f}(G)$ (this is not true when $\operatorname{char}_R = p$). We denote by $\operatorname{Gr}_R^\infty(G)$ the Grothendieck group of $\operatorname{Rep}_R^{\infty,f}(G)$, and

$$\pi \mapsto [\pi] : \operatorname{Rep}_R^\infty(G) \to \operatorname{Gr}_R^\infty(G)$$

the natural homomorphism. The map $\chi \mapsto \chi \circ \operatorname{nrd}$ is a bijection from the smooth characters $F^* \to R^*$ onto the smooth characters $G \to R^*$.

For a set X and a function f on X with value in $\mathbb{Z}$ or in R, the support $\operatorname{Supp} f$ of f is the set of $x \in X$ with $f(x) \neq 0$ and 1_Y will denote the characteristic function of a subset Y of X.

3. Nilpotent orbits

3.1. An element $X \in \mathfrak{g}$ is nilpotent if and only if $X^r = 0$ for some $r \in \mathbb{N}$. The set $\mathfrak{N}$ of nilpotent elements in $\mathfrak{g}$ is stable by G-conjugation. A G-orbit in $\mathfrak{N}$ is called a **nilpotent orbit of** G. The set $G \backslash \mathfrak{N}$ of nilpotent orbits of G is finite, in bijection with the set $\mathfrak{P}(n)$ of partitions of n ([Bushnell-Henniart-Lemaire10] §2.4-2.6).

3.2. Let V be the right D-vector space D^n. The group G identifies with $\mathrm{Aut}_D(V)$ and its Lie algebra $\mathfrak{g}$ with $\mathrm{End}_D V$. Let $X \in \mathrm{End}_D V$ be nilpotent. The composition $\lambda = (\lambda_1, \dots)$ of n,

$$\lambda_i = \dim_D \mathrm{Ker}\, X^i - \dim_D \mathrm{Ker}\, X^{i-1} \quad \textit{for } i \geq 1, \tag{3.1}$$

is a partition because the multiplication by X induces an injection from $\mathrm{Ker}\, X^i / \mathrm{Ker}\, X^{i+1}$ to $\mathrm{Ker}\, X^{i-1} / \mathrm{Ker}\, X^i$. We get a canonical map $\mathfrak{N} \to \mathfrak{P}(n)$ sending 0 to (n). The map is bijective. Let $\mathfrak{O}_\lambda$ denote the nilpotent orbit of G containing X. The dual partition of λ is $\hat{\lambda} = (\hat{\lambda}_i = |\{j \mid \lambda_j \geq i\}|)$. There is a partial order on $\mathfrak{P}(n)$

$$\mu \leq \lambda \ \Leftrightarrow \ \hat{\lambda} \leq \hat{\mu} \ \Leftrightarrow \ \sum_{i=1}^{j} \mu_i \leq \sum_{i=1}^{j} \lambda_i \text{ for all } j.$$

There is also a partial order on $G \backslash \mathfrak{N}$

$$\mathfrak{O}' \leq \mathfrak{O} \ \Leftrightarrow \mathfrak{O}' \subset \overline{\mathfrak{O}} \text{ where } \overline{\mathfrak{O}} \text{ is the closure of } \mathfrak{O} \text{ in } \mathfrak{g}.$$

The bijection reverses the partial order.

$$\overline{\mathfrak{O}}_\lambda = \cup_{\hat{\mu} \leq \hat{\lambda}} \mathfrak{O}_\mu. \tag{3.2}$$

The unique maximal partition (n) corresponds the null orbit $\{0\} = \mathfrak{O}_{(n)}$. The unique minimal partition $(1, \dots, 1)$ corresponds to the unique maximal nilpotent orbit $\mathfrak{O}_{(1,\dots,1)}$, called regular, of closure $\overline{\mathfrak{O}}_{(1,\dots,1)} = \mathfrak{N}$. The parabolic subgroup P of $\mathrm{Aut}_D(V)$ preserving the flag $(\mathrm{Ker}\, X^i)_i$ of the iterated kernels of X, is associated to P_λ. The intersection $\mathfrak{O}_\lambda \cap \mathfrak{n}_\lambda$ is open dense in $\mathfrak{n}_\lambda$ [Jantzen04, §13.17]. The dimension of $\mathfrak{O}_\lambda$ as an F-variety is even and equal to (loc.cit.)

$$\dim_F \mathfrak{O}_\lambda = 2 \dim_F \mathfrak{n}_\lambda = 2d^2 \dim_D \mathfrak{n}_\lambda, \tag{3.3}$$

$$\dim_D \mathfrak{n}_\lambda = \sum_{i<j} \lambda_i \lambda_j. \tag{3.4}$$

We denote $d_\lambda = \sum_{i<j} \lambda_i \lambda_j$ and $d(\mathfrak{P}(n)) = \{d_\lambda \mid \lambda \in \mathfrak{P}(n)\}$,

$$d(\mathfrak{P}(n)) = \{d_{(n)} = 0 < d_{(n-1,1)} = n-1 < \dots < d_{(1,\dots,1)} = n(n-1)/2\}. \tag{3.5}$$

The map $\lambda \mapsto d_\lambda : \mathfrak{P}(n) \to \mathbb{N}$ is injective only when $n \leq 5$.

$d(\mathfrak{P}(2)) = \{0 < 1\}$.
$d(\mathfrak{P}(3)) = \{0 < 2 = d_{(2,1)} < 3\}$.
$d(\mathfrak{P}(4)) = \{0 < 3 = d_{(3,1)} < 4 = d_{(2,2)} < 5 = d_{(2,1,1)} < 6\}$.
$d(\mathfrak{P}(5)) = \{0 < 4 = d_{(4,1)} < 6 = d_{(3,2)} < 7 = d_{(3,1,1)} < 8 = d_{(2,2,1)} < 9 = d_{(2,1,1,1)} < 10\}$.
$d(\mathfrak{P}(6)) = \{0 < 5 = d_{(5,1)} < 8 = d_{(4,2)} < 9 = d_{(4,1,1)} = d_{(3,3)} < \dots < 15\}$.

4. Nilpotent orbital integrals

Assume $R = \mathbb{C}$. The nilpotent orbital integral of the zero nilpotent orbit $\{0\}$ is the value at 0,

$$\mu_{\{0\}}(\varphi) = \varphi(0) \quad (\varphi \in C_c^\infty(\mathfrak{g}; \mathbb{C})).$$

Let $\mathfrak{O}$ be a non-zero nilpotent orbit of G and $\lambda \in \mathfrak{P}(n) \setminus \{(n)\}$ such that $\mathfrak{O} = \mathfrak{O}_\lambda$. The nilpotent orbital integral of $\mathfrak{O}$ is a linear form sending $\varphi \in C_c^\infty(\mathfrak{g};\mathbb{C})$ to

$$\mu_{\mathfrak{O}_\lambda}(\varphi) = \int_{\mathfrak{n}_\lambda} \varphi_{K_0}(Y)\, dY \tag{4.1}$$

$$\varphi_{K_0}(Z) = \int_{K_0} \varphi(kZk^{-1})dk \quad \text{for } Z \in \mathfrak{g} \tag{4.2}$$

dY and dk are the Haar measures on $\mathfrak{n}_\lambda$ and K_0 given in §2. For (4.1), see [Howe74] when $D = F$, [Lemaire04] for D general.

4.1. Homogeneity. For $t \in F^*, \varphi \in C_c^\infty(\mathfrak{g})$, write $\varphi_t(Z) = \varphi(t^{-1}Z)$ for $Z \in \mathfrak{g}$.

Proposition 4.1. *The nilpotent integral orbital of $\mathfrak{O}$ satisfies the homogeneity relation:*

$$\mu_{\mathfrak{O}}(\varphi_t) = |t|_F^{d(\mathfrak{O})}\, \mu_{\mathfrak{O}}(\varphi), \quad \dim_F(\mathfrak{O}) = 2d(\mathfrak{O}).$$

Proof. For $\lambda \in \mathfrak{P}(n) \setminus \{(n)\}$, we have $d(\mathfrak{O}_\lambda) = \dim_F \mathfrak{n}_\lambda$ by (3.4) and by (4.1)

$$\mu_{\mathfrak{O}_\lambda}(\varphi) = \int_{\mathfrak{n}_\lambda} \varphi_{K_0}(Y)\, dY = |t|_F^{\dim_F \mathfrak{n}_\lambda} \int_{\mathfrak{n}_\lambda} (\varphi_{K_0}(tY)\, dY = |t|_F^{\dim_F \mathfrak{n}_\lambda} \mu_{\mathfrak{O}_\lambda}(\varphi_{t^{-1}}).$$

□

For a nilpotent orbit $\mathfrak{O}$ of G and a lattice $\mathfrak{L}$ in $\mathfrak{g}$, we denote by $\mu_{\mathfrak{O},\mathfrak{L}}$ the restriction of $\mu_{\mathfrak{O}}$ to $C_c^\infty(\mathfrak{g}/\mathfrak{L};\mathbb{C})$ (identified to the functions on $\mathfrak{g}$ invariant by translation by $\mathfrak{L}$). The homogeneity implies ([Harish-Chandra78] Lemma 14 when the characteristic of F is 0):

Corollary 4.2. *For any lattice $\mathfrak{L}$ in $\mathfrak{g}$, the linear forms $\mu_{\mathfrak{O},\mathfrak{L}}$ of $C_c^\infty(\mathfrak{g}/\mathfrak{L};\mathbb{C})$ for the nilpotent orbits $\mathfrak{O}$ of G are linearly independent.*

Proof. For any $d \in \mathbb{N}$, let $\mathfrak{N}_d$ denote the union of nilpotent orbits of dimension $\leq d$. Any nilpotent orbit $\mathfrak{O}$ of dimension $d > 1$ is open in $\mathfrak{N}_d$ and $\mathfrak{O} \cup \mathfrak{N}_{d-1}$ is closed. We choose:

a) $\varphi_{\mathfrak{O}} \in C_{\mathbb{C}}^\infty(\mathfrak{g};\mathbb{C})$ such that

$$\mu_{\mathfrak{O}}(\varphi_{\mathfrak{O}'}) = \begin{cases} 1 \text{ if } \mathfrak{O} = \mathfrak{O}' \\ 0 \text{ if } \mathfrak{O} \neq \mathfrak{O}' \end{cases},$$

by induction on $\dim \mathfrak{O}$.

b) a lattice $\mathfrak{L}_0$ in $\mathfrak{g}$ such that $\varphi_{\mathfrak{O}} \in C_c^\infty(\mathfrak{g}/\mathfrak{L}_0;\mathbb{C})$ for each $\mathfrak{O} \in G\backslash\mathfrak{N}$,

c) $t \in F^*$ such that $\mathfrak{L} \subset t\mathfrak{L}_0$.

Then, $(\varphi_{\mathfrak{O}})_t$ belongs to $C_c^\infty(\mathfrak{g}/\mathfrak{L};\mathbb{C})$ and by homogeneity.

$$\mu_{\mathfrak{O}}((\varphi_{\mathfrak{O}'})_t) = |t|^{d(\mathfrak{O})}\mu_{\mathfrak{O}}(\varphi_{\mathfrak{O}'}) = \begin{cases} |t|^{d(\mathfrak{O})} & \text{if } \mathfrak{O} = \mathfrak{O}' \\ 0 & \text{if } \mathfrak{O} \neq \mathfrak{O}' \end{cases}.$$

□

4.2. Fourier transform. The bilinear map $(Z,Y)\mapsto \mathrm{trd}(ZY):\mathfrak{g}\times\mathfrak{g}\to F$ is non degenerate. Let $\psi: F\to\mathbb{C}^*$ be a non-trivial additive character on F. The Fourier transform in $C_c^\infty(\mathfrak{g};\mathbb{C})$ with respect to ψ and the Haar measure dZ (fixed in §1) is the endomorphism of $C_c^\infty(\mathfrak{g};\mathbb{C})$:

$$\varphi\mapsto\hat\varphi(Y)=\int_{\mathfrak{g}}\varphi(Z)\,\psi(\mathrm{trd}(ZY))\,dZ\quad (Y\in\mathfrak{g},\ \varphi\in C_c^\infty(\mathfrak{g};\mathbb{C})). \tag{4.3}$$

There exists a positive real number $c_\psi>0$ such that $\hat{\hat\varphi}(Z)=c_\psi\varphi(-Z)$ for $Z\in\mathfrak{g}$ [3]. In particular

$$\int_{\mathfrak{g}}\hat\varphi(Y)dY=c_\psi\varphi(0). \tag{4.4}$$

For an O_F-lattice $\mathfrak{L}$ in $\mathfrak{g}$, the Fourier transform of $1_{\mathfrak{L}}$ is $\mathrm{vol}(\mathfrak{L},dZ)\,1_{\mathfrak{L}^*_\psi}$ where

$$\mathfrak{L}^*_\psi=\{Z\in M_n(D)\mid \psi(\mathrm{trd}(Z\mathfrak{L})))=1\}=\{Z\in M_n(D)\mid\ \mathrm{trd}(Z\mathfrak{L})))\subset\mathrm{Ker}(\psi)\}.$$

Example 4.3. When ψ is trivial on P_F and not on O_F, $M_n(O_D)^*_\psi=M_n(P_D)$ ([Weil67] X, §2, Proposition 5).

For an open subset $\mathfrak{C}$ of $\mathfrak{g}$, the extension by zero embeds $C_c^\infty(\mathfrak{C};\mathbb{C})$ into $C_c^\infty(\mathfrak{g};\mathbb{C})$.

Proposition 4.4. *Let $\mathfrak{C}$ be an open neighborhood of zero in $\mathfrak{g}$. The linear forms*

$$\varphi\mapsto\mu_{\mathfrak{O}}(\hat\varphi):C_c^\infty(\mathfrak{C};\mathbb{C})\to\mathbb{C}$$

for $\mathfrak{O}\in G\backslash\mathfrak{N}$, are linearly independent.

Proof. This follows from the linear independence of the $\mu_{\mathfrak{O},\mathfrak{L}}$ for any lattice $\mathfrak{L}$ (Corollary 4.2) ([Harish-Chandra78] corollary of Lemma 14). □

4.3. Let $\mathfrak{O}$ be a nilpotent orbit of G and ψ a non-trivial smooth character of F. We compute the nilpotent orbital integral $\mu_{\mathfrak{O}}(\hat\varphi)$ (4.1) of the Fourier transform $\hat\varphi$ with respect to ψ of $\varphi\in C_c^\infty(\mathfrak{g};\mathbb{C})$. Let λ be the partition of n such that $\mathfrak{O}=\mathfrak{O}_\lambda$. Write (P,M,N) for $(P_\lambda,M_\lambda,N_\lambda)$. The bilinear map $(Y,Y^-)\mapsto\mathrm{trd}(YY^-):\mathfrak{n}\times\mathfrak{n}^-\to\mathbb{C}$ is non degenerate because $\mathrm{trd}(YW)=0$ for $Y\in\mathfrak{n},W\in\mathfrak{p}$. The corresponding Fourier transform with respect to ψ is the linear map :

$$\varphi_2\mapsto\hat\varphi_2(Y)=\int_{\mathfrak{n}^-}\varphi_2(Y^-)\psi(\mathrm{trd}(YY^-))\,dY^-:C_c^\infty(\mathfrak{n}^-;\mathbb{C})\to C_c^\infty(\mathfrak{n};\mathbb{C}).$$

There exists a positive real number $c_{\psi,\mathfrak{n}}$ such that

$$\int_{\mathfrak{n}}\int_{\mathfrak{n}^-}\varphi_2(Y^-)\psi(\mathrm{trd}(YY^-))\,dY^-\,dY=\int_{\mathfrak{n}}\hat\varphi_2(Y)dY=c_{\psi,\mathfrak{n}}\,\varphi_2(0).$$

For $\varphi\in C_c^\infty(\mathfrak{g};\mathbb{C})$ of Fourier transform $\hat\varphi$ with respect to ψ, put

$$\hat\mu_{\mathfrak{O}}(\varphi)=\mu_{\mathfrak{O}}(\mathbf{c}_{\psi,\mathfrak{n}}^{-1}\,\hat\varphi). \tag{4.5}$$

Proposition 4.5. *We have* $\hat\mu_{\mathfrak{O}}(\varphi)=\int_{\mathfrak{p}}\varphi_{K_0}(W)\,dW$.

[3]The non-trivial additive characters $F\to\mathbb{C}^*$ are $\psi^a(x)=\psi(ax),x\in F$ for $a\in F^*$. As $d(aZ)=|a|_F^{d^2n^2}dZ$, we have $c_{\psi^a}=|a|_F^{-n^2d^2}c_\psi$

This was proved only "for some Haar measures" when $D = F$ and the characteristic of F is 0 [Howe74]. The proposition follows from the next three lemmas where $\varphi \in C_c^\infty(\mathfrak{g}; \mathbb{C})$.

Lemma 4.6. $\int_{\mathfrak{p}} \int_{\mathfrak{n}} \int_{\mathfrak{n}^-} \varphi((Y^- + W))\psi(\mathrm{trd}(YY^-))\, dY^-\, dY\, dW = c_{\psi,\mathfrak{n}} \int_{\mathfrak{p}} \varphi(W)\ dW.$

Proof. We have $C_c^\infty(\mathfrak{g}; \mathbb{C}) = C_c^\infty(\mathfrak{p}; \mathbb{C}) \otimes C_c^\infty(\mathfrak{n}^-; \mathbb{C})$. For $\varphi_1 \in C_c^\infty(\mathfrak{p}; \mathbb{C}), \varphi_2 \in C_c^\infty(\mathfrak{n}^-; \mathbb{C})$ and $\varphi \in C_c^\infty(\mathfrak{g}; \mathbb{C})$ such that $\varphi(Y^- + W) = \varphi_1(W)\varphi_2(Y^-)$ for $Y^- \in \mathfrak{n}, W \in \mathfrak{p}$, we have

$$\int_{\mathfrak{p}} \int_{\mathfrak{n}} \int_{\mathfrak{n}^-} \varphi((Y^- + W))\psi(\mathrm{trd}(YY^-))\, dY^-\, dY\, dW = c_{\psi,\mathfrak{n}} \int_{\mathfrak{p}} \varphi_1(W)\varphi_2(0)\, dW = c_{\psi,\mathfrak{n}} \int_{\mathfrak{p}} \varphi(W)\, dW.$$

□

Lemma 4.7. *The integration over* $\mathfrak{n}$ *of the Fourier transform is integration over* $\mathfrak{p}$*:*

$$\int_{\mathfrak{n}} \hat{\varphi}(Y)\, dY = c_{\psi,\mathfrak{n}} \int_{\mathfrak{p}} \varphi(W)\, dW. \tag{4.6}$$

Proof. The left hand side of (4.6) is

$$\int_{\mathfrak{n}} \int_{\mathfrak{g}} \varphi(Z)\psi(\mathrm{trd}(YZ))\, dZ\, dY = \int_{\mathfrak{n}} \int_{\mathfrak{p}} \int_{\mathfrak{n}^-} \varphi(Y^- + W)\psi(\mathrm{trd}(Y(Y^- + W))\, dY^-\, dW\, dY$$

because $dZ = dY^-\, dW$, and as $\mathrm{trd}(YW) = 0$ for $Y \in \mathfrak{n}, W \in \mathfrak{p}$

$$= \int_{\mathfrak{n}} \int_{\mathfrak{p}} \int_{\mathfrak{n}^-} \varphi(Y^- + W)\psi(\mathrm{trd}(YY^-))\, dY^-\, dW\, dY = c_{\psi,\mathfrak{n}} \int_{\mathfrak{p}} \varphi(W)\ dW$$

because we can invert the integrals on $\mathfrak{n}$ and on $\mathfrak{p}$ [4] and by Lemma 4.6. □

Lemma 4.8. *The Fourier transform of* φ_{K_0} *is* $(\hat{\varphi})_{K_0}$ *for* $\varphi \in C_c^\infty(\mathfrak{g}; \mathbb{C})$*.*

Proof. Write $K = K_0$. Then $(\hat{\varphi})_K(Y) = \int_K \hat{\varphi}(kYk^{-1})\, dk$ for $Y \in \mathfrak{g}$ is equal to

$$\int_K \int_{\mathfrak{g}} \varphi(Z)\psi(\mathrm{trd}(kYk^{-1}Z))\, dZ\, dk = \int_{K \times \mathfrak{g}} \varphi(kZk^{-1})\psi(\mathrm{trd}(kYk^{-1}\, kZk^{-1}))\, dZ,\, dk$$

because dZ is K-invariant. This is

$$\int_{K \times \mathfrak{g}} \varphi(kZk^{-1})\psi(\mathrm{trd}(kYZk^{-1}))\, dZ\, dk = \int_{K \times \mathfrak{g}} \varphi(kZk^{-1})\psi(\mathrm{trd}(YZ))\, dZ\, dk$$
$$= \int_{\mathfrak{g}} \varphi_K(Z)\psi(\mathrm{trd}(YZ))\, dZ.$$

□

[4] taking $\varphi = \varphi_1\varphi_2$ as above one wants to compute the integral on $\mathfrak{n}$ then on $\mathfrak{p}$ of $\varphi_1(W)\hat{\varphi}_2(Y)$ and we can exchange the integrals because both functions have compact support

5. Trace of an admissible representation and parabolic induction

5.1. Let R be a field of characteristic $\mathrm{char}_R \neq p$ and dg a Haar measure on G with values in R. Let $\pi \in \mathrm{Rep}_R^\infty(G)$ be an admissible representation of G on an R-vector space V. The linear endomorphism of V

$$\pi(f(g)dg) = \int_G f(g)\pi(g)dg \tag{5.1}$$

has a finite rank. Its trace is an invariant R-distribution on G

$$\mathrm{trace}(\pi) : f \mapsto \mathrm{trace}(\pi(f(g)dg), \quad f \in C_c^\infty(G; R),$$

called the character of π.

