PARABOLIC INDUCTION IN CHARACTERISTIC p

RACHEL OLLIVIER AND MARIE-FRANCE VIGNÉRAS

ABSTRACT. Let F (resp. $\mathbb{F}$) be a nonarchimedean locally compact field with residue characteristic p (resp. a finite field with characteristic p). For $k =$ F or $k = \mathbb{F}$, let $\mathbf{G}$ be a connected reductive group over k and R be a commutative ring. We denote by $\mathrm{Rep}(\mathbf{G}(k))$ the category of smooth R-representations of $\mathbf{G}(k)$. To a parabolic k-subgroup $\mathbf{P} = \mathbf{MN}$ of $\mathbf{G}$ corresponds the parabolic induction functor $\mathrm{Ind}_{\mathbf{P}(k)}^{\mathbf{G}(k)} : \mathrm{Rep}(\mathbf{M}(k)) \to \mathrm{Rep}(\mathbf{G}(k))$. This functor has a left and a right adjoint.

Let $\mathcal{U}$ (resp. $\mathbb{U}$) be a pro-p Iwahori (resp. a p-Sylow) subgroup of $\mathbf{G}(k)$ compatible with $\mathbf{P}(k)$ when $k =$ F (resp. $\mathbb{F}$). Let $H_{\mathbf{G}(k)}$ denote the pro-p Iwahori (resp. unipotent) Hecke algebra of $\mathbf{G}(k)$ over R and $\mathrm{Mod}(H_{\mathbf{G}(k)})$ the category of right modules over $H_{\mathbf{G}(k)}$. There is a functor $\mathrm{Ind}_{H_{\mathbf{M}(k)}}^{H_{\mathbf{G}(k)}} : \mathrm{Mod}(H_{\mathbf{M}(k)}) \to \mathrm{Mod}(H_{\mathbf{G}(k)})$ called parabolic induction for Hecke modules; it has a left and a right adjoint.

We prove that the pro-p Iwahori (resp. unipotent) invariant functors commute with the parabolic induction functors, namely that $\mathrm{Ind}_{\mathbf{P}(k)}^{\mathbf{G}(k)}$ and $\mathrm{Ind}_{H_{\mathbf{M}(k)}}^{H_{\mathbf{G}(k)}}$ form a commutative diagram with the $\mathcal{U}$ and $\mathcal{U} \cap \mathbf{M}(\mathrm{F})$ (resp. $\mathbb{U}$ and $\mathbb{U} \cap \mathbf{M}(\mathbb{F})$) invariant functors. We prove that the pro-p Iwahori (resp. unipotent) invariant functors also commute with the right adjoints of the parabolic induction functors. However, they do not commute with the left adjoints of the parabolic induction functors in general; they do if p is invertible in R.

When R is an algebraically closed field of characteristic p, we show that an irreducible admissible R-representation of $\mathbf{G}(\mathrm{F})$ is supercuspidal (or equivalently supersingular) if and only if the $H_{\mathbf{G}(\mathrm{F})}$-module $\mathfrak{m}$ of its $\mathcal{U}$-invariants admits a supersingular subquotient, if and only if $\mathfrak{m}$ is supersingular.

CONTENTS

1. INTRODUCTION

1.1. Let F be a nonarchimedean locally compact field with residue characteristic p, let **G** be a connected reductive group over F. We set G := **G**(F) and call it a p-adic reductive group. Let R be a commutative ring. We denote by Rep(G) the category of smooth R-representations of G. When R is a field, we denote by Irr(G) the set of irreducible admissible R-representations of G modulo isomorphism. For A a ring we denote by Mod(A) the category of right A-modules.

When R is the complex field $\mathbb{C}$, the category Rep(G) has been studied because of its connection to the theory of automorphic forms. The classification of Irr(G) (known as the conjectures of Langlands and Arthur) is now understood for many groups but not yet for a general p-adic reductive group G.

For an algebraically closed field R of characteristic p, the description of Irr(G) is reduced to the classification of the irreducible admissible supercuspidal representations ([4]), using the parabolic induction and a process of reduction which is similar to the one established for complex representations of GL(n, F) by Bernstein and Zelevinski ([9], [67]).

The study of the smooth R-representations of G (called mod ℓ representations if the characteristic of R is a prime number ℓ) for a general R, is motivated by the congruences between automorphic forms and by number theory. The theory of Harish-Chandra to study complex representations cannot be extended to the case when R is a field of characteristic p because G does not admit a Haar measure with values in such a field. However, certain tools such as the parabolic induction or the Hecke algebras of types remain available for any R. Note that when R is a field of characteristic p, there is only one natural type, namely the trivial R-representation of a pro-p Iwahori subgroup $\mathcal{U}$ of G. This is because any non zero smooth R-representation of G admits a non zero invariant vector by any pro-p subgroup of G.

In this article, we compare parabolic induction functors (and their adjoints) for representations of G and for modules of the pro-p Iwahori Hecke algebra of G. We work over an arbitrary ring R, and emphasize how things depend on whether p is invertible in R. See Theorem 1 in this introduction.

We also provide the analogous statements in the case of a finite reductive group $\mathbb{G}$ and the corresponding unipotent Hecke algebra (Theorem 2). In that case and in characteristic zero, induction and restriction operators are biadjoint to each other – a structure which leads to an action of a Heisenberg algebra ([35]). On the other hand, in characteristic $\ell \neq p$, these operators can be used in the proof of Broué's abelian defect conjecture ([23]). In characteristic p, induction and restriction are no longer biadjoint to each other and things are more complicated. For this reason we work out in some detail the relationship between these functors for $\mathbb{G}$. We also hope that the clarification of the finite case will shed some light on the induction-restriction functors for the p-adic reductive group G. A summary table is provided in Section 6.

In the case of the p-adic reductive group G and when R is an algebraically closed field of characteristic p, recall that the irreducible admissible supercuspidal representations of G are still mysterious in general.

However, we describe their $\mathcal{U}$-invariant subspace as pro-p Iwahori Hecke modules: this result, the proof of which uses Theorem 1, is given in Theorem 3 in this introduction.

We now describe our results in more detail.

1.2. We fix a minimal parabolic subgroup $\mathbf{B}$ of $\mathbf{G}$ with a Levi decomposition $\mathbf{B} = \mathbf{ZU}$ of Levi subgroup $\mathbf{Z}$ and unipotent radical $\mathbf{U}$ and a compatible Iwahori subgroup of G of pro-p Sylow $\mathcal{U}$. Let $R[\mathcal{U}\backslash\mathrm{G}]$ the natural smooth representation of G on the space of left $\mathcal{U}$-invariant functions $\mathrm{G} \to R$ with compact support. Let $\mathcal{H}$ be the algebra of $R[\mathrm{G}]$-intertwiners of $R[\mathcal{U}\backslash\mathrm{G}]$. It is the pro-$p$ Iwahori Hecke R-algebra of G. We denote by $\mathrm{Mod}(\mathcal{H})$ the category of right $\mathcal{H}$-modules and when R is a field, by $\mathrm{Irr}(\mathcal{H})$ the set of simple right $\mathcal{H}$-modules modulo isomorphism. The subspace $V^{\mathcal{U}}$ of $\mathcal{U}$-invariant vectors in $V \in \mathrm{Rep}(\mathrm{G})$ is a right $\mathcal{H}$-module and we consider the functor $(-)^{\mathcal{U}} : V \mapsto V^{\mathcal{U}} = \mathrm{Hom}_{\mathrm{G}}(R[\mathcal{U}\backslash\mathrm{G}], V)$ from $\mathrm{Rep}(\mathrm{G})$ to $\mathrm{Mod}(\mathcal{H})$. We call it the $\mathcal{U}$-invariant functor. It has a left adjoint given by the tensor product functor $- \otimes_{\mathcal{H}} R[\mathcal{U}\backslash\mathrm{G}]$.

When R is a field, the $\mathcal{U}$-invariant functor induces a bijection from the set of $V \in \mathrm{Irr}(\mathrm{G})$ with $V^{\mathcal{U}} \neq 0$ onto $\mathrm{Irr}(\mathcal{H})$ if the characteristic of R is $\ell \neq p$ ([54, I.6.3] applied to the global Hecke R-algebra of G which is the algebra of compact and locally constant functions $\mathrm{G} \to R$ with the convolution product given by a left Haar measure on G with values in R) or if $\mathrm{G} = \mathrm{GL}(2, \mathbb{Q}_p)$ ([13], [56]). When R is a field of characteristic p, the $\mathcal{U}$-invariant functor does not always send a simple object in $\mathrm{Rep}(\mathrm{G})$ onto a simple object in $\mathrm{Mod}(\mathcal{H})$ (counter examples are built in [14] in the case of $\mathrm{G} = \mathrm{GL}(2, \mathrm{F})$ when F is a strict unramified extension of $\mathbb{Q}_p$).

We refer to this setup as *the p-adic case.*

1.3. Replace F by a finite field $\mathbb{F}$ of characteristic p, the p-adic reductive group G by the finite reductive group $\mathbb{G} = \mathbf{G}(\mathbb{F})$, the pro-$p$ Iwahori group $\mathcal{U}$ by the unipotent radical $\mathbb{U}$ of $\mathbb{B} = \mathbb{B}(\mathbb{F})$, and the pro-$p$ Iwahori Hecke R-algebra by the algebra H of all $\mathbb{G}$-intertwiners of $R[\mathbb{U}\backslash\mathbb{G}]$ (called the unipotent Hecke R-algebra). We consider the $\mathbb{U}$-invariant functor $(-)^{\mathbb{U}} : V \mapsto V^{\mathbb{U}} = \mathrm{Hom}_{\mathbb{G}}(R[\mathbb{U}\backslash\mathbb{G}], V)$ from $\mathrm{Rep}(\mathbb{G})$ to $\mathrm{Mod}(\mathrm{H})$. Its left adjoint is the tensor product functor $- \otimes_{\mathrm{H}} R[\mathbb{U}\backslash\mathbb{G}]$.

Suppose that R is a field and denote by $\mathrm{Irr}(\mathbb{G})$ the set of irreducible R-representations of $\mathbb{G}$ modulo isomorphism. If R has characteristic different from p, the $\mathbb{U}$-invariant functor $(-)^{\mathbb{U}}$ induces a bijection between the set of $V \in \mathrm{Irr}(\mathbb{G})$ with $V^{\mathbb{U}} \neq 0$ and the set $\mathrm{Irr}(\mathrm{H})$ ([54, I.6.3] applied to the group algebra $R[\mathbb{G}]$). When R is a field of characteristic p, under certain mild hypotheses on R (see for example[20, Thms. 6.10 and 6.12]), it yields a bijection between $\mathrm{Irr}(\mathbb{G})$ and $\mathrm{Irr}(\mathrm{H})$.

We refer to this setup as *the finite case.*

1.4. We recall the definition of the functor of parabolic induction in the p-adic case. Consider a parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$ containing $\mathbf{B}$ with Levi decomposition $\mathbf{P} = \mathbf{MN}$ where $\mathbf{Z} \subset \mathbf{M}$. We set $\mathrm{P} := \mathbf{P}(\mathrm{F}), \mathrm{M} := \mathbf{M}(\mathrm{F})$ and $\mathrm{N} := \mathbf{N}(\mathrm{F})$. The parabolic induction functor $\mathrm{Ind}_{\mathrm{P}}^{\mathrm{G}} : \mathrm{Rep}(\mathrm{M}) \to \mathrm{Rep}(\mathrm{G})$ sends $W \in \mathrm{Rep}(\mathrm{M})$ to the representation of G by right translation on the R-module $\{f : \mathrm{G} \to W \mid f(mnxk) = mf(x) \,\forall\, (m, n, x, k) \in (\mathrm{M}, \mathrm{N}, \mathrm{G}, K_f)\}$ where K_f is some open subgroup of G depending on f. When R is a field, the parabolic induction respects admissibility because $\mathrm{P}\backslash\mathrm{G}$ is compact; an admissible irreducible

R-representation of G is called supercuspidal when, for any $P \neq G$ and any admissible $W \in \mathrm{Irr}(M)$, it is not a subquotient of $\mathrm{Ind}_P^G W$.

The functor Ind_P^G is faithful. It has a left adjoint given by the N-coinvariant functor $\mathbf{L}_P^G = (-)_N$. It also has a right adjoint denoted by $\mathbf{R}_P^G$ [61]. The functor $\mathbf{R}_P^G$ has been computed by Casselman and Bernstein when $R = \mathbb{C}$ and by Dat when p is invertible in R for many groups G but not yet for a general p-adic reductive group G ([25]). When R is a field of characteristic p, the parabolic induction Ind_P^G is fully faithful and the right adjoint $\mathbf{R}_P^G$ is equal to Emerton's functor on admissible representations ([61]) because $\mathbf{R}_P^G$ respects admissibility ([5]).

Before stating our first result, we recall that $\mathcal{U}_M := \mathcal{U} \cap M$ is a pro-p Iwahori subgroup of M compatible with $\mathbf{B_M} = \mathbf{B} \cap \mathbf{M}$. There exists a functor $\mathrm{Ind}_{\mathcal{H}_M}^{\mathcal{H}} : \mathrm{Mod}_{\mathcal{H}_M} \to \mathrm{Mod}_{\mathcal{H}}$ defined in [66] (in the current article, we refer to §4.2 for precise statements and references). We call it parabolic induction for Hecke modules. It has a left adjoint $\mathrm{L}_{\mathcal{H}_M}^{\mathcal{H}}$ and a right adjoint $\mathrm{R}_{\mathcal{H}_M}^{\mathcal{H}}$.

Theorem 1. *The functor* $\mathrm{Ind}_{\mathcal{H}_M}^{\mathcal{H}} : \mathrm{Mod}_{\mathcal{H}_M} \to \mathrm{Mod}_{\mathcal{H}}$ *and its adjoints satisfy:*

$$(-)^{\mathcal{U}} \circ \mathrm{Ind}_P^G \cong \mathrm{Ind}_{\mathcal{H}_M}^{\mathcal{H}} \circ (-)^{\mathcal{U}_M}, \quad (-)^{\mathcal{U}_M} \circ \mathbf{R}_P^G \cong \mathrm{R}_{\mathcal{H}_M}^{\mathcal{H}} \circ (-)^{\mathcal{U}}.$$

When p is invertible in R, we also have $(-)^{\mathcal{U}_M} \circ \mathbf{L}_P^G \cong \mathrm{L}_{\mathcal{H}_M}^{\mathcal{H}} \circ (-)^{\mathcal{U}}$. *But in general there is no functor* $F : \mathrm{Mod}_{\mathcal{H}_M} \to \mathrm{Mod}_{\mathcal{H}}$ *such that* $(-)^{\mathcal{U}_M} \circ \mathbf{L}_P^G \cong F \circ (-)^{\mathcal{U}}$.

The theorem is proved in this article by answering Questions 4 and 5 and 6 in §4.3.4.

In the finite case, we let $\mathbb{P} := \mathbf{P}(\mathbb{F}), \mathbb{M} := \mathbf{M}(\mathbb{F}), \mathbb{N} := \mathbf{N}(\mathbb{F})$ and recall the definition of the parabolic induction functor $\mathrm{Ind}_{\mathbb{P}}^{\mathbb{G}} : \mathrm{Rep}(\mathbb{M}) \to \mathrm{Rep}(\mathbb{G})$. It sends $W \in \mathrm{Rep}(\mathbb{M})$ to the representation of $\mathbb{G}$ by right translation on the R-module $\{f : \mathbb{G} \to W \mid f(mnx) = mf(x) \, \forall (m, n, x) \in (\mathbb{M}, \mathbb{N}, \mathbb{G})\}$. When R is a field, a representation $V \in \mathrm{Irr}(\mathbb{G})$ is finite dimensional and supercuspidality is defined as in the p-adic case. The functor $\mathrm{Ind}_{\mathbb{P}}^{\mathbb{G}}$ is faithful with left adjoint the $\mathbb{N}$-coinvariant functor $\mathbf{L}_{\mathbb{P}}^{\mathbb{G}} := (-)_{\mathbb{N}}$ and right adjoint the $\mathbb{N}$-invariant functor $\mathbf{R}_{\mathbb{P}}^{\mathbb{G}} : (-)^{\mathbb{N}}$ (see §3.1). The parabolic induction for Hecke modules is defined to be the functor $\mathrm{Ind}_{H_{\mathbb{M}}}^{H} = - \otimes_{H_{\mathbb{M}}} H : \mathrm{Mod}_{H_{\mathbb{M}}} \to \mathrm{Mod}_H$. Its right adjoint $\mathrm{R}_{H_{\mathbb{M}}}^{H}$ is the natural restriction functor $\mathrm{Mod}(H) \to \mathrm{Mod}(H_{\mathbb{M}})$. Its left adjoint is computed in Prop. 3.4 (see also Remark 3.6). We prove an analog of Theorem 1:

Theorem 2. *The functor* $\mathrm{Ind}_{H_{\mathbb{M}}}^{H} : \mathrm{Mod}_{H_{\mathbb{M}}} \to \mathrm{Mod}_H$ *and its adjoints satisfy:*

$$(-)^{\mathbb{U}} \circ \mathrm{Ind}_{\mathbb{P}}^{\mathbb{G}} \cong \mathrm{Ind}_{H_{\mathbb{M}}}^{H} \circ (-)^{\mathbb{U}_{\mathbb{M}}}, \quad (-)^{\mathbb{U}_{\mathbb{M}}} \circ \mathbf{R}_{\mathbb{P}}^{\mathbb{G}} \cong \mathrm{R}_{H_{\mathbb{M}}}^{H} \circ (-)^{\mathbb{U}}.$$

When p is invertible in R, we also have $(-)^{\mathbb{U}_{\mathbb{M}}} \circ \mathbf{L}_{\mathbb{P}}^{\mathbb{G}} \cong \mathrm{L}_{H_{\mathbb{M}}}^{H} \circ (-)^{\mathbb{U}}$. *But in general the functors* $(-)^{\mathbb{U}_{\mathbb{M}}} \circ \mathbf{L}_{\mathbb{P}}^{\mathbb{G}}$ *and* $\mathrm{L}_{H_{\mathbb{M}}}^{H} \circ (-)^{\mathbb{U}}$ *are not isomorphic.*

The theorem is proved by answering Questions 1, 2 and 3 in §3.3.4.

1.5. A table comparing the parabolic induction functors

$$\mathrm{Ind}_{\mathcal{H}_M}^{\mathcal{H}} : \mathrm{Mod}(\mathcal{H}_M) \to \mathrm{Mod}(\mathcal{H}) \text{ and } \mathrm{Ind}_{H_{\mathbb{M}}}^{H} : \mathrm{Mod}(H_{\mathbb{M}}) \to \mathrm{Mod}(H)$$

and their respective left and right adjoints $\mathrm{L}_{\mathcal{H}_M}^{\mathcal{H}}$, $\mathrm{R}_{\mathcal{H}_M}^{\mathcal{H}}$, $\mathrm{L}_{H_{\mathbb{M}}}^{H}$ and $\mathrm{R}_{H_{\mathbb{M}}}^{H}$ (when R is an arbitrary ring and when p is invertible in R) is provided in Section 6.

1.6. In the p-adic case, let $\mathbf{T}$ be the maximal split central subtorus of $\mathbf{Z}$. Let $\mathrm{T} := \mathbf{T}(\mathrm{F})$ and denote by $\mathrm{X}^+_*(\mathbf{T})$ the monoid of dominant cocharacters of $\mathbf{T}$ with respect to $\mathbf{B}$. The algebra $\mathcal{H}$ is a finitely generated module over its center $\mathcal{Z}$ and $\mathcal{Z}$ is a finitely generated module over a subalgebra $\mathcal{Z}_{\mathrm{T}}$ isomorphic to $R[\mathrm{X}^+_*(\mathbf{T})]$ ([42] when $\mathbf{G}$ is F-split; generalized by [63]).