The characters of the irreducible smooth complex representations of G are linearly independent ([Vigneras96] I.6.13 where $c = 0$ should be 0).

For any exact sequence $0 \to \pi_1 \to \pi \to \pi_2 \to 0$ of admissible R-representations of G, $\mathrm{trace}(\pi) = \mathrm{trace}(\pi_1) + \mathrm{trace}(\pi_2)$. Any finite length smooth R-representation of G is admissible. By the universal property of Grothendieck groups, the character induces a linear map from the Grothendieck group $\mathrm{Gr}_R^\infty(G)$ of $\mathrm{Rep}_R^{\infty,f}(G)$ to the space of invariant R-distributions on G.

For any open compact subgroup K of G, the restriction to K induces a linear map

$$\nu \mapsto \nu|_K : \mathrm{Gr}_R^\infty(G) \to \mathrm{Gr}_R^\infty(K) \tag{5.2}$$

from $\mathrm{Gr}_R^\infty(G)$ to the Grothendieck group $\mathrm{Gr}_R^\infty(K)$ of admissible smooth R-representations of K. When K is a pro-p group, the category $\mathrm{Rep}_R^\infty(K)$ is semi-simple.

5.2. Parabolic induction. Let R be a field and P a parabolic subgroup of G of Levi subgroup M and unipotent radical N. The parabolic induction $\mathrm{ind}_P^G : \mathrm{Rep}_R^\infty(M) \to \mathrm{Rep}_R^\infty(G)$ sends $(\sigma, W) \in \mathrm{Rep}_R^\infty(M)$ to $(\mathrm{ind}_P^G(\sigma), V) \in \mathrm{Rep}_R^\infty(G)$ where V is the space of functions $f : G \to W$ right invariant by some open subgroup of G and satisfying $f(pg) = \tilde\sigma(p)f(g)$ for $(p, g) \in P \times G$ and $\tilde\sigma$ is the inflation to P of σ. It is an exact functor respecting admissibility and finite length.

Replacing P by a G-conjugate does not change the isomorphism class of $\mathrm{ind}_P^G(\sigma)$ and a G-conjugate of P contains B.

We suppose in this section that $B \subset P$. This implies $G = K_0P = PK_0 = K_0P^- = P^-K_0$ where $K_0 = GL_n(O_D)$ and $P^- = MN^-$ the opposite parabolic subgroup with respect to M.

The parabolic induction of the trivial R-character of M

$$\pi_P = \mathrm{ind}_P^G 1$$

will play an important role. As our parabolic induction is not normalized, $[\pi_P] \in \mathrm{Gr}_R^\infty(G)$ depends on the choice of P of Levi M.

Lemma 5.1. *Assume* $\mathrm{char}_R \neq p$ *and let* P' *be a parabolic subgroup of* G *associated to* P*. The representation* π_P *has the same restriction to* K_0 *as* $\pi_{P'}$*.*

Proof. [5] Let R^{ac} be an algebraic closure of R. In the group of unramified smooth R^{ac}-characters of M, the set of χ such that $\mathrm{ind}_P^G \chi$ is irreducible is Zariski dense [Dat05, Theorem 1.2]. There exist unramified smooth R^{ac}-characters χ and χ' of M such that the R^{ac}-representations $\mathrm{ind}_P^G \chi$ and $\mathrm{ind}_{P'}^G \chi'$ are irreducible and isomorphic [Dat09, Lemma 4.13]. Let R' be the finite extension of R generated the values of χ and χ'. The R'-representations $\mathrm{ind}_P^G \chi$ and $\mathrm{ind}_{P'}^G \chi'$ are irreducible and isomorphic. We deduce that the restriction to K_0 of the R'-representations π_P and $\pi_{P'}$ are isomorphic. As R-representations of K_0, $\oplus^r \pi_{P'} \simeq \oplus^r \pi_P$ where $r = [R : R']$. For any $j \geq 1$, taking the invariants by K_j, the finite dimensional representations $\oplus^r (\pi_{P'})^{K_j}$ and $\oplus^r (\pi_P)^{K_j}$ of the finite group K_0/K_j are isomorphic. By Krull-Remak-Schmidt, $(\pi_P)^{K_j} \simeq (\pi_P)^{K_j}$. As this is true for any j, we have $\pi_{P'} \simeq \pi_P$. ☐

5.2.1. When $R = \mathbb{C}$ and $\sigma \in \mathrm{Rep}_{\mathbb{C}}^\infty(M)$ is admissible, we compute the character of $\mathrm{ind}_P^G(\sigma)$ in terms of the character of σ.

Lemma 5.2. *For $f \in C_c^\infty(G, \mathbb{C})$, the function $Sf(m) = \int_N \int_{K_0} f(kmnk^{-1}) dkdn$ on M belongs to $C_c^\infty(M, \mathbb{C})$.*

Proof. The normal open compact subgroups K of K_0 form a fundamental system of neighborhoods of 1 in G and for $g \in G$ the open compact sets KgK form a fundamental system of neighborhoods of g in G. For $g \in G$ and $m \in M$, $m^{-1}KgK \cap N$ is open in N. The set of $m \in M$ such that $m^{-1}KgK \cap N \neq \emptyset$ is open compact in M [6], $S1_{KgK}$ is 0 outside of this set and $S1_{KgK}(m) = \mathrm{vol}(m^{-1}KgK \cap N, dn)$ for $m^{-1}KgK \cap N \neq \emptyset$. ☐

Remark 5.3. For a normal open compact subgroup K of K_0 such that $K \cap P = (K \cap M)(K \cap N)$, $S1_K = \mathrm{vol}(K \cap N, dn) 1_{M \cap K}$. For $f \in C_c^\infty(G, \mathbb{C})$ with $\mathrm{Supp}\, f \subset K$, then $\mathrm{Supp}\, Sf \subset K \cap M$.

Proposition 5.4. *We have* $\mathrm{trace}(\pi(f(g)dg)) = trace(\sigma(Sf(m)dm))$ *for $\sigma \in \mathrm{Rep}_{\mathbb{C}}^\infty(M)$ admissible, $\pi = \mathrm{ind}_P^G(\sigma)$, and (f, Sf) as in Lemma 5.2.*

Proof. a) Preliminaries. As $G = PK_0$, a function in the space V of π is determined by its restriction to K_0, and $\pi|_{K_0} \simeq \mathrm{ind}_{P\cap K_0}^{K_0}(\sigma|_{M\cap K_0})$. Denote $V|_{K_0}$ the restrictions to K_0 of the functions in V. Let W denote the space of σ and ρ the action of K_0 on $C^\infty(K_0; W)$ by right translation. We identify $C^\infty(K_0; W)$ and $C^\infty(K_0; R) \otimes_R W$. Then $(\mathrm{ind}_{P\cap K_0}^{K_0}(\sigma|_{M\cap K_0}), V|_{K_0})$ is a subrepresentation of $(\rho, C^\infty(K_0; R) \otimes_R W)$. Let dx denote the restriction to $P \cap K_0$ of dp (equal to the restriction of dk). The map B : $(\rho, C^\infty(K_0; R) \otimes_R W) \to (\mathrm{ind}_{P\cap K_0}^{K_0}(\sigma|_{M\cap K_0}, V|_{K_0})$

$$B(h \otimes w)(k) = \mathrm{vol}(P \cap K_0, dx)^{-1} \int_{P\cap K_0} h(x^{-1}k) \tilde{\sigma}(x)(w) dx \quad (h \in C^\infty(K_0; R), w \in W, k \in K_0),$$

is a K_0-equivariant projection. The function $B(h \otimes w)$ on K_0 extends to a function $F_{h,w} \in V$

$$F_{h,w}(pk) = \mathrm{vol}(P \cap K_0, dx)^{-1} \int_{P\cap K_0} h(x^{-1}k) \tilde{\sigma}(px)(w) dx \quad ((p, k) \in P \times K_0).$$

[5] This proof suggested by the referee simplifies our original proof using [Minguez-Sécherre14]

[6] $P \cap KgK$ is compact and the quotient map $P \to M$ is continuous

b) Choose a normal open pro-p subgroup K of K_0 such that f is binvariant by K. The endomorphism $\pi(f(g)dg)$ of V restricted to V^K is an endomorphism A of V^K of trace $\mathrm{trace}(A) = \mathrm{trace}(\pi(f(g)dg)$. Choose a disjoint decomposition $K_0 = \sqcup_i y_i K$. The 1_{y_iK} form a basis of $C^\infty(K_0; R)^K$, the support of $B(1_{y_iK} \otimes w)$ is in y_iK, and $\mathrm{trace}(A)$ is the trace of the endomorphism $w \mapsto \sum_i B(F_{1_{y_iK},w})(y_i)$ of W. For $y \in K_0$, $B(F_{1_{yK},w})(y)$ is equal to

$$\int_G f(g)F_{1_{yK},w}(yg)dg = \int_G f(y^{-1}g)F_{1_{yK},w}(g)dg = \int_{K_0\times P} f(y^{-1}p^{-1}k)F_{1_{yK},w}(p^{-1}k)dk\,dp$$
$$= \mathrm{vol}(P\cap K_0, dx)^{-1}\int_{K_0\times P\times P\cap K_0} f(y^{-1}p^{-1}k)h_y(x^{-1}k)\sigma(p^{-1}x)(w)\,dk\,dp\,dx$$
$$= \int_{K_0\times P} f(y^{-1}p^{-1}k)h_y(k)\sigma(p^{-1})(w)\,dk\,dp = \int_{K_0\times P} f(y^{-1}pk)1_{yK'}(k)\tilde\sigma(p)(w)\,dk\,dp$$
$$= \mathrm{vol}(K', dk)\int_P f(y^{-1}py)\tilde\sigma(p)(w)\,dp.$$

Therefore

$$\sum_i B(F_{1_{y_iK},w})(y_i) = \mathrm{vol}(K, dk)\int_P \sum_i f(y_i^{-1}py_i)\tilde\sigma(p)(w)\,dp = \int_{K_0\times P} f(k^{-1}pk)\tilde\sigma(p)(w)dk\,dp$$
$$= \int_{K_0\times M\times N} f(k^{-1}mnk)\sigma(m)(w)dk\,dm\,dn = \sigma(Sf(m)dm)(w).$$

We deduce that the trace of $\pi(f(g)dg)$ is the trace of $\sigma(Sf(m)dm)$. □

The set $\{P_\lambda \mid \lambda \in \mathfrak{P}(n)\}$ represents the parabolic subgroups of G modulo association.

Proposition 5.5. *When P is a parabolic subgroup of G associated to P_λ, we have*

$$\mathrm{trace}(\pi_P(f(g)\,dg)) = \hat\mu_{\mathfrak{O}_\lambda}(\varphi).$$

for $f \in C_c^\infty(K_1; \mathbb{C})$ and $\varphi \in C_c^\infty(M_n(P_D); \mathbb{C})$ such that $f(1+X) = \varphi(X)$ for $X \in M_n(P_D)$, and $\hat\mu_{\mathfrak{O}_\lambda}$ as in (4.5).

Proof. For (f, φ) as in the proposition, the functions

$$(5.3)\quad f_{K_0}(g) = \int_{K_0} f(kgk^{-1})\,dk \quad (g \in G), \quad \varphi_{K_0}(X) = \int_{K_0} \varphi(kXk^{-1})\,dk \quad (X \in M_n(D)),$$

belong also to $C_c^\infty(K_1; \mathbb{C}), C_c^\infty(M_n(P_D); \mathbb{C})$ and $f_{K_0}(1 + X) = \varphi_{K_0}(X)$ for $X \in M_n(P_D)$,

$$\int_P f(p)\,dp = \int_{\mathfrak{p}} \varphi(W)\,dW,$$

$\mathrm{trace}(\pi_P(f(g)\,dg)) = \mathrm{trace}(\pi_{P_\lambda}(f(g)\,dg))$ as $\pi_P = \pi_{P_\lambda}$ on K_0(Lemma 5.1), and

$$\mathrm{trace}(\pi_P(f(g)\,dg)) = \int_M Sf(m)\,dm = \int_P f_{K_0}(p)\,dp = \int_{\mathfrak{p}} \varphi_{K_0}(W)\,dW = \hat\mu_{\mathfrak{O}}(\varphi).$$

for $P = P_\lambda, \mathfrak{O} = \mathfrak{O}_\lambda$, by Propositions 5.2 and 4.5. □

Corollary 5.6. *For any non zero map $c : \mathfrak{P}(n) \to \mathbb{C}$, the restriction of*

$$\sum_{\lambda\in\mathfrak{P}(n)} c(\lambda)\,[\pi_{P_\lambda}] \in \mathrm{Gr}_{\mathbb{C}}^\infty(G)$$

to an arbitrary open compact subgroup K of G is not 0.

Corollary 5.7. *For any non zero map $c : \mathfrak{P}(n) \to \mathbb{C}$, the restriction of the invariant $\mathbb{C}$-distribution on G*

$$\sum_{\lambda \in \mathfrak{P}(n)} c(\lambda)\ \mathrm{trace}(\pi_{P_\lambda})$$

to an arbitrary open compact subgroup K of G is not 0.

Proof. By Propositions 5.5 and 4.4, the characters of π_{P_λ} are linearly independant on any neighborhood of 1, because their values on $f \in C_c^\infty(K_1;\mathbb{C})$ are the Fourier transforms of the nilpotent orbital integrals of $\mathfrak{O}_\lambda$ on $\varphi \in C_c^\infty(M_n(P_D);\mathbb{C})$ when $f(1+X) = \varphi(X)$. □

6. Complex representations of G near the identity

6.1. By [Harish-Chandra78] when $\mathrm{char}_F = 0$ (for any reductive p-adic group) and ([Lemaire04] Proposition 4.3 with $E = F$), any non-zero representation $\pi \in \mathrm{Rep}_{\mathbb{C}}^{\infty,f} G$ non-zero π has a **germ expansion** of map c_π on K_π, meaning that:

There exists a map $c_\pi : G\backslash\mathfrak{N} \to \mathbb{C}$ (the coefficient map) and an open subgroup K_π of $K_1 = 1 + M_n(P_D)$ such that

$$\mathrm{trace}(\pi(f(g)dg)) = \sum_{\mathfrak{O} \in G\backslash\mathfrak{N}} c_\pi(\mathfrak{O})\, \hat{\mu}_{\mathfrak{O}}(\varphi) \tag{6.1}$$

for $f \in C_c^\infty(K_\pi;\mathbb{C}), \varphi \in C_c^\infty(M_n(P_D);\mathbb{C})$ such that $f(1+X) = \varphi(X)$ for $X \in M_n(P_D)$.

It is convenient to see c_π as a map on the set $\mathfrak{P}(n)$ of partitions of n, or on the set of parabolic subgroups P of G,

$$c_\pi(\lambda) = c_\pi(\mathfrak{O}_\lambda) = c_\pi(P) \text{ for } \lambda \in \mathfrak{P}(n) \text{ and } P \text{ associated to } P_\lambda. \tag{6.2}$$

For example, $c_\pi((n)) = c_\pi(\{0\}) = c_\pi(G)$. By Proposition 5.5, we have for $f \in C_c^\infty(K_\pi;\mathbb{C})$,

$$\mathrm{trace}(\pi(f(g)dg)) = \sum_{\lambda \in \mathfrak{P}(n)} c_\pi(\lambda)\ \mathrm{trace}(\pi_{P_\lambda}(fdg)) = \sum_P c_\pi(P)\ \mathrm{trace}(\pi_P(fdg)). \tag{6.3}$$

the last sum is over a system of representatives P of the parabolic subgroups of G modulo association. We list some properties of the map c_π for $\pi \in \mathrm{Rep}_{\mathbb{C}}^{\infty,f}(G)$.

- The map c_π is unique by Corollary 5.7 and is not 0 because

$$\dim_{\mathbb{C}} \pi^K = trace(\pi(1_K \,\mathrm{vol}(K,dg)^{-1}\, dg) \neq 0 \quad \text{for small open subgroups } K \text{ of } K_\pi. \tag{6.4}$$

- Two representations $\pi, \pi' \in \mathrm{Rep}_{\mathbb{C}}^{\infty,f}(G)$ have the same coefficient map if and only if their restrictions to some open compact subgroup of G are isomorphic, because the linear forms $\hat{\mu}_{\mathfrak{O}}$ restricted to $C_c^\infty(-1+K_\pi;\mathfrak{C})$ are linearly independent (Proposition 4.4).
- In particular,

$$c_\pi = c_{\pi\otimes\chi} \tag{6.5}$$

for any smooth character χ of G, because χ is trivial on some open compact subgroup.

- The map c_π depends only on the image $[\pi]$ of π in the Grothendieck group $\mathrm{Gr}^\infty_{\mathbb{C}}(G)$. It passes to a linear map $\nu \mapsto c_\nu$ on the Grothendieck group $\mathrm{Gr}^\infty_{\mathbb{C}}(G)$ such that $c_\pi = c_{[\pi]}$ for $\pi \in \mathrm{Rep}^{\infty,f}_{\mathbb{C}}(G)$. But $c_\nu = 0$ does not imply $\nu = 0$. For example, $c_\nu = 0$ for $\nu = [\mathrm{ind}^G_{P_\lambda} 1] - [\mathrm{ind}^G_{P_\lambda} \theta]$ when θ is any unramified character of M_λ.
- When π is finite dimensional, it is trivial on some $K_\pi \subset K_1$ hence

$$c_\pi((n)) = \dim_{\mathbb{C}} \pi, \quad c_\pi(\lambda) = 0 \ \text{ for } \lambda \neq (n). \tag{6.6}$$

Conversely, if $c_\pi(\lambda) = 0$ for $\lambda \neq (n)$ then

$$\mathrm{trace}(\pi(f(g)dg)) = c_\pi(\{0\})\, \hat{\mu}_{\{0\}}(\varphi) = c_\pi((n)) \int_G f(g)dg \tag{6.7}$$

for (f, φ) as in (6.1). Hence $\dim_{\mathbb{C}} \pi^K = c_\pi((n))$ for any open subgroup K of K_π, so π is finite dimensional.
- When $D \neq F$, a finite dimensional irreducible smooth representation of D^* may have dimension > 1, but:

Lemma 6.1. *When R an algebraically closed field, $D = F$ or $n > 1$, then a finite dimensional irreducible R-representation of G is of the form $\pi = \chi \circ nrd$ for some R-character χ of F^*.*

Proof. This clear when $G = F^*$ because F^* is commutative and the Schur's lemma is valid for G. When $n > 1$, then $\mathrm{Ker}(\pi)$ is an open subgroup of G, and in particular contains an open subgroup of U. But $\mathrm{Ker}(\pi)$ is also normal in G, so it contains U, and all the conjugates of U. Those conjugates generate $\mathrm{Ker}(\mathrm{nrd})$, so π factors through nrd implying the lemma. ☐

6.2. We revert to $R = \mathbb{C}$ and show that the values of c_π are integers (proved in [Howe74] when $D = F$ has characteristic 0 and π is irreducible supercuspidal). The key of the proof is the next lemma 6.2 inspired by Howe ([Howe74] Lemma 6).

For a partition $\lambda = (\lambda_1, \dots, \lambda_r)$ of n, let A_λ be the matrix of the endomorphism of the right D-vector space D^n operating on the canonical basis $e_1, \dots, e_n$ by sending $e_1, \dots, e_{\lambda_1}$ to 0, $e_{\lambda_1+1}, \dots, e_{\lambda_1+\lambda_2}$ to $e_1, \dots, e_{\lambda_2}$, and $e_{\lambda_1+\ldots+\lambda_i+j}$ to $e_{\lambda_1+\ldots+\lambda_{i-1}+j}$ for $i = 2, \dots, r-1, j = 1, \dots, \lambda_{i+1}$. Then, $\mathrm{Ker}\, A^i_\lambda$ is the D-subspace generated by $e_1, \dots, e_{\lambda_1+\ldots+\lambda_i}$. The parabolic subgroup of G stabilizing the flag $(\mathrm{Ker}\, A^i_\lambda)_i$ is P_λ, and $A_\lambda \in \mathfrak{n}_\lambda$. Fixing a character ψ of F trivial on P_F and not on O_F, for an integer $j \geq 1$, let ξ_λ be the character of $K_j = 1 + M_n(P^j_D)$ trivial on K_{2j} defined by

$$\xi_\lambda(1 + x) = \psi \circ \mathrm{trd}(A_\lambda\, p_D^{1-2j} x) \quad \text{for } x \in M_n(P^j_D). \tag{6.8}$$

Lemma 6.2. *For $\mu \in \mathfrak{P}(n)$, the multiplicity $m(\xi_\lambda, \pi_{P_\mu})$ of ξ_λ in π_{P_μ} is 0 unless $\lambda \geq \mu$. We have $m(\xi_\lambda, \pi_{P_\lambda}) = 1$.*

Proof. For $\mu \in \mathfrak{P}(n)$, $m(\xi_\lambda, \pi_{P_\mu})$ is the cardinality of

$$(P_\mu \cap GL_n(O_D)) \backslash \{k \in GL_n(O_D) \mid \xi_\lambda(k^{-1}(P_\mu \cap K_j)k) = 1\} / K_j.$$

Let $k \in K_0 = GL_n(O_D)$. We have $\xi_\lambda(k^{-1}(P_\mu \cap K_j)k) = 1$ if and only if

$$\xi_\lambda(k^{-1}(1 + \mathfrak{p}_\mu(P_D^j))k) = 1, \tag{6.9}$$

where $\mathfrak{p}_\mu(P_D^j) = \mathfrak{p}_\mu \cap M_n(P_D^j)$. The weaker condition $\xi_\lambda(k^{-1}(1 + \mathfrak{p}_\mu(P_D^{2j-1}))k) = 1$ already implies $m(\xi_\lambda, \pi_{P_\mu}) = 0$ unless $\lambda \geq \mu$. Indeed, it reads $\psi \circ \mathrm{trd}(A_\lambda\, k^{-1}\mathfrak{p}_\mu(O_D)k) = 1$. It depends on the images $\overline{k}, \overline{A}_\lambda$ of k, A_λ in $GL_n(k_D)$ and says that $\mathrm{trd}(\overline{k}\,\overline{A}_\lambda\,\overline{k}^{-1}\,\mathfrak{p}_\mu(k_D)) = 0$, that is, $\overline{k}\,\overline{A}_\lambda\,\overline{k}^{-1} \in \mathfrak{n}_\mu(k_D)$. Let $0 \subset W_1 \subset \ldots$ be the flag of k_D^n whose stabilizer is $P_\mu(k_D)$. Then $\overline{k}\,\overline{A}_\lambda\,\overline{k}^{-1} \in \mathfrak{n}_\mu(k_D)$ means $\overline{k}\,\overline{A}_\lambda\,\overline{k}^{-1}(W_i) \subset W_{i-1}$ for $i \geq 1$, and in particular that $\mathrm{Ker}(\overline{k}\,(\overline{A}_\lambda)^i\,\overline{k}^{-1}) = \overline{k}(\mathrm{Ker}(\overline{A}_\lambda)^i)$ contains W_i. As $\dim_D W_{i+1} - \dim_D W_i = \mu_i$, one obtains $\lambda_1 + \ldots + \lambda_i \geq \mu_1 + \ldots + \mu_i$ for each i, that is $\lambda \geq \mu$.

Suppose now $\mu = \lambda$. We prove that (6.9) is equivalent to $k \in P_\lambda(O_D)K_j$. By its definition ξ_λ is trivial on $1 + \mathfrak{p}_\lambda(P_D^j)$ because $A_\lambda \in \mathfrak{n}_\lambda$ hence $\mathrm{trd}(A_\lambda \mathfrak{p}_\lambda) = 0$, so $P_\lambda(O_D)K_j$ does satisfy (6.9). Conversely, $B_\lambda = A_\lambda p_D^{1-2j} \in \mathfrak{n}_\lambda(P_D^{1-2j})$. The condition (6.9) means that $\mathrm{trd}(B_\lambda k^{-1}\mathfrak{p}_\lambda(P_D^j)k) \in P_F$ and implies

$$B_\lambda = k^{-1}Xk + Y, \text{ where } X \in \mathfrak{n}_\lambda, Y \in M_n(P_D^{1-j}).$$

Indeed, writing $kB_\lambda k^{-1} = X + Y$ with $X \in \mathfrak{n}_\lambda, Y \in \mathfrak{p}_\lambda^-$, we have:

$$\begin{aligned}
&\mathrm{trd}(B_\lambda k^{-1}\mathfrak{p}_\lambda(P_D^j)k) = \mathrm{trd}(kB_\lambda k^{-1}\mathfrak{p}_\lambda(P_D^j)) = \mathrm{trd}(Y\mathfrak{p}_\lambda(P_D^j)) = \mathrm{trd}(YM_n(P_D^j)),\\
&\mathrm{trd}(YM_n(P_D^j)) \in P_F \Leftrightarrow \mathrm{trd}(P_D^{j-d}YM_n(O_D)) \in O_F \Leftrightarrow Y \in M_n(P_D^{1-j}).
\end{aligned}$$

See Example 4.3 for the last equivalence. One gets $B_\lambda k^{-1} = k^{-1}X + Y_1$ with $Y_1 \in M_n(P_D^{1-j})$. Note that $B_\lambda \in M_n(P_D^{1-2j})$ hence also X. We get $B_\lambda^2 k^{-1} = B_\lambda k^{-1}X + B_\lambda Y_1 = k^{-1}X^2 + Y_1X + B_\lambda Y_1 = k^{-1}X^2 + Y_2$ with $Y_2 \in M_n(P_D^{j+2(1-2j)})$. By induction $B_\lambda^i k^{-1} = k^{-1}X^i + Y_i$ with $Y_i \in M_n(P_D^{j+i(1-2j)})$ for $1 \leq i \leq r$. For a basis vector $e \in \mathrm{Ker}\, A_\lambda^i$, we have $X^i e = 0$ because $X \in \mathfrak{n}_\lambda$, and $B_\lambda^i k^{-1}e = k^{-1}X^i e + Y^i e = Y^i e$. As $(A_\lambda p_D^{1-2j})^i k^{-1}e \in M_n(P_D^{j+i(1-2j)})e \Leftrightarrow A_\lambda^i k^{-1}e \in M_n(P_D^j)e$, the coefficients of $k^{-1}e$ on the basis vectors which are not in $\mathrm{Ker}\, A_\lambda^i$ are in P_D^j. This means $k^{-1} \in K_jP_\lambda(O_D)$, what we wanted. □

We shall need more properties of ξ_λ in the section on Whittaker spaces.

Lemma 6.3. *The normalizer of ξ_λ in $K_0 = GL_n(O_D)$ is $P_\lambda(O_D)K_j$.*

Proof. For $k \in K_0$, the property $\xi_\lambda(1 + x) = \xi_\lambda(1 + kxk^{-1})$ for all $x \in M_n(P_D^j)$ means $k^{-1}B_\lambda k - B_\lambda \in M_n(P_D^{1-j})$. As in the proof of Lemma 6.2 one deduces $B^ik - kB^i \in M_n(P_D^{j-i(1-2j)})$ for $i \geq 1$ and one sees that $k \in P(O_D)K_j$. □

Remark 6.4. There is a unique function in π_{P_λ} with support $P_\lambda K_j$ and restriction ξ_λ to K_j since ξ_λ is trivial on $1 + \mathfrak{p}_\lambda(P_D^j)$. That function is a basis of the line of vectors in π_{P_λ} transforming according to ξ_λ under the action of K_j.

We prove now that the $c_\pi(\lambda)$ are integers. By (6.3), when $K_j = 1 + M_n(P_D^j) \subset K_\pi$ and $\delta \in \mathrm{Rep}_{\mathbb{C}}^\infty(K_j)$ irreducible, the multiplicity $m(\delta,\pi)$ of δ in $\pi \in \mathrm{Rep}_{\mathbb{C}}^\infty(G)$ satisfies

$$m(\delta,\pi) = \sum_{\mu\in\mathfrak{P}(n)} c_\pi(\mu)\, m(\delta,\pi_{P_\mu}). \tag{6.10}$$

Lemma 6.2 and (6.10) imply:

$$c_\pi(\lambda) = m(\xi_\lambda,\pi) - \sum_{\mu\in\mathfrak{P}(n),\mu<\lambda} c_\pi(\mu)\, m(\xi_\lambda,\pi_{P_\mu}). \tag{6.11}$$

In particular when λ is minimal in $\mathrm{Supp}\, c_\pi$, $c_\pi(\lambda) = m(\xi_\lambda,\pi)$ is positive and independent of the choice of j such that $K_j = 1 + M_n(P_D^j) \subset K_\pi$. By upwards induction on $\mathfrak{P}(n)$ (downwards induction on the nilpotent orbits), we obtain that the $c_\pi(\lambda)$ are integers.

As the values of the map c_π are integers, we get more properties:

- $c_\pi = c_{\sigma(\pi)}$ when σ is an automorphism of $\mathbb{C}$.
- For $\nu \in \mathrm{Gr}_R^\infty(G)$, there exists a map $c_\nu : \mathfrak{P}(n) \to \mathbb{Z}$ and an open subgroup K_ν of G such that ν and $\sum_{\lambda\in\mathfrak{P}(n)} c_\nu(\lambda)\,[\pi_{P_\lambda}] \in \mathrm{Gr}_R^\infty(G)$ have isomorphic restrictions to K_ν.

When $R = \mathbb{C}$, the first part of Theorem 1.3 is a version of the germ expansion. For any R, when π satisfies the first part of Theorem 1.3 we say sometimes that π has a germ expansion with map c_π on K_π.

7. Parabolic induction

In this section R is a field and $\mathrm{char}_R \neq p$. We prove now that the first part of Theorem 1.3 implies Theorem 1.5. Let $P, M, (n_i), \sigma_i, r, \sigma, \pi$ as in Theorem 1.5. Write $pr : P \to M$ for the projection of kernel N. Given partitions λ_i of n_i for $1 \leq i \leq r$, we have the parabolic subgroup $P_{(\lambda_i)}$ of M corresponding to the parabolic subgroups P_{λ_i} of $GL_{n_i}(D)$. Given functions $c_i : \mathfrak{P}(n_i) \to \mathbb{Z}$ for $1 \leq i \leq r$, the function $c : \mathfrak{P}(n) \to \mathbb{Z}$ defined by

$$c(\lambda) = \sum \prod_{i=1,\ldots,r} c_i(\lambda_i),$$

where the sum is over $r-$tuples of partitions $(\lambda_1,\ldots,\lambda_r)$ inducing to λ before Theorem 1.5, is called induced by $(c_1,\ldots,c_r)$.

Theorem 7.1. *Assume that for $i = 1,\ldots,r$, there exists a function $c_{\sigma_i} : \mathfrak{P}(n_i) \to \mathbb{Z}$ and an open compact subgroup K_{σ_i} of $GL_{n_i}(D)$ such that $\sigma_i = \sum_{\lambda_i\in\mathfrak{P}(n_i)} c_{\sigma_i}(\lambda_i)\, \mathrm{ind}_{P_{\lambda_i}}^{GL_{n_i}(D)} 1$ on K_{σ_i}. Then*

$$\pi = \sum_{\lambda\in\mathfrak{P}(n)} c_\pi(\lambda)\, \mathrm{ind}_{P_\lambda}^G 1$$

on K_π, where $c_\pi : \mathfrak{P}(n) \to \mathbb{Z}$ is the function induced by $(c_{\sigma_1},\ldots,c_{\sigma_r})$ and K_π is any open compact subgroup of G such that $\cup_{g\in P\backslash G/K_\pi} pr(P \cap gK_\pi g^{-1})$ is contained in $K_{\sigma_1} \times \ldots \times K_{\sigma_r}$.

Proof. The theorem follows from the fact that for any field R, $\mathrm{ind}_P^G(\mathrm{ind}_{P_{(\lambda_i)}}^M 1)$ has the same restriction to K_0 than $\mathrm{ind}_{P_\lambda}^G 1$ by Lemma 5.1, and for given a open compact subgroup C_M of M, there exists an open compact subgroup C of G such that

$$\cup_{g\in P\backslash G/C}\, pr(P\cap gCg^{-1}) \ \subset \ C_M. \tag{7.1}$$

The existence of K_π follows from (7.1) applied to $C_M = K_{\sigma_1}\times\ldots\times K_{\sigma_r}$.

The restriction of a smooth R-representation σ of M to C_M determines the restriction of $\mathrm{ind}_P^G\sigma$ to C,

$$(\mathrm{ind}_P^G\sigma)|_C \simeq \oplus_{g\in P\backslash G/C}\,\mathrm{ind}_{C\cap g^{-1}Pg}^C(\sigma^g)$$

where $\sigma^g(k) = \sigma(gkg^{-1})$ for $g\in G, k\in g^{-1}Pg\cap C$, and σ^g depends only on the restriction of σ to $pr(P\cap gCg^{-1})$. If $\sigma'\in \mathrm{Rep}_R^{\infty,f}(M)$ is isomorphic to σ on C_M, then $\mathrm{ind}_P^G\sigma'$ and $\mathrm{ind}_P^G\sigma$ are isomorphic on C. The same holds true for virtual representations ν,ν' of M. Take $\nu=\sigma_1\otimes\ldots\otimes\sigma_r$ and $\nu'=\nu'_1\otimes\ldots\otimes\nu'_r$ with $\nu'_i=\sum_{\lambda_i\in\mathfrak{P}(n_i)} c_{\sigma_i}(\lambda_i)\,\mathrm{ind}_{P_{\lambda_i}}^{GL_{n_i}(D)} 1$. □

Corollary 7.2. (Variant of Theorem 7.1) *Assume that $GL_{n_i}(D)$ satisfies the first part of Theorem 1.3 for $i=1,\ldots,r$. Then for $\sigma\in\mathrm{Rep}_R^{\infty,f}(M)$, there exists an open compact subgroup K_σ of M and a unique map $c_\sigma:\mathfrak{P}(n_1)\times\ldots\times P(n_r)\to\mathbb{Z}$ such that $\sigma=\sum_{(\lambda_i)\in(\mathfrak{P}(n_i))} c_\sigma((\lambda)_i)\pi_{P_{(\lambda_i)}}$ on K_σ, and $\pi=\mathrm{ind}_P^G\sigma$ is equal to $\sum_\lambda c_\pi(\lambda)\pi_{P_\lambda}$ on any open compact subgroup K_π of G such that $K_\sigma\subset\cap_{g\in P\backslash G/K_\pi} M\cap gK_\pi g^{-1}$ and $c_\pi:\mathfrak{P}(n)\to\mathbb{Z}$ is induced by c_σ.*

Remark 7.3. When $G=GL_n(F)$, given partitions λ_i of n_i for $i=1,\ldots,r$, and $\lambda\in\mathfrak{P}(n)$ induced by the λ_i, the nilpotent orbit $\mathfrak{O}_\lambda$ is the nilpotent orbit induced by the nilpotent orbit $\mathfrak{O}_{(\lambda_i)}$ of M corresponding to the λ_i, in the sense of [Lusztig-Spaltenstein79] (see [Jantzen04]). If $R=\mathbb{C}$, $\mathrm{char}_F=0$, $p\neq 2$, $D=F$, the formula for c_π follows from ([Moeglin-Waldspurger87] §II.1.3 where G is a classical group).

8. Whittaker spaces

Our purpose in this section is to relate the coefficient map c_π to the dimensions of the different Whittaker spaces of π when $\pi\in\mathrm{Rep}_{\mathbb{C}}^\infty(G)$ is irreducible. We first introduce those subspaces.

The commutator subgroup of the group U of upper unipotent matrices is the group U' of upper unipotent matrices with coefficients $u_{i,i+1}=0$ for $i=1,\ldots,n-1$ (use the identities $E_{a,b}E_{c,d}=E_{a,d}$ if $b=c$ and 0 otherwise). The map sending $(u_{i,j})\in U$ to $(u_{1,2},\ldots,u_{n-1,n})$ induces an isomorphism from U/U' to the additive group D^{n-1}. The action of the group $T\simeq(D^*)^n$ of diagonal matrices by conjugation on U and on U' induces an action on D^{n-1}, the diagonal matrix $\mathrm{diag}(a_1,\ldots,a_n)\in T$ sends $(d_1,\ldots,d_{n-1})\in D^{n-1}$ to $(a_1d_1a_2^{-1},\ldots,a_{n-1}d_{n-1}a_n^{-1})$.

Let us fix a non-trivial smooth character ψ of F. Then $\psi_D=\psi\circ\mathrm{trd}$ is a non-trivial character of D. Sending $y\in D$ to the character $\psi_D^y(x)=\psi_D(yx)$ for $x\in D$, is an isomorphism from the additive group D to its group of smooth characters. Sending $y=(y_1\ldots y_{n-1})\in D^{n-1}$ to $(\psi_D^{y_1},\ldots,\psi_D^{y_{n-1}})$, is an isomorphism from D^{n-1} to its group of smooth characters. The above action of T on D^{n-1} induces an action on its groups of characters, the diagonal matrix $\mathrm{diag}(a_1,\ldots,a_n)$ sends $y=(y_1,\ldots,y_{n-1})\in D^{n-1}$ to $(a_2^{-1}y_1a_1,\ldots,a_n^{-1}y_{n-1}a_{n-1})$.

Let $y=(y_1,\ldots,y_{n-1})\in D^{n-1}$, r be the number of indices i where $y_i=0$, and

$$I=I(y)=\begin{cases}\emptyset & \text{if } r=0,\\ \{i_1<\ldots<i_r\} & \text{the set of indices } i \text{ where } y_i=0 \text{ if } r\neq 0.\end{cases}\tag{8.1}$$

The smooth character of U corresponding to y is

$$\theta_y(u)=\psi\circ\mathrm{trd}(X_yv)\quad u=1+v\in U,$$

where $X_y\in M_n(D)$ is the nilpotent matrix with $(y_1,\ldots,y_{n-1})$ just below the diagonal and 0 elsewhere. The character θ_y is called **non-degenerate** if $I(y)=\emptyset$, and **degenerate** otherwise. The character θ_y is trivial if and only if $I(y)=\{1,\ldots,n-1\}$. The group $B=TU$ is its own normalizer in G, so the G-normalizer of θ is of the form $T_{\theta_y}U$ where T_{θ_y} is the T-normalizer of θ_y. It is the intersection of B with the commutant of X_y.

The element y is conjugate under T to the element $\delta_I\in D^{n-1}$ with coefficient 0 in I and 1 elsewhere. The nilpotent matrix X_{δ_I} is a diagonal of Jordan blocks of sizes forming a composition λ_I of n,

$$\lambda_I=\begin{cases}(n) & \text{when } I=\emptyset,\\ (i_1,i_2-i_1,\ldots,n-i_r) & \text{when } I\neq\emptyset.\end{cases}\tag{8.2}$$

Any composition λ of n is equal to λ_I for a unique subset I of $I(y)=\{1,\ldots,n-1\}$. Put $X_\lambda=X_{\delta_I}$,

$$\theta_\lambda(u)=\psi\circ\mathrm{trd}(X_\lambda v)\quad u=1+v\in U,\tag{8.3}$$

and T_λ the T-normalizer of θ_λ. The group T_λ contains the group $T_{(n)}=\{\mathrm{diag}(d,\ldots,d)\mid d\in D^*\}$ isomorphic to D^*.