We suppose that R is an algebraically closed field of characteristic p. The supercuspidality of $V \in \mathrm{Irr}(\mathrm{G})$ can be seen on the action of $\mathcal{Z}$ on $V^{\mathcal{U}}$. A $\mathcal{Z}$-module $\mathcal{M}$ is called supersingular if for all $v \in \mathcal{M}$ and all non invertible $x \in \mathrm{X}^+_*(\mathbf{T})$, corresponding to $z_x \in \mathcal{Z}_{\mathrm{T}}$, there exists a positive integer n such that $z_x^n v = 0$. A right $\mathcal{H}$-module is called supersingular when it is supersingular as a $\mathcal{Z}$-module. Abe described $\mathrm{Irr}(\mathcal{H})$ via parabolic induction from supersingular simple modules ([1]). When $\mathrm{G} = \mathrm{Z}$, all finite dimensional $\mathcal{H}$-modules are supersingular and all irreducible admissible R-representation of G are supercuspidal. We establish the following:

Theorem 3. *Suppose that R is an algebraically closed field of characteristic p. Let V be an irreducible admissible R-representation of* G. *Then the following are equivalent:*

(1) *V is supercuspidal.*
(2) *the finite dimensional $\mathcal{H}$-module $V^{\mathcal{U}}$ is supersingular.*
(3) *the finite dimensional $\mathcal{H}$-module $V^{\mathcal{U}}$ admits a supersingular subquotient.*

We choose a special parahoric subgroup $\mathcal{K}$ of pro-p Sylow $\mathcal{U}$. For an irreducible smooth representation W of $\mathcal{K}$, the center $\mathcal{Z}(W)$ of the algebra of intertwiners of the compactly induced representation $\mathrm{ind}^{\mathrm{G}}_{\mathcal{K}}(W)$ and the center of $\mathcal{H}$ have a similar structure, and there is a similar notion of a supersingular $\mathcal{Z}(W)$-module. A representation $V \in \mathrm{Irr}(\mathrm{G})$ is supersingular when the $\mathcal{Z}(W)$-module $\mathrm{Hom}_{\mathcal{K}}(W, V)$ is non zero and supersingular for some W. A corollary of the description of $\mathrm{Irr}(\mathrm{G})$ via parabolic induction from supersingular irreducible admissible representations [4], is that for $V \in \mathrm{Irr}(\mathrm{G})$, supercuspidality is equivalent to supersingularity. If $V^{\mathcal{U}}$ is supersingular, then V is supersingular (see Remark 5.4). The converse $(1) \Rightarrow (2)$ follows from the commutativity of Diagram (4.12) (which is contained in Theorem 1 by passing to left adjoints in the identity involving $\mathrm{R}^{\mathrm{G}}_{\mathrm{P}}$ and $(-)^{\mathcal{U}}$) and from the computation of $\mathcal{M} \otimes_{\mathcal{H}} R[\mathcal{U}\backslash\mathrm{G}]$ for a simple $\mathcal{H}$-module $\mathcal{M}$ done in [6, Cor. 5.13]. Lastly, the argument to prove $(3) \Rightarrow (1)$ follows from the computation of $V^{\mathcal{U}}$ done in [6, Thm. 4.17]. Our first proof did not use [6] and started with a reduction to the case where G is almost simple, simply connected and isotropic. But this allowed us to prove $(3) \Rightarrow (1)$ only when the index $[\mathrm{G} : \mathrm{G}^{der}C]$ is finite, where $\mathbf{G}^{der}$ and $\mathbf{C}$ are the derived group and the center of $\mathbf{G}$, $\mathrm{G}^{der} = \mathbf{G}^{der}(\mathrm{F})$ and $C = \mathbf{G}(\mathrm{F})$.

2. Notation and Preliminaries

Let F be a nonarchimedean locally compact field with residue characteristic p. The ring of integers of F is denoted by O_F. The residue field of F is a finite field $\mathbb{F}_q$ with q elements, where q is a power of the prime number p. We choose a uniformizer ϖ and denote by val_F the valuation on F normalized by $val_F(\varpi) = 1$.

We introduce the following objects and notation, keeping in mind that throughout the article, the group of F-points of an algebraic F-group $\mathbf{Y}$ will be denoted by Y.

- $\mathbf{G}$ a connected reductive group over F;
- $\mathbf{T}$ a maximal F-split torus of $\mathbf{G}$, of $\mathbf{G}$-centralizer $\mathbf{Z}$ and $\mathbf{G}$-normalizer $\mathbf{N_G}$;
- Φ the relative root system of $\mathbf{T}$ in $\mathbf{G}$. To $\alpha \in \Phi$ is associated the root subgroup $\mathbf{U}_\alpha$. We have $\mathbf{U}_{2\alpha} \subset \mathbf{U}_\alpha$ if $\alpha, 2\alpha \in \Phi$.
- $\mathbf{B}$ is a minimal parabolic F-subgroup of $\mathbf{G}$ with Levi decomposition $\mathbf{B} = \mathbf{ZU}$; we have B = ZU;
- $\varphi = (\varphi_\alpha)_{\alpha \in \Phi}$ a valuation of the root datum $(Z, (U_\alpha)_{\alpha \in \Phi})$ of G of type Φ generating G [15, Def. (6.2.1)] which is compatible with the valuation val_F [16, (5.1.22) formula (2)], discrete [15, Def. (6.2.21)] and special [15, (6.2.13) Def.].

To $(\mathbf{T}, \mathbf{B}, \varphi)$ is associated a reduced root system Σ with a basis Δ, an apartment $\mathscr{A}$ in the semisimple building of G, a special vertex $x_0 \in \mathscr{A}$ [53, 1.9], and an alcove $\mathscr{C} \subset \mathscr{A}$ of vertex x_0. The fixer of x_0, resp. $\mathscr{C}$, in the kernel of the Kottwitz homomorphism κ_G [36, 7.1 to 7.4], is a special maximal parahoric group $\mathcal{K}$ of G of pro-unipotent radical $\mathcal{K}^1$, resp. an Iwahori subgroup $\mathcal{B}$ of unique pro-p Sylow subgroup $\mathcal{U}$.

2.1. Root system and Weyl groups attached to G.

2.1.1. *Relative root system.* The relative root system Φ of T in G is related to Σ by a map $e : \Phi \to \mathbb{N}_{>0}$ such that $\Sigma = \{e(\beta)\beta \mid \beta \in \Phi\}$. For a root $\beta \in \Phi$ such that $\beta/2$ does not belong to Φ, the image of the homomorphism $U_\beta - \{1\} \xrightarrow{\varphi_\beta} \mathbb{Q}$ given by the valuation φ is $e(\beta)^{-1}\mathbb{Z}$ [62, §3.5 (39)]. When $\mathbf{G}$ is F-split, $\Phi = \Sigma$ is a reduced root system and $e(\beta) = 1$ for all $\beta \in \Phi$. The map $\beta \mapsto e(\beta)\beta$ induces a bijection from the set Φ_{red} of reduced roots of Φ onto Σ, and from Δ onto a basis Π of Φ.

To $\alpha \in \Sigma$ we associate the group $U_\alpha := U_\beta$ where β is the unique reduced root of Φ such that $\alpha = e(\beta)\beta$. To an affine root $(\alpha, r) \in \Sigma_{aff} := \Sigma \times \mathbb{Z}$ corresponds a compact open subgroup

$$\mathcal{U}_{(\alpha,r)} = \mathcal{U}_{(\beta, e(\beta)^{-1}r)} = \{1\} \cup \{x \in U_\beta - \{1\} \mid \varphi_\beta(x) \geq e(\beta)^{-1}r\}.$$

We identify Σ with the subset $\Sigma \times \{0\}$ of Σ_{aff}. In particular, we write $\mathcal{U}_\alpha$ instead of $\mathcal{U}_{\alpha,0}$.

2.1.2. *Iwahori decompositions.* We have the inclusions $\mathcal{K}^1 \subset \mathcal{U} \subset \mathcal{B} \subset \mathcal{K}$. These groups are generated by their intersections with, respectively, Z, U, and $U^{op} := \mathbf{U}^{op}(F)$ where $\mathbf{B}^{op} = \mathbf{ZU}^{op}$ is the opposite parabolic subgroup of $\mathbf{B}$. We describe these intersections:

- $Z \cap \mathcal{K}^1 = Z \cap \mathcal{U} = Z^1$ is the unique pro-p Sylow subgroup of the unique parahoric sugroup $Z^0 := \operatorname{Ker} \kappa_Z$ of Z. We have $Z \cap \mathcal{K} = Z \cap \mathcal{B} = Z^0$. The intersection $T^0 := T \cap Z^0$, resp. $T^1 := T \cap Z^1$, is the maximal compact subgroup, resp. the pro-p Sylow subgroup of T^0. The group Z^0 is an open subgroup of the maximal compact subgroup of Z.

- $U \cap \mathcal{K}^1$ (resp. $\mathcal{U}_0 := U \cap \mathcal{U} = U \cap \mathcal{B} = U \cap \mathcal{K}$) is the image of $\prod_{\alpha \in \Sigma^+} \mathcal{U}_{(\alpha,1)}$ (resp. $\prod_{\alpha \in \Sigma^+} \mathcal{U}_\alpha$) by the multiplication map.
- $U^{op} \cap \mathcal{K}^1 = U^{op} \cap \mathcal{U} = U^{op} \cap \mathcal{B}$ (resp. $\mathcal{U}_0^{op} := U^{op} \cap \mathcal{K}$) is the image of $\prod_{\alpha \in \Sigma^-} \mathcal{U}_{(\alpha,1)}$ (resp. $\prod_{\alpha \in \Sigma^-} \mathcal{U}_\alpha$) by the multiplication map.

The products above are ordered in some arbitrary chosen way, $\Sigma^+ \subset \Sigma$ is the subset of positive roots and $\Sigma^- = -\Sigma^+$. The groups $\mathcal{K}^1 \subset \mathcal{U} \subset \mathcal{B}$ (but not $\mathcal{K}$) have an Iwahori decomposition: they are the product (in any order) of their intersections with U^{op}, Z, U. For the pro-p Iwahori subgroup $\mathcal{U} \subset G$ this means that the product map

$$\tag{2.1} \prod_{\alpha \in \Sigma^-} \mathcal{U}_{(\alpha,1)} \times Z^1 \times \prod_{\alpha \in \Sigma^+} \mathcal{U}_\alpha \xrightarrow{\sim} \mathcal{U}$$

is a bijection.

The subgroups Z^0 and Z^1 of Z are normalized by $\mathcal{N}_G := \mathbf{N}_{\mathbf{G}}(F)$. The quotients $W_0 := \mathcal{N}_G/Z$, $W := \mathcal{N}_G/Z^0$, and $W(1) := \mathcal{N}_G/Z^1$ of $\mathcal{N}_G$ by Z, Z^0 and Z^1 respectively, are the finite, Iwahori, pro-p Iwahori Weyl groups.

The finite Weyl group W_0, the Weyl group of the root system Φ and the Weyl group of the reduced root system Σ are canonically isomorphic. The subgroup $\Lambda := Z/Z^0$ of W is commutative and finitely generated, the subgroup $\Lambda(1) = Z/Z^1$ of $W(1)$ is finitely generated but not always commutative. The subgroup $\Lambda_T := T/T^0$ of Λ has finite index and identifies with the group $X_*(\mathbf{T})$ of algebraic cocharacters of $\mathbf{T}$.

Notation 2.1. *Let G' (resp. G'_α for $\alpha \in \Phi$) denote the subgroup of G generated by U, U^{op} (resp. $U_\alpha, U_{-\alpha}$). It is normal in G and $G = ZG'$. The group $G_{aff} = Z^0 G'$ is generated by all the parahoric subgroups of G. For any subset $X \subset G$ we put $X' := X \cap G'$.*

The quotient $W' := \mathcal{N}_G'/(Z^0)' = (\mathcal{N}_G \cap G_{aff})/Z^0 \subset W$ is called the affine Iwahori Weyl group (Notation 2.1). The quotient $_1W' := \mathcal{N}_G'/(Z^1)' \subset W(1)$ is the pro-p-affine Iwahori Weyl group. It is contained in the inverse image $W'(1) = (\mathcal{N}_G \cap G_{aff})/Z^1$ of W' in $W(1)$, and is usually different.

Notation 2.2. *For a subset $X \subset W$ we denote by $X(1)$ its preimage in $W(1)$.*

2.1.3. *Lifts of the elements in the (pro-p) Iwahori Weyl group.* For $w \in W$, we will sometimes introduce a lift $\tilde{w} \in W(1)$ for w. For w in W or $W(1)$, we will sometimes introduce a lift $\hat{w} \in \mathcal{N}_G$ for w. Given $w \in W$, the notation $\hat{\tilde{w}}$ will correspond to the choice of an element in $\mathcal{N}_G$ with projection $\tilde{w}$ in $W(1)$ and w in W. We represent this choice by the following diagram

$$\tag{2.2} \begin{array}{ccccc} \mathcal{N}_G & \to & W(1) & \to & W \\ \hat{\tilde{w}} & \mapsto & \tilde{w} & \mapsto & w \end{array}$$

2.1.4. *Bruhat decompositions of G and G'.* Throughout the article, we will sometimes omit the notation $\hat{w}$, $\hat{\tilde{w}}$ or $\tilde{w}$ and we will just use w in the formulas that do not depend on the choice of the lift. This is the case in the following statements. As a set, G is the disjoint union of the double cosets (see [15, 1.2.7, 4.2.2 (iiii)], [62, Prop. 3.34, Prop. 3.35]):

$$\tag{2.3} G = \sqcup_{w \in W} \mathcal{B}w\mathcal{B} = \sqcup_{w \in W(1)} \mathcal{U}w\mathcal{U},$$

and the group G' is the disjoint union of double cosets

$$(2.4) \qquad G' = \sqcup_{w \in W'} \mathcal{B}' w \mathcal{B}' = \sqcup_{w \in {}_1 W'} \mathcal{U}' w \mathcal{U}'$$

where again we do not specify the choice of the lifts in $\mathcal{N}'_G$ of w in W' (resp. $w \in {}_1 W'$) since the double cosets above do not depend on this choice. The equalities (2.4) follow easily from the analogous equalities for $G_{aff} = Z^0 G'$ proved in [62, Prop. 3.34].

2.1.5. *Affine roots and decompositions of the Iwahori-Weyl group* W. Choosing x_0 as its origin, the apartment $\mathscr{A}$ identifies with the real vector space $V := \mathbb{R} \otimes_{\mathbb{Z}} (X_*(\mathbf{T})/X_*(\mathbf{C}))$ where $X_*(\mathbf{C})$ is the group of algebraic cocharacters of the connected center $\mathbf{C}$ of $\mathbf{G}$. We fix a W_0-invariant scalar product $\langle\,.\,,.\,\rangle$ of V. Let $\mathfrak{H}$ denote the set of the affine hyperplanes $\{x \in V \mid \alpha(x) = -r\}$ associated to the affine roots $(\alpha, r) \in \Sigma_{aff}$. The alcove $\mathscr{C}$ identifies with the alcove of $(V, \mathfrak{H})$ of vertex 0 contained in the dominant Weyl chamber $\mathscr{D}$ determined by Δ.

For any affine root $A = (\alpha, r) \in \Sigma_{aff}$, we have the orthogonal reflection s_A at the associated affine hyperplane and the subgroup $G'_{s_A} \subset G'$ generated by $\mathcal{U}_A$ and $\mathcal{U}_{-A}$. The Weyl group W_0 of Σ is generated by the set $S := \{s_\alpha \mid \alpha \in \Delta\}$.

There is a partial order on Σ given by $\alpha \preceq \beta$ if and only if $\beta - \alpha$ is a linear combination with integral nonnegative coefficients of simple roots. Let Σ_m be the set of roots in Σ that are minimal elements for $\preceq$. The walls of the alcove $\mathscr{C}$ are the affine hyperplanes associated to the set $\Delta_{aff} := \Delta \cup \{(\alpha, 1), \alpha \in \Sigma_m\}$ of simple affine roots. The affine Weyl group W_{aff} of Σ is generated by the set $S_{aff} := \{s_A, A \in \Delta_{aff}\}$. The set of W_{aff}-conjugates of S_{aff} is the set $\mathfrak{S} := \{s_A, A \in \Sigma_{aff}\}$ of reflections of W_{aff}.

Let $s = s_A \in S_{aff}$ be a simple affine reflection, with $A = (\alpha, r) \in \Delta_{aff}$. We associate to s a positive root $\beta \in \Phi^+$ such that $\alpha = e(\beta)\beta$. When the root system Φ is reduced, β is determined by this relation. When the root system Φ is not reduced, we choose β to be the unique root such that, either β is reduced and not multipliable, or β is multipliable and $\mathcal{U}_A \neq \mathcal{U}_{2\beta, e(2\beta)^{-1}r} \mathcal{U}_{\alpha, r+1}$, or β is not reduced and $\mathcal{U}_A = \mathcal{U}_{\beta, e(\beta)^{-1}r} \mathcal{U}_{\alpha, r+1}$ [62, §4.2]. We then set $\beta_s := \beta$, we denote by $G'_s := G'_{\beta_s}$ the subgroup generated by $\mathcal{U}_{\beta_s} \cup \mathcal{U}_{-\beta_s}$, we let $\mathscr{U}_s := \mathcal{U}_{\beta_s, e(\beta_s)^{-1}r}$, $\mathscr{U}_s^{op} := \mathcal{U}_{-\beta_s, -e(\beta_s)^{-1}r}$ and consider the compact open subgroup of G'_s

$$\mathscr{G}'_s \quad \text{generated by} \quad \mathscr{U}_s \cup \mathscr{U}_s^{op}.$$

Later, we will also consider the groups $\mathscr{U}_{s,+} := \mathcal{U}_{\beta_s, e(\beta_s)^{-1}(r+1)}$, $\mathscr{U}_{s,+}^{op} := \mathcal{U}_{-\beta_s, -e(\beta_s)^{-1}(r-1)}$, $\mathbb{U}_s := \mathscr{U}_s/\mathscr{U}_{s,+}$ and $\mathbb{U}_s^{op} := \mathscr{U}_s^{op}/\mathscr{U}_{s,+}^{op}$. The set Π of simple roots of Φ is contained in the set $\Pi_{aff} := \{(\beta_s, e_{\beta_s^{-1}r} \mid s \in S_{aff}\}$ only if $U_{\beta,0} \neq U_{2\beta,0} U_{\beta, e(\beta)^{-1}}$ when $\beta \in \Pi$ is multipliable.

To an element $u \in \mathscr{U}_s - \{1\}$ is associated a lift of s in $\mathcal{N}_G$. It is the single element $n_s(u)$ in the intersection $\mathcal{N}_G \cap U_{-\beta_s} u U_{-\beta_s}$. As this intersection is equal to $\mathcal{N}_G \cap \mathscr{U}_s^{op} u \mathscr{U}_s^{op}$ [15, 6.2.1 (V5)] [62, §3.3 (19)], we have $n_s(u) \in \mathscr{G}'_s$.

$$(2.5) \qquad \text{The elements } n_s(u) \in \mathcal{N}_G \cap \mathscr{G}'_s \text{ for } u \in \mathscr{U}_s - \{1\}, \text{ are called the admissible lifts of } s.$$

Lemma 2.3. *When* $s \in S$, *we have* $\mathscr{G}'_s \subset G'_s \cap \mathcal{K}$.