We fix a representation $\pi\in\mathrm{Rep}^\infty_{\mathbb{C}}(G)$ of space V. Given a smooth character θ of U, we look at the space V_θ of θ-coinvariants of U in V, or at its dual, the (Whittaker) space of linear forms Λ on V such that $\Lambda(uv)=\theta(u)\Lambda(v)$ for $u\in U, v\in V$. It is customary to say that π has a **Whittaker model** with respect to θ if $V_\theta\neq 0$. Indeed any choice of non-zero linear form Λ on V_θ gives a non-zero intertwining from π to $\mathrm{Ind}_U^G(\theta)$ by sending $v\in V$ to the function taking value $\Lambda(gv)$ at $g\in G$; that intertwining is an embedding if π is irreducible, hence the name "model". We say that π has a **non-degenerate Whittaker model**, or that π is **generic** if $V_\theta\neq 0$ for some (equivalently all) non-degenerate characters θ of U. We say that π has a Whittaker model if it has a **Whittaker model** with respect to some choice of θ.

Using the action of T on U by conjugation, we see that to analyse the V_θ for all choices of θ, it is enough to consider the θ_λ associated to the compositions λ of n.

Remark 8.1. 1) It is known that if π is irreducible then V_θ is finite dimensional (when θ is not degenerate [Bushnell-Henniart02], in general [Aizenbud-BS22]; these papers treat the case of a general reductive group G). The group T_θ acts on V_θ; since T_θ is not commutative if $D \neq F$, we cannot expect V_θ to have always dimension 0 or 1 (as when $D = F$ and θ not degenerate).

2) Moeglin and Waldspurger [Moeglin-Waldspurger87] consider more general Whittaker spaces, but ours are enough for our purpose (Theorem 8.2 below). Also they use the exponential map, which is not available when F has positive characteristic. Instead we use the map $X \mapsto 1 + X : M_n(P_D) \to 1 + M_n(P_D)$, as in [Howe74] and [Rodier74] when $D = F$.

3) If π is irreducible cuspidal, π can only have non-degenerate Whittaker models because θ_I is trivial on the unipotent radical N_{λ_I} of the parabolic group P_{λ_I}. Hence π_{θ_I} is a quotient of the N_I-coinvariant space $\pi_{N_{\lambda_I}}$ of π. If $\pi_{N_{\lambda_I}} = 0$ then $\pi_{\theta_I} = 0$, and N_{λ_I} is trivial if and only if $I = \emptyset$.

4) It is possible to extend to $GL_n(D)$ the theory of [Bernstein-Zelevinski 77] 5.1 to 5.15 to show that a non-zero π has a Whittaker model (see [Abe-Herzig23] 3.4). But that is a consequence of our theorem below (Corollary 8.3).

We now prove Theorem 1.6 (for $R = \mathbb{C}$). We can assume that π is irreducible. We want to relate the coefficient map $c_\pi : \mathfrak{P}(n) \to \mathbb{Z}$ of the germ expansion of π with the dimensions of the spaces V_{θ_λ} for the compositions λ of n, following [Moeglin-Waldspurger87]. We define the **Whittaker support** of π as the set of partitions μ of n such that $V_{\theta_\lambda} \neq 0$ for some composition λ of n with associated partition $\hat{\mu}$ (the partition dual to μ).

Theorem 8.2. *The minimal elements in* $\operatorname{Supp} c_\pi$ *and in the Whittaker support of* π *are the same.*

Let μ *be a partition of* n *minimal in* $\operatorname{Supp} c_\pi$ *and let* λ *be a composition of* n *with associate partition* $\hat{\mu}$. *Then* $c_\pi(\mu) = \dim_{\mathbb{C}} V_{\theta_\lambda}$.

Since π has a non-zero germ expansion, the theorem implies:

Corollary 8.3. *Any irreducible smooth complex representation* π *of* G *has a Whittaker model.*

Remark 8.4. 1) By the theorem $(1, \dots, 1)$ is minimal in $\operatorname{Supp} c_\pi$ if and only if V has a non-degenerate Whittaker model. This was proved when $D = F$ [Rodier74].

2) (n) is minimal in $\operatorname{Supp} c_\pi$ if and only if $\dim_{\mathbb{C}}(V)$ is finite. By the theorem that happens if and only if V has only the trivial Whittaker model.

3) In part 2 of the theorem, $\dim(V_{\theta_\lambda})$ does not depend on the choice of the composition λ with associated partition $\hat{\mu}$. It is the multiplicity in π of the character ξ_μ of K_j defined in (6.8), if j is large enough.

We turn back to the proof of the theorem. As said at the beginning of this section, our proof is based on the method of [Moeglin-Waldspurger87], replacing the exponential by $X \to 1 + X$. The starting idea is already in [Rodier74], but that paper is restricted to the non-degenerate Whittaker models, and $D = F$. Compared to those works, we work with the germ expansion of π in terms of the π_{P_λ} rather than with Fourier transforms of nilpotent orbits. We find that it simplifies matter a bit, and it is coherent with our approach.

Proof. We fix a composition $\lambda = (\lambda_1, ..., \lambda_r)$ of n. We write θ for the character θ_λ of U and X for the lower triangular nilpotent matrix in Jordan blocks of size $\lambda_1, ..., \lambda_r$ down the diagonal (if I is the subset of $\{1, \dots, n-1\}$ such that $\lambda = \lambda_I$, then $X = X_{\delta_I}$). For each positive integer j we define a character ψ_j of $K_j = 1 + M_n(P_D^j)$ trivial on K_{2j},

$$\psi_j(1+x) = \psi \circ \mathrm{trd}(X p_D^{1-2j} x), \quad x \in M_n(P_D^j), \tag{8.4}$$

where ψ is a character of F trivial on P_F but not on O_F. In other words, ψ_j is obtained, in the formula (6.8) for ξ_λ by replacing the matrix A_λ there with the matrix X. We let λ' the partition of n obtained from λ by putting its parts in decreasing order, and C the matrix $A_{\lambda'^\wedge}$ associated as in Lemma 6.2 to the partition $\lambda'^\wedge$.

Lemma 8.5. *The matrices C and X are conjugate by permutation matrices (corresponding to permutations of the canonical basis of D^n).*

Proof. A suitable permutation of the canonical basis puts the blocks of X in decreasing size order, and we get the matrix X' analogous to X but corresponding to λ'. Let us describe a permutation of the basis which conjugates X' to C. Let d be the size of the largest blocks of X'. Put at the end the first vectors of the blocks of X' of size d. Before them, put a bunch of vectors: the images under X' of the previous ones, completed with the first vectors of the blocks of size $d-1$ of X', if any. Once you have the vectors corresponding to size i, put before them the images under X' of the already chosen vectors, completed with the first vectors of the blocks of size $i-1$. Reaching $i = 1$ completes the process. □

Remark 8.6. By this lemma, we can apply Lemma 6.2 to ψ_j. Hence, For any positive integer j, one has $m(\psi_j, \pi_{P_{\lambda'^\wedge}}) = 1$ and $m(\psi_j, \pi_{P_\mu}) = 0$ unless $\lambda'^\wedge \geq \mu$. If $\lambda'^\wedge$ is minimal in $\mathrm{Supp}\, c_\pi$, then we have $c_\pi(\lambda'^\wedge) = m(\psi_j, \pi)$ for any positive integer j such that the germ expansion of π is valid on K_j.

We now turn to the Whittaker quotient V_θ, approaching it (following Rodier's initial idea) by a suitable conjugate ψ'_j of ψ_j and letting j go to infinity.

The diagonal matrix $t = \mathrm{diag}(1, p_D, ..., p_D^{n-1})$ acts by conjugation on $M_n(D)$, multiplying the (a, b)-coefficient x of a matrix by $p_D^a x p_D^{-b}$. Conjugating ψ_j yields a character ψ'_j of the group $K'_j = t^{2j-1} K_j t^{-2j+1}$ which satisfies also Remark 8.6. The group U is the increasing union of $U \cap K'_j$ over j, whereas the decreasing subgroups $B^- \cap K'_j$ have trivial intersection. The restriction of ψ'_j to $K'_j \cap U$ is equal to that of θ, whereas its restriction to $K'_j \cap B^-$ is trivial. The multiplication induces a bijection (an Iwahori decomposition):

$$(K'_j \cap U) \times (K'_j \cap B^-) \to K'_j$$

The projector $e'_j : V \to V(\psi'_j)$ of V onto its ψ'_j-isotypic space $V(\psi'_j)$ (which has dimension $m(\psi'_j, \pi) = m(\psi_j, \pi)$) can be obtained by first projecting onto vectors fixed by $K'_j \cap B^-$, and then applying the projector f_j

$$f_j(v) = \int_{K'_j \cap U} \theta(u)^{-1} \pi(u) v \, du, \quad v \in V,$$

with respect to the Haar measure du giving measure 1 to $K'_j \cap U$.

We write $p : V \to V_\theta$ for the projection of V onto V_θ and $p_j : V(\psi'_j) \to V_\theta$ for its restriction to $V(\psi'_j)$.

Lemma 8.7. *The map $p_j : V(\psi'_j) \to V_\theta$ is surjective for large j.*

Proof. Let $v \in V$. For large enough j, $v \in V^{K'_j \cap B^-}$ hence $e'_j(v) = f_j(v)$ and $p(e'_j(v)) = p(v)$. Lifting in that way a basis of the finite-dimensional space V_θ gives the result. □

Lemma 8.8. *Il $V_\theta \neq 0$, then there is a partition μ in* $\operatorname{Supp} c_\pi$ *with $\mu \leq \lambda'\hat{}$.*

Proof. Il $V_\theta \neq 0$ is not 0, then by Lemma 8.7, $V(\psi'_j) \neq 0$ for large j, so $\operatorname{tr}(\pi(e'_j)) \neq 0$. Applying the germ expansion of π to e'_j there is a minimal partition μ of n in $\operatorname{Supp} c_\pi$. By Remark 8.6, $c_\pi(\mu) = m(\psi_j, \pi_{P_\mu})$ and $\mu \leq \hat{\lambda}'$. □

Lemma 8.9. *Let j_0 be a positive integer such that π has a germ expansion on K_{j_0}, and $j'_0 = j_0 + 2n - 2$. If $\lambda'\hat{}$ is minimal in* $\operatorname{Supp} c_\pi$ *and $j \geq j'_0$, then the endomorphism $v \to e'_j e'_{j+1} v$ of $V(\psi'_j)$ is a non-zero homothety.*

In [Moeglin-Waldspurger87], that Lemma is given for unspecified large j by their Lemmas I.13 and I.15. They are rather more involved than Lemme 4 in [Rodier74], which however applies only to non-degenerate Whittaker models and $D = F$. The proof of Lemma 8.9 will be given later.

Proposition 8.10. *If $\lambda'\hat{}$ is minimal in* $\operatorname{Supp} c_\pi$ *and $j \geq j'_0$, then p_j is an isomorphism, so that* $\dim_{\mathbb{C}}(V_\theta) = \dim_{\mathbb{C}} V(\psi'_j)$.

Proof. We already know by Lemma 8.7 that p_j is surjective for j large. We also know by Remark 8.6 that $\dim_{\mathbb{C}} V(\psi'_j) = m(\psi'_j, \pi)$ is constant for$j \geq j_0$. The main point is Lemma 8.9 which implies that for $j \geq j'_0$, the linear map $q_j : V(\psi'_j) \to V(\psi'_{j+1}), v \to v_1 = e'_{j+1} v$ is injective, hence is an isomorphism because the two spaces have the same dimension. Moreover a vector $v \in V(\psi'_j)$ is already invariant under $K'_{j+1} \cap B$ so what was said before Lemme 7.7 we have $e'_{j+1} v = f_{j+1} v$, and $v_1 = e'_{j+1} v$ has the same image in V_λ as v. Iterating the process we get for positive integers k, vectors $v_k = e'_{j+k} v_{k-1} = f_{j+k} v_{k-1}$. By definition of the projector f_j, we have $f_{j+k} f_{j+k-1} = f_{j+k}$ and consequently $v_k = f_{j+k} v$. But $p(v) = 0$ if and only if $f_{j+k} v = 0$ for large k (Bernstein-Zelevinsky xyz). As $v_k = 0$ implies $v_{k-1} = 0$ by the injectivity already established, we get $\operatorname{Ker}(p_j) = 0$. But for large j, p_j is surjective so is an isomorphism, and $\dim_{\mathbb{C}}(V(\psi'_j) = \dim_{\mathbb{C}}(V_\theta)$. But for $j \geq j'_0$, the dimension of $V(\psi'_j)$ is constant so p_j is an isomorphism and the Proposition follows. □

Proposition 8.10 implies Part 2 of Theorem 8.2 and that a partition of n which is minimal in $\operatorname{Supp} c_\pi$ belongs to the Whittaker support of π. Conversely, let $\mu \in \mathfrak{P}(n)$ minimal in the Whittaker support of π. Then by Lemma 8.8, there is a partition μ' in $\operatorname{Supp} c_\pi$ with $\mu' \leq \mu$, and we may assume that μ' is minimal in $\operatorname{Supp} c_\pi$. But by Proposition 8.10, that implies that μ' belongs to the Whittaker support of π, so $\mu' = \mu$. Assuming Lemma 8.9, Theorem 8.2 is proved. □

It remains to prove Lemma 8.9. We can conjugate by t^{1-2j} to transform ψ'_j back to ψ_j, and even further conjugate (Lemma 8.5) by a permutation matrix σ to transform ψ_j into the character ξ_j attached to the matrix B. We need to prove that the endomorphism of eV sending v to efv is a non-zero homothety, where e is the K_j-projector onto the one dimensional space $eV = V(\xi_j)$ and f is integration on the group $J = \sigma(t^2)(K_j \cap U)(\sigma(t^2)^{-1}$ against its character $(1+x) \mapsto \psi \circ \mathrm{trd}(-B.(p_D)^{-1-2j}x)$. Clearly efe is an element of eHe where H is the full Hecke algebra of G, so we may restrict the mentioned integration to elements in the support of the Hecke algebra eHe. Also if $j \geq 2n-2$, the group J is contained in K_{j-2n+2} so it normalizes K_j, and the support of efe is contained in the normalizer of ξ_j in K_{j-2n+2}.

By Lemma 6.3, the normalizer of ξ_λ in $K_0 = GL_n(O_D)$ is $P_{\lambda'}(O_D)K_j$. Take $j-2n+2 \geq j_0$ and $g = 1+x$ be in the support of efe. The trace of ege in eV can be computed using the germ expansion of π as the sum over $\mu \in \mathfrak{P}(n)$ of $c_\pi(\mu)$ times the trace of efe in π_{P_μ}. By our choice $\lambda'\hat{}$ is minimal in $\mathrm{Supp}\, c_\pi$, so the only contribution is $c_\pi(\lambda'\hat{})$. Applying that to any ege in the support of efe gives Lemma 8.9, and even that the homothety is via a positive integer.

9. Jacquet-Langlands correspondence

The Jacquet-Langlands correspondence extended by Badulescu ([Badulescu07] Théorème 3.1), is a surjective morphism LJ with a section JL

$$LJ : \mathrm{Gr}^\infty_{\mathbb{C}}(GL_{dn}(F)) \to \mathrm{Gr}^\infty_{\mathbb{C}}(G), \quad JL : \mathrm{Gr}^\infty_{\mathbb{C}}(G) \to \mathrm{Gr}^\infty_{\mathbb{C}}(GL_{dn}(F))$$

which is an injective morphism of $\mathbb{Z}$-modules extending the classical Jacquet-Langlands correspondence between essentially square integrable representations.

Theorem 9.1. *For $\nu \in \mathrm{Gr}^\infty_{\mathbb{C}}(GL_{dn}(F))$ and $\lambda \in \mathfrak{P}(n)$, we have $(-1)^n c_{LJ(\nu)}(\lambda) = (-1)^{dn} c_\nu(d\lambda)$.*

Corollary 9.2. *For $\nu \in \mathrm{Gr}^\infty_{\mathbb{C}}(G)$ and $\lambda \in \mathfrak{P}(n)$, we have $(-1)^n c_\nu(\lambda) = (-1)^{dn} c_{JL(\nu)}(d\lambda)$.*

The remainer of this section gives the proof of the theorem.

9.1. Badulescu-Jacquet-Langlands correspondence.

9.1.1. *Preliminaries.* Let $\mathrm{Irr}^2_{\mathbb{C}}(G)$ denote the set of isomorphism classes of essentially square integrable irreducible smooth complex representations of G. Any irreducible smooth complex representation of D^* is essentially square integrable.

As in §1, $P_\lambda = M_\lambda N_\lambda$ is a parabolic subgroup of G for $\lambda \in \mathfrak{P}(n)$. For $\mu \in \mathfrak{P}(dn)$, we denote now by $P_\mu = P_\mu N_\mu$.

A basis of the Grothendieck group $\mathrm{Gr}^\infty_{\mathbb{C}}(G)$ is

$$\mathfrak{B}_G = \{[n.\mathrm{ind}_{P_\lambda}^G \sigma] \mid \sigma \in \mathrm{Irr}^2_{\mathbb{C}}(M_\lambda), \lambda \in \mathfrak{P}(n)\}$$

where $n.\mathrm{ind}_{P_\lambda}^G$ the normalized parabolic induction ([Badulescu07] Proposition 2.2). As $\mathrm{Irr}^2_{\mathbb{C}}(G)$ is stable by the twist by a smooth character of G,

$$\mathfrak{B}'_G = \{[\mathrm{ind}_{P_\lambda}^G \sigma] \mid \sigma \in \mathrm{Irr}^2_{\mathbb{C}}(M_\lambda),\ \lambda \in \mathfrak{P}(n)\}.$$

is also a basis of $\mathrm{Gr}^\infty_{\mathbb{C}}(G)$). Let C_d be the submodule of $\mathrm{Gr}^\infty_{\mathbb{C}}(GL_{dn}(F))$ of basis the set

$$\mathfrak{B}'_d = \{[\mathrm{ind}_{P_\mu}^{GL_{dn}(F))} \sigma] \mid \sigma \in \mathrm{Irr}^2_{\mathbb{C}}(M_\mu), \mu \in \mathfrak{P}(dn) \text{ but } \mu \notin d\mathfrak{P}(n)\}.$$

The Aubert involution ι of $\mathrm{Gr}^\infty_{\mathbb{C}}(G)$ sends an irreducible representation π to an irreducible representation modulo a sign [Aubert95]:

$$\iota(\pi) = (-1)^{n-r}|\iota(\pi)| \tag{9.1}$$

where $|\iota(\pi)|$ is irreducible and r is the number of elements of the cuspidal support of π, meaning that $\pi \subset \mathrm{ind}^G_{P_\lambda} \sigma$ for $\lambda = (\lambda_1, \dots, \lambda_r) \in \mathfrak{P}(n)$ and $\sigma \in \mathrm{Irr}^2_{\mathbb{C}}(M_\lambda)$ cuspidal ([Badulescu07] (3.4), [Tadic90] §1).

Let λ be a partition of n and δ_λ the modulus of the parabolic subgroup $P_\lambda = M_\lambda N_\lambda$ of G, $\delta_\lambda(g) = |(\det \mathrm{Ad}(g)|_{\mathrm{Lie}\, N_\lambda})|_F$ for $g \in P_\lambda$. For a partition μ of dn, let δ'_μ denote the modulus of the parabolic subgroup $P'_\mu = M'_\mu N'_\mu$ of $GL_{dn}(F)$.

Lemma 9.3. *Let L/F be an extension splitting D. We have $\delta_\lambda = \delta'_{d\lambda}$ on $P_\lambda(L) = P'_{d\lambda}(L)$.*

Proof. We have $G(L) = GL_{dn}(F)$ and $P_\lambda(L) = P'_{d\lambda}(L)$. The modulus δ_λ is an algebraic character, and can also be computed in $P_\lambda(L)$. Similarly for $\delta'_{d\lambda}$. The reduced norm on G becomes the determinant on $G(L)$. □

Let L/F be an extension splitting D. The reduced characteristic polynomial P_a of $a \in M_n(D)$ is the characteristic polynomial of $a \otimes 1 \in M_n(D) \otimes_F L \simeq M_{nd}(L)$, which belongs to $F[X]$, does not depend on the choice of L, and $P_a(a) = 0$ [BourbakiA-8, §17 page 333 Définition 1, page 336 Corollaire 2, (34)], [Badulescu18, §2 Propositions 2.1 and 2.2].

Lemma 9.4. *The reduced characteristic polynomial of a matrix in $M_n(D)$ belongs to $O_F[X]$ if and only if the matrix is $GL_n(D)$-conjugate to an element of $M_n(O_D)$.*

Proof. We have $M_n(D) \simeq \mathrm{End}_D D^n$ where D^n is seen as a right D-module. Let $e_1, \dots, e_n$ be a basis of D^n over D. When $P_a \in O_F[X]$, the O_D-module generated by the $a^i e_1, \dots, a^i e_n$ for the positive integers i, is finitely generated because $P_a(a) = 0$, hence a stabilizing an O_D-lattice of D^n is $GL_n(D)$-conjugate to an element of $M_n(O_D)$. Conversely, if $a \in M_n(O_D)$ then $a \otimes 1 \in M_{nd}(O_L)$ hence its characteristic polynomial P_a belongs to $O_F[X]$; for $g \in GL_n(D)$ we have $P_{gag^{-1}} = P_a \in O_F[X]$. □

We identify the space S of unitary polynomials in $F[T]$ of degree dn with F^{dn} by taking the non-dominant coeffficients. The map sending $X \in M_n(D)$ to its reduced characteristic polynomial P_X which belongs to S, is continuous ([BourbakiA-8] §17 Définition 1, [Reiner75] §9a).

We recall from [Badulescu18, Chapter 2, §2 to §6]:

An element $g \in G$ is called regular semi-simple when the roots of P_g in an algebraic closure F^{ac} of F have multiplicity 1. The set G^{rs} of regular semi-simple elements of G is open dense in G. The conjugacy class of $g \in G^{rs}$ is the set of elements $g' \in G$ with $P_{g'} = P_g$. Note that $g = 1 + p_F^j X \in G^{rs}$ is conjugate to an element of $1 + p_F^j M_n(O_D)$ if and only if X is conjugate to an element of $M_n(O_D)$ if and only the coefficients of $P_X(T) = p_F^{-jdn} P_g(T p_F^j + 1)$ belong to O_F. The set $\{P_g \mid g \in G^{rs}\}$ consists of the monic polynomials in $F[T]$ of degree dn without multiple roots in F^{ac}, with a non-zero constant term and with all irreducible factors of degree divisible by d. Let $GL_{dn}(F)^{rs,d}$ be the set of

$h \in GL_{dn}(F)^{rs}$ such that $P_h \in \{P_g \mid g \in G^{rs}\}$. We say that $g \in G^{rs}$ and $h \in GL_{dn}(F)^{rs,d}$ correspond and we write $g \leftrightarrow h$ when $P_g = P_h$.

Let $g \in G^{rs}$. The G-centralizer T_g of g is a maximal torus, isomorphic to the group of units of $F[T]/(P_g)$. We put on G/T_g the quotient measure dx^* of the Haar measure on G (§1) and on the Haar measure on T_g giving the value 1 to the maximal torus. The orbital integral of $f \in C_c^\infty(G;\mathbb{C})$ at g is

$$\Phi(f,g) = \int_{G/T_g} f(xgx^{-1})\, dx^*. \tag{9.2}$$

Let $C_c^\infty(GL_{dn}(F)^{rs};\mathbb{C})^{(d)}$ be the set of $\varphi \in C_c^\infty(GL_{nd}(F)^{rs};\mathbb{C})$ with $\Phi(\varphi,h)=0$ when h is not in $GL_{nd}(F)^{rs,d}$. We say that $f \in C_c^\infty(G^{rs};\mathbb{C})$ and $\varphi \in C_c^\infty(GL_{nd}(F)^{rs};\mathbb{C})^{(d)}$ correspond and we write $f \leftrightarrow \varphi$ when $\Phi(f,g) = \Phi(\varphi,h)$ if $g \in G^{rs}$ and $h \in GL_{nd}(F)^{rs,d}$ correspond. For $f \in C_c^\infty(G^{rs};\mathbb{C})$ there exists $\varphi \in C_c^\infty(GL_{nd}(F)^{rs};\mathbb{C})^{(d)}$ such that $f \leftrightarrow \varphi$, and conversely ([Badulescu18] Proposition 5.1).

9.1.2. *Jacquet-Langlands correspondence.* The classical Jacquet-Langlands correspondence ([DKV84], [Badulescu02]) is the unique bijective map

$$JL : \mathrm{Irr}^2_{\mathbb{C}}(G)) \to \mathrm{Irr}^2_{\mathbb{C}}(GL_{dn}(F)) \quad \text{such that for } \ \pi \in \mathrm{Irr}^2_{\mathbb{C}}(G),$$
$$(-1)^n \ \mathrm{trace}(\pi(f(g)dg)) = (-1)^{dn}\, trace(JL(\pi)(\varphi(h)dh))$$

when $f \in C_c^\infty(G);\mathbb{C})^{rs}$, $\varphi \in C_c^\infty(GL_{dn}(F);\mathbb{C})^{rs,d}$, $f \leftrightarrow \varphi$. The image by JL of the Steinberg representation of G is the Steinberg representation of $GL_{dn}(F)$. The maps JL extends to

1) a bijective map

$$JL : \mathrm{Irr}^2_{\mathbb{C}}(M_\lambda)) \to \mathrm{Irr}^2_{\mathbb{C}}(M'_{d\lambda}) \text{ for any composition } \lambda \text{ of } n.$$

2) an injective map

$$JL : \mathfrak{B}_G \to \mathfrak{B}_{GL_{dn}(F)}$$

$$JL([n.\,\mathrm{ind}_{P_\lambda}^G\, \sigma] = [n.\,\mathrm{ind}_{P_{d\lambda}}^{GL_{dn}(F)}\, JL(\sigma)] \text{ for } \sigma \in \mathrm{Irr}^2_{\mathbb{C}}(M_\lambda), \lambda \in \mathfrak{P}(n), \tag{9.3}$$

and by linearity to an injective homomorphism

$$JL : \mathrm{Gr}^\infty_{\mathbb{C}}(G) \to \mathrm{Gr}^\infty_{\mathbb{C}}(GL_{dn}(F)),$$

satisfying ([Badulescu07] Théorème 3.1):

$$(-1)^n \ \mathrm{trace}\, \nu(f(g)dg) = (-1)^{dn} \ \mathrm{trace}\, JL(\nu)(\varphi(h)dh) \tag{9.4}$$

when $\nu \in \mathrm{Gr}^\infty_{\mathbb{C}}(G)$, $f \in C_c^\infty(GL_n(D)^{rs};\mathbb{C})$, $\varphi \in C_c^\infty(GL_{dn}(F)^{rs};\mathbb{C})^{(d)}$, $f \leftrightarrow \varphi$. We have

$$\mathrm{Gr}^\infty_{\mathbb{C}}(GL_{dn}(F)) = JL(\mathrm{Gr}^\infty_{\mathbb{C}}(G)) \oplus C_d.$$

The homomorphism JL commutes with

a) the twist by smooth characters:
$JL((\chi \circ \mathrm{nrd}) \otimes \nu) = (\chi \circ \det) \otimes JL(\nu)$ when χ is a smooth character of F^*,

b) the normalized parabolic induction ([Badulescu07] Théorème 3.6):
$JL(\mathrm{ind}_{P_\lambda}^G(\delta_\lambda^{1/2}\nu) = \mathrm{ind}_{P_{d\lambda}}^{GL_{dn}(F)}({\delta'_{d\lambda}}^{1/2} JL(\nu))).$

3) a surjective homorphism extending the inverse LJ of the classical Jacquet-Langlands correspondence JL for the Levi subgroups :

$$LJ : \mathfrak{B}_{GL_{dn}(F)} \to \mathfrak{B}_G$$

(9.5)

$$LJ([n.\operatorname{ind}_{P_\mu}^{GL_{dn}(F)} \sigma] = \begin{cases} [n.\operatorname{ind}_{P_\lambda}^G LJ(\sigma)] & \text{for } \sigma \in \operatorname{Irr}^2_{\mathbb{C}}(M_\mu),\ \mu = d\lambda \in d\mathfrak{P}(n), \\ 0 & \text{for } \sigma \in \operatorname{Irr}^2_{\mathbb{C}}(M_\mu),\ \mu \in \mathfrak{P}(dn) \text{ but } \mu \notin d\mathfrak{P}(n) \end{cases}$$

giving by linearity a surjective homomorphism (the Badulescu-Jacquet-Langlands correspondence):

$$LJ : \operatorname{Gr}^\infty_{\mathbb{C}}(GL_{dn}(F)) \to \operatorname{Gr}^\infty_{\mathbb{C}}(G)$$

of kernel C_d, section JL, satisfying

$$(9.6) \qquad (-1)^{dn} \operatorname{trace} \nu(f(g)dg) = (-1)^n \operatorname{trace} LJ(\nu)(\varphi(h)dh)$$

when $\nu \in \operatorname{Gr}^\infty_{\mathbb{C}}(GL_{dn}(F))$, $f \in C_c^\infty(GL_{dn}(F)^{rs};\mathbb{C})$, $\varphi \in C_c^\infty(G^{rs};\mathbb{C})^{(d)}$, $f \leftrightarrow \varphi$. The homorphism LJ commutes with the twist by smooth characters: if χ is a smooth character of F^* and $\nu \in \operatorname{Gr}^\infty_{\mathbb{C}}(GL_{dn}(F))$,

$$(9.7) \qquad LJ((\chi \circ \det) \otimes \nu) = (\chi \circ \operatorname{nrd}) \otimes LJ(\nu),$$

the normalized parabolic induction: if δ'_μ the modulus of P'_μ and $\nu \in \operatorname{Gr}^\infty_{\mathbb{C}}(M'_\mu)$, still denoting $JL : \operatorname{Gr}^\infty_{\mathbb{C}}(M'_\mu) \to \operatorname{Gr}^\infty_{\mathbb{C}}(M_\lambda)$ the natural morphism, we have

$$(9.8) \qquad LJ(\operatorname{ind}_{P'_\mu}^{GL_{dn}(F)}({\delta'_\mu}^{1/2}\nu)) = \begin{cases} 0 & \text{if } \mu \notin d\mathfrak{P}(n) \\ \operatorname{ind}_{P_\lambda}^G(\delta_\lambda^{1/2} LJ(\nu)) & \text{if } \mu = d\lambda,\ \lambda \in \mathfrak{P}(n) \end{cases}$$

and is compatible with the Aubert involution ι up to a sign ([Badulescu07] Proposition 3.16):

$$(9.9) \qquad (-1)^n \iota \circ LJ = LJ \circ (-1)^{dn} \iota.$$

As LJ sends the Steinberg representation of $GL_{dn}(F)$ to the Steinberg representation of G, the Aubert involution of the Steinberg representation is the trivial representation up to a sign, and LJ commutes with the parabolic induction, we have:

$$(9.10) \qquad (-1)^{nd} LJ(\pi_{P'_\mu}) = \begin{cases} (-1)^n \pi_{P_\lambda} & \text{if } \mu = d\lambda, \\ 0 & \text{otherwise.} \end{cases}$$

9.2. The theorem 9.1 is an easy consequence of (9.6), (9.10), and of the linear independance of the restrictions to $K \cap GL_n(D)^{rs}$ of the characters of the representations π_{P_μ} of $GL_{dn}(F)$ for $\mu \in \mathfrak{P}(dn)$, for any open compact subgroup K of $GL_{dn}(F)$. We give the details.

Let $P = MN$ be a parabolic subgroup of G of Levi M, $\sigma \in \operatorname{Irr}^2_{\mathbb{C}}(M)$, $\pi = \operatorname{ind}_P^G \sigma$. Let $c_\pi, c_{JL(\pi)}$ be the maps and $K_\pi, K_{JL(\pi)}$ groups in the germ expansions (6.1) of $[\pi], JL([\pi])$, such that for any $g \in K_\pi \cap G^{rs}$ there exists $h \in K_{JL(\pi)} \cap GL_{dn}(F)^{rs,d}$ with $g \leftrightarrow h$, as we can because for $g \in GL_n(D)^{rs}$, $h \in GL_{dn}(F)^{rs,d}$ with the same reduced characteristic polynomial $P(T)$, the coefficients of $p_F^{-jdn} P(Tp_F^j + 1)$ belong to O_F if and only if g is conjugate to an element of $1+p_F^j M_n(O_D)$ if and only if h is conjugate to element of $1+p_F^j M_{dn}(O_F)$ (Lemma 9.4).

Let $f \in C_c^\infty(K_{LJ(\pi)} \cap G^{rs};\mathbb{C}), \varphi \in C_c^\infty(K_\pi \cap GL_{dn}(F)^{rs};\mathbb{C})^{(d)}, \ f \leftrightarrow \varphi$. The germ expansion (6.1) applied to (9.6) $(-1)^n$ trace $LJ(\pi)(f(g)dg) = (-1)^{dn}$ trace $\pi(\varphi(g)dg)$ gives

$$(-1)^n \sum_{\lambda\in\mathfrak{P}(n)} c_{LJ(\pi)}(\lambda) \ \mathrm{trace}\, \pi_{P_\lambda}(f(g)dg) = (-1)^{dn} \sum_{\mu\in\mathfrak{P}(dn)} c_\pi(\mu) \ \mathrm{trace}\, \pi_{P'_\mu}(\varphi(g)dg),$$

and applying (9.6), then (9.10) to the RHS,

$$= (-1)^n \sum_{\mu\in\mathfrak{P}(dn)} c_\pi(\mu) \ \mathrm{trace}\, LJ(\pi_{P'_\mu})(f(g)dg) = (-1)^{dn} \sum_{\lambda\in\mathfrak{P}(n)} c_\pi(d\lambda) \ \mathrm{trace}\, \pi_{P_\lambda}(f(g)dg).$$

So, $(-1)^n \sum_{\lambda\in\mathfrak{P}(n)} c_{LJ(\pi)}(\lambda)$ trace $\pi_{P_\lambda}(f(g)dg) = (-1)^{dn} \sum_{\lambda\in\mathfrak{P}(n)} c_\pi(d\lambda)$ trace $\pi_{P_\lambda}(f(g)dg)$.

The linear independence of the characters of π_{P_λ} on $K_{LJ(\pi)}$ for $\lambda \in \mathfrak{P}(n)$ (Corollary 5.7) and the local integrability of characters imply the [7] linear independence of the characters of π_{P_λ} on $K_{LJ(\pi)} \cap G^{rs}$ for $\lambda \in \mathfrak{P}(n)$ and

$$(-1)^{dn} c_\pi(\lambda) = (-1)^n c_{LJ(\pi)}(d\lambda) \quad \text{for} \quad \lambda \in \mathfrak{P}(n).$$

for any $[\pi]$ in the basis $\mathfrak{B}_G$ of $\mathrm{Gr}_{\mathbb{C}}^\infty(G)$. This ends the proof of the theorem 9.1.

9.3. Applications to $c_\pi((n))$ For $\pi \in \mathrm{Irr}^2_{\mathbb{C}}(G)$ and a division central F-algebra D_{dn} of reduced degree dn, there exists a unique $\pi_{dn} \in \mathrm{Irr}_{\mathbb{C}}(D^*_{dn})$ such that their images by the classical Jacquet-Langlands correspondence in $\mathrm{Irr}^2_{\mathbb{C}}(GL_{dn}(F))$ are equal. The dimension of π_{dn} is finite and by Theorem 9.1) $(-1)^n c_\pi(n) = -c_{JL(\pi_{dn})}(dn) = -\dim_{\mathbb{C}} \pi_{dn}$. An irreducible smooth complex representation π of G is tempered if and only if $\pi = \mathrm{ind}_P^G \sigma$ for a parabolic subgroups $P = MN$ of G and $\sigma \in \mathrm{Irr}^2_{\mathbb{C}}(M)$ ([Lapid-Minguez-Tadic16] A.11).

For $\nu \in \mathrm{Gr}_{\mathbb{C}}^\infty(GL_{dn}(F))$ and $\lambda \in \mathfrak{P}(n)$, we have $(-1)^n c_{LJ(\nu)}(\lambda) = (-1)^{dn} c_\nu(d\lambda)$.

Corollary 9.5. *Let* $\pi \in \mathrm{Rep}_{\mathbb{C}}^\infty(G)$ *irreducible and tempered. Then*

$$c_\pi((n)) = \begin{cases} (-1)^{n-1} \dim_{\mathbb{C}} \pi_{dn} & \text{if } \pi \in \mathrm{Irr}^2_{\mathbb{C}}(G) \\ 0 & \text{if } \pi \notin \mathrm{Irr}^2_{\mathbb{C}}(G) \end{cases}.$$

10. Coefficient field of characteristic different from p

Let R be a field. Our goal is to show that Theorem 1.3 proved using the Harish-Chandra germ expansion remain valid for R-representations when the characteristic of R is not p. There are two simple reasons:

a) For a parabolic subgroup P of G, the representation $\mathrm{ind}_P^G 1$ is defined over $\mathbb{Z}$.

b) For a field extension R'/R, the scalar extension from R to R' of smooth representations of a profinite group H respects finite length, and is an injection at the level of Grothendieck

[7] Put $K = K_{LJ(\pi)}$. Any $f \in C_c^\infty(G;\mathbb{C})$ with support in K is a limit of (uniformly bounded) functions f_n with support in $K \cap G^{rs}$, so by the local integrability of characters and the Lebesgue dominated convergence theorem, trace $\pi_{P_\lambda}(f(g)dg) = \lim_n$ trace $\pi_{P_\lambda}(f_n(g)dg)$.

groups [Henniart-Vignéras19]. For an irreducible smooth R-representation π of H, the R'-representation $R' \otimes_R \pi$ considered as an R-representation is π-isotypic (a direct sum of representations isomorphic to π).

From now on, $\mathrm{char}_R \neq p$. When $\pi \in \mathrm{Rep}_R^{\infty,f}$ is equal to $\sum_{\lambda\in\mathfrak{P}(n)} c_\pi(\lambda)\,\mathrm{ind}_{P_\lambda}^G 1$ on K_π as in Theorem 1.3, the map c_π is unique because:

Proposition 10.1 (Corollary 5.7)**.** *Let K be an open pro-p subgroup of G. For any non zero map $c : \mathfrak{P}(n) \to \mathbb{Z}$, the restriction to K of*

$$\sum_{\lambda\in\mathfrak{P}(n)} c(\lambda)\,[\pi_{P_\lambda}] \in \mathrm{Gr}_R^\infty(G)$$

is not 0*.*

Proof. We can suppose R algebraically closed by b). The categories $\mathrm{Rep}_R^\infty(K)$ and $\mathrm{Rep}_{\mathbb{C}}^\infty(K)$ are equivalent and the Grothendieck groups $\mathrm{Gr}_R^\infty(K)$ and $\mathrm{Gr}_{\mathbb{C}}^\infty(K)$ are isomorphic because K is a pro-p group and $\mathrm{char}_R \neq p$. The proposition is true when $R = \mathbb{C}$ (Corollary 5.7) and the representations π_{P_λ} correspond. Hence the proposition is true for any R. □

We list other properties which will be used in the proof of the theorem 1.3.

10.1. Twist by a character, image by an automorphism

Assume that $\pi \in \mathrm{Rep}_R^{\infty,f}(G)$ has a germ expansion of map c_π on K_π (the first part of Theorem 1.3), χ is a smooth R-character of G and σ is an automorphism of R. Then the representations $\pi \otimes \chi$ and $\sigma(\pi)$ have a germ expansion of maps $c_{\pi\otimes\chi} = c_{\sigma(\pi)} = c_\pi$ on $K_{\pi\otimes\chi} = K_{\sigma(\pi)} = K_\pi$ if χ is trivial on K_π. The reason is a) ($\sum_{\lambda\in\mathfrak{P}(n)} c_\pi(\lambda)\,[\pi_{P_\lambda}]$ is defined over $\mathbb{Z}$).