Proof. When $s \in S$ we have $r = 0$; the groups $\mathcal{U}_{\beta_s, 0}$ and $\mathcal{U}_{-\beta_s, 0}$, hence also $\mathscr{G}'_s$, are contained in $G'_s \cap \mathcal{K}$. $\qquad \square$

Notation 2.4. *To $s \in S_{aff}$ we associate*

- *the facet of the alcove $\mathscr{C}$ fixed by s, and the parahoric subgroup $\mathcal{K}_s$ of G fixing that facet that is to say the fixer of the facet in the kernel $\ker \kappa_G$ of the Kottwitz homomorphism; we have $Z \cap \mathcal{K}_s = Z^0$ [62, Prop. 3.15] and $\mathscr{G}'_s \subset \mathcal{K}_s$ [62, formulas (4.4), (4.9)].*
- *the pro-unipotent radical $\mathcal{K}^1_s$ of $\mathcal{K}_s$ and the finite reductive group $\mathbb{K}_s := \mathcal{K}_s/\mathcal{K}^1_s$;*
- *the order q_s of the unipotent radical of a proper parabolic subgroup of $\mathbb{K}_s$; it is a positive power of q;*
- *the subgroup $\mathbb{G}'_s$ of $\mathbb{K}_s$ generated by $\mathbb{U}_s \cup \mathbb{U}^{op}_s$ and the group $\mathbb{Z}'_s := \mathbb{G}'_s \cap \mathbb{Z}$ where $\mathbb{Z} := Z^0/Z^1$;*
- *some admissible lift $n_s \in \mathscr{N}_G \cap \mathscr{G}'_s$ of s.*
- *the image $s(x)$ in $\mathbb{Z}$ of the conjugate $n_s \hat{x} n_s^{-1}$ by n_s of a lift $\hat{x}$ in Z^0 of $x \in \mathbb{Z}$*

Remark 2.5. The group $\mathbb{Z}'_s = \mathbb{Z} \cap \mathbb{G}'_s$ is equal to $Z^0 \cap \mathscr{G}'_s$ (Notation 2.4); the images in $\mathbb{K}_s = \mathcal{K}_s/\mathcal{K}^1_s$ of $\mathscr{U}_s, \mathscr{U}^{op}_s$ are respectively $\mathbb{U}_s, \mathbb{U}^{op}_s$ [62, Prop. 3.23]; hence the image in $\mathbb{K}_s$ of $\mathscr{G}'_s$ is $\mathbb{G}'_s$ and the image in $\mathbb{K}_s$ of Z'_s is contained in $\mathbb{Z}'_s$; the group $\mathbb{Z}'_s$ is normalized by the image of n_s in $\mathbb{K}_s$.

An element $z \in Z$ acts by translation on V via the unique homomorphism $Z \xrightarrow{\nu} V$ defined by

$$(2.6) \qquad \langle \nu(t), \chi \rangle = - val_F(\chi(t)) \qquad \text{for any } \chi \in \Phi \text{ and } t \in T.$$

The action of W_0, seen as the Weyl group of Σ, and of Z on V combine to define an action, also denoted by ν, of $\mathscr{N}_G$ on V by affine automorphisms. The action of $\mathscr{N}_G$ on V respects the set $\mathfrak{H}$ of affine hyperplanes. Being trivial on Z_0, it identifies with an action of the Iwahori Weyl group W, also denoted by ν. The action of W' on V gives an isomorphism $W' \simeq W_{aff}$. This allows to identify $W_0 \subset W_{aff}$ with the subgroup $W'_0 \subset W'$ of elements fixing 0. The subgroup $\Omega \subset W$ normalizing $\mathscr{C}$ is isomorphic to the image of the Kottwitz morphism κ_G; it is a commutative finitely generated group.

These considerations imply that W admits two decompositions $W = W' \rtimes \Omega = \Lambda \rtimes W'_0$ as a semidirect product. Therefore,

$$(2.7) \qquad W \simeq W_{aff} \rtimes \Omega \simeq \Lambda \rtimes W_0.$$

We inflate the length function of the Coxeter system (W_{aff}, S_{aff}) to a map $\ell : W \to \mathbb{N}$, called the length of W, such that $\Omega \subset W$ is the subset of elements of length 0.

The positive roots Σ^+ and Φ^+ take nonnegative values on the dominant Weyl chamber $\mathscr{D}$. The image of $x \in V$ by an affine root $(\alpha, r) \in \Sigma_{aff}$ is $\alpha(x) + r$. The affine roots $\Sigma^+_{aff} := \{(\alpha, r), \alpha \in \Sigma, r > 0\} \cup \Sigma^+$ take nonnegative value on $\mathscr{C}$. We set $\Sigma^-_{aff} = -\Sigma^+_{aff}$. The action of $w = w_0 \lambda \in W = W'_0 \ltimes \Lambda$ on V induces an action

$$(2.8) \qquad (\alpha, r) \mapsto (w_0(\alpha), r - \langle \nu(\lambda), \alpha \rangle)$$

on Σ_{aff}. For $w \in W$, the group $\hat{w} \mathcal{U}_{(\alpha,r)} \hat{w}^{-1}$ does not depend on the lift $\hat{w} \in \mathscr{N}_G$ and is equal to $\mathcal{U}_{w(\alpha,r)}$. The length $\ell(w)$ of $w \in W$ is the number of $A \in \Sigma^+_{aff}$ such that $w(A) \in \Sigma^-_{aff}$ is negative. The length function $\ell : W \to \mathbb{N}$ inflates to a length function $\ell : W(1) \to \mathbb{N}$.

2.2. **The finite reductive group** $\mathbb{G}$. Let $\mathbf{G}_{x_0}$ and $\mathbf{G}_{\mathscr{C}}$ denote the connected Bruhat-Tits group schemes over O_F whose O_F-valued points are $\mathcal{K}$ and $\mathcal{B}$ respectively. Their reductions over the residue field $\mathbb{F}_q$ are denoted by $\overline{\mathbf{G}}_{x_0}$ and $\overline{\mathbf{G}}_{\mathscr{C}}$. Note that $\mathrm{G} = \mathbf{G}_{x_0}(F) = \mathbf{G}_{\mathscr{C}}(F)$. By [53, 3.4.2, 3.7 and 3.8], $\overline{\mathbf{G}}_{x_0}$ is a connected reductive $\mathbb{F}_q$-group. Let $\overline{\mathbf{T}}$ and $\overline{\mathbf{Z}}$ be the reduction over $\mathbb{F}_q$ of the connected group schemes over O_F whose O_F-valued points are T^0 and Z^0 respectively. The group $\overline{\mathbf{T}}$ is a maximal $\mathbb{F}_q$-split subtorus of $\overline{\mathbf{G}}_{x_0}$ of centralizer $\overline{\mathbf{Z}}$ which is a $\mathbb{F}_q$-torus.

The quotient $\mathbb{G} := \mathcal{K}/\mathcal{K}^1$ of $\mathcal{K}$ by $\mathcal{K}^1$ is a finite reductive group isomorphic to $\overline{\mathbf{G}}_{x_0}(\mathbb{F}_q)$. The Iwahori subgroup $\mathcal{B} \subseteq \mathcal{K}$ is the preimage in $\mathcal{K}$ of the minimal parabolic subgroup $\mathbb{B}$ of $\mathbb{G}$ of unipotent radical $\mathbb{U} := (\mathcal{K} \cap U)/(\mathcal{K}^1 \cap U)$ and Levi decomposition $\mathbb{B} = \mathbb{Z}\mathbb{U}$ where $\mathbb{Z} = \mathrm{Z}^0/\mathrm{Z}^1 \cong \overline{\mathbf{Z}}(\mathbb{F}_q)$ contains the maximal split torus $\mathbb{T} := \mathrm{T}^0/\mathrm{T}^1 \cong \overline{\mathbf{T}}(\mathbb{F}_q)$; the group $\mathbb{Z}$ is a maximal torus of $\mathbb{G}$. The pro-p Iwahori subgroup $\mathcal{U} \subseteq \mathcal{K}$ is the preimage in $\mathcal{K}$ of $\mathbb{U}$; the $\mathbb{G}$-normalizer of $\mathbb{T}$ is denoted by $\mathscr{N}_{\mathbb{G}}$. The relative root system of $\mathbb{G}$ attached to $\mathbb{T}$ identifies with the set $\Phi_{\mathbb{G}} := \{\beta_s \mid s \in S\}$ (notation in §2.1.5) of roots $\beta \in \Phi$ which are not multipliable, *i.e.* such that $2\beta \notin \Phi$ or such that $\mathcal{U}_\beta \neq \mathcal{U}_{2\beta} \mathcal{U}_{\beta + e(\beta)}$ if $2\beta \in \Phi$. The unipotent subgroup $\mathbb{U}_{\beta_s} = \mathbb{U}_s \subset \mathbb{G}$ attached to $\beta_s \in \Phi_{\mathbb{G}}$ is $\mathcal{U}_{\beta_s}/\mathcal{U}_{\beta_s + e(\beta_s)}$. The set $\Pi_{\mathbb{G}} = \{\beta \in \Phi_{\mathbb{G}} \mid \beta/2 \notin \Phi_{\mathbb{G}}, \, e(\beta)\beta \in \Delta\}$ is a basis of $\Phi_{\mathbb{G}}$. The map sending $\beta \in \Pi_{\mathbb{G}}$ to the unique positive reduced root $\alpha \in \Phi$ in $\mathbb{Q}\beta$ induces a bijection between $\Pi_{\mathbb{G}}$ and the basis Π of Φ, giving a bijection between the set $S_{\mathbb{G}} := \{s_\beta \mid \beta \in \Pi_{\mathbb{G}}\}$ of reflections in the Weyl group of $\Phi_{\mathbb{G}}$ and S. The root systems $\Phi_{\mathbb{G}}$ and Φ have isomorphic Weyl groups [53, 3.5.1].

The definition of a strongly split BN-pair of characteristic p is given in [20, Def. 2.20]. By [62, §3.5 Prop. 3.25], we have:

(2.9) $(\mathbb{G}, \mathbb{B}, \mathscr{N}_{\mathbb{G}}, S)$ *and the decomposition* $\mathbb{B} = \mathbb{Z}\mathbb{U}$ *form a strongly split BN-pair of characteristic p.*

where S is seen as a subset of $\mathbb{W} = \mathscr{N}_{\mathbb{G}}/\mathbb{Z}$ (isomorphic to the Weyl group of Σ). We have $\mathscr{N}_{\mathbb{G}} = (\mathscr{N}_{\mathrm{G}} \cap \mathcal{K})/(\mathscr{N}_{\mathrm{G}} \cap \mathcal{K}^1) = (\mathscr{N}_{\mathrm{G}} \cap \mathcal{K})/\mathrm{Z}^1$ and the reduction map $\mathscr{N}_{\mathrm{G}} \cap \mathcal{K} \to \mathscr{N}_{\mathbb{G}}$ induces an isomorphism $(\mathscr{N}_{\mathrm{G}} \cap \mathcal{K})/\mathrm{Z}^0 \xrightarrow{\cong} \mathbb{W}$. The group $(\mathscr{N}_{\mathrm{G}} \cap \mathcal{K})/\mathrm{Z}^0$ is contained in W' and its action on V fixes 0. We recall that the fixer W'_0 of 0 in W defined after (2.6) has the same order as W_0, hence as $\mathbb{W}$ and as $(\mathscr{N}_{\mathrm{G}} \cap \mathcal{K})/\mathrm{Z}^0$. We deduce $\mathrm{W}'_0 = (\mathscr{N}_{\mathrm{G}} \cap \mathcal{K})/\mathrm{Z}^0$. The natural surjective map $\mathrm{W} \to \mathrm{W}_0$ induces an isomorphism $(\mathscr{N}_{\mathrm{G}} \cap \mathcal{K})/\mathrm{Z}^0 = \mathrm{W}'_0 \xrightarrow{\cong} \mathrm{W}_0$. The group $\mathrm{W}'_0(1)$ is the preimage of W'_0 in $\mathrm{W}(1)$ (Notation 2.2). In summary, we have

(2.10) $\mathbb{W} = \mathscr{N}_{\mathbb{G}}/\mathbb{Z} \cong \mathrm{W}_0 = \mathscr{N}_{\mathrm{G}}/\mathrm{Z} \cong (\mathscr{N}_{\mathrm{G}} \cap \mathcal{K})/\mathrm{Z}^0 = \mathrm{W}'_0 \subset \mathrm{W}; \quad \mathscr{N}_{\mathbb{G}} = (\mathscr{N}_{\mathrm{G}} \cap \mathcal{K})/\mathrm{Z}^1 = \mathrm{W}'_0(1) \subset \mathrm{W}(1).$

The action of $\mathbb{W}$ on Σ via the identification $\mathbb{W} \cong \mathrm{W}'_0$ coincides with the natural action of $\mathbb{W}$ on Σ, and the length function on $\mathbb{W}$ coincides with the restriction to W'_0 of $\ell : \mathrm{W} \to \mathbb{N}$. The length function on $\mathbb{W}$ is still denoted by ℓ. It inflates to a map $\mathscr{N}_{\mathbb{G}} \to \mathbb{N}$ which we also denote by ℓ.

Because of (2.9), we have the Bruhat decompositions $\mathbb{G} = \sqcup_{w \in \mathbb{W}} \mathbb{B}w\mathbb{B} = \sqcup_{n \in \mathscr{N}_{\mathbb{G}}} \mathbb{U}n\mathbb{U}$ [20, Prop. 6.6], corresponding to the disjoint unions $\mathcal{K} = \sqcup_{w \in \mathrm{W}'_0} \mathcal{B}w\mathcal{B} = \sqcup_{w \in \mathrm{W}'_0(1)} \mathcal{U}w\mathcal{U}$. We do not specify the choice of a lift for $w \in \mathbb{W}$ in $\mathscr{N}_{\mathbb{G}}$ since the double coset $\mathbb{B}w\mathbb{B}$ does not depend on it (compare with the decompositions in 2.1.4).

2.3. **Lifts of the elements in the finite Weyl group.**

Remark 2.6. We will keep in mind the following diagram:

(2.11)
$$\mathscr{N}_{\mathrm{G}} \cap \mathcal{K} \longrightarrow (\mathscr{N}_{\mathrm{G}} \cap \mathcal{K})/\mathrm{Z}^0 = \mathrm{W}_0' \subset \mathrm{W}$$
$$\mathrm{mod}\ \mathcal{K}^1 \downarrow \cong$$
$$\mathscr{N}_{\mathrm{G}}/\mathrm{Z} = \mathbb{W}.$$

Given an element w in $\mathbb{W}$, we may pick a lift for w in $\mathscr{N}_{\mathrm{G}}$. We may also identify w with an element of W_0' and pick a lift in $\mathscr{N}_{\mathrm{G}} \cap \mathcal{K}$. In §2.4.3 in particular, we will go back and forth between lifts in $\mathscr{N}_{\mathrm{G}}$ and in $\mathscr{N}_{\mathrm{G}} \cap \mathcal{K}$ for elements in $\mathbb{W}$.

For $s \in S$ choose $n_s \in \mathscr{N}_{\mathrm{G}} \cap \mathcal{G}_s'$ an admissible lift of s. This yields a map $s \mapsto n_s : S \to \mathscr{N}_{\mathrm{G}} \cap \mathcal{K}$ (Lemma 2.3).

Proposition 2.7. *There is a unique extension of the map* $s \mapsto n_s : S \to \mathscr{N}_{\mathrm{G}} \cap \mathcal{K}$ *to a map* $w \mapsto n_w :$ $\mathbb{W} \to \mathscr{N}_{\mathrm{G}} \cap \mathcal{K}$ *such that* $n_{ww'} = n_w n_{w'}$ *for* $w, w' \in \mathbb{W}$ *such that* $\ell(ww') = \ell(w) + \ell(w')$.

This will be useful in §2.6.2.

Proof. This result is proved in [4, IV.6 Prop.] for the group $\mathbb{G}$. The arguments in loc. cit. are valid for G. The unicity follows from the reduced decomposition of $w \in \mathbb{W}$ as a product of elements of S. The existence follows from [12, IV.1.5,Prop. 5] once we know that for s, s' distinct in S, and m the order of ss', then $(n_s n_{s'})^r = (n_{s'} n_s)^r$ if $m = 2r$ and $(n_s n_{s'})^r n_s = (n_{s'} n_s)^r n_{s'}$ $m = 2r + 1$. That follows from [15, Prop. (6.1.8) (9)], because we may assume that $\mathbf{G}$ is semisimple simply connected of relative rank 2, by replacing $\mathbf{G}$ by the simply connected covering $\mathbf{M^{sc}}$ of the derived group of $\mathbf{M} = \mathrm{Ker}\alpha \cap \mathrm{Ker}\alpha'$ where $\alpha, \alpha' \in \Pi$ correspond to s, s'. $\square$

Notation 2.8. *For* $w \in \mathbb{W}$, *we denote by* $\mathbf{n}_w$ *the image of* n_w *in* $\mathbb{G}$ *via the map* $\mathcal{K} \to \mathcal{K}/\mathcal{K}^1 = \mathbb{G}$. *We have*
$$\mathbf{n}_{ww'} = \mathbf{n}_w \mathbf{n}_{w'} \text{ for } w, w' \in \mathbb{W} \text{ such that } \ell(ww') = \ell(w) + \ell(w').$$

Remark 2.9. If $\mathbf{w}$ is the longest element of $\mathbb{W}$ and $s \in S$, then $s' := \mathbf{w}s\mathbf{w}^{-1}$ is an element of S and we have

(2.12)
$$n_{\mathbf{w}} n_s = n_{s'} n_{\mathbf{w}} \text{ and } \mathbf{n}_{\mathbf{w}} \mathbf{n}_s = \mathbf{n}_{s'} \mathbf{n}_{\mathbf{w}}.$$

We have indeed $\mathbf{w} = s'\mathbf{w}s$ with $\ell(\mathbf{w}s) = \ell(\mathbf{w}) - 1$. So $n_{\mathbf{w}} = n_{s'} n_{\mathbf{w}s}$ and $n_{\mathbf{w}} n_s = n_{s'} n_{\mathbf{w}s} n_s = n_{s'} n_{\mathbf{w}}$. Note that (2.12) implies $n_{\mathbf{w}} n_w = n_{\mathbf{w}w\mathbf{w}^{-1}} n_{\mathbf{w}}$ and $n_w n_{\mathbf{w}} = n_{\mathbf{w}} n_{\mathbf{w}w\mathbf{w}^{-1}}$ for all $w \in \mathbb{W}$, by induction on the length of w and using $\mathbf{w}^2 = 1$.

Example. For example, when $\mathbf{G}$ is F-split, we fix an épinglage for G as in SGA3 Exp. XXIII, 1.1. In particular, to $\alpha \in \Phi$ is attached a central isogeny $\phi_\alpha : \mathrm{SL}_2(F) \to \mathrm{G}_{s_\alpha}$ where G_{s_α} is the subgroup of G generated by $\mathcal{U}_\alpha$ and $\mathcal{U}_{-\alpha}$. With the notation of 2.1.5, $\Pi = \Delta$ and $\Pi_m = \Delta_{aff} \setminus \Delta$. For $\alpha \in \Pi$, set $n_{s_\alpha} := \phi_\alpha \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. For $\alpha \in \Pi_m$, set $n_{s_{(\alpha,1)}} := \phi_\alpha \begin{pmatrix} 0 & -\varpi^{-1} \\ \varpi & 0 \end{pmatrix}$. We have T = Z and $q_s = q$ for any $s \in S_{aff}$ (Notation 2.4).

2.4. Parabolic subgroups.

2.4.1. *Standard parabolic subgroups.* Let $J \subseteq \Pi$ and $\Phi_J \subseteq \Phi$ the subset of all linear combinations of elements in J. To J we attach the following subgroups of $\mathbf{G}$: the subtorus $\mathbf{T}_J$ of $\mathbf{T}$ with dimension $dim(\mathbf{T}) - |J|$ equal to the connected component of $\bigcap_{\alpha \in J} \ker \alpha \subseteq \mathbf{T}$ and the Levi subgroup $\mathbf{M}_J$ of $\mathbf{G}$ defined to be the centralizer of $\mathbf{T}_J$. The group $\mathbf{M}_J$ is a reductive connected algebraic F-group of maximal F-split torus $\mathbf{T}$ and minimal parabolic $\mathbf{B}_J := \mathbf{B} \cap \mathbf{M}_J = \mathbf{Z}(\mathbf{U} \cap \mathbf{M}_J)$; we have $\mathbf{N}_{\mathbf{M}_J}(\mathbf{T}) = \mathbf{N}_{\mathbf{G}}(\mathbf{T}) \cap \mathbf{M}_J$. The restriction φ_J of φ to the root datum $(Z, (U_\alpha)_{\alpha \in \Phi_J})$ of M_J associated to T is a special discrete valuation compatible with val_F (with the definition of Bruhat-Tits of a root datum [15, 6.1.8]).

The same objects as the ones we attached to G in §2.1 and §2.2 can be attached to M_J. We introduce an index J for the objects attached to M_J. When the set $J = \{\alpha\}$ contains a single element α we denote M_J by M_α.

In particular, we associate to $(\mathbf{T}, \mathbf{B}_J, \varphi_J)$ a reduced root subsystem Σ_J with basis Δ_J, a special maximal parahoric subgroup $\mathcal{K}_J$ of M_J of pro-unipotent radical $\mathcal{K}_J^1$, an Iwahori subgroup $\mathcal{B}_J$ of unique pro-p Sylow subgroup $\mathcal{U}_J$, a finite, Iwahori, pro-p Iwahori Weyl group of M_J denoted respectively by $W_{J,0}$, W_J, and $W_J(1)$, and a real vector space V_J. We note that M_J and G have the same Z and $\mathscr{N}_{M_J} = \mathscr{N}_G \cap M_J$. We have:

- $\Sigma_J = \{e_\alpha \alpha \mid \alpha \in \Phi_J\} \subset \Sigma, \quad \Delta_J = \{e_\alpha \alpha \mid \alpha \in J\} \subset \Delta$,
- $\mathcal{K}_J = \mathcal{K} \cap M_J, \quad \mathcal{K}_J^1 = \mathcal{K}^1 \cap M_J, \quad \mathcal{B}_J = \mathcal{B} \cap M_J, \quad \mathcal{U}_J = \mathcal{U} \cap M_J$ (these equalities are justified in [65, Proposition 4.2]),
- the pro-p Iwahori Weyl group $W_J(1)$ of M coincides with the preimage of W_J in $W(1)$ by the quotient map $W(1) \to W$. Therefore the notation $W_J(1)$ is consistent with the one introduced in 2.2.

Note that $\mathcal{U}_\emptyset = Z^1$ and $V_\emptyset = \{0\}$. When $J \neq \Pi$, the real vector space V_J is only a strict quotient of V (see §2.1.5). This difficulty arises for the semisimple Bruhat-Tits building but not for the extended Bruhat-Tits buildings of G and of M_J. We put on V_J the $W_{J,0}$-scalar product image of the W_0-scalar product on V by the surjective linear map

$$V \xrightarrow{p_J} V_J \quad \alpha(v) = \alpha(p_J(v)) \quad (v \in V, \alpha \in \Sigma_J).$$

For $(\alpha, n) \in \Sigma_{J,aff}$, the inverse image of the affine hyperplane $\mathrm{Ker}_{V_J}(\alpha + n) \in \mathfrak{H}_J$ is $\mathrm{Ker}_V(\alpha + n) \in \mathfrak{H}$ (defined after (2.4)). The image $p_J(\mathscr{C})$ of the alcove $\mathscr{C}$ is contained in the alcove $\mathscr{C}_J$ of $(V_J, \mathfrak{H}_J)$. Recall that $\Sigma_{J,aff}$ is contained in Σ_{aff}. For $A \in \Sigma_{J,aff}$, we attach to the orthogonal reflection $s_{A,J}$ of V_J with respect to $\mathrm{Ker}_{V_J}(A)$ the element s_A. This defines a map $\mathfrak{S}_J \to \mathfrak{S}$ (definition in §2.1.5) which induces an injective homomorphism $W_{J,aff} \hookrightarrow W_{aff}$ (see the notations in 2.1.5, those relative to M_J have an index J).

The image of $S_J = \{s_{\alpha,J}, \alpha \in \Delta_J\}$ is contained in S, but the image of $S_{J,aff}$ is not contained in S_{aff}. The map p_J is W_J-equivariant [65, Lemma 4.1]. We have

$$W_J \simeq \Lambda \rtimes W_{J,0} \simeq W_{J,aff} \rtimes \Omega_J.$$

Remark 2.10. For $s \in S_J$ (but not in $S_{J,aff}$), the positive root $\beta_s \in \Phi_J$, the groups $\mathscr{U}_s$ and $\mathscr{G}'_s$ associated to s, and the admissible lifts of s (as in §2.1.5) for the group M_J are the same as those for G.

Let N_J be the subgroup of G generated by all U_α for $\alpha \in \Phi^+ \setminus \Phi_J$ and N_J^{op} be the subgroup of G generated by all U_α for $\alpha \in \Phi^- \setminus \Phi_J$. Then $\mathrm{P}_J := \mathrm{M}_J \mathrm{N}_J$ is a parabolic subgroup of G. If $J = \emptyset$, then $\mathrm{P}_\emptyset = \mathrm{B}$, $\mathrm{M}_\emptyset = \mathrm{Z}$ and $\mathrm{N}_\emptyset = \mathrm{U}$. A parabolic subgroup of the form P_J for $J \subseteq \Pi$ is called *standard* and $\mathrm{P}_J^{op} = \mathrm{M}_J \mathrm{N}_J^{op}$ is its opposite parabolic subgroup.

As before in §2.2, Π identifies canonically with the set $\Pi_{\mathbb{G}}$ of simple roots of $\mathbb{G}$ with respect to $\mathbb{B}$ in the root system $\Phi_{\mathbb{G}}$ of $\mathbb{G}$ with respect to $\mathbb{T}$. The subset $J \subset \Pi$ identifies with a subset $J_{\mathbb{G}} \subset \Pi_{\mathbb{G}}$ and the root system Φ_J generated by J in $\Phi_{\mathbb{G}}$ with the the root system $\Phi_{\mathbb{M}_J}$ generated by $J_{\mathbb{G}}$ in $\Phi_{\mathbb{G}}$. The image $\mathbb{P}_J$ of $\mathrm{P}_J \cap \mathcal{K}$ in $\mathbb{G}$ coincides with the valued points of the standard parabolic subgroup of $\overline{\mathbf{G}}_{x_0}$ corresponding to $J_{\mathbb{G}}$ [62, Proposition 3.26]. It has Levi decomposition $\mathbb{P}_J = \mathbb{M}_J \mathbb{N}_J$ which coincides with the image of $\mathrm{P}_J \cap \mathcal{K} = (\mathrm{M}_J \cap \mathcal{K})(\mathrm{N}_J \cap \mathcal{K})$.

If $J = \emptyset$, then $\mathbb{P}_\emptyset = \mathbb{B}$. The $\mathbb{M}_J$-normalizer of $\mathbb{T}$ is $\mathscr{N}_{\mathbb{M}_J} = \mathbb{M}_J \cap \mathscr{N}_{\mathbb{G}}$.

The set $S_{J_{\mathbb{G}}}$ corresponding to $J_{\mathbb{G}}$ in $S_{\mathbb{G}}$ identifies canonically with S_J and the group $\mathscr{N}_{\mathbb{M}_J}/\mathbb{Z}$ with the subgroup $\mathbb{W}_J$ of $\mathbb{W}$ generated by S_J. As in (2.10), the finite Weyl group W_J identifies with a subgroup of $\mathbb{W}_J$, and $\mathscr{N}_{\mathbb{M}_J}$ as a subgroup of $\mathbb{W}(1)$. As in (2.9),

(2.13)
$(\mathbb{M}_J, \mathbb{B}_J, \mathscr{N}_{\mathbb{M}_J}, S_J)$ *and the decomposition* $\mathbb{B}_J = \mathbb{Z}\mathbb{U}_J$ *form a strongly split BN-pair of characteristic p*

where $\mathbb{B}_J = \mathbb{B} \cap \mathbb{M}_J$ and $\mathbb{U}_J = \mathbb{U} \cap \mathbb{M}_J$. We have the decompositions $\mathbb{M}_J = \sqcup_{w \in \mathbb{W}_J} \mathbb{B}_J w \mathbb{B}_J = \sqcup_{n \in \mathscr{N}_{\mathbb{M}_J}} \mathbb{U}_J n \mathbb{U}_J$ and $\mathbb{P}_J = \sqcup_{w \in \mathbb{W}_J} \mathbb{B} w \mathbb{B}$.

2.4.2. *Iwahori decomposition and positive (resp. negative) monoids.* Let $J \subset \Pi$. We will write from now on P instead of P_J, M instead of M_J and N instead of N_J and will otherwise replace the index J for the objects introduced in §2.4.1 by the index M. In particular, the finite, Iwahori, and pro-p-Iwahori Weyl groups of M are now respectively denoted by $\mathrm{W}_{0,\mathrm{M}}$, W_{M} and $\mathrm{W}_{\mathrm{M}}(1)$.

The group $\mathcal{B}_{\mathrm{N}} = \mathcal{B} \cap \mathrm{N} = \mathcal{U} \cap \mathrm{N} = \mathcal{U}_{\mathrm{N}}$ (resp. $\mathcal{B}_{\mathrm{N}^{op}} = \mathcal{B} \cap \mathrm{N}^{op} = \mathcal{U} \cap \mathrm{N}^{op} = \mathcal{U}_{\mathrm{N}^{op}}$) is the image by the multiplication map of $\prod_{\alpha \in \Sigma^+ - \Sigma_{\mathrm{M}}^+} \mathcal{U}_{(\alpha,0)}$ (resp. $\prod_{\alpha \in \Sigma^- - \Sigma_{\mathrm{M}}^-} \mathcal{U}_{(\alpha,1)}$), as in §2.1.

Similarly, the group $\mathcal{B}_{\mathrm{M}}$, resp. $\mathcal{U}_{\mathrm{M}}$ (which was denoted by $\mathcal{B}_J$, resp. $\mathcal{U}_J$ in §2.4.1) is the image of $\prod_{\alpha \in \Sigma_{\mathrm{M}}^-} \mathcal{U}_{(\alpha,1)} \times \mathrm{T}^0 \times \prod_{\alpha \in \Sigma_{\mathrm{M}}^+} \mathcal{U}_{(\alpha,0)}$ (resp. $\prod_{\alpha \in \Sigma_{\mathrm{M}}^-} \mathcal{U}_{(\alpha,1)} \times \mathrm{T}^1 \times \prod_{\alpha \in \Sigma_{\mathrm{M}}^+} \mathcal{U}_{(\alpha,0)}$).

The groups $\mathcal{B}$ and $\mathcal{U}$ have an Iwahori decomposition with respect to P:

(2.14) $$\mathcal{B} = \mathcal{B}_{\mathrm{N}} \, \mathcal{B}_{\mathrm{M}} \, \mathcal{B}_{\mathrm{N}^{op}} \text{ and } \mathcal{U} = \mathcal{U}_{\mathrm{N}} \, \mathcal{U}_{\mathrm{M}} \, \mathcal{U}_{\mathrm{N}^{op}}$$

with any order of the products.

An element $m \in \mathrm{M}$ contracts $\mathcal{U}_{\mathrm{N}}$ and dilates $\mathcal{U}_{\mathrm{N}^{op}}$ if it satisfies the condition: $m\mathcal{U}_{\mathrm{N}}m^{-1} \subseteq \mathcal{U}_{\mathrm{N}}$ and $m^{-1}\mathcal{U}_{\mathrm{N}^{op}}m \subseteq \mathcal{U}_{\mathrm{N}^{op}}$ (compare with [55] II.4 or [17, (6.5)]). This property of an element $m \in \mathrm{M}$ depends only on the double coset $\mathcal{U}_{\mathrm{M}} m \mathcal{U}_{\mathrm{M}}$. Such an element will be called M-positive. Note that if $m \in \mathcal{K}_{\mathrm{M}}$ then $m\mathcal{U}_{\mathrm{N}}m^{-1} = \mathcal{U}_{\mathrm{N}}$ and $m^{-1}\mathcal{U}_{\mathrm{N}^{op}}m = \mathcal{U}_{\mathrm{N}^{op}}$.

The monoid of M-positive elements in M will be denoted by M^+. The monoid $(M^+)^{-1} \subseteq M$ will be denoted by M^-. Its elements contract $\mathcal{U}_{N^{op}}$ and dilate $\mathcal{U}_N$ and are called M-negative.

The elements w in W_M satisfying $w(\Sigma^+ - \Sigma_M^+) \subset \Sigma_{aff}^+$ are called M-positive and form a monoid denoted by W_{M+}. Denote by W_{M-} the monoid $(W_{M+})^{-1}$.

Remark 2.11. We have

(1) The monoid W_{M+} is isomorphic to the semi-direct product $W_M \ltimes \Lambda_{M+}$ where

$$\Lambda_{M+} := \{\lambda \in \Lambda \mid -\alpha(\nu(\lambda)) \geq 0 \ \text{ for all } \alpha \in \Sigma^+ - \Sigma_M^+\}.$$

(2) The set W_{M+} is a system of representatives of the double cosets $\mathcal{B}_M \backslash M^+ / \mathcal{B}_M$. The preimage $W_{M+}(1)$ of W_{M+} in $W(1)$ is a system of representatives of the double cosets $\mathcal{U}_M \backslash M^+ / \mathcal{U}_M$.

(3) The set W_{M-} is a system of representatives of the double cosets $\mathcal{B}_M \backslash M^- / \mathcal{B}_M$. The preimage $W_{M-}(1)$ of W_{M-} in $W(1)$ is a system of representatives of the double cosets $\mathcal{U}_M \backslash M^- / \mathcal{U}_M$.

To justify the above, we refer to [42, Lemma 5.21] when G is split, but this is general: in particular for (1) see [66, §2.1].

2.4.3. *Distinguished cosets representatives in* $\mathbb{W}$ *and* W_0'. Let $J \subseteq \Pi$ and $P := P_J$, $\mathbb{P} := \mathbb{P}_J$ the corresponding standard parabolic subgroups of G and $\mathbb{G}$ respectively. The following lemma is proved for example in [21, 2.3.3]. The reduced root system Σ_M attached to M was previously denoted by Σ_J in §2.4.1, and the Weyl group $W_{\mathbb{M}}$ of $\mathbb{M}$ by W_J .

Lemma 2.12. *The set* $^{\mathbb{M}}\mathbb{W}$ *of elements* $d \in \mathbb{W}$ *satisfying* $d^{-1}\Sigma_M^+ \subset \Sigma^+$ *is a system of representatives of the right cosets* $W_{\mathbb{M}} \backslash \mathbb{W}$. *It satisfies* $\ell(wd) = \ell(w) + \ell(d)$ *for any* $w \in W_{\mathbb{M}}$ *and* $d \in {}^{\mathbb{M}}\mathbb{W}$. *In particular,* d *is the unique element with minimal length in* $W_{\mathbb{M}}d$.

Remark 2.13.   (1) Recall that $\mathbb{W}$ is naturally isomorphic to the subgroup W_0' of W. The set $^{\mathbb{M}}\mathbb{W}$ may therefore be seen as the subset of the elements in W_0' satisfying $d^{-1}\Sigma_M^+ \subset \Sigma^+$.

(2) For $d \in \mathbb{W}$ seen in W_0', the condition $d^{-1}\Sigma_M^+ \subset \Sigma^+$ is equivalent to $d^{-1}(\mathcal{U} \cap M)d \subset \mathcal{U}$ (the latter condition does not depend on the choice of a lift for d in $\mathcal{N}_G$). This follows immediately from the decomposition of $\mathcal{U}_M = \mathcal{U} \cap M$ given in §2.4.2. By reduction modulo $\mathcal{K}^1$, we also have $d^{-1}(\mathbb{U} \cap \mathbb{M})d \subset \mathbb{U}$ for any $d \in {}^{\mathbb{M}}\mathbb{W}$ (this does not depend on the choice of a lift for d in $\mathcal{N}_G$).

(3) By (2.14), point (2) implies that

(2.15) $\mathcal{U}\hat{w}\mathcal{U}\hat{d}\mathcal{U} = \mathcal{U}\hat{w}\hat{d}\mathcal{U}$ for $w \in W_{\mathbb{M}}$, $d \in {}^{\mathbb{M}}\mathbb{W}$ seen in W and with respective lifts $\hat{w}$, $\hat{d}$ in $\mathcal{N}_G \cap \mathcal{K}$ and

(2.16) $\mathbb{U}\hat{w}\mathbb{U}\hat{d}\mathbb{U} = \mathbb{U}\hat{w}\hat{d}\mathbb{U}$ for $w \in W_{\mathbb{M}}$, $d \in {}^{\mathbb{M}}\mathbb{W}$ with respective lifts $\hat{w}$, $\hat{d}$ in $\mathcal{N}_{\mathbb{G}}$.

Lemma 2.14. *For* $w \in W_{M+}$ *and* $d \in {}^{\mathbb{M}}\mathbb{W}$, *we have* $\ell(w) + \ell(d) = \ell(wd)$. *In particular, for* $\hat{w}, \hat{d} \in \mathcal{N}_G$ *lifting* w *and* d *respectively, we have* $\mathcal{U}\hat{w}\mathcal{U}\hat{d}\mathcal{U} = \mathcal{U}\hat{w}\hat{d}\mathcal{U}$.

Proof. This is proved for $G = GL(n, F)$ in [41, Lemma 2.3]. The proof is general as noted in [66, Lemma 2.22]. $\square$

2.4.4. *Double and simple cosets decompositions in* $\mathbb{G}$ *and* G. Let $J \subseteq \Pi$ and P $:= P_J$, $\mathbb{P} := \mathbb{P}_J$ the corresponding standard parabolic subgroups of G and $\mathbb{G}$ respectively, with Levi decompositions P $=$ MN and $\mathbb{P} = \mathbb{MN}$. In part (1) of the lemma below, the lifts $\hat{d}$ of $d \in {}^{\mathbb{M}}\mathbb{W}$ are taken in $\mathbb{G}$. In part (2) they are taken in $\mathcal{K}$.

Lemma 2.15.   (1) *We have* $\mathbb{G} = \sqcup_{d \in {}^{\mathbb{M}}\mathbb{W}} \mathbb{P}\hat{d}\mathbb{U}$ *and for any* $d \in {}^{\mathbb{M}}\mathbb{W}$:

       (a) $(\hat{d}\mathbb{U}\hat{d}^{-1}\mathbb{N}) \cap \mathbb{M} \subseteq \mathbb{U} \cap \mathbb{M}$,

       (b) $(\mathbb{P} \cap \hat{d}\mathbb{U}\hat{d}^{-1})\mathbb{N} = (\mathbb{U} \cap \mathbb{M})\mathbb{N} = \mathbb{U}$.