10.2. Germ expansion on the Grothendieck group Assume that any $\pi \in \mathrm{Rep}_R^{\infty,f}(G)$ has a germ expansion of map c_π on some open compact subgroup K_π of G. Then, the linear map $\nu \mapsto c_\nu : \mathrm{Gr}_R^\infty(G) \to \{\mathfrak{P}(n) \to \mathbb{Z}\}$ such that $c_{[\pi]} = c_\pi$ for $\pi \in \mathrm{Rep}_R^{\infty,f}(G)$, has the property that the restrictions to some open compact subgroup K_ν of G of ν and of $\sum_{\lambda\in\mathfrak{P}(n)} c_\nu(\lambda)\,[\pi_{P_\lambda}]$ are isomorphic.

Parabolic induction: For a parabolic subgroup P of G of Levi M, the parabolic induction ind_P^G is exact and respects finite length and passes to a linear map between the Grothendieck groups:

$$\mathrm{ind}_P^G : \mathrm{Gr}_R^\infty(M) \to \mathrm{Gr}_R^\infty(G), \quad \mathrm{ind}_P^G[\sigma] = [\mathrm{ind}_P^G\,\sigma] \text{ for } \sigma \in \mathrm{Rep}_R^{\infty,f}(M).$$

When $\nu \in \mathrm{Gr}_R^\infty(M)$ has a germ expansion of map c_ν, then $\mathrm{ind}_P^G\,\nu$ has a germ expansion of map induced by c_ν (Theorem 7.1).

10.3. For $j \in \mathbb{N}_{>0}$ and λ is a composition of n, the values of the character ξ_λ of $K_j = 1 + M_n(P_D^j)$ defined by (6.8) and of the character θ_λ of U defined by (8.3) are roots of 1 of order powers of p. Assume that the field R contains roots of unity of any p-power order,

$$\text{we write } \mu_{p^\infty} \subset R, \ \text{ implying } \mathrm{char}_R \neq p.$$

We can define ξ_λ and θ_λ over R as before, and the Whittaker support of an irreducible smooth R-representation of G as before Theorem 8.2.

Let $\pi \in \mathrm{Rep}_R^{\infty,f}(G)$ having a germ expansion of map $c_\pi : \mathfrak{P}(n) \mapsto \mathbb{Z}$: for a positive integer j_0 the restriction of π and of $\sum_{\lambda\in\mathfrak{P}(n)} c_\pi(\lambda)\,\mathrm{ind}_{P_\lambda}^G 1$ to K_{j_0} are equal. With the same proofs as for $R = \mathbb{C}$, we have:

Theorem 10.2. *1) For any integer $j \geq j_0$ and any λ partition of n, we have*

$$c_\pi(\lambda) = m(\xi_\lambda, \pi) - \sum_{\mu\in\mathfrak{P}(n),\mu<\lambda} c_\pi(\mu)\, m(\xi_\lambda, \pi_{P_\mu}). \tag{10.1}$$

In particular when λ is minimal in the support of c_π, $c_\pi(\lambda) = m(\xi_\lambda, \pi)$ is positive and independent of $j \geq j_0$.

2) Theorem 8.2 is valid.

An algebraically closed field R with $\mathrm{char}_R \neq p$ contains μ_p^∞. To prove Theorem 1.3 for R algebraically closed, by Theorem 10.2 and Proposition 10.1, we have only to prove that any $\pi \in \mathrm{Rep}_R^{\infty,f}(G)$ has a germ expansion: there exists a map $c_\pi : \mathfrak{P}(n) \to \mathbb{Z}$ such that π and $\sum_\lambda c_\pi(\lambda)\,\mathrm{ind}_{P_\lambda}^G 1$ have equal on some open compact subgroup K_π of G.

We prove now Theorem 1.3 going from $R = \mathbb{C}$ to $R = \mathbb{Q}_\ell^{ac}$ to $R = \mathbb{F}_\ell^{ac}$, $\ell \neq p$, to an algebraically closed field R, and finally to a not necessarily algebraically closed field R.

10.4. $R \simeq R'$**.** For any prime number ℓ, the fields $\mathbb{C}$ and $\mathbb{Q}_\ell^{ac}$ are isomorphic. It is easy to see that if Theorem 1.3 is true for a field R, it is also true for an isomorphic field R'. Indeed, a field isomorphism $j : R \to R'$ induces isomorphisms of categories $j_G : \mathrm{Rep}_R^\infty(G) \to \mathrm{Rep}_{R'}^\infty(G)$ and $j_K : \mathrm{Rep}_R^\infty(K) \to \mathrm{Rep}_{R'}^\infty(K)$ for any open compact subgroup K of G. The isomorphisms commute with the restriction to K and the parabolic induction ind_P^G. For $\pi \in \mathrm{Rep}_R^\infty(G)$ and $\sigma \in \mathrm{Rep}_R^\infty(M)$,

$$j_K(\pi|_K) = j_G(\pi)|_K, \quad \mathrm{ind}_P^G(j_M(\sigma)) = j_G(\mathrm{ind}_P^G \sigma).$$

When the theorems are true for R they are also true for R'. For $\pi \in \mathrm{Rep}_R^{\infty,f}(G)$, then $c_\pi = c_{j_G(\pi)}$ and we can take $K_{j_G(\pi)} = K_\pi$.

10.5. $R \simeq \mathbb{F}_\ell^{ac}$ **for** $\ell \neq p$**.**

The theorems over $\mathbb{Q}_\ell^{ac}$ imply the theorem over $\mathbb{F}_\ell^{ac}$ by reduction modulo ℓ for $\ell \neq p$. We denote by $\mathbb{Z}_\ell^{ac}$ the ring of integers of $\mathbb{Q}_\ell^{ac}$. A lattice in a $\mathbb{Q}_\ell^{ac}$-vector space V is a free $\mathbb{Z}_\ell^{ac}$-submodule generated by a $\mathbb{Q}_\ell^{ac}$-basis of V.

Let $\pi \in \mathrm{Rep}_{\mathbb{Q}_\ell^{ac}}^{\infty,f}(G)$. One says that π is integral when the space of π contains a G-stable lattice $\mathfrak{L}_\pi$. Then, the reduction modulo ℓ of $\mathfrak{L}_\pi$ equal to $\mathbb{F}_\ell^{ac} \otimes_{\mathbb{Z}_\ell^{ac}} \mathfrak{L}_\pi$ belongs to $\mathrm{Rep}_{\mathbb{F}_\ell^{ac}}^{\infty,f}(G)$ and its image in the Grothendieck group $\mathrm{Gr}_{\mathbb{F}_\ell^{ac}}^\infty(G)$ does not depend on the choice of $\mathfrak{L}_\pi$. It is called the **reduction modulo** ℓ of π, and denoted by $r_\ell(\pi)$. The subcategory of integral representations $\mathrm{Rep}_{\mathbb{Q}_\ell^{ac}}^{\infty,f,int}(G)$ in $\mathrm{Rep}_{\mathbb{Q}_\ell^{ac}}^{\infty,f}(G)$ is abelian [Vigneras96]; let $\mathrm{Gr}_{\mathbb{Q}_\ell^{ac}}^{\infty,int}(G)$ be its

Grothendieck group. The reduction modulo ℓ passes to a surjective (not injective) map between the Grothendieck groups:

$$r_\ell : \mathrm{Gr}^{\infty,int}_{\mathbb{Q}_\ell^{ac}}(G) \to \mathrm{Gr}^{\infty}_{\mathbb{F}_\ell^{ac}}(G),$$

and there is an explicit subset $E(G)$ of $\mathrm{Rep}^{\infty,f,int}_{\mathbb{Q}_\ell^{ac}}(G)$ such that the set $\{r_\ell(\pi) \mid \pi \in E(G)\}$ is a basis of the Grothendieck group $\mathrm{Gr}^{\infty}_{\mathbb{F}_\ell^{ac}}(G)$ ([Minguez-Sécherre14] Théorème 9.35).

For a parabolic subgroup P of G with Levi M, the parabolic induction $\mathrm{ind}_P^G : \mathrm{Rep}^{\infty}_{\mathbb{Q}_\ell^{ac}}(M) \to \mathrm{Rep}^{\infty}_{\mathbb{Q}_\ell^{ac}}(G)$ is exact, respects finite length and integrality hence passes to the Grothendieck groups and $r_\ell \circ \mathrm{ind}_P^G = \mathrm{ind}_P^G \circ r_\ell$ on $\mathrm{Rep}^{\infty,f,int}_{\mathbb{Q}_\ell^{ac}}(M)$.

The representation π_P over $\mathbb{Q}_\ell^{ac}$ are integral, with a canonical integral structure (the functions with values in $\mathbb{Z}_\ell^{ac}$: π_P over $\mathbb{Z}_\ell^{ac}$) of reduction modulo ℓ the representation π_P over $\mathbb{F}_\ell^{ac}$.

If $\pi \in \mathrm{Rep}^{\infty,f,int}_{\mathbb{Q}_\ell^{ac}}(G)$ has a germ expansion of map c_π on K_π, then $r_\ell(\pi) \in \mathrm{Gr}^{\infty}_{\mathbb{F}_\ell^{ac}}(G)$ has a germ expansion of map c_π on K_π.

Lemma 10.3. *Let $\pi, \pi' \in \mathrm{Rep}^{\infty,f,int}_{\mathbb{Q}_\ell^{ac}}(G)$ with $r_\ell(\pi) = r_\ell(\pi')$. Then $c_\pi = c_{\pi'}$.*

Proof. When j is large, we have (10.1) for π and π'. As K_j is a pro-p group, $m(\xi_\lambda, \pi) = m(r_\ell(\xi_\lambda), r_\ell(\pi))$. Therefore $r_\ell(\pi) = r_\ell(\pi')$ implies $m(\xi_\lambda, \pi) = m(\xi_\lambda, \pi')$. By induction on λ we deduce $c_\pi = c_{\pi'}$. ☐

As the $r_\ell(\pi)$ for $\pi \in E(G)$ generate $\mathrm{Gr}^{\infty}_{\mathbb{F}_\ell^{ac}}(G)$, Lemma 10.3 gives the existence of a linear map

$$c : \mathrm{Gr}^{\infty}_{\mathbb{F}_\ell^{ac}}(G) \to \{\mathfrak{P}(n) \to \mathbb{Z}\} \text{ defined by } c_{r_\ell(\pi)} = c_\pi \text{ for } \pi \in \mathrm{Rep}^{\infty,f,int}_{\mathbb{Q}_\ell^{ac}}(G).$$

For $\pi \in \mathrm{Rep}^{\infty,f}_{\mathbb{F}_\ell^{ac}}(G)$, the restrictions of π and of $\sum_{\lambda \in \mathfrak{P}(n)} c_\pi(\lambda)\, r_\ell(\pi_{P_\lambda})$ to some open pro-p group K_π of G are isomorphic. Theorem 1.3 when $R = \mathbb{F}_\ell^{ac}$ is proved.

10.6. R'/R **algebraically closed fields** Given an extension R'/R of algebraically closed fields of characteristic different from p, we prove that the germ expansion over R for all $n \geq 1$ is equivalent to the germ expansion over R' for all $n \geq 1$. Therefore we get Theorem 1.3 over any algebraically closed field R, because we already proved for $R = \mathbb{C}$ and $R = \mathbb{F}_\ell^{ac}$ when $\ell \neq p$.

The proof relies on properties, that we now recall, of the scalar extension $\pi \mapsto R' \otimes_R \pi : \mathrm{Rep}^{\infty}_R(G) \to \mathrm{Rep}^{\infty}_{R'}(G)$ from R to R' and of the representations of G parabolically induced from Speh representations of the Levi subgroups of G. Fix the same square root of $q = p^f$ in R and in R'.

The scalar extension from R to R' respects irreducible smooth representations and cuspidality, is exact and passes to an injective linear map $\nu \mapsto R' \otimes_R \nu : \mathrm{Gr}^{\infty}_R(G) \to \mathrm{Gr}^{\infty}_{R'}(G)$ between the Grothendieck groups, commutes with the parabolic induction and for any open pro-p subgroup K of G is an isomorphism of categories $\delta \mapsto R' \otimes_R \delta : \mathrm{Rep}^{\infty}_R(K) \to \mathrm{Rep}^{\infty}_{R'}(K)$ [Henniart-Vignéras19]. When $\pi \in \mathrm{Rep}^{\infty,f}_R(G)$ the multiplicity $m(\delta, \pi)$ in π of $\delta \in \mathrm{Rep}^{\infty}_R(K)$ irreducible is equal to $m(R' \otimes_R \delta, R' \otimes_R \pi)$. Any irreducible cuspidal R'-representation ρ'

of G is the twist by an unramified smooth R'-character χ of G of an irreducible cuspidal R-representation ρ of G, $\rho' = \chi \otimes (R' \otimes_R \rho) = \chi \otimes_R \rho$ [Vigneras96]. By Lemma 10.4 below, this is also true for Speh representations.

Let m be a divisor of $n = mr, r \geq 1$, and ρ an irreducible cuspidal R-representation of $GL_m(D))$. To (ρ, n) are attached in [Minguez-Sécherre14]:

- an unramified smooth R-character ν_ρ of $GL_m(D)$ depending only on the inertia class of ρ (loc.cit. §5.2).
- a cuspidal R-segment $\Delta_{\rho,n} = (\rho, \nu_\rho \otimes \rho, \ldots, \nu_\rho^{-1+r} \otimes \rho)$ of length r, denoted $[0, -1+r]_\rho$ in (loc.cit. §7.2).
- an irreducible subrepresentation $Z(\Delta_{\rho,n}) \in \mathrm{Rep}_R^\infty(GL_n(D))$ (a Speh representation) of the normalized parabolic induction $\rho \times \ldots \times (\nu_\rho^{-1+r} \otimes \rho)$ of $\rho \otimes \ldots \otimes (\nu_\rho^{-1+r} \otimes \rho) \in \mathrm{Rep}_R^\infty M_\lambda$ for $\lambda = (m, \ldots, m) \in \mathfrak{P}(n)$ (loc.cit. §7.2).

Lemma 10.4. *For each unramified smooth R'-character χ of F^*,*

$$(\chi \circ \mathrm{nrd}) \otimes_R Z(\Delta_{\rho,n}) \simeq Z(\Delta_{(\chi \circ \mathrm{nrd}) \otimes_R \rho, n}).$$

This important property is stated in [Minguez-Sécherre17][(8.1.2)] (c.f.[DS23, Lemme 5.9]).

To a composition $(n_1, \ldots, n_r)$ of n, a divisor m_i of n_i and an irreducible cuspidal R-representation ρ_i of $GL_{m_i}(D))$ for $1 \leq i \leq r$, are associated

- a cuspidal R-multisegment $\mathfrak{M} = (\Delta_{\rho_1,n_1}, \ldots, \Delta_{\rho_r,n_r})$,
- a Speh R-representation $Z(\mathfrak{M}) = Z(\Delta_{\rho_1,n_1}) \otimes \ldots Z(\Delta_{\rho_r,n_r})$ of $M = M_{(n_1,\ldots,n_r)}$,
- the normalized parabolic induction $n.I(\mathfrak{M}) = \mathrm{ind}_P^G(Z(\mathfrak{M})\delta_P^{1/2})$ of $Z(\mathfrak{M})$ where $P = P_{(n_1,\ldots,n_r)}$ and δ_P is the module of P.

The Grothendieck group $\mathrm{Gr}_R^\infty(G)$ is generated by the $[n.I(\mathfrak{M})]$ for the cuspidal R-multisegments $\mathfrak{M}$ of $GL_n(D)$ ([Minguez-Sécherre14] proof of Lemma 9.36 with Proposition 9.29).

But $Z(\mathfrak{M})\delta_P^{1/2}$ is also a Speh representation $Z(\mathfrak{M}') = Z(\Delta_{\rho'_1,n_1}) \otimes \ldots Z(\Delta_{\rho'_r,n_r})$ where ρ'_i is the twist of ρ_i by an unramified character. Therefore $\mathrm{Gr}_R^\infty(G)$ is also generated by the images of the parabolic induction $I(\mathfrak{M}) = \mathrm{ind}_P^G(Z(\mathfrak{M}))$ for the cuspidal R-multisegments $\mathfrak{M}$. If the Speh R-representations $Z(\mathfrak{M})$ of G have a germ expansion then the $I(\mathfrak{M})$ have a germ expansion (Theorem 7.1) and any $\pi \in \mathrm{Rep}_R^{\infty,f}(G)$ has a germ expansion.

We are now ready to prove that the existence of a germ expansion over R is equivalent to the existence of a germ expansion over R'. Let $\mathfrak{M}' = (\Delta_{\rho'_1,n_1}, \ldots, \Delta_{\rho'_r,n_r})$ be a cuspidal R'-multisegment of $GL_n(D)$. For $i = 1 \ldots, r$, ρ'_i is an irreducible smooth cuspidal R'-representation of $GL_{m_i}(D)$ for a divisor m_i of n_i; there exists an unramified smooth R'-character χ'_i and an irreducible smooth cuspidal R'-representation of $GL_{m_i}(D)$ such that $\rho'_i = \rho_i \chi_i$ and $Z(\Delta_{\rho'_i,n_i}) = \chi'_i Z(\Delta_{\rho_i,n_i})$. Let $\mathfrak{M} = (\Delta_{\rho_1,n_1}, \ldots, \Delta_{\rho_r,n_r})$ and χ' the unramified R'-character of $M_{n_1,\ldots,n_r}$ corresponding to the χ'_i. Then $Z(\mathfrak{M}') = \chi' Z(\mathfrak{M})$. The Speh R'-representation $Z(\mathfrak{M}')$ has a germ expansion if and only if the Speh R-representation $Z(\mathfrak{M})$ has a germ expansion.

10.7. ***R* not necessarily algebraically closed** Let R be a field of characteristic different from p. We prove that there is a germ expansion over R when there is a germ expansion over an algebraic closure R^{ac} of R, using the following properties of the scalar extension from R to R^{ac} [Henniart-Vignéras19]:

For $\pi \in \mathrm{Rep}_R^\infty(G)$ irreducible, the R^{ac}-representaton $R^{ac} \otimes_R \pi$ has finite length because π is admissible as the characteristic of R is different from p. Assume that there is a map $c : \mathfrak{P}(n) \to \mathbb{Z}$ such that $R^{ac} \otimes_R \pi = R^{ac} \otimes_R (\sum_\lambda c(\lambda)\pi_{P_\lambda}$ on an open compact subgroup K of G. The scalar extension $\mathrm{Gr}_R^\infty(K) \to \mathrm{Gr}_{R^{ac}}^\infty(K)$ from R to R^{ac} is injective.

Therefore $\pi = \sum_\lambda c(\lambda)\pi_{P_\lambda}$ on K. The representation π has a germ expansion with the same map $c_\pi = c_{R^{ac}\otimes_R\pi} = c$. The above properties of the scalar extension from R to R^{ac} imply:

For any irreducible subquotient π' of $R^{ac} \otimes_R \pi$, we have

$$c_\pi = \ell_\pi c_{\pi'} \quad \text{where } \ell_\pi \text{ is the length of } R^{ac} \otimes_R \pi. \tag{10.2}$$

Therefore c_π and $c_{\pi'}$ have the same support. As $c_{\pi'}(\lambda) > 0$ when λ is minimal in the support of $c_{\pi'}$ (Theorem 10.2), $c_\pi(\lambda) > 0$. This ends the proof of Theorem 1.3.

10.8. The Jacquet-Langlands correspondence

The classical Jacquet-Langlands correspondence JL between essentially square integrable representations on both sides, is compatible with character twists and equivariant under the action of $\mathrm{Aut}(\mathbb{C})$. Transported to $\mathbb{Q}_\ell^{ac}$ [8],

$$JL : \mathrm{Irr}^2_{\mathbb{Q}_\ell^{ac}}(G) \to \mathrm{Irr}^2_{\mathbb{Q}_\ell^{ac}}(GL_{dn}(F))$$

preserves the property of being integral, and two integrals representations of G are congruent modulo ℓ if and only if their images under JL are congruent modulo ℓ ([Minguez-Sécherre17] Theorem 1.1). Once a square root of $q = p^f$ in $\mathbb{Q}_\ell^{ac}$ has been chosen when f is odd to normalize parabolic induction, the Jacquet-Langlands correspondence LJ transported to the Grothendieck groups of $\mathbb{Q}_\ell^{ac}$-representations does reduce modulo ℓ thus yielding a map for $\mathbb{F}_\ell^{ac}$-representations ([Minguez-Sécherre17] Theorem 1.16)

$$LJ : \mathrm{Gr}^\infty_{\mathbb{F}_\ell^{ac}}(GL_{dn}(F)) \to \mathrm{Gr}^\infty_{\mathbb{F}_\ell^{ac}}(G).$$

By our argument of reduction modulo ℓ in §10.5 we see that Theorem 9.1 is valid for $\mathbb{F}_\ell^{ac}$-representations. When R is an algebraically closed field of characteristic different from p, the reasoning of §10.6 then gives a map

$$LJ : \mathrm{Gr}^\infty_R(GL_{dn}(F)) \to \mathrm{Gr}^\infty_R(G)$$

satisfying Theorem 9.1 for R-representations.