    (2) *We have* G $= \sqcup_{d \in {}^{\mathbb{M}}\mathbb{W}} \mathrm{P}\mathcal{U}\hat{d}\mathcal{U} = \sqcup_{d \in {}^{\mathbb{M}}\mathbb{W}} \mathrm{P}\hat{d}\mathcal{U}$ *and for any* $d \in {}^{\mathbb{M}}\mathbb{W}$:

       (a) $\hat{d}^{-1}(\mathcal{U} \cap \mathrm{M})\hat{d} \subseteq \mathcal{U}$.

       (b) $\hat{d}\mathcal{U}\hat{d}^{-1} \cap \mathrm{P}\mathcal{U} = \hat{d}\mathcal{U}\hat{d}^{-1} \cap \mathcal{U}$,

       (c) $(\mathrm{P} \cap \hat{d}\mathcal{U}\hat{d}^{-1})\mathrm{N} = (\mathcal{U} \cap \mathrm{M})\mathrm{N}$.

Proof.   (1) That $\mathbb{G}$ is the union of all $\mathbb{P}\hat{d}\mathbb{B} = \mathbb{P}\hat{d}\mathbb{U}$ for $d \in {}^{\mathbb{M}}\mathbb{W}$ follows from the Bruhat decomposition for $\mathbb{G}$ and the decomposition of $\mathbb{P}$ as the disjoint union of all $\mathbb{B}\hat{w}\mathbb{B}$ for $w \in \mathbb{W}_J$ (see (2.9) and (2.13)). Let $d \in {}^{\mathbb{M}}\mathbb{W}$. By Remark 2.13(2), we have $\mathbb{U} \cap \mathbb{M} \subset \hat{d}\mathbb{U}\hat{d}^{-1}$ and $\mathbb{U}^{op} \cap \mathbb{M} \subset \hat{d}\mathbb{U}^{op}\hat{d}^{-1}$. Since $\hat{d}\mathbb{U}\hat{d}^{-1} \cap \mathbb{U}^{op} \cap \mathbb{P} \subset \hat{d}\mathbb{U}\hat{d}^{-1} \cap \mathbb{U}^{op} \cap \mathbb{M}$, it implies that $\hat{d}\mathbb{U}\hat{d}^{-1} \cap \mathbb{U}^{op} \cap \mathbb{P} = \{1\}$. Since an element in $\mathbb{U}$ can be written uniquely as the product of an element in $\mathbb{U} \cap \hat{d}^{-1}\mathbb{U}\hat{d}$ by an element in $\mathbb{U} \cap \hat{d}^{-1}\mathbb{U}^{op}\hat{d}$ ([21], 2.5.12), we deduce that $\hat{d}\mathbb{U}\hat{d}^{-1} \cap \mathbb{P} \subset \mathbb{U}$. Points (1)(a) and (1)(b) then follow easily (for (1)(b) also use $\mathbb{U} \cap \mathbb{M} \subset \hat{d}\mathbb{U}\hat{d}^{-1}$).

    (2) Recall that here the lifts for $d \in {}^{\mathbb{M}}\mathbb{W}$ are in $\mathcal{K}$. The first identity is proved for G $= \mathrm{GL}(n, \mathrm{F})$ in [41], 6A1] and for G split in [42], Lemma 5.18].

We review the arguments to check that they generalize to the case of an arbitrary reductive group. Recall that $\mathbb{W} \cong \mathrm{W}'_0$ indexes the doubles cosets of $\mathcal{K}$ mod $\mathcal{B}$, that ${}^{\mathbb{M}}\mathbb{W}$ is a system of representatives of the right cosets $\mathbb{W}_{\mathbb{M}} \backslash \mathbb{W}$, and that for $w \in \mathbb{W}$ with lift $\hat{w} \in \mathcal{K}$, we have $\mathrm{P}\mathcal{U}\hat{w}\mathcal{U} = \mathrm{P}\hat{w}\mathcal{U}$ since $\hat{w}^{-1}\mathcal{U}_{\mathrm{N^{op}}}\hat{w} \subset \mathcal{U}$. From the Iwasawa decomposition G $= \mathrm{P}\mathcal{K}$, we get that G is the union of all $\mathrm{P}\hat{d}\mathcal{U}$ for $d \in {}^{\mathbb{M}}\mathbb{W}$. It is a disjoint union because, given $d, d' \in {}^{\mathbb{M}}\mathbb{W}$, if $\hat{d}' \in \mathrm{P}\hat{d}\mathcal{U}$ then $\hat{d}' \in (\mathcal{K} \cap \mathrm{P})\hat{d}\mathcal{U}$. Reducing mod $\mathcal{K}^1$ and using (1) gives $d' = d$ (we recall that the image of $\mathcal{K} \cap \mathrm{P}$ in $\mathbb{G}$ is $\mathbb{P}$). For $d \in {}^{\mathbb{M}}\mathbb{W}$, we have $\hat{d}^{-1}\mathcal{U}_{\mathrm{N^{op}}}\hat{d} \subset \mathcal{K}^1 \subset \mathcal{U}$ since $\hat{d}$ lies in $\mathcal{K}$, so it is easy to see that $\mathrm{P}\mathcal{U}\hat{d}\mathcal{U} = \mathrm{P}\mathcal{U}_{\mathrm{N^{op}}}\hat{d}\mathcal{U} = \mathrm{P}\hat{d}\mathcal{U}$.

(a) This is Remark 2.13(2).

(b) Note that $\hat{d}\mathcal{U}\hat{d}^{-1}$ is contained in $\mathcal{K}$. An element in $\mathrm{P}\mathcal{U}$ is of the form pu for $p \in \mathrm{P}$ and $u \in \mathcal{U}_{\mathrm{N^{op}}}$. Suppose that it lies in $\hat{d}\mathcal{U}\hat{d}^{-1}$. Then $p \in \mathcal{K} \cap \mathrm{P}$ and the projection of pu in $\mathcal{K}/\mathcal{K}^1 = \mathbb{G}$ is equal to $\bar{p} \in \hat{d}\mathbb{U}\hat{d}^{-1}$ where $\bar{p} \in \mathbb{P}$ is the projection of p. But $\mathbb{P} \cap \hat{d}\mathbb{U}\hat{d}^{-1} \subset \mathbb{U}$ by (1)(b) so $\bar{p} \in \mathbb{U}$ and $p \in \mathcal{U}$. We have checked $\mathrm{P}\mathcal{U} \cap \hat{d}\mathcal{U}\hat{d}^{-1} \subset \mathcal{U}$ which gives (2)(b).

(c) By (b), we have $\mathrm{P} \cap \hat{d}\mathcal{U}\hat{d}^{-1} \subset \mathrm{P} \cap \mathcal{U}$ which is contained in $(\mathrm{M} \cap \mathcal{U})\mathrm{N}$. It gives one inclusion. The other inclusion comes from (a).

                                                            $\square$

Lemma 2.16. *Let* $w \in \mathrm{W}_{\mathrm{M}^+}(1)$ *with lift* $\hat{w} \in \mathrm{M}^+$ *and* $d \in {}^{\mathbb{M}}\mathbb{W}$ *with lift* $\hat{d} \in \mathcal{K}$.

(1) *Consider a decomposition into simple cosets $\mathcal{U}_{\mathrm{M}} \hat{w} \mathcal{U}_{\mathrm{M}} = \sqcup_x \mathcal{U}_{\mathrm{M}} \hat{w} x$ for $x \in \hat{w}^{-1} \mathcal{U}_{\mathrm{M}} \hat{w} \cap \mathcal{U}_{\mathrm{M}} \backslash \mathcal{U}_{\mathrm{M}}$. Then*

$$\mathcal{U} \hat{w} \mathcal{U} = \sqcup_x \mathcal{U} \hat{w} x \mathcal{U}_{\mathrm{N^{op}}}.$$

Furthermore, we have $\mathrm{P}\mathcal{U}\hat{w}\mathcal{U} = \mathrm{P}\mathcal{U}$. For $u \in \mathcal{U}_{\mathrm{N^{op}}}$ and $x \in \mathcal{U}_{\mathrm{M}}$, we have $u \in \mathrm{P}\mathcal{U}\hat{w}x$ if and only if $\mathcal{U}\hat{w}x = \mathcal{U}\hat{w}xu$.

(2) *Consider a decomposition into simple cosets $\mathcal{U}\hat{d}\mathcal{U} = \sqcup_y \mathcal{U}\hat{d}y$ for $y \in \mathcal{U} \cap \hat{d}^{-1}\mathcal{U}\hat{d} \backslash \mathcal{U}$. Then*

$$\mathrm{P}\hat{d}\mathcal{U} = \mathrm{P}\mathcal{U}\hat{d}\mathcal{U} = \sqcup_y \mathrm{P}\mathcal{U}\hat{d}y.$$

(3) *We have $\mathcal{U}\hat{w}\hat{d}\mathcal{U} = \mathcal{U}\hat{w}\mathcal{U}\hat{d}\mathcal{U} = \sqcup_{x,u_x,y}\mathcal{U}\hat{w}xu_x\hat{d}y$ for*

$$x \in \hat{w}^{-1}\mathcal{U}_{\mathrm{M}}\hat{w} \cap \mathcal{U}_{\mathrm{M}}\backslash\mathcal{U}_{\mathrm{M}}, \ u_x \in (\hat{w}x)^{-1}\mathcal{U}\hat{w}x \cap \mathcal{U}_{\mathrm{N^{op}}}\backslash\mathcal{U}_{\mathrm{N^{op}}}, \ y \in \mathcal{U} \cap \hat{d}^{-1}\mathcal{U}\hat{d}\backslash\mathcal{U}.$$

Proof. (1) The first fact is proved in [55, II.4], but we give a proof for the convenience of the reader. We have $\mathcal{U}\hat{w}\mathcal{U} = \mathcal{U}\hat{w}\mathcal{U}_{\mathrm{M}}\mathcal{U}_{\mathrm{N^{op}}}$ because $\mathcal{U}\hat{w}\mathcal{U} = \mathcal{U}\hat{w}\mathcal{U}_{\mathrm{N}}\mathcal{U}_{\mathrm{M}}\mathcal{U}_{\mathrm{N^{op}}}$ and $\hat{w}\mathcal{U}_{\mathrm{N}}\hat{w}^{-1} \subset \mathcal{U}_{\mathrm{N}}$ since w is M-positive. The decomposition of $\mathcal{U}\hat{w}\mathcal{U}$ is disjoint because for $x \in \mathcal{U}_{\mathrm{M}}$ we have $\mathcal{U}\hat{w}x\mathcal{U}_{\mathrm{N^{op}}} = \mathcal{U}_{\mathrm{N}}\mathcal{U}_{\mathrm{M}}\hat{w}x\mathcal{U}_{\mathrm{N^{op}}}$ (since w is M-positive and x normalizes $\mathcal{U}_{\mathrm{N^{op}}}$) and the decomposition in $\mathrm{NMN^{op}}$ is unique. We have $\mathrm{P}\mathcal{U}\hat{w}\mathcal{U} = \mathrm{P}\mathcal{U}$ because $\mathrm{P}\mathcal{U}\hat{w}\mathcal{U} = \mathrm{P}\mathcal{U}_{\mathrm{N^{op}}}\hat{w}\mathcal{U} = \mathrm{P}\hat{w}^{-1}\mathcal{U}_{\mathrm{N^{op}}}\hat{w}\mathcal{U}$ and $\hat{w}^{-1}\mathcal{U}_{\mathrm{N^{op}}}\hat{w} \subset \mathcal{U}_{\mathrm{N^{op}}}$ since w is M-positive. The last fact is drawn from [48, Proposition 7 p.77] and we recall the argument. Let $x \in \mathcal{U}_{\mathrm{M}}$. Note first that $1 \in \mathrm{P}\mathcal{U}\hat{w}x = \mathrm{P}\hat{w}^{-1}\mathcal{U}_{\mathrm{N^{op}}}\hat{w}x$ since $x \in \mathrm{P}$. So, given $u \in \mathcal{U}_{\mathrm{N^{op}}}$, if $\mathcal{U}\hat{w}x = \mathcal{U}\hat{w}xu$ then $u \in \mathrm{P}\mathcal{U}\hat{w}x$. Now let $u \in \mathcal{U}_{\mathrm{N^{op}}}$ and suppose that $u \in \mathrm{P}\mathcal{U}\hat{w}x = \mathrm{P}\hat{w}^{-1}\mathcal{U}_{\mathrm{N^{op}}}\hat{w}x$. Then there is $p \in \mathrm{P}$ and $k \in \hat{w}^{-1}\mathcal{U}_{\mathrm{N^{op}}}\hat{w} \subseteq \mathcal{U}_{\mathrm{N^{op}}}$ such that $u = pkx = px(x^{-1}kx)$. Since $x \in \mathcal{U}_{\mathrm{M}}$ normalizes $\mathcal{U}_{\mathrm{N^{op}}}$, it implies that $px \in \mathcal{U}_{\mathrm{N^{op}}} \cap \mathrm{P} = \{1\}$. Therefore $xux^{-1} \in \hat{w}^{-1}\mathcal{U}_{\mathrm{N^{op}}}\hat{w}$ and $\mathcal{U}\hat{w}xu = \mathcal{U}\hat{w}x$.

(2) The first equality $\mathrm{P}\hat{d}\mathcal{U} = \mathrm{P}\mathcal{U}\hat{d}\mathcal{U}$ is given by Lemma 2.15(2). We have to prove that for $y \in \mathcal{U}$, if $\mathrm{P}\mathcal{U}\hat{d}y \cap \mathrm{P}\mathcal{U}\hat{d} \neq \emptyset$ then $\mathcal{U}\hat{d}y = \mathcal{U}\hat{d}$. So let $y \in \mathcal{U}$ and suppose that $\mathrm{P}\mathcal{U}\hat{d}y \cap \mathrm{P}\mathcal{U}\hat{d} \neq \emptyset$. This means that there is $p \in \mathrm{P}$, $u,v \in \mathcal{U}_{\mathrm{N^{op}}}$ such that $pv = u\hat{d}y\hat{d}^{-1}$ and therefore $p \in \mathcal{K} \cap \mathrm{P}$. Reducing mod $\mathcal{K}^1$, recalling $\mathcal{U}_{\mathrm{N^{op}}} \subset \mathcal{K}^1$, and denoting respectively by $\bar{p}$, $\bar{\hat{d}}$ and $\bar{y}$ the images of p, $\hat{d}$ and y in $\mathbb{G}$, we get $\bar{\hat{d}}\bar{y}\bar{\hat{d}}^{-1} \in \mathbb{P} \cap \bar{\hat{d}}\mathbb{U}\bar{\hat{d}}^{-1}$ which implies $\bar{\hat{d}}\bar{y}\bar{\hat{d}}^{-1} \in \mathbb{U}$ by Lemma (2.15)(1)(b). Therefore, $\hat{d}y\hat{d}^{-1} \in \mathcal{U}$ and $\mathcal{U}\hat{d}y = \mathcal{U}\hat{d}$.

(3) The first equality $\mathcal{U}\hat{w}\hat{d}\mathcal{U} = \mathcal{U}\hat{w}\mathcal{U}\hat{d}\mathcal{U}$ comes from Lemma 2.14. From (1) we deduce $\mathcal{U}\hat{w}\mathcal{U}\hat{d}\mathcal{U} = \cup_{x,y}\mathcal{U}\hat{w}x\mathcal{U}\hat{d}y = \cup_{x,y}\mathcal{U}\hat{w}x\mathcal{U}_{\mathrm{N^{op}}}\hat{d}y = \cup_{x,u_x,y}\mathcal{U}\hat{w}xu_x\hat{d}y$ for x,u_x,y ranging over the sets described in the lemma. It remains to prove that this is a disjoint union: it is a disjoint union on y by (2) because $\mathcal{U}\hat{w}xu_x \subseteq \mathcal{U}\hat{w}\mathcal{U} \subseteq \mathrm{P}\mathcal{U}$ by (1). Fixing y, it is a disjoint union on x by (1). Fixing y,x, the equality $\mathcal{U}\hat{w}xu\hat{d}y = \mathcal{U}\hat{w}xu'\hat{d}y$ for $u,u' \in \mathcal{U}_{\mathrm{N^{op}}}$, is equivalent to $u' \in (\hat{w}x)^{-1}\mathcal{U}\hat{w}xu$. $\qquad\square$

2.5. Pro-p Hecke algebras and universal representations.

2.5.1. *Presentation of the pro-p Iwahori Hecke algebra of* G. Let R be a commutative ring. The universal R-representation $\mathbf{X} := \mathrm{ind}_{\mathcal{U}}^{\mathrm{G}}(1)$ of G is the compact induction of the trivial character of $\mathcal{U}$ with values in R. We see it as the space of R-valued functions with compact support in $\mathcal{U}\backslash\mathrm{G}$, endowed with the action of G by right translation. The pro-p Iwahori Hecke R-algebra of G is the R-algebra of the G-equivariant

endomorphisms of $\mathbf{X}$. It will be denoted by $\mathcal{H}$. It is naturally isomorphic to the convolution algebra $R[\mathcal{U}\backslash\mathrm{G}/\mathcal{U}]$ of the $\mathcal{U}$-biinvariant functions with compact support in G, with product defined by

$$f \star f' : \mathrm{G} \to R, g \mapsto \sum_{u \in \mathcal{U}\backslash\mathrm{G}} f(gu^{-1})f'(u)$$

for $f, f' \in R[\mathcal{U}\backslash\mathrm{G}/\mathcal{U}]$. We will identify $\mathcal{H}$ and $R[\mathcal{U}\backslash\mathrm{G}/\mathcal{U}]$ without further notice. For $g \in \mathrm{G}$, we denote by $\tau_g \in \mathcal{H}$ the characteristic function of $\mathcal{U}g\mathcal{U}$. For $w \in \mathrm{W}(1)$, the double coset $\mathcal{U}\hat{w}\mathcal{U}$ does not depend on the chosen lift $\hat{w} \in \mathscr{N}_{\mathrm{G}}$ and we simply write τ_w for the corresponding element of $\mathcal{H}$. The set of all τ_w for $w \in \mathrm{W}(1)$ is a basis for $\mathcal{H}$.

For $s \in S_{aff}$, let $n_s \in \mathscr{N}_{\mathrm{G}}$ be an admissible lift of s (see (2.5)) and $\mathbb{Z}'_s$ the subgroup of $\mathbb{Z}$ defined in Notation 2.4.

The product in $\mathcal{H}$ is given by the following braid and, respectively, quadratic relations [62, thm.2.2]:

(2.17) $\tau_{ww'} = \tau_w \tau_{w'}$ for $w, w' \in \mathrm{W}(1)$ satisfying $\ell(ww') = \ell(w) + \ell(w')$,

(2.18) $\tau_{n_s}^2 = q_s \tau_{n_s^2} + c_{n_s}\tau_{n_s}$ for $s \in S_{aff}$, where $c_{n_s} = \sum_{z \in \mathbb{Z}'_s} c_{n_s}(z)\tau_z$,

for positive integers $c_{n_s}(z)$ satisfying: $c_{n_s}(z) = c_{n_s}(z^{-1}) = c_{n_s}(zxs(x)^{-1})$ if $x \in \mathbb{Z}$ and $z \in \mathbb{Z}'_s$ (for $s(x)$ see Notation 2.4), the precise definition of the integers $c_{n_s}(z)$ is given in [62, step 2 of Prop. 4.4]. If we set $c_{n_s}(z) = 0$ for $z \in \mathbb{Z} \backslash \mathbb{Z}'_s$, the latter relations remain valid for all $z \in \mathbb{Z}$. Furthermore, $c_{n_s}\tau_{n_s} = \tau_{n_s}c_{n_s}$ and the sum over $z \in \mathbb{Z}$ of all $c_{n_s}(z)$ is equal to $q_s - 1$. We identify the positive integers q_s and $c_{n_s}(z)$ with their natural image in R.

If R has characteristic p, then in $\mathcal{H}$ we have $c_{n_s} = c_s$ where

(2.19) $c_s := (q_s - 1)|\mathbb{Z}'_s|^{-1} \sum_{z \in \mathbb{Z}'_s} \tau_z \ \in \mathcal{H}.$

Example. Following up the example at the end of §2.3, suppose $\mathbf{G}$ is F-split. Then we have, $q_s = q$ and $c_{n_s} = c_s$ for any $s \in S_{aff}$.

Indeed, in Notation 2.4, the group G_s is $\mathrm{SL}(2, \mathbb{F}_q)$ or $\mathrm{PGL}(2, \mathbb{F}_q)$, and $|\mathbb{Z}'_s| = q - 1$ or $(q-1)/2$. When $|\mathbb{Z}'_s| = (q-1)/2$, we have $\mathbb{Z}'_s = \{xs(x)^{-1}, \ x \in \mathbb{Z}\}$. The positive integers $c_{n_s}(z)$ for $z \in \mathbb{Z}'_s$ of sum $q-1$ are equal to 1 when $|\mathbb{Z}'_s| = q-1$, and to 2 when $|\mathbb{Z}'_s| = (q-1)/2$ because they are constant.[1]

Remark 2.17. In $\mathcal{H}$, the quadratic relation (2.18) satisfied by τ_{n_s} extends to $\tau_{\hat{s}}$ for an arbitrary lift $\hat{s} \in \mathscr{N}_{\mathrm{G}}$ of $s \in S_{aff}$. The quadratic relation satisfied by $\tau_{\hat{s}}$ is $\tau_{\hat{s}}^2 = q_s \tau_{\hat{s}^2} + c_{\hat{s}}\tau_{\hat{s}}$ where $c_{\hat{s}} := c_{n_s}\tau_t$ and $t := n_s^{-1}\hat{s} \in \mathbb{Z}$, because

$$\tau_{\hat{s}}^2 = \tau_{n_s}\tau_t\tau_{n_s}\tau_t = \tau_{n_s}^2\tau_{n_s^{-1}tn_s}\tau_t = (q_s\tau_{n_s^2} + c_{n_s}\tau_{n_s})\tau_{n_s^{-1}tn_s}\tau_t = q_s\tau_{\hat{s}^2} + c_{n_s}\tau_t\tau_{\hat{s}}.$$

When $s_1, s_2 \in S_{aff}$ admit lifts $\tilde{s}_1, \tilde{s}_2$ in $\mathrm{W}(1)$ which are conjugate $\tilde{s}_2 = w\tilde{s}_1w^{-1}$ by $w \in \mathrm{W}(1)$, then for arbitrary lifts $\hat{s}_1, \hat{s}_2 \in \mathscr{N}_{\mathrm{G}}$ of $\tilde{s}_1, \tilde{s}_2$ we have $c_{\hat{s}_2} = wc_{\hat{s}_1}w^{-1}$ (we note $w(\sum_{z \in \mathbb{Z}} c(z)\tau_z)w^{-1} := \sum_{z \in \mathbb{Z}} c(z)\tau_{wzw^{-1}}$ for $c(z) \in \mathbb{Z}$). This follows from the equivalence of the properties (A) and (B) in [62, Thm 4.7] applied to $R = \mathbb{Z}$.

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[1]This should replace [62, Prop. 4.4 Proof (6)]. In [62, Prop. 4.4] when G is split "For example" should be "or". It is wrongly claimed that $\mathbb{F}_q^* \simeq \mathbb{T}'_s$ in [58, page 6 and in §2.2], as noticed by P. Schneider.

Remark 2.18. Suppose that p is invertible in R. The elements τ_w for $w \in W(1)$ are invertible in $\mathcal{H}$. For $s \in S_{aff}$ the element $q_s \tau_{n_s^2}$ is indeed invertible with inverse $q_s^{-1} \tau_{n_s^{-2}}$ because q_s is a power of p (Notation 2.4) and $n_s^2 \in \mathbb{Z}$. The inverse of τ_{n_s} in $\mathcal{H}$ is $q_s^{-1} \tau_{n_s^{-2}}(\tau_{n_s} - c_{n_s})$.

Notation 2.19. For $w \in W_{aff}$, denote by $v_s(w)$ the multiplicity of $s \in S_{aff}$ in a reduced decomposition of w and let
$$q_w := \prod_{s \in S_{aff}} q_s^{v_s(w)}.$$
Set $q_w := q_{w_{aff}}$ for any $w \in \Omega w_{aff} \Omega$ and $w_{aff} \in W_{aff}$. The element q_w for $w \in W$ is well defined and does not depend on a reduced expression of $w \in W$ ([60, Chap. 1]). The function $w \mapsto q_w$ on W inflates to $W(1)$. By [62, Lemma 4.12, Prop. 4.13], there is a unique family $(\tau_w^*)_{w \in W(1)}$ in $\mathcal{H}$ satisfying the relations

- $\tau_{w_1}^* \tau_{w_2}^* = \tau_{w_1 w_2}^*$ if $\ell(w_1) + \ell(w_2) = \ell(w_1 w_2)$,
- $\tau_u = \tau_u^*$ if $\ell(u) = 0$,
- $\tau_{n_s}^* := \tau_{n_s} - c_{n_s}$ if $s \in S_{aff}$.

The set of all τ_w^* for $w \in W(1)$ form another basis of $\mathcal{H}$. We have $(\tau_w^*)^{-1} = q_w \tau_{w^{-1}}$.

2.5.2. *Pro-p Iwahori Hecke algebra attached to a Levi subgroup.* Let $J \subset \Delta$. We consider the associated standard parabolic subgroup P with Levi decomposition P = MN. Recall that $\mathcal{B}_M = \mathcal{B} \cap M$ is an Iwahori subgroup in M with pro-p Sylow subgroup is $\mathcal{U}_M = \mathcal{U} \cap M$ (notation introduced in §2.4.2, see also §2.4.1). The R-algebra of the M-endomorphisms of the universal R-representation $\mathbf{X}_M = \mathrm{ind}_{\mathcal{U}_M}^M(1)$ is the pro-p Iwahori Hecke R-algebra $\mathcal{H}_M$ of M. For $g \in M$ (resp. $w \in W_M(1)$), we denote by τ_g^M (resp. τ_w^M) the characteristic function of $\mathcal{U}_M g \mathcal{U}_M$ (resp. $\mathcal{U}_M \hat{w} \mathcal{U}_M$). A basis for $\mathcal{H}_M$ is given by the set of all τ_w^M for $w \in W_M(1)$. Another basis is given by the set of all $\tau_w^{M,*}$ for $w \in W_M(1)$, where $(\tau_w^{M,*})_{w \in W_M(1)}$ is defined for $\mathcal{H}_M$ as it was for $\mathcal{H}$ in Notation 2.19.

Denote by $\mathcal{H}_{M+}$ (resp. $\mathcal{H}_{M-}$) the subspace of $\mathcal{H}_M$ with basis the set of all τ_w^M for $w \in W_{M+}(1)$ (resp. $w \in W_{M-}(1)$) as defined in Remark 2.11. The algebra $\mathcal{H}_M$ does not inject in $\mathcal{H}$ in general, but $\mathcal{H}_{M+}$ (resp. $\mathcal{H}_{M-}$) does: the linear maps

$$(2.20) \qquad \mathcal{H}_M \xrightarrow{\theta} \mathcal{H}, \quad \tau_m^M \longmapsto \tau_m \ (m \in M), \quad \mathcal{H}_M \xrightarrow{\theta^*} \mathcal{H}, \quad \tau_m^{M,*} \longmapsto \tau_m^* \ (m \in M),$$

restricted to $\mathcal{H}_{M+}$ or to $\mathcal{H}_{M-}$ respect the products, and identify $\mathcal{H}_{M+}$ and $\mathcal{H}_{M-}$ with four subalgebras of $\mathcal{H}$. The algebra $\mathcal{H}_M$ is a localization of $\mathcal{H}_{M+}$ (resp. $\mathcal{H}_{M-}$) at a central element (see [55, II.4], [17, (6.12)], [66, Thm. 1.4]).

Remark 2.20. When p is invertible in R, the homomorphisms $\theta|_{\mathcal{H}_{M+}}, \theta|_{\mathcal{H}_{M-}}, \theta^*|_{\mathcal{H}_{M+}}$ and $\theta^*|_{\mathcal{H}_{M-}}$ extend uniquely to four embeddings from $\mathcal{H}_M$ into $\mathcal{H}$, denoted by $\theta^+, \theta^-, \theta^{*+}$ and by θ^{*-}, determined by the formula

$$\theta^+(\tau_m^M) = \tau_a^{-n} \tau_{a^n m}, \ \theta^-(\tau_m^M) = \tau_{a^{-1}}^{-n} \tau_{a^{-n} m}, \ \theta^{*+}(\tau_m^{M,*}) = (\tau_a^*)^{-n} \tau_{a^n m}^*, \ \theta^{*-}(\tau_m^{M,*}) = (\tau_{a^{-1}}^*)^{-n} \tau_{a^{-n} m}^*$$

for $m \in M$, where $a \in C_M$ is a strictly M-positive element (recall that $\mathbf{C_M}$ is the connected center of $\mathbf{M}$): $\cap_{n \in \mathbb{N}} a^n \mathcal{U}_N a^{-n} = \{1\}$ and $\cap_{n \in \mathbb{N}} a^{-n} \mathcal{U}_{N^{op}} a^n = \{1\}$), $n \in \mathbb{N}$ such that $a^n m \in M^+$ and $a^{-n} m \in M^-$

(in particular a is M-positive). This follows from Remark 2.18 and from [55, II.6)]. For the convenience of the reader, we give the proof for $\theta^*|_{\mathcal{H}_{\mathrm{M}^+}}$. The arguments are the same for the other homomorphisms.

1. $\theta^{*+}(\tau_m^{\mathrm{M},*})$ is well defined: $(\tau_a^*)^{-n}\tau_{a^n m}^*$ does not depend on the choice of the pair (a,n). We show $(\tau_a^*)^{-n}\tau_{a^n m}^* = (\tau_b^*)^{-r}\tau_{b^r m}^*$ for any other pair (b,r). Using that for $x,y \in \mathrm{M}^+$ such that $\tau_x^{\mathrm{M},*}$ and $\tau_y^{\mathrm{M},*}$ commute, the elements τ_x^* and τ_y^* commute as $\theta^*|_{\mathcal{H}_{\mathrm{M}^+}}$ respects the product, the equality is equivalent to $(\tau_b^*)^r \tau_{a^n m}^* = (\tau_a^*)^n \tau_{b^r m}^*$. It is also equivalent to $(\tau_b^{\mathrm{M},*})^r \tau_{a^n m}^{\mathrm{M},*} = (\tau_a^{\mathrm{M},*})^n \tau_{b^r m}^{\mathrm{M},*}$ because $\theta^*|_{\mathcal{H}_{\mathrm{M}^+}}$ is injective and respects the product. Then it is equivalent to $\tau_{b^r a^n m}^{\mathrm{M},*} = \tau_{a^n b^r m}^{\mathrm{M},*}$ because the image of a,b in W_{M} have length 0. This equality is true because a and b commute.

2. θ^{*+} respects the product: $\theta^{*+}(\tau_m^{\mathrm{M},*})\theta^{*+}(\tau_{m'}^{\mathrm{M},*}) = \theta^{*+}(\tau_m^{\mathrm{M},*}\tau_{m'}^{\mathrm{M},*})$ for $m,m' \in \mathrm{M}$. We choose the same pair (a,n) for m and m'. Using the arguments given in 1. we show that $(\tau_a^*)^{-n}$ and $\tau_{a^n m}^*$ commute: $(\tau_a^*)^{-n}\tau_{a^n m}^* = \tau_{a^n m}^*(\tau_a^*)^{-n} \Leftrightarrow (\tau_a^*)^n\tau_{a^n m}^* = \tau_{a^n m}^*(\tau_a^*)^n \Leftrightarrow (\tau_a^{\mathrm{M},*})^n\tau_{a^n m}^{\mathrm{M},*} = \tau_{a^n m}^{\mathrm{M},*}(\tau_a^{\mathrm{M},*})^n \Leftrightarrow \tau_{a^{2n} m}^{\mathrm{M},*} = \tau_{a^n m a^n}^{\mathrm{M},*}$, which is true because a and m commute. So $(\tau_a^*)^{-n}\tau_{a^n m}^*(\tau_a^*)^{-n}\tau_{a^n m'}^* = (\tau_a^*)^{-2n}\tau_{a^n m}^*\tau_{a^n m'}^* = (\tau_a^*)^{-2n}\tau_{a^n m a^n m'}^* = (\tau_a^*)^{-2n}\tau_{a^{2n} m m'}^*$.

The modulus δ_{P} of the parabolic group $\mathrm{P} = \mathrm{MN}$ is the generalized index $\delta_P(x) = [x\mathcal{U}_{\mathrm{P}}x^{-1} : \mathcal{U}_{\mathrm{P}}]$ for $x \in \mathrm{P}$ and $\mathcal{U}_{\mathrm{P}} = \mathcal{U} \cap \mathrm{P}$ [54, I.2.6]. The reductive p-adic group M is unimodular [54, I.2.7 a)], hence

$$\delta_{\mathrm{P}}(x) = [x\mathcal{U}_{\mathrm{N}}x^{-1} : \mathcal{U}_{\mathrm{N}}] \quad (x \in \mathrm{P})$$

where $\mathcal{U}_{\mathrm{N}} = \mathcal{U} \cap \mathrm{N}$. When p is invertible in R, the modulus of P over R is the character $x \to \delta_{\mathrm{P}}(x)1_R$ of P with values in R.

Lemma 2.21. *When p is invertible in R, we have $\theta^{*+}(\tau_m^{\mathrm{M}}) = \theta^-(\tau_m^{\mathrm{M}})\delta_{\mathrm{P}}(m)$ for all $m \in \mathrm{M}$.*

Proof. As θ^{*+} respects the product, it respects the inverse: $\theta^{*+}((\tau_m^{\mathrm{M},*})^{-1}) = ((\tau_a^*)^{-n}\tau_{a^n m}^*)^{-1} = (\tau_{a^n m}^*)^{-1}(\tau_a^*)^n$ for all $m \in \mathrm{M}$ and $a \in C_{\mathrm{M}}$ a strictly M-positive element such that $a^n m \in \mathrm{M}^+$. Let $\mu = m^{-1}$. As $(\tau_w^*)^{-1} = \tau_{w^{-1}}q_w$ and $q_w = q_{w^{-1}}$, we obtain $\theta^{*+}(\tau_\mu^{\mathrm{M}})q_{M,\mu} = \tau_{\mu a^{-n}}q_{\mu a^{-n}}(\tau_{a^{-1}}q_{a^{-1}})^{-n}$. This is valid for all $\mu \in \mathrm{M}$. We replace μ by m to get $\theta^{*+}(\tau_m^{\mathrm{M}}) = \tau_{m a^{-n}}(\tau_{a^{-1}})^{-n}q_{m a^{-n}}(q_{a^{-1}})^{-n}q_{M,m}^{-1}$ for all $m \in \mathrm{M}$ and $m a^{-n} \in \mathrm{M}^-$ as the inverse map sends M^+ onto M^-. We have $\tau_m^{\mathrm{M}} = \tau_{m a^{-n}}^{\mathrm{M}}\tau_{a^n}^{\mathrm{M}} = \tau_{m a^{-n}}^{\mathrm{M}}(\tau_{a^{-1}}^{\mathrm{M}})^{-n}$ and θ^- respects the product: $\theta^-(\tau_m^{\mathrm{M}}) = \theta^-(\tau_{m a^{-n}}^{\mathrm{M}})\theta^-(\tau_{a^{-1}}^{\mathrm{M}})^{-n} = \tau_{m a^{-n}}(\tau_{a^{-1}})^{-n}$. We deduce that $\theta^{*+}(\tau_m^{\mathrm{M}}) = \theta^-(\tau_m^{\mathrm{M}})\delta(m)$ where $\delta(m) = q_{m a^{-n}}(q_{a^{-1}})^{-n}q_{M,m}^{-1}$ for $m \in \mathrm{M}$, $a \in C_{\mathrm{M}}$ strictly M-positive, $m a^{-n} \in \mathrm{M}^-$.

It remains to show $\delta = \delta_{\mathrm{P}}$ on M. It is well known that δ_{P} respects the product on M [54, I.2.6]. The restriction of δ to the monoid M^- is $\delta(w) = q_w q_{M,w}^{-1}$. It respects the product by applying [66, Lemma 2.7]. Furthermore, we have $\delta(m) = \delta(m a^{-n})\delta(a^n)$ for $m \in \mathrm{M}$, $a \in C_{\mathrm{M}}$ strictly M-positive and $m a^{-n} \in \mathrm{M}^-$ (this is because $\delta(a^n) = (q_{a^{-1}})^{-n}$ and $q_{\mathrm{M},m a^{-n}} = q_{\mathrm{M},m}$). Therefore δ respects the product on M. The two homomorphisms δ and δ_{P} from M to R are trivial on $\mathcal{K} \cap \mathrm{M}$; as $\mathrm{M} = (\mathcal{K} \cap \mathrm{M})\mathrm{Z}(\mathcal{K} \cap \mathrm{M})$ by the Bruhat decomposition and $\mathrm{Z} = \cup_{n \in \mathbb{N}}(\mathrm{Z} \cap \mathrm{M}^-)a^n$ for $a \in C_{\mathrm{M}}$ strictly M-positive, δ and δ_{P} are equal on M if they are equal on $\mathrm{Z} \cap \mathrm{M}^-$. Let $z \in \mathrm{Z} \cap \mathrm{M}^-$. We have $q_z = [\mathcal{U}z\mathcal{U} : \mathcal{U}] = [\mathcal{U}z\mathcal{U}_{\mathrm{P}} : \mathcal{U}] = [\mathcal{U}_{\mathrm{P}}z\mathcal{U}_{\mathrm{P}} : \mathcal{U}_{\mathrm{P}}] = [\mathcal{U}_{\mathrm{M}}z\mathcal{U}_{\mathrm{M}} : \mathcal{U}_{\mathrm{M}}][\mathcal{U}_{\mathrm{N}}z\mathcal{U}_{\mathrm{N}} : \mathcal{U}_{\mathrm{N}}] = q_{\mathrm{M},z}[\mathcal{U}_{\mathrm{N}} : (\mathcal{U}_{\mathrm{N}} \cap z^{-1}\mathcal{U}_{\mathrm{N}}z)] = q_{\mathrm{M},z}\delta_{\mathrm{P}}(z)$. Hence $\delta(z) = q_z q_{\mathrm{M},z}^{-1} = \delta_{\mathrm{P}}(z)$ for $z \in \mathrm{Z} \cap \mathrm{M}^-$. Therefore $\delta = \delta_{\mathrm{P}}$ on M. $\qquad\square$

2.5.3. *Iwahori Hecke algebra.* Considering the Iwahori subgroup $\mathcal{B}$ instead of its pro-p radical $\mathcal{U}$ in §2.5.1, one defines the Iwahori Hecke algebra $\mathcal{H}_\mathcal{B}$ of the G-endomorphisms of the universal R-representation $\mathbf{X}_\mathcal{B} := \mathrm{ind}_\mathcal{B}^\mathrm{G}(1)$. It is naturally isomorphic to the convolution algebra $R[\mathcal{B}\backslash\mathrm{G}/\mathcal{B}]$ of the $\mathcal{B}$-biinvariant functions with compact support in G. For $g \in \mathrm{G}$, denote by τ_g the characteristic function of $\mathcal{B}g\mathcal{B}$. For $w \in \mathrm{W}$, denote by τ_w the characteristic function of $\mathcal{B}\hat{w}\mathcal{B}$ which does not depend on the choice of a lift $\hat{w} \in \mathscr{N}_\mathrm{G}$ for w. The set of all τ_w for $w \in \mathrm{W}$ is a basis for $\mathcal{H}_\mathcal{B}$. The product in $\mathcal{H}_\mathcal{B}$ is given by the following braid and, respectively, quadratic relations:

$$(2.21) \qquad \tau_{ww'} = \tau_w\tau_{w'} \text{ for } w, w' \in \mathrm{W} \text{ satisfying } \ell(ww') = \ell(w) + \ell(w').$$

$$(2.22) \qquad \tau_s^2 = q_s\tau_1 + (q_s - 1)\tau_s \text{ for any } s \in S_{aff}.$$

These relations are well known by [32], [10], [38] under certain hypotheses on **G**. It has been checked by Vignéras [62, Thm 2.1 and Rmk 2.6] that they hold for a general connected F-group **G**. The key result for the quadratic relation is the following ([32, Prop. 2.8], [16, 5.2.12], [62, Theorem 33]): for $s \in S_{aff}$, we have

$$(2.23) \qquad \mathcal{B}s\mathcal{B}s\mathcal{B} = \mathcal{B}s\mathcal{B} \sqcup \mathcal{B}.$$

Note that conversely, once we admit the quadratic relations, this identity can be deduced from them using the definition of the convolution product on $R[\mathcal{B}\backslash\mathrm{G}/\mathcal{B}]$ (choose for example $R = \mathbb{C}$).