Theorem 10.5. *(Theorem 9.1). When R is an algebraically closed field of characteristic different from p, for $\nu \in \mathrm{Gr}^\infty_R(GL_{dn}(F))$ and $\lambda \in \mathfrak{P}(n)$, we have $(-1)^n c_{LJ(\nu)}(\lambda) = (-1)^{dn} c_\nu(d\lambda)$.*

[8](for the root of q in $\mathbb{Q}_\ell^{ac}$ image of $\sqrt{q} \in \mathbb{C}$ via the isomorphism)

Remark 10.6. When $D \neq F$, there are cuspidal complex representations of $GL_n(D)$ that are isomorphic to their complex conjugate, and not the scalar extension of a real representation. So the Jacquet-Langlands correspondence does not descend to an arbitrary fied R.

A counter-example occurs already for D^* and D is a quaternion field over F with $q \equiv 3 \mod 4$. Take a regular complex character χ of k_D^* of order 4, seen as a character of O_D^* and extended by -1 on a uniformizer p_F of F. The induced representation $\mathrm{ind}_{F^*O_D^*}^{D^*} \chi$ has dimension 2 and its image is the quaternion group of order 8 which is not defined over $\mathbb{R}$. The irreducible representation $\pi^0 = JL(\mathrm{ind}_{F^*O_D^*}^{D^*} \chi)$ of $GL_2(F)$ is cuspidal of level 0 and can be explicited. For example for $F = \mathbb{Q}_3$, the irreducible cuspidal representation σ^0 of $GL_2(\mathbb{F}_3)$ corresponding to π^0 has dimension 2 and is defined over $\mathbb{R}$. As the central character of π^0 is trivial on O_F^*, σ^0 factorizes by $PGL_2(\mathbb{F}_3) = S_4$ which has all its irreducible representations defined over $\mathbb{R}$ and even over $\mathbb{Q}$.

11. Invariant vectors by Moy-Prasad subgroups

We prove in this section Theorem 1.4. Let R be a field, P a parabolic subgroup of G of Levi M and K an open compact subgroup of G. The positive integer

$$\dim_R(\pi_P)^K = |P\backslash G/K|$$

depends only on $[\pi_P]$, hence only on the conjugacy class of M and of K. We can suppose that $P = P_\lambda$ for $\lambda \in \mathfrak{P}(n)$ and $K \subset K_0$. We have $G = P_\lambda K_0$ and $P_\lambda\backslash G/K \simeq (P_\lambda \cap K_0)\backslash K_0/K$.

Example 11.1. We have $(P_\lambda \cap K_0)\backslash K_0/1+M_n(P_D) \simeq P_\lambda(k_D)\backslash GL_n(k_D)$ where $k_D = O_D/P_D$ is the residue field of D, q_D its cardinality. We deduce

$$|P_\lambda\backslash G/1 + M_n(P_D)| = [n!]_{q_D} / \prod_i [\lambda_i!]_{q_D},$$

where $[n!]_q = \prod_{m=1}^n [m]_q,\ [m]_q = (q^m - 1)/(q - 1)$ ([Suzuki22] Lemma 1.13).

Proposition 11.2. *Let $G_{x,r}$ denote the a Moy-Prasad subgroup of G fixing an element x of the building of the adjoint group $\mathcal{BT}$ of G, and r is a positive real number, and $j \in \mathbb{N}$. We have*

$$|P\backslash G/G_{x,r+j/d}| = |P\backslash G/G_{x,r}|\, q^{d\, d_\lambda j}. \tag{11.1}$$

When K' is a normal open subgroup of K,

$$|P\backslash G/K'| = \sum_{g \in P\backslash G/K} |P\backslash PgK/K'|, \quad |P\backslash PgK/K'| = \frac{[K : K']}{[(K \cap g^{-1}Pg) : (K' \cap g^{-1}Pg)]}.$$

The group $G_{x,r+j/d}$ is normal in $G_{x,r}$, and (11.1) follows from :

Proposition 11.3. *We have $[G_{x,r} \cap P : G_{x,r+1/d} \cap P] = q^{d\,(n^2 - d_\lambda)}$.*

Note that the index is the same for all (x, r). The D-dimension of the Lie algebra $\mathfrak{p}$ of P is $n^2 - d_\lambda$ where $\lambda \in \mathfrak{P}(n)$ is the partition such that P is associated to P_λ.

Example 11.4. When $P = G$, then $\lambda = (n), d_{(n)} = 0$, $[G_{x,r} : G_{x,r+1/d}] = q^{d\,n^2}$.

When $P = B$, then $\lambda = (1,\ldots,1), d_{(1,\ldots,1)} = n(n-1)/2$, $[G_{x,r} \cap B : G_{x,r+1/d} \cap B] = q^{d\,(n(n+1)/2)}$.

Proof. It is more convenient to use lattice functions rather than points in the Bruhat-Tits building $\mathcal{BT}$. For that we follow [Broussous-Lemaire02] denoted here by [BL]. Recall that a lattice function is a map Φ from $\mathbb{R}$ to O_D-lattices in D^n satisfying the conditions of ([BL] Definition 2.1); in particular

$$\Phi(s+1/d) = P_D\,\Phi(s) \quad \text{for any } s \in \mathbb{R}. \tag{11.2}$$

The group $\mathbb{R}$ acts on lattice functions by translations, and to a lattice function is associated a point in $\mathcal{BT}$. That point is the same for a translate, and one gets in that way a G-equivariant bijection from the set of lattice functions up to translation onto $\mathcal{BT}$. For any lattice function Φ and any $r \in \mathbb{R}$, one defines a lattice in $M_n(D)$

$$\mathfrak{g}_{\Phi,r} = \{A \in M_n(D) \mid A(\Phi(s)) = \Phi(r+s) \ \text{ for any } s \in \mathbb{R}\}.$$

In their introduction [BL] indicate that $\mathfrak{g}_{\Phi,r} = \mathfrak{g}_{x,r}$ where $x \in \mathcal{BT}$ corresponds to Φ and $\mathfrak{g}_{x,r}$ is the lattice in $M_n(D)$ defined by Moy and Prasad. They also say that the subgroup $G_{x,r}$ for $r \geq 0$, of G defined by Moy and Prasad satisfies:

$$G_{x,0} = (\mathfrak{g}_{\Phi,0})^*, \quad G_{x,r} = 1 + \mathfrak{g}_{\Phi,r} \ \text{ if } r > 0.$$

They refer to their Appendix A, written by B.Lemaire; the relevant comments are in the lines before their Proposition A.3.6.

An immediate consequence of condition (11.2) is that $\mathfrak{g}_{\Phi,r+1/d} = P_D\,\mathfrak{g}_{\Phi,r}$. That implies in particular that

$$[\mathfrak{g}_{\Phi,r} : \mathfrak{g}_{\Phi,r+1/d}] = q^{d\,n^2} \ \text{ for any } r > 0.$$

More generally, if W is a sub-D-vector space of $M_n(D)$, $\mathfrak{g}_{\Phi,r+1/d} \cap W = P_D\,(\mathfrak{g}_{\Phi,r} \cap W)$. Applying that to $\mathfrak{p}$, we get

$$[\mathfrak{g}_{\Phi,r} \cap \mathfrak{p} : \mathfrak{g}_{\Phi,r+1/d} \cap \mathfrak{p}] = q^{d\,dim_D(\mathfrak{p})} \ \text{ for any } r > 0.$$

This proves the proposition because $[G_{x,r} \cap P : G_{x,r+1/d} \cap P] = [\mathfrak{g}_{\Phi,r} \cap \mathfrak{p} : \mathfrak{g}_{\Phi,r+1/d} \cap \mathfrak{p}]$ for $r > 0$ and $\dim_D(\mathfrak{p}) = n^2 - d_\lambda$. □

We deduce:

Corollary 11.5. *Let P be a parabolic subgroup of G associated to P_λ for $\lambda \in \mathfrak{P}(n)$, and $G_{x,r+j/d}$ a Moy-Prasad subgroup for $x \in \mathcal{BT}, r \in \mathbb{R}, r > 0$ and $j \in \mathbb{N}$. We have for $g \in G$,*

$$|P\backslash PgG_{x,r}/G_{x,r+1/d}| = \frac{[G_{x,r} : G_{x,r+1/d}]}{[(G_{x,r} \cap P) : (G_{x,r+1/d} \cap P)]} = q^{d\,d_\lambda}. \tag{11.3}$$

Clearly, (11.1) follows from (11.3).

Example 11.6. 1) For a vertex x of $\mathcal{BT}$, the Moy-Prasad group $G_{x,0}$ is conjugate to $K_0 = GL_n(O_D)$ and $G_{x,r}$ is conjugate to $K_1 = 1 + p_D M_n(O_D)$ for $0 < r \leq 1/d$. Hence

$$|P_\lambda \backslash G/G_{x,r}| = \begin{cases} |P_\lambda \backslash G/K_0| = 1 \quad \text{if} \quad r = 0, \\ |P_\lambda \backslash G/K_1| = \frac{[n]_{q^d}!}{\prod_k [\lambda_k]_{q^d}!} \quad \text{if} \quad 0 < r \leq 1/d. \end{cases}$$

where $[n]_q! = \frac{q-1}{q-1} \dots \frac{q^n-1}{q-1}$. Indeed $|P_\lambda \backslash G/K_0| = 1$ because $G = P_\lambda K_0$, and $|P_\lambda \backslash G/K_1| = [GL_n(\mathbb{F}_{q^d}) : P_\lambda(\mathbb{F}_{q^d})]$.

2) For the barycenter x of an alcove, $G_{x,0}$ is conjugate to the Iwahori group I, inverse image in K_0 of the upper triangular group of $GL_n(k_D)$, and $G_{x,r}$ is conjugate to the pro-Iwahori group $I_{1/d}$, inverse image of the strictly upper triangular group of $GL_n(k_D)$, for $0 < r \leq 1/d$. Write $\mathfrak{J}$ for the lattice of $(x_{i,j}) \in M_n(O_D)$ with $x_{i,j} \in P_D$ when $i > j$, and $\mathfrak{J}_{1/d}$ for the lattice of $(x_{i,j}) \in M_n(O_D)$ with $x_{i,j} \in P_D$ when $i \geq j$. Then,

$$I = \mathfrak{J}^*, \quad I_{1/d} = 1 + \mathfrak{J}_{1/d} \quad \text{for } 0 < r \leq 1/d.$$

We have $P_\lambda \backslash G/I \simeq P_\lambda \backslash G/I_{1/d} \simeq (S_{\lambda_1} \times \ldots \times S_{\lambda_r}) \backslash S_n$ hence

$$|P_\lambda \backslash G/G_{x,r}| = |P_\lambda \backslash G/I| = |P_\lambda \backslash G/I_{1/d}| = \frac{n!}{\prod_k \lambda_k!}.$$

Remark 11.7. Proposition 11.3 reduces the computation of $|P_\lambda \backslash G/G_{x,r}|$ for $r > 0$ to the case $0 < r < 1/d$. For $g \in G, x \in \mathcal{BT}, r \geq 0$, we have $gG_{x,r}g^{-1} = G_{g(x),r}$; this reduces the computation of $|P_\lambda \backslash G/G_{x,r}|$ for $x \in \mathcal{BT}$ to the case where x belongs to the the closed alcove $\mathcal{A}$ of $\mathcal{BT}$ determined by B.

Theorem 1.3 implies for $\pi \in \mathrm{Rep}_R^{\infty,f}(G)$,

$$\dim_R \pi^{G_{x,r+j/d}} = \sum_{\lambda \in \mathfrak{P}(n)} c_\pi(\lambda) \, |P_\lambda \backslash G/G_{x,r+j/d}|. \tag{11.4}$$

and the integer $c_\pi(\lambda)$ is positive if $d_\lambda = d(\pi)$ then λ is minimal in the support of c_π. Applying (11.1), we deduce Theorem 1.4.

Remark 11.8. (i) The polynomial $P_{\pi,G_{x,r}}(X)$ is determined by those where x is in a closed alcove of $\mathcal{B}T$ and $0 < r < 1/d$ because

$$P_{\pi,G_{x,r+j/d}}(X) = P_{\pi,G_{x,r}}(q^{dj}X) \quad \text{for} \quad 0 < r < 1/d, \ j \in \mathbb{N}.$$

$$P_{\pi,G_{x,r}}(X) = P_{\pi,G_{g(x),r}}(X) \quad \text{for} \quad 0 \leq r, \ g \in G.$$

(ii) For $\pi \in \mathrm{Rep}_R^{\infty,f}(G)$, and any Moy-Prasad pro-p group $G_{x,r}$ of G

$$\dim_R \pi^{G_{x,r+j/d}} \ \sim \ a_{\pi,G_{x,r}} \, q^{d(\pi)dj} \quad \text{when } j \in \mathbb{N} \text{ goes to infinity.}$$

The integer $d(\pi)$ can be called the **Gelfand-Kirillov dimension** of π.

12. $G = GL_2(D)$

In this section we assume that $G = GL_2(D)$, R is a field of characteristic different from p except in §12.5 where its characteristic is p, and we give more details on the polynomial $P_{\pi,K}(X)$ attached to $\pi \in \mathrm{Rep}_R^{\infty,f}(G)$ and a Moy-Prasad subgroup K.

12.1. The Moy-Prasad open compact subgroups of G are conjugate to the open compact subgroups

$$K_0 \supset I_0 \supset I_{1/2} \supset K_1 \supset I_1 \supset I_{3/2} \supset K_2 \supset I_2 \supset \dots,$$

where $K_0 = GL_2(O_D)$, $I_0 = \mathfrak{j}^* = \mathrm{red}^{-1} B(k_D)$ an Iwahori subgroup, $I_{1/2} = 1 + \mathfrak{j}_{1/2} = \mathrm{red}^{-1} U(k_D)$ a pro-p Iwahori subgroup, for $j \in \mathbb{N}$,

$$I_{j+1/2} = 1 + p_D^j \mathfrak{j}_{1/2}, \quad K_{j+1} = 1 + p_D^{j+1} M_2(O_D), \quad I_{j+1} = 1 + p_D^{j+1}\mathfrak{j},$$

where $\mathfrak{j}$ is the lattice of $(x_{i,j}) \in M_2(O_D)$ with $x_{2,1} \in P_D$, $\mathfrak{j}_{1/2}$ is the lattice of $(x_{i,j}) \in \mathfrak{j}$ with $x_{1,1}, x_{2,2} \in P_D$, and $\mathrm{red} : GL_2(O_D) \to GL_2(k_D)$ is the reduction modulo p_D.

The parahoric subgroups of G are conjugate to K_0 and I_0. The Moy-Prasad subgroups of G which are pro-p groups are conjugate of $I_{j+1/2}, K_{j+1}, I_{j+1}$ for $j \in \mathbb{N}$ [9].

To justify the preceding assertions, it is convenient to use of lattice functions Φ from $\mathbb{R}$ to in $D \oplus D$, as in the proof of Proposition 11.3. The lattice function Φ_0 with value $L_0 = O_D \oplus O_D$ at 0 and $P_D L_0$ at s for $0 < s < 1/d$ gives a vertex x_0 in the Bruhat-Tits tree $\mathcal{BT}$ of G, and $G_{x_0,0} = \mathfrak{g}^*_{\Phi_0,0}$ is the stabilizer K_0 of L_0, whereas $G_{x_0,r} = 1 + \mathfrak{g}_{\Phi_0,r}$ for $r > 0$ so that $G_{x_0,r} = K_{j+1}$ if $dr = j + s$ with $0 < s \leq 1$. This gives the groups K_j in the list and accounts for all Moy-Prasad subgroups associated to the vertices of $\mathcal{BT}$.

Any point in $\mathcal{BT}$ is sent by G to a point in the segment with ends x_0 and the vertex x_1 corresponding to $L_1 = O_D \oplus P_D$ so it suffices to look at the Moy-Prasad subgroups $G_{x_\alpha,r}$ when x_α is a barycenter $\alpha x_0 + (1-\alpha)x_1$ with $0 < \alpha < 1$. Since there is an element of G exchanging x_0 and x_1, we need only look at $0 < \alpha < 1/2$ which we now assume. A lattice function Φ_α giving x_α takes value L_0 at 0, L_1 at s if $0 < s \leq \alpha/d$ and $P_D L_0$ if $\alpha/d < s \leq 1/d$. Then $G_{x_\alpha,0}$ is the intersection of the stabilizers of L_0 and L_1, that is I_0.
For $0 < dr \leq \alpha$, $G_{x_\alpha,r+j/d} = I_{j+1/2}$ for any $j \in \mathbb{N}$, as $\mathfrak{g}_{\Phi_\alpha,r}$ is the set of $X \in M_2(D)$ sending L_0 in L_1, and L_1 in $P_D L_0$.
For $\alpha < dr \leq 1-\alpha$ (which cannot happen if $\alpha = 1/2$), $G_{x?,r+j/d} = K_{j+1}$ for any $j \in \mathbb{N}$, as $\mathfrak{g}_{\Phi_\alpha,r}$ is the set of $X \in M_2(D)$ sending L_0 and L_1 in $P_D L_0$.
When $1-\alpha < dr < 1$ we find similarly that $G_{x_\alpha,r+j/d} = I_{j+1}$ for any $j \in \mathbb{N}$.

The indices between two consecutive groups are

$[K : I] = q+1$, $[I : I_{1/2}] = (q-1)^2$, $[I_{1/2} : K_1] = q$, $[K_1 : I_1] = q$, $[I_1 : I_{3/2}] = q^2$, $[I_{3/2} : K_2] = q$,

and so on. Proposition 11.3, Corollary 11.3 and Remark 11.7 give the integers

- $|B\backslash G/K_0| = 1$ as $G = BK_0$,
- $|B\backslash G/I_0| = |B\backslash G/I_{1/2}| = 2$ as $G = BI \sqcup BsI = BI_{1/2} \sqcup BsI_{1/2}$, where s is the antidiagonal matrix with coefficients 1.
- $|B\backslash G/K_1| = (q^{2d}-1)(q^{2d}-q^d)/q^d(q^d-1)^2 = q^d + 1$.
- $|B\backslash G/I_1| = 2q^d$ because $B\backslash G/I_1 = B\backslash BI/I_1 \sqcup B\backslash BsI/I_1$ and
  $B\backslash BI/I_1 \simeq (B \cap I)\backslash I/I_1 \simeq ((B\cap I)/(B\cap I)_1)\backslash(I/I_1)$,
  $B\backslash BsI/I_1 \simeq B^-\backslash G/I_1 \simeq ((B^-\cap I)/(B^-\cap I)_1)\backslash(I/I_1)$,
  $|(I_1 \cap B)\backslash(I \cap B)| = |(I_1 \cap B^-)\backslash(I \cap B^-)| = (q^d-1)^2 q^d$ and $[I : I_1] = (q^d-1)^2 q^{2d}$.

[9]The indices of the preceding section have been multiplied by d

- $|B\backslash G/I_{j+1/2}| = 2q^{dj}$.
- $|B\backslash G/K_{j+1}| = (q^d+1)q^{dj}$.
- $|B\backslash G/I_{j+1}| = 2q^{d(j+1)}$.

12.2. There are only two nilpotent orbits $\{0\}$ and $\mathfrak{O} \neq \{0\}$ corresponding to the partitions (2) and $(1,1)$ of 2. By the germ expansion for $\pi \in \mathrm{Rep}_R^{\infty,f}(G)$ (Theorem 1.3), there exists $a_\pi, b_\pi \in \mathbb{Z}$ and an integer $j_\pi \geq 0$ such that for any integer $j \geq j_\pi$

- $\dim_{\mathbb{C}} \pi^{I_{1/2+j}} = a_\pi + 2\, b_\pi\, q^{dj}$,
- $\dim_{\mathbb{C}} \pi^{K_{1+j}} = a_\pi + (q^d+1)\, b_\pi\, q^{dj}$,
- $\dim_{\mathbb{C}} \pi^{I_{1+j}} = a_\pi + 2q^d\, b_\pi\, q^{dj}$.