Lemma 2.22. *For $s \in S_{aff}$, we have $\mathcal{B}s\mathcal{B}s\mathcal{B} = \mathcal{B}s\mathcal{B}s \sqcup \mathcal{B}s$.*

Proof. The double coset $\mathcal{B}s\mathcal{B}$ decomposes into q_s simple right cosets [62, Prop. 3.38]. The set $\mathcal{B}s\mathcal{B}s$ is the disjoint union of q_s right cosets contained in $\mathcal{B}s\mathcal{B}s\mathcal{B}$. It does not contain $\mathcal{B}s$ since $\mathcal{B}s\mathcal{B}$ and $\mathcal{B}$ are disjoint. Therefore, and since $\mathcal{B}s\mathcal{B}s\mathcal{B}$ is the disjoint union of $q_s + 1$ simple cosets (by (2.23)), we have $\mathcal{B}s\mathcal{B}s = \mathcal{B}s\mathcal{B}s\mathcal{B} - \mathcal{B}s$. $\qquad\square$

Remark 2.23. The formulas, for $w \in \mathrm{W}$,

$$\mathrm{Triv}_{\mathcal{H}_\mathcal{B}}(\tau_w) := q_w.1_R, \quad \mathrm{Sign}_{\mathcal{H}_\mathcal{B}}(\tau_w) := (-1)^{\ell(w)}.1_R$$

define respectively the trivial character $\mathrm{Triv}_{\mathcal{H}_\mathcal{B}}$ and the sign character $\mathrm{Sign}_{\mathcal{H}_\mathcal{B}}$ of $\mathcal{H}_\mathcal{B}$ with values in R.

Remark 2.24. (1) The Iwahori Hecke R-algebra $\mathcal{H}_\mathcal{B}$ is a quotient of the pro-p Iwahori Hecke R-algebra $\mathcal{H}$ by the augmentation map sending to 1_R the elements of the form τ_t for $t \in \mathrm{Z}^0/\mathrm{Z}_1 = \mathbb{Z}$. In particular, $\mathrm{Triv}_{\mathcal{H}_\mathcal{B}}$ and $\mathrm{Sign}_{\mathcal{H}_\mathcal{B}}$ extend to characters of $\mathcal{H}$ that we denote respectively by $\mathrm{Triv}_\mathcal{H}$ and $\mathrm{Sign}_\mathcal{H}$ and call the trivial and sign characters of $\mathcal{H}$ (compare with [64, Def. 2.7]).

(2) Suppose that $|\mathrm{Z}^0/\mathrm{Z}^1| = |\mathbb{Z}|$ is invertible in the commutative ring R, then one can define the following central idempotent of $\mathcal{H}$ (see [58, §1.3] in the split case):

$$\varepsilon_\mathbf{1} = \frac{1}{|\mathbb{Z}|} \sum_{t \in \mathbb{Z}} \tau_t.$$

We have $\varepsilon_\mathbf{1}\mathbf{X} \cong \mathbf{X}_\mathcal{B}$ and the algebra $\varepsilon_\mathbf{1}\mathcal{H}$ with unit $\varepsilon_\mathbf{1}$ is isomorphic to $\mathcal{H}_\mathcal{B}$ (see [42, §2.3.3] in the split case, but the proof is general).

2.5.4. *Trivial representation of* G *and trivial character of the pro-p Iwahori Hecke algebra.* Let R be a commutative ring. Consider the trivial character $\mathrm{Triv}_{\mathcal{H}}$ of $\mathcal{H}$ (Remark 2.24(1)). It is defined by $\tau_{\tilde{w}} \longmapsto q_w.1_R$, for all $\tilde{w} \in \mathrm{W}(1)$ with projection $w \in \mathrm{W}$. Let $\mathrm{Triv}_{\mathrm{G}} : \mathrm{G} \to R^*$ denotes the trivial character of G. As a right $\mathcal{H}$-module, we have $(\mathrm{Triv}_{\mathrm{G}})^{\mathcal{U}} \cong \mathrm{Triv}_{\mathcal{H}}$. This is because, for all $\tilde{w} \in \mathrm{W}(1)$ as above, we have $q_w = [\mathcal{B}w\mathcal{B}, \mathcal{B}] = [\mathcal{U}\tilde{w}\mathcal{U}, \mathcal{U}]$ ([62, Cor. 3.30]) (recall that for $X = \mathcal{B}$ or $\mathcal{U}$, we denote by $[XwX, X]$ the number of left cosets mod X in XwX. It is equal to the number of right cosets mod X in XwX).

Lemma 2.25. *Let R be a commutative ring with $q_s + 1$ invertible in R for all $s \in S_{aff}$. We have an isomorphism of representations of* G

$$(2.24) \qquad \qquad \mathrm{Triv}_{\mathcal{H}} \otimes_{\mathcal{H}} \mathbf{X} \cong \mathrm{Triv}_{\mathrm{G}}.$$

Proof. By Remark 2.24(1) we have $\mathcal{H}_{\mathcal{B}} = R \otimes_{R[\mathbb{Z}]} \mathcal{H}$ and $\mathbf{X}_{\mathcal{B}} = R \otimes_{R[\mathbb{Z}]} \mathbf{X}$, for the augmentation map $R[\mathbb{Z}] \to R$ sending the elements $t \in \mathbb{Z}$ to 1_R and for the linear map $R[\mathbb{Z}] \to \mathcal{H}$ sending the elements $t \in \mathbb{Z}$ to τ_t. Therefore, we have a natural isomorphism of G-representations $\mathrm{Triv}_{\mathcal{H}} \otimes_{\mathcal{H}} \mathbf{X} \cong \mathrm{Triv}_{\mathcal{H}_{\mathcal{B}}} \otimes_{\mathcal{H}_{\mathcal{B}}} \mathbf{X}_{\mathcal{B}}$ (without any hypothesis on the q_s). We show here that $\mathrm{Triv}_{\mathcal{H}_{\mathcal{B}}} \otimes_{\mathcal{H}_{\mathcal{B}}} \mathbf{X}_{\mathcal{B}} \cong \mathrm{Triv}_{\mathrm{G}}$ when is $q_s + 1$ invertible in R for all $s \in S_{aff}$. From the remark before the lemma, one gets immediately that the right $\mathcal{H}_{\mathcal{B}}$-module $(\mathrm{Triv}_{\mathrm{G}})^{\mathcal{B}}$ is isomorphic to $\mathrm{Triv}_{\mathcal{H}_{\mathcal{B}}}$ so by adjunction, $\mathrm{Triv}_{\mathrm{G}}$ is a quotient of the representation $\mathrm{Triv}_{\mathcal{H}_{\mathcal{B}}} \otimes_{\mathcal{H}_{\mathcal{B}}} \mathbf{X}_{\mathcal{B}}$: the map is given by $\lambda \otimes \mathbf{1}_{\mathcal{B}} \mapsto \lambda$ where $\mathbf{1}_{\mathcal{B}}$ is the characteristic function of $\mathcal{B}$. We want to check that this map is injective and it will follow from the fact that we have a decomposition of R-modules

$$(2.25) \qquad \qquad \mathbf{X}_{\mathcal{B}} = R\mathbf{1}_{\mathcal{B}} + \ker(\mathrm{Triv}_{\mathcal{H}_{\mathcal{B}}})\mathbf{X}_{\mathcal{B}}.$$

The R-module $\mathbf{X}_{\mathcal{B}}$ is free and decomposes as the direct sum of all $\mathbf{X}_{\mathcal{B}}(w)$ for $w \in \mathrm{W}$ where we denote by $\mathbf{X}_{\mathcal{B}}(w)$ the R-module of all functions with support in $\mathcal{B}w\mathcal{B}$.

We show by induction on $\ell(w)$ that $\mathbf{X}_{\mathcal{B}}(w)$ is contained in $R\mathbf{1}_{\mathcal{B}} + \ker(\mathrm{Triv}_{\mathcal{H}_{\mathcal{B}}})\mathbf{X}_{\mathcal{B}}$. If $\ell(w) = 0$ then $\mathrm{Triv}_{\mathcal{H}_{\mathcal{B}}}(\tau_w) = 1$ and $\mathbf{X}_{\mathcal{B}}(w) = R\mathbf{1}_{\mathcal{B}w}$ is contained in $R\mathbf{1}_{\mathcal{B}} + \ker(\mathrm{Triv}_{\mathcal{H}_{\mathcal{B}}})\mathbf{X}_{\mathcal{B}}$ since the characteristic function $\mathbf{1}_{\mathcal{B}w}$ satisfies $\mathbf{1}_{\mathcal{B}w} = \tau_w \mathbf{1}_{\mathcal{B}} = \mathbf{1}_{\mathcal{B}} + (\tau_w - 1)\mathbf{1}_{\mathcal{B}} \in R\mathbf{1}_{\mathcal{B}} + \ker(\mathrm{Triv}_{\mathcal{H}_{\mathcal{B}}})\mathbf{X}_{\mathcal{B}}$. Now suppose $\ell(w) \geq 0$ and let $s \in S_{aff}$ such that $\ell(sw) = \ell(w) + 1$. Note that $\mathbf{X}_{\mathcal{B}}(sw)$ is generated as a representation of $\mathcal{B}$ by the characteristic function $f := \mathbf{1}_{\mathcal{B}n_s\hat{w}} = \hat{w}^{-1}\mathbf{1}_{\mathcal{B}n_s}$, where $\hat{w}$ is a chosen lift for w in $\mathcal{N}_{\mathrm{G}}$ and n_s is defined as in Notation 2.4. Therefore, to prove that $\mathbf{X}_{\mathcal{B}}(sw)$ is contained in $R\mathbf{1}_{\mathcal{B}} + \ker(\mathrm{Triv}_{\mathcal{H}_{\mathcal{B}}})\mathbf{X}_{\mathcal{B}}$, it is enough to consider the case of f. Since τ_{n_s} corresponds to the double coset $\mathcal{B}n_s\mathcal{B}$ and commutes to the action of $n_s^{-1} \in \mathrm{G}$, we have $\tau_{n_s}\mathbf{1}_{\mathcal{B}n_s} = \mathbf{1}_{\mathcal{B}n_s\mathcal{B}n_s}$ so, by Lemma 2.22, we have $\tau_{n_s}\mathbf{1}_{\mathcal{B}n_s} = \mathbf{1}_{\mathcal{B}n_s\mathcal{B}n_s\mathcal{B}} - \mathbf{1}_{\mathcal{B}n_s}$. On the other hand, $\tau_{n_s}\mathbf{1}_{\mathcal{B}} = \mathbf{1}_{\mathcal{B}n_s\mathcal{B}} = \mathbf{1}_{\mathcal{B}n_s\mathcal{B}n_s\mathcal{B}} - \mathbf{1}_{\mathcal{B}}$ by (2.23). We deduce

$$(2.26) \qquad \qquad \mathbf{1}_{\mathcal{B}n_s} - \mathbf{1}_{\mathcal{B}} = \tau_{n_s}(\mathbf{1}_{\mathcal{B}} - \mathbf{1}_{\mathcal{B}n_s}).$$

By translation by $\hat{w}^{-1}$ which commutes with the action of $\mathcal{H}$, we get:

$$f - \mathbf{1}_{\mathcal{B}\hat{w}} = \tau_{n_s}(\mathbf{1}_{\mathcal{B}\hat{w}} - f) = (\tau_{n_s} - q_s)(\mathbf{1}_{\mathcal{B}\hat{w}} - f) - q_s(f - \mathbf{1}_{\mathcal{B}\hat{w}}).$$

If $1 + q_s$ is invertible in R, this means that f lies in $\mathbf{1}_{\mathcal{B}\hat{w}} + \ker(\mathrm{Triv}_{\mathcal{H}_{\mathcal{B}}})\mathbf{X}_{\mathcal{B}}$. Conclude by induction. This proves the decomposition (2.25) which is enough to prove that the natural surjective map $\mathrm{Triv}_{\mathcal{H}_{\mathcal{B}}} \otimes_{\mathcal{H}_{\mathcal{B}}}$

$\mathbf{X}_{\mathcal{B}} \to \mathrm{Triv}_{\mathrm{G}}$ is injective. Note in passing that from the surjectivity of this map we also get that the decomposition (2.25) is a direct decomposition. $\qquad \square$

2.6. **Finite Hecke algebras and universal representations.**

2.6.1. *Definitions.* Let $J \subseteq \Pi$. Let P be the associated standard parabolic subgroup of G, and $\mathbb{P}$ with Levi decomposition $\mathbb{P} = \mathbb{M}\mathbb{N}$ the corresponding parabolic subgroup of $\mathbb{G}$. Let $\mathcal{P}$ (resp. $\mathcal{N}$) be the preimage in $\mathcal{K}$ of $\mathbb{P}$ (resp. $\mathbb{N}$). The group $\mathcal{P}$ is a parahoric subgroup of G containing $\mathcal{U}$. We have $J = \Pi$ if and only if $\mathcal{P} = \mathcal{K}$. Extending by zero functions on $\mathcal{P}$ to $\mathcal{K}$, and respectively on $\mathcal{K}$ to G, induces embeddings which are respectively $\mathcal{P}$ and $\mathcal{K}$ equivariant:

$$(2.27) \qquad \mathrm{ind}_{\mathcal{U}}^{\mathcal{P}}(1) \hookrightarrow \mathrm{ind}_{\mathcal{U}}^{\mathcal{K}}(1) \hookrightarrow \mathbf{X} \ .$$

Passing to $\mathcal{U}$-invariants induces naturally embeddings of the convolutions algebras

$$(2.28) \qquad (\mathrm{ind}_{\mathcal{U}}^{\mathcal{P}}(1))^{\mathcal{U}} \hookrightarrow (\mathrm{ind}_{\mathcal{U}}^{\mathcal{K}}(1))^{\mathcal{U}} \hookrightarrow \mathbf{X}^{\mathcal{U}} \ .$$

Let $\mathbb{U}_{\mathbb{M}} := \mathbb{M} \cap \mathbb{U}$ (it was previously denoted by $\mathbb{U}_J$ (see (2.13))). The universal R-representation

$$\mathrm{X}_{\mathbb{M}} := \mathrm{ind}_{\mathbb{U}_{\mathbb{M}}}^{\mathbb{M}}(1),$$

of $\mathbb{M}$ is the induction of the trivial character of $\mathbb{U}_{\mathbb{M}}$ with values in R. The associated finite unipotent Hecke R-algebra is

$$\mathrm{H}_{\mathbb{M}} := \mathrm{End}_{k[\mathbb{M}]}(\mathrm{X}_{\mathbb{M}}) \cong [\mathrm{ind}_{\mathbb{U}_{\mathbb{M}}}^{\mathbb{M}}(1)]^{\mathbb{U}_{\mathbb{M}}}.$$

It has basis the set of all characteristic functions of the double cosets $\mathbb{U}_{\mathbb{M}} n \mathbb{U}_{\mathbb{M}}$ for $n \in \mathscr{N}_{\mathbb{M}}$ (see the decomposition after (2.13)). When $J = \Pi$ and therefore $\mathbb{M} = \mathbb{G}$, we omit the subscript and simply write X and H instead of $\mathrm{X}_{\mathbb{G}}$ and $\mathrm{H}_{\mathbb{G}}$ respectively.

Since $\mathcal{P}/\mathcal{U} \simeq \mathbb{P}/\mathbb{U} = \mathbb{M}/\mathbb{U}_{\mathbb{M}}$, $\mathcal{U}\backslash\mathcal{P}/\mathcal{U} \simeq \mathbb{U}_{\mathbb{M}}\backslash\mathbb{M}/\mathbb{U}_{\mathbb{M}}$, and $\mathcal{P}/\mathcal{N} \simeq \mathbb{M}$, the representation $\mathrm{X}_{\mathbb{M}}$ can also be seen as a representation of $\mathcal{P}$ trivial on $\mathcal{N}$ and as such it is isomorphic to $\mathrm{ind}_{\mathcal{U}}^{\mathcal{P}}(1)$. Likewise X is isomorphic to $\mathrm{ind}_{\mathcal{U}}^{\mathcal{K}}(1)$ when seen as a representation of $\mathcal{K}$ trivial on $\mathcal{K}^1$. Via these isomorphisms, (2.27) gives embedddings

$$(2.29) \qquad \mathrm{X}_{\mathbb{M}} \hookrightarrow \mathrm{X} \hookrightarrow \mathbf{X},$$

where the first one is the $\mathbb{M}$-equivariant morphism sending $\mathbf{1}_{\mathbb{U}_{\mathbb{M}}}$ onto $\mathbf{1}_{\mathbb{U}}$. The embeddings of algebras (2.28) can be reformulated as embeddings

$$(2.30) \qquad \mathrm{H}_{\mathbb{M}} \hookrightarrow \mathrm{H} \hookrightarrow \mathcal{H}.$$

The first one sends, for $m \in \mathbb{M}$, the characteristic function of $\mathbb{U}_{\mathbb{M}} m \mathbb{U}_{\mathbb{M}}$ to $\mathbb{U} m \mathbb{U}$. The second one sends, for $g \in \mathcal{K}$, the characteristic function of $\mathbb{U}(g \bmod \mathcal{K}^1)\mathbb{U}$ to $\mathcal{U} g \mathcal{U}$. The algebra $\mathrm{H}_{\mathbb{M}}$ can therefore be seen as a subalgebra of H and of $\mathcal{H}_{\mathrm{M}}$.