12.3. The maps $\pi \mapsto a_\pi$ and $\pi \mapsto b_\pi$ are additive hence determined by their values on irreducible representations. For $\pi \in \mathrm{Rep}_R^\infty(G)$ irreducible,

- $a_\pi = \dim_R \pi,\ b_\pi = 0$ if the dimension of π is finite ($\dim_R \pi = 1$ if R is algebraically closed),
- $b_\pi > 0$ if the dimension of π is infinite.

The dimension of $\sigma \in \mathrm{Rep}_{\mathbb{C}}^{\infty,f}(T)$ is finite and by Theorem 7.1 for $\pi = \mathrm{ind}_B^G \sigma$,

- $a_\pi = 0,\ b_\pi = \dim_R \sigma$.

The R-representation $\mathrm{ind}_B^G 1$ contains the trivial representation 1 of G and the quotient St is called the Steinberg representation. By additivity, $a_1 + a_{\mathrm{St}} = a_{\mathrm{ind}_B^G 1},\ b_1 + b_{\mathrm{St}} = b_{\mathrm{ind}_B^G 1}$ hence

- $a_{\mathrm{St}} = -1,\ b_{\mathrm{St}} = 1$.

For $g \in G$, let $v_D(g)$ be the integer such that $|\,\mathrm{nrd}(g)| = q^{v_D(g)}$.

Proposition 12.1. *The Steinberg R-representation* St *of G is reducible if and only if* St *is indecomposable of length 2, with a cuspidal subrepresentation c*St *and the character $(-1)^{v_D(g)}$ as a quotient, if and only if* $\mathrm{char}_R = \ell > 0$ *divides q^d+1.*

The representation $\mathrm{ind}_B^G 1$ *is indecomposable except when* $\mathrm{char}_R = \ell$ *is odd and divides $q^d - 1$.*

Proof. This is proved in [Vigneras96] if $D = F$, and follows from [Minguez-Sécherre14] in general. We indicate how to get the result using techniques of [Vigneras96]. The restriction of $\mathrm{ind}_B^G(1)$ to B is the direct sum $\mathrm{ind}_B^G 1 = 1 \oplus \tau$ of the trivial representation 1 on the line of constant functions and of the representation τ on the space of functions vanishing at 1, i.e. with support in BsN, isomorphic to the representation of B by conjugation on $C_c^\infty(N;R)$. Integrating such functions on N against a Haar measure (that is taking coinvariants) gives that the modulus δ_B of B is a quotient of τ. Moreover δ_B does not split as a subrepresentation of τ since δ_B is trivial on N and obviously the restriction of τ to N has no trivial subrepresentation. One proves as in ([Bushnell-Henniart06] 8.2) that the corresponding subrepresentation τ^0 of B is irreducible, so τ is indecomposable of length 2 with quotient δ_B.

Thus $\mathrm{ind}_B^G(1)$ has length ≤ 3, and it can have length 3 only if δ_B extends to an R-character G. This latter property is equivalent to $q^{2d} = 1$ in R because δ_B is the inflation of the character $\nu^d \otimes \nu^{-d}$ of T where ν is the character $\nu(x) = |\operatorname{nrd}(x)|$ of D^*. If $\mathrm{char}_R = 0$ or $\mathrm{char}_R = \ell > 0$ not dividing $q^{2d} - 1$, then St is irreducible. Otherwise, δ_B extends to the the character ν^d of G where $\nu(g) = |\operatorname{nrd}(g)|$ for $g \in G$, the contragredient $\operatorname{ind}_B^G(\delta_B) = \nu^d \otimes \operatorname{ind}_B^G 1$ of $\operatorname{ind}_B^G(1)$ has a unique one-dimensional subrepresentation ν^d, which is consequently a quotient of $\operatorname{ind}_B^G(1)$. If ℓ divides $q^d + 1$ but not $q^d - 1$, the character $\nu^d = (-1)^{\mathrm{val}_D}$ is not trivial, then $\operatorname{ind}_B^G(1)$ is indecomposable of length 3 and St_G is indecomposable of length 2 with quotient $(-1)^{\mathrm{val}_D}$.

If ℓ divides $q^d - 1$, δ_B is trivial and $B\backslash G$ admits a G-invariant measure giving volume 0 to $B\backslash G$ if ℓ divides also $q^d + 1$ (which means $\ell = 2$) and 1 otherwise. Integration on $B\backslash G$ implements the duality between $\operatorname{ind}_B^G(1)$ and itself. The integration on $B\backslash G$ is 0 on the constant functions if ℓ divides $q^d + 1$ and the identity otherwise. Therefore if ℓ divides $q^d + 1$, the space of constant functions is isotropic, so its orthogonal has codimension 1, and again $\operatorname{ind}_B^G(1)$ is indecomposable of length 3 and St is indecomposable of length 2 with quotient the trivial representation. But if ℓ does not divides $q^d + 1$, $\operatorname{ind}_B^G(1) = 1 \oplus \mathrm{St}$ and St is irreducible otherwise it would have a cuspidal subquotient which would be contained in $\operatorname{ind}_B^G 1$ (autodual) which is impossible by Frobenius. □

By additivity,

- $a_{c\,\mathrm{St}} = -2,\ b_{c\,\mathrm{St}} = 1.$

When $\mu_{p^\infty} \subset R$, there are two kinds of Whittaker spaces for π: the trivial one, dual of the U-coinvariants π_U of π, and the non-degenerate one, dual of the coinvariants $\pi_{U,\theta}$ of π by a non trivial character θ of U. By Theorem 8.2 we have for π irreducible

- $b_\pi = \dim_R(\pi_{U,\theta})$,

This equality is valid when π has finite length because the θ-coinvariant functor is exact. In particular for $\sigma \in \mathrm{Rep}_{\mathbb{C}}^{\infty,f}(T)$

- $\dim_R(\operatorname{ind}_B^G \sigma)_{U,\theta} = \dim_R \sigma$

12.4. Assume $R = \mathbb{C}$ and $\sigma \in \mathrm{Rep}_{\mathbb{C}}^{\infty}(T)$ irreducible. The normalized parabolic induction $\operatorname{ind}_B^G(\delta_B^{1/2} \otimes \sigma)$ of σ is reducible if and only if $\sigma = \rho \otimes (\chi_\rho \otimes \rho)$ where ρ is an irreducible representation of D^*, and χ_ρ the unramified character of D^* giving the cuspidal segment $\Delta_{\rho,2} = \{\rho, \chi_\rho \otimes \rho\}$ ([Lapid-Minguez-Tadic16] for a proof which does not use the Jacquet-Langlands correspondence). In this case, $\operatorname{ind}_B^G(\delta_B^{1/2} \otimes \sigma)$ is indecomposable of length 2, one irreducible subquotient is the Speh representation $Z(\Delta_{\rho,2})$ and the other subquotient is an essentially square integrable representation $L(\Delta_{\rho,2})$.

The Speh subrepresentation $Z(\Delta_{\rho,2})$ is a character if and only if $\dim_{\mathbb{C}} \rho = 1$. In that case, $L(\Delta_{\rho,2})$ is the twist of the Steinberg representation St by this character. The twist of π by a character does not change the value of the a_π, b_π. Hence

- $a_{L(\Delta_{\rho,2})} = -1,\ b_{L(\Delta_{\rho,2})} = 1$ if $\dim_{\mathbb{C}} \rho = 1$,

by unicity of the Whittaker model as $b_{L(\Delta_{\rho,2})} = \dim_{\mathbb{C}}(L(\Delta_{\rho,2})_{U,\theta} > 0$.

When $D \neq F$, there are irreducible complex representations ρ of D^* of dimension > 1. In that case, the Speh representation $Z(\Delta_{\rho,2})$ is infinite dimensional hence is generic. The essentially square integrable representation $L(\Delta_{\rho,2})$ is also infinite dimensional hence generic; it corresponds by Jacquet-Langlands to an irreducible representation $\pi_{\rho,2}$ of the multiplicative group D_{2d}^* of a central division F-algebra of reduced dimension $2d$. Recalling Corollary 9.5, we have when $\dim_{\mathbb{C}} \rho > 1$:

- $a_{Z(\rho,2)} = -a_{L(\rho,2)} = \dim_{\mathbb{C}} \pi_{\rho,2}$,
- $b_{Z(\rho,2)} + b_{L(\rho,2)} = \dim_{\mathbb{C}} (\operatorname{ind}_B^G \sigma)_{U,\theta} = \dim_{\mathbb{C}} \sigma$,
  $b_{Z(\rho,2)} = \dim_{\mathbb{C}} Z(\Delta_{\rho,2})_{U,\theta} > 0, \ \ b_{L(\rho,2)} = \dim_{\mathbb{C}} L(\Delta_{\rho,2})_{U,\theta} > 0$.

The T-stabilizer of the non-trivial character $\theta(u) = \psi \circ \operatorname{trd}(v)$ for $u = 1 + v$ in U,

$$T_\theta = \{\operatorname{diag}(d, d) \mid d \in D^*\},$$

acts naturally on the θ-coinvariants of a representation of G. How does one identify the two factors of $(\operatorname{ind}_B^G \sigma)_{U,\theta} = Z(\rho, 2)_{U,\theta} \oplus L(\rho, 2)_{U,\theta}$? We shall come back to that question in future work.[10]

When $\pi \in \operatorname{Rep}_{\mathbb{C}}^\infty(G)$ irreducible is not isomorphic to a subquotient of $\operatorname{ind}_B^G \sigma$ for $\sigma \in \operatorname{Rep}_{\mathbb{C}}^\infty(T)$ irreducible, it is called supercuspidal. Its dimension is infinite, it is essentially square integrable and corresponds by Jacquet-Langlands to an irreducible representation π_2 of D_{2d}^*. We have for $\pi \in \operatorname{Rep}_{\mathbb{C}}^\infty(G)$ irreducible supercuspidal (Corollary 9.5):

- $a_\pi = -\dim_{\mathbb{C}} \pi_2, \quad b_\pi = \dim_{\mathbb{C}} (\pi)_{U,\theta} > 0$.

For some supercuspidal representation π, D. Prasad and A. Raghuram computed $\dim_{\mathbb{C}} (\pi)_{U,\theta}$ [Prasad-Raghuram00]. When $D = F$, $b_\pi = 1$ by the unicity of the non-degenerate Whittaker model. The explicit classification of the irreducible cuspidal representations of $GL_2(F)$ or the explicit Jacquet-Langlands correspondence alllows to compute explicitely a_π. The normalized level $\ell(\pi)$ of $\pi \in \operatorname{Rep}_{\mathbb{C}}^\infty(GL_2(F))$ irreducible defined in ([Bushnell-Henniart06] 12.6) is the minimum of two half-integers: the smallest integer j such that $\pi^{K_{j+1}} \neq 0$ and the smallest element $j \in 1/2\mathbb{Z}$ such that $\pi^{I_{j+1/2}} \neq 0$. It is equal to the depth of π defined in [Moy-Prasad96]. Since a_π stays the same if we twist π by a character, we may assume that π is minimal in the sense that $\ell(\pi) \leq \ell(\pi \otimes \chi)$ for any character χ of $GL_2(F)$.

Proposition 12.2. *For $\pi \in \operatorname{Rep}_{\mathbb{C}}^\infty(GL_2(F))$ irreducible cuspidal and minimal, we have $a_\pi = -2q^{\ell(\pi)}$ if $\ell(\pi)$ is an integer, and $a_\pi = -(q+1)q^{\ell(\pi)-1/2}$ otherwise.*

Proof. It is easier to use the Jacquet-Langlands correspondence. We compute $\dim_{\mathbb{C}} \pi_2$, where π_2 is the irreducible smooth representation of D_2^* corresponding to π. The level $\ell(\pi_2)$ of π_2 is the smallest integer j such that π_2 is trivial on $1+P_{D_2}^{j+1}$, and shows that $\ell(\pi_2) = 2\ell(\pi)$ ([Bushnell-Henniart06] 56.1). Since the Jacquet-Langlands correspondence is compatible with character twists, π_2 is minimal. By ([Bushnell-Henniart06] 56.4 Proposition) π_2 is induced from a representation Λ of a subgroup J of D_2^* described in ([Bushnell-Henniart06] 56.5 Lemmas 1 and 2). If $\ell(\pi_2) = 2j + 1$ is odd, then $J = E^*(1 + P_{D_2}^{j+1})$ where E/F is a ramified quadratic extension in the quaternion division algebra D_2, and Λ is a character, so that $\dim_{\mathbb{C}} \pi_2 = (q + 1)q^j$, confirming the second case in the proposition. If $\ell(\pi_2) = 2j$ is a multiple of 4, then $J = E^*(1 + P_{D_2}^{j+1})$ where E/F is now unramified and Λ is again

[10]After this paper was written, we received a paper of S. Nadimpalli and M. Sheth [Nadimpalli-Sheth23] calculating the dimensions of the two factors for certain ρ

a character, so that $\dim_{\mathbb{C}} \pi_2 = 2q^j$. Finally if $\ell(\pi_2) = 2j$ is not a multiple of 4, then J contains $E^*(1+P_{D_2}^{j+1})$ with index q^2, where again E/F is unramified, but Λ has dimension q, so that $\dim_{\mathbb{C}} \pi_2 = 2q.q^{2j}/q^{j-1}q^2 = 2q^j$ as expected. □

Remark 12.3. 1) If π is cuspidal and minimal, and $\pi^{I_j} = 0$ for an integer $j > 0$ then $\pi^{K_j} = 0$, so that the exponent of q in the proposition is the smallest integer such that $\pi^{K_{j+1}} \neq 0$.

2) As pointed out in ([Bushnell-Henniart06] Chapter 13, 56.9: Comments), the Jacquet-Langlands correspondence there is characterized by its compatibility with character twists and preservation of the ϵ-factors. But since the Jacquet-Langlands characterized by equality of characters possesses those properties, both correspondences are the same.

3) Instead of using the Jacquet-Langlands correspondence in the proof we could have used the known fact that the character of π is constant, equal to $-\delta(\pi)/\delta(\mathrm{St}_G)$, on elliptic regular elements close to identity, where δ denotes the formal degree ([Howe74] when $\mathrm{char}_F = 0$, [Bushnell-Henniart-Lemaire10] when $\mathrm{char}_F = p$). The quotient $\delta(\pi)/\delta(\mathrm{St}_G)$ has been computed for $GL_n(F)$ when n is prime in ([Carayol84] Section 5).

12.5. Coefficient field of characteristic p Up to now the characteristic of the coefficient field R was p. But some results may remain true for a field R of characteristic p, for example the dimension of the invariants of an irreducible admissible non-supersingular R-representation of $G = GL_2(D)$, by congruence subgroups of Moy-Prasad subgroups of G (Theorem 1.4).

Let R be a field of characteristic p and $\sigma = \rho \otimes \rho' \in \mathrm{Rep}_R^\infty(T)$ irreducible, $\rho, \rho' \in \mathrm{Rep}_R^\infty D^*$. If the inflation $\tilde{\sigma}$ of σ to B does not extend to G, the parabolically induced representation $\mathrm{ind}_B^G \sigma$ is irreducible. Otherwise, $\rho \simeq \rho'$, $\mathrm{ind}_B^G \sigma$ is indecomposable of length 2, contains the (unique) finite dimensional representation σ_G extending $\tilde{\sigma}$, of quotient $\sigma_G \otimes \mathrm{St}$ where $\mathrm{St} = \mathrm{ind}_B^G 1/1$ is the Steimberg representation. Those are the not supersingular irreducible representations ([AbeHenniartHerzigVignéras17] when R is algebraically closed and [Henniart-Vignéras19] in general).

Lemma 12.4. *When $\tilde{\sigma}$ extends to a representation σ_G of G, we have $\sigma_G = \tau \otimes \mathrm{nrd}_{G/F^*}$, and $\sigma \simeq \rho \otimes \rho$ with $\rho \simeq \tau \otimes \mathrm{nrd}_{D^*/F^*}$ for $\tau \in \mathrm{Rep}_R^\infty F^*$ irreducible.*

Proof. When R is algebraically closed, this follows from Lemma 6.1. In general, let R^{ac}/R be an an algebraic closure. There exists a character $\chi \in \mathrm{Rep}_{R^{ac}}^\infty F^*$ such that $\chi \otimes \mathrm{nrd}_{G/F^*}, \chi \otimes \mathrm{nrd}_{D^*/F^*}$ is a subquotient to $R^{ac} \otimes_R \sigma_G, R^{ac} \otimes_R \rho \simeq R^{ac} \otimes_R \rho'$. Let $\tau \in \mathrm{Rep}_R^\infty F^*$ irreducible such that χ is a subquotient to $R^{ac} \otimes_R \tau$. Then $\sigma_G = \tau \otimes \mathrm{nrd}_{G/F^*}$, $\rho \simeq \rho' \simeq \tau \otimes \mathrm{nrd}_{D^*/F^*}$. □

Proposition 12.5. *Let $\pi \in \mathrm{Rep}_R^\infty(G)$ irreducible not supersingular. For $j \geq 0$, we have*

- $\dim_R \pi^{I_{1/2+j}} = a_\pi + 2\, b_\pi\, q^{dj}$,
- $\dim_R \pi^{K_{1+j}} = a_\pi + (q^d + 1)\, b_\pi\, q^{dj}$,
- $\dim_R \pi^{I_{1+j}} = a_\pi + 2q^d\, b_\pi\, q^{dj}$,

where

$$(a_\pi, b_\pi) = \begin{cases} (0, \dim_R \sigma) & \textit{if } \pi = \operatorname{ind}_B^G \sigma \\ (\dim_R \sigma, 0) & \textit{if } \pi = \sigma_G \\ (-\dim_R \sigma, 1) & \textit{if } \pi = \sigma_G \otimes \mathrm{St} \end{cases},$$

and $\sigma \in \mathrm{Rep}_R^\infty(T^*)$.

Proof. The formulas for a finite dimensional representation and for $\operatorname{ind}_B^G \sigma$ are clear. They imply the formula for the twisted Steinberg representations by the next proposition. ☐

Proposition 12.6. *Let R be a field, and K a Moy-Prasad pro-p subgroup of G. The natural map $(\operatorname{ind}_B^G 1)^K \to \mathrm{St}^K$ is surjective.*

Proof. When $\mathrm{char}_R \neq p$, the K-invariant functor is exact and the surjectivity is clear. When $\mathrm{char}_R = p$, one can argue as follows.

The image of $f \in \operatorname{ind}_B^G 1$ in St is K-invariant if and only if there exists a map $c_f : K \to R$ such that $f(gk) = f(g) + c_f(k)$ for any $g \in G, k \in K$. As $f(gkk') = f(g) + c_f(kk') = f(gk) + c_f(k') = f(g) + c_f(k) + c_f(k')$ for $k, k' \in K$, the map c_f is an homomorphism. For $k \in K \cap B$ we have $f(k) = f(1)$ hence $c_f(k) = 0$. For $k \in K \cap sBs$ we have $f(sk) = f(skss) = f(s)$ because $sks \in B$, hence $c_f(k) = 0$. As $K \cap B$ and $K \cap sBs$ generate K, we deduce that $c_f = 0$. ☐

Let $G = GL(2, \mathbb{Q}_p)$ and $\pi \in \mathrm{Rep}_{\mathbb{F}_p^{ac}}(G)$ irreducible supersingular. By ([Morra13] Proposition 4.9, Corollary 4.15), for p odd and $j \geq 0$, we have:

- $\dim_{\mathbb{C}} \pi^{I_{1/2+j}} = a_\pi + 2b_\pi\, p^j$, where $(a_\pi, b_\pi) = (-2, 2)$,
- $\dim_{\mathbb{C}} \pi^{K_{1+j}} = a'_\pi + (p+1)\, b_\pi\, p^j$, where

$$a'_\pi = \begin{cases} -3 & \text{if } \pi = \pi_0 \otimes (\chi \circ \det) \text{ for a character } \chi \in \mathrm{Rep}^\infty_{\mathbb{F}_p^{ac}} F^* \\ -4 & \text{otherwise} \end{cases}.$$

Here π_0 denote the supersingular irreducible quotient of $\mathbb{F}_p^{ac}[GL(2, \mathbb{Z}_p) Z \backslash G]$, Z the center of G.

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