Proposition 2.26. *As a left (resp. right)* $\mathrm{H}_{\mathbb{M}}$*-module,* H *is free with basis the set of all* $\tau_{\tilde{d}}$ *(resp.* $\tau_{\tilde{d}^{-1}}$*) for* $d \in {}^{\mathbb{M}}\mathbb{W}$*, where* $\tilde{d}$ *denotes a lift for* d *in* $\mathscr{N}_{\mathbb{G}}$*.*

Proof. This is a consequence of the relation $\ell(w) + \ell(d) = \ell(wd)$ for $w \in \mathbb{W}_{\mathrm{M}}, d \in {}^{\mathbb{M}}\mathbb{W}$ (Lemma 2.14) and of the braid relations (2.17). $\qquad \square$

When $w \in \mathscr{N}_{\mathbb{M}}$ we will consider τ_w^{M} as an element of $\mathrm{H}_{\mathbb{M}}$ or of $\mathcal{H}_{\mathrm{M}}$ depending on the context. In particular, when $s \in S_J$ (defined in §2.4.1) and $\mathbf{n}_s \in \mathscr{N}_{\mathbb{M}}$ is the image of an admissible lift $n_s \in \mathrm{M} \cap \mathcal{K}$ in $\mathbb{M}$ as in Notation 2.8 (see also Remark 2.10) then $\tau_{\mathbf{n}_s}$ satisfies the same quadratic relation (2.18) as τ_{n_s}.

Recall that $\mathrm{M} \cap \mathcal{K}$ is contained in the positive (resp. negative) submonoid M^+ (resp. M^-) of M. The embedding (2.30) of $\mathrm{H}_{\mathbb{M}}$ into $\mathcal{H}$ is the restriction of θ defined in (2.20) to $\mathrm{H}_{\mathbb{M}}$, i.e. the map

$$\mathrm{H}_{\mathbb{M}} \longrightarrow \mathcal{H}, \quad \tau_w^{\mathrm{M}}, \, w \in \mathscr{N}_{\mathbb{M}} \longmapsto \tau_w$$

where we identify $\mathscr{N}_{\mathbb{M}}$ as a subgroup of $\mathrm{W}(1)$ via the isomorphisms (2.10).

We will often identify $\mathrm{H}_{\mathbb{M}}$ with its image in H or in $\mathcal{H}$.

2.6.2. *Trace maps and involutions on the finite Hecke algebras.* The commutative ring R is arbitrary unless otherwise mentioned. A R-linear form $\mathrm{H}_{\mathbb{M}} \to R$ is called non degenerate if its kernel does not contain any non zero left or right ideal. In the next proposition, we define such a non degenerate form $\delta_{\mathbb{M}}$ and associate to it an automorphism $\iota_{\mathbb{M}}$ of $\mathrm{H}_{\mathbb{M}}$ such that $\delta_{\mathbb{M}}(ab) = \delta_{\mathbb{M}}(\iota_{\mathbb{M}}(b)a)$. When $\mathbb{M} = \mathbb{G}$ we will omit the subscript and simply write δ and ι.

Proposition 2.27. (1) *Pick* $\widetilde{\mathbf{w}_{\mathbb{M}}} \in \mathscr{N}_{\mathbb{M}}$ *a lift for the longest element* $\mathbf{w}_{\mathbb{M}}$ *in* $\mathbb{W}_{\mathbb{M}} \cong \mathscr{N}_{\mathbb{M}}/\mathbb{Z}$. *Let* $\delta_{\mathbb{M}} : \mathrm{H}_{\mathbb{M}} \to R$ *be the linear map sending* $\tau_{\widetilde{\mathbf{w}_{\mathbb{M}}}}$ *to 1 and* τ_n *to zero for all* $n \in \mathscr{N}_{\mathbb{M}} - \{\widetilde{\mathbf{w}_{\mathbb{M}}}\}$. *Then* $\delta_{\mathbb{M}}$ *is non degenerate. The automorphism of* $\mathrm{H}_{\mathbb{M}}$ *defined by*

$$\iota_{\mathbb{M}} : \tau_n \mapsto \tau_{\widetilde{\mathbf{w}_{\mathbb{M}}} n \widetilde{\mathbf{w}_{\mathbb{M}}}^{-1}} \text{ for all } n \in \mathscr{N}_{\mathbb{M}}$$

satisfies $\delta_{\mathbb{M}}(ab) = \delta_{\mathbb{M}}(\iota_{\mathbb{M}}(b)a)$ *for any* $a, b \in \mathrm{H}_{\mathbb{M}}$.

(2) *Suppose that* R *is a ring where* p *is not a zero divisor. The linear* R-*form* $\delta'_{\mathbb{M}} : \mathrm{H}_{\mathbb{M}} \to R$ *sending* τ_1 *to 1 and* τ_w *to zero for all* $w \in \mathscr{N}_{\mathbb{M}} - \{1\}$ *is non degenerate and symmetric: it satisfies* $\delta'_{\mathbb{M}}(ab) = \delta'_{\mathbb{M}}(ba)$ *for any* $a, b \in \mathrm{H}_{\mathbb{M}}$. *The corresponding automorphism* $\iota'_{\mathbb{M}}$ *of* $\mathrm{H}_{\mathbb{M}}$ *is the identity map.*

We comment on this proposition and give its proof in the case when $\mathbb{M} = \mathbb{G}$ so as to avoid the subscripts $\mathbb{M}$.

Remark 2.28. (1) For $t \in \mathbb{Z}$ and $\widetilde{\mathbf{w}}_1 \in \mathscr{N}_{\mathbb{G}}$ a lift of the longest element $\mathbf{w}$ in $\mathbb{W}$, $\widetilde{\mathbf{w}}_2 := t\widetilde{\mathbf{w}}_1$ is another lift of $\mathbf{w}$. Denote by ι_1 and ι_2 the corresponding automorphisms of H as in Prop.2.27(1). Then for any right H-module $\mathfrak{m}$, denote by $\mathfrak{m}\iota_1$, resp. $\mathfrak{m}\iota_2$, the space $\mathfrak{m}$ endowed with the right action of H given by $(m, h) \mapsto m\iota_1(h)$ resp. $(m, h) \mapsto m\iota_2(h)$. Define analogously, for a left H-module $\mathfrak{m}$, the twisted modules $\iota_1\mathfrak{m}$ and $\iota_2\mathfrak{m}$. The map $m \mapsto m\tau_{t^{-1}}$ (resp. $m \mapsto \tau_t m$) yields an isomorphism of right H-modules $\mathfrak{m}\iota_1 \xrightarrow{\cong} \mathfrak{m}\iota_2$ (resp. $\iota_1\mathfrak{m} \xrightarrow{\cong} \iota_2\mathfrak{m}$).

(2) The map ι of Prop. 2.27(1) is an involution if and only if the chosen lift $\widetilde{\mathbf{w}}$ for $\mathbf{w}$ is such that $\widetilde{\mathbf{w}}^2$ is central in $\mathscr{N}_{\mathbb{G}}$. Since $\mathbf{w}^2 = 1$, we know that $\widetilde{\mathbf{w}}^2 \in \mathbb{Z}$. We pick $\widetilde{\mathbf{w}} := \mathbf{n}_{\mathbf{w}}$ as in Notation 2.8 and check that ι is then an involution. Since $\mathbb{Z}$ is commutative, it remains to show that $\mathbf{n}_{\mathbf{w}}^2$ commutes with $\mathbf{n}_s$ for any $s \in S$, and this follows from the formula (2.12).

Proof of Proposition 2.27. (1) If R is a field of characteristic p, it is proved in [47, Lemma 2.2] that the kernel of δ as defined in the lemma does not contain any non zero left or right ideal (see also [20, Prop. 6.11], the necessary hypotheses being satisfied by (2.13)). One easily checks that the proofs therein work over a ring of arbitrary characteristic.

The R-linear map ι given in the lemma is bijective. It remains to check that ι respects the product in H and that $\delta(\tau_u \tau_v) = \delta(\iota(\tau_v)\tau_u)$ for all $u, v \in \mathscr{N}_{\mathrm{G}}$. First we show that we only need to do it for one choice of lift in $\mathscr{N}_{\mathrm{G}}$ for $\mathbf{w}$. If n and $n' := tn$ with $t \in \mathbb{Z}$ are two such lifts, we denote respectively by (δ, ι) and (δ', ι') the pairs attached to them as in the proposition and we suppose that (δ, ι) satisfies the properties above. The conjugation by the invertible element τ_t is an automorphism of the algebra H. We have $\delta' = \delta(\tau_{t^{-1}}\_)$ and $\iota' = \iota(\tau_t \_ \tau_{t^{-1}})$. It implies that ι' also respects the product in H. Furthermore, for $u, v \in \mathscr{N}_{\mathrm{G}}$, we have $\delta'(\iota'(\tau_v)\tau_u) = \delta(\tau_{t^{-1}}\tau_{tnv(tn)^{-1}}\tau_u) = \delta(\iota(\tau_v)\tau_{t^{-1}u}) = \delta(\tau_{t^{-1}u}\tau_v) = \delta'(\tau_u\tau_v)$. So the pair (δ', ι') also satisfies the properties above. Consequently, it suffices to write the proof for a specific choice of lift for $\mathbf{w}$ and we choose $\mathbf{n_w} \in \mathscr{N}_{\mathrm{G}}$ as defined in Notation 2.8.

With this choice, we first check that ι respects the product. Let $u, v \in \mathscr{N}_{\mathrm{G}}$ and suppose that $\ell(u) + \ell(v) = \ell(uv)$. Conjugating by $\mathbf{n_w}$ preserves the length in $\mathscr{N}_{\mathrm{G}}$ since $\mathbf{w}(\Sigma^+) = \Sigma^-$. Therefore, we also have $\ell(\mathbf{n_w}u\mathbf{n_w}^{-1}) + \ell(\mathbf{n_w}v\mathbf{n_w}^{-1}) = \ell(\mathbf{n_w}uv\mathbf{n_w}^{-1})$ and $\iota(\tau_u\tau_v) = \iota(\tau_{uv}) = \tau_{\mathbf{n_w}uv\mathbf{n_w}^{-1}} = \tau_{\mathbf{n_w}u\mathbf{n_w}^{-1}}\tau_{\mathbf{n_w}v\mathbf{n_w}^{-1}} = \iota(\tau_u)\iota(\tau_v)$. Now let $s \in S$ and consider the quadratic relation $\tau_{\mathbf{n}_s}^2 = q_s\tau_{\mathbf{n}_s^2} + c_{\mathbf{n}_s}\tau_{\mathbf{n}_s}$ satisfied by $\tau_{\mathbf{n}_s}$ (see (2.18) and §2.6). First note that in fact we have $\mathbf{w}(\Pi) = -\Pi$, therefore $s' = \mathbf{w}s\mathbf{w}^{-1} \in S$ and $q_s = q_{s'}$ ([62, §2.1]). Furthermore, $\mathbf{n_w}\mathbf{n}_s\mathbf{n_w}^{-1} = \mathbf{n}_{s'}$ by (2.12) and $\mathbf{n_w}\mathbb{U}_s\mathbf{n_w}^{-1} = \mathbb{U}_{s'}$, $\mathbf{n_w}\mathbb{U}_s^{op}\mathbf{n_w}^{-1} = \mathbb{U}_{s'}^{op}$, $\mathbf{n_w}\mathbb{Z}'_s\mathbf{n_w}^{-1} = \mathbb{Z}'_{s'}$. In the formula (2.18) we have $c_{\mathbf{n}_s}(z) = |\mathbf{n}_s\mathbb{U}_s\mathbf{n}_s \cap \mathbb{U}_s\mathbf{n}_szU_s|$ for $z \in \mathbb{Z}'_s$ [62, Proposition 4.4, (62)], therefore $c_{\mathbf{n}_{s'}}(\mathbf{n_w} . \mathbf{n_w}^{-1}) = c_{\mathbf{n}_s}(.)$ as in Remark 2.17 hence $\iota(c_{\mathbf{n}_s}) = c_{\mathbf{n}_{s'}}$. We have

$$(\iota(\tau_{\mathbf{n}_s}))^2 = (\tau_{\mathbf{n}_{s'}})^2 = q_{s'}\tau_{\mathbf{n}_{s'}^2} + c_{\mathbf{n}_{s'}}\tau_{\mathbf{n}_{s'}}$$

while

$$\iota(\tau_{\mathbf{n}_s}^2) = \iota(q_s\tau_{\mathbf{n}_s^2} + c_{\mathbf{n}_s}\tau_{\mathbf{n}_s}) = q_s\tau_{\mathbf{n}_{s'}^2} + \iota(c_{\mathbf{n}_s})\iota(\tau_{\mathbf{n}_s}) = q_s\tau_{\mathbf{n}_{s'}^2} + c_{\mathbf{n}_{s'}}\tau_{\mathbf{n}_{s'}},$$

the second identity in the line above being true because $c_{\mathbf{n}_s}$ is a linear combination of τ_t for $t \in \mathbb{Z}$ and $\ell(t) = 0$. We have proved that $(\iota(\tau_{\mathbf{n}_s}))^2 = \iota(\tau_{\mathbf{n}_s}^2)$. This concludes the proof of the fact that ι respects the product.

Lastly, we show that $\delta(\tau_u\tau_v) = \delta(\iota(\tau_v)\tau_u)$ for all $u, v \in \mathscr{N}_{\mathrm{G}}$. The property is easy to check when $\ell(v) = 0$. We now show it when v has length 1 and then proceed by induction on $\ell(v)$. When v has length 1, it is enough to treat the case $v = \mathbf{n}_s$. Then as above, there is $s' \in S$ such that $\mathbf{n}_{s'} = \mathbf{n_w}\mathbf{n}_s\mathbf{n_w}^{-1}$. First suppose that $u\mathbf{n}_s = \mathbf{n_w}$. Then necessarily $\ell(\mathbf{n_w}) = \ell(u) + 1$ and $\tau_u\tau_{\mathbf{n}_s} = \tau_{\mathbf{n_w}}$ has image 1 by δ. On the other hand $\mathbf{n}_{s'}u = \mathbf{n_w}$ with $1 + \ell(u) = \ell(\mathbf{n_w})$, so the image of $\tau_{\mathbf{n}_{s'}}\tau_u = \iota(\tau_{\mathbf{n}_s})\tau_u$ by δ is also 1. Now suppose that $u\mathbf{n}_s \neq \mathbf{n_w}$ which implies $\mathbf{n}_{s'}u \neq \mathbf{n_w}$. This means in particular that if $\ell(u\mathbf{n}_s) = \ell(u) + 1$ (resp. $\ell(\mathbf{n}_{s'}u) = 1 + \ell(u)$), then $\delta(\tau_u\tau_{\mathbf{n}_s}) = 0$ (resp. $\delta(\tau_{\mathbf{n}_{s'}}\tau_u) = 0$).

> 1/ If $\ell(u\mathbf{n}_s) = \ell(u) + 1$ then the only case left to examine is $\ell(\mathbf{n}_{s'}u) = \ell(u) - 1$. The quadratic relation (2.18) implies that $\tau_{\mathbf{n}_{s'}}\tau_u$ is a R-linear combination of the elements τ_w for $w \in \mathscr{N}_{\mathrm{G}}$ with $\ell(w) \leq \ell(u) < \ell(u\mathbf{n}_s)$. Since none of these w is $\mathbf{n_w}$ we have $\delta(\tau_{\mathbf{n}_{s'}}\tau_u) = 0$.

2/ If $\ell(u\mathbf{n}_s) = \ell(u) - 1$, then again by (2.18) we know that $\tau_u \tau_{\mathbf{n}_s}$ is a R-linear combination of the elements τ_w for $w \in \mathscr{N}_{\mathrm{G}}$ with $\ell(w) \leq \ell(u)$.

2a/ If u does not have maximal length in $\mathscr{N}_{\mathrm{G}}$, then $\delta(\tau_u \tau_{\mathbf{n}_s}) = 0$ and the only case left to examine is $\ell(\mathbf{n}_{s'} u) = \ell(u) - 1$. But then by the same argument about the decomposition of $\tau_{\mathbf{n}_{s'}} \tau_u$, we get $\delta(\tau_{\mathbf{n}_{s'}} \tau_u) = 0$.

2b/ If u has maximal length, then $u = \mathbf{n}_{\mathbf{w}} t$ for $t \in \mathbb{Z}$.

On the one hand, $\tau_u \tau_{\mathbf{n}_s} = \tau_{\mathbf{n}_{\mathbf{w}}} \tau_t \tau_{\mathbf{n}_s} = \tau_{\mathbf{n}_{\mathbf{w}}} \tau_{\mathbf{n}_s} \tau_{\mathbf{n}_s^{-1} t \mathbf{n}_s} = \tau_{\mathbf{n}_{\mathbf{w}} \mathbf{n}_s^{-1}} \tau_{\mathbf{n}_s}^2 \tau_{\mathbf{n}_s^{-1} t \mathbf{n}_s}$ is equal to $\tau_{\mathbf{n}_{\mathbf{w}} \mathbf{n}_s^{-1}} (q_s \tau_{\mathbf{n}_s^2} + \tau_{\mathbf{n}_s} c_{n_s}) \tau_{\mathbf{n}_s^{-1} t \mathbf{n}_s}$ as $\tau_{\mathbf{n}_s}$ and c_{n_s} commute, hence $\tau_u \tau_{\mathbf{n}_s} = q_s \tau_{\mathbf{n}_{\mathbf{w}} t \mathbf{n}_s} + \tau_{\mathbf{n}_{\mathbf{w}}} c_{n_s} \tau_{\mathbf{n}_s^{-1} t \mathbf{n}_s}$. So

$$\delta(\tau_u \tau_{\mathbf{n}_s}) = c_{n_s}(\mathbf{n}_s^{-1} t^{-1} \mathbf{n}_s) = c_{n_s}(t^{-1}) = c_{n_s}(t).$$

The first equality above comes from the fact that c_{n_s} and $\tau_{\mathbf{n}_s}$ commute and the second equality is recalled after (2.18).

On the other hand, $\iota(\tau_{\mathbf{n}_s}) \tau_u = \tau_{\mathbf{n}_{s'}} \tau_u = \tau_{\mathbf{n}_{s'}} \tau_{\mathbf{n}_{\mathbf{w}}} \tau_t = \tau_{\mathbf{n}_{s'}}^2 \tau_{\mathbf{n}_{s'}^{-1} \mathbf{n}_{\mathbf{w}}} \tau_t$, so $\iota(\tau_{\mathbf{n}_s}) \tau_u = (q_{s'} \tau_{\mathbf{n}_{s'}^2} + c_{n_{s'}} \tau_{\mathbf{n}_{s'}}) \tau_{\mathbf{n}_{s'}^{-1} \mathbf{n}_{\mathbf{w}}} \tau_t = q_{s'} \tau_{\mathbf{n}_{s'} \mathbf{n}_{\mathbf{w}}} t + c_{n_{s'}} \tau_{\mathbf{n}_{\mathbf{w}}} \tau_t$. So,

$$\delta(\iota(\tau_{\mathbf{n}_s}) \tau_u) = c_{n_{s'}}(\mathbf{n}_{\mathbf{w}} t^{-1} \mathbf{n}_{\mathbf{w}}^{-1}) = c_{n_s}(t^{-1}) = c_{n_s}(t).$$

because $\mathbf{n}_{\mathbf{w}} c_{n_s} \mathbf{n}_{\mathbf{w}}^{-1} = c_{n_{s'}}$ (Remark 2.17), in other terms $\sum_{t \in \mathbb{Z}} c_{n_s}(t) \mathbf{n}_{\mathbf{w}} t \mathbf{n}_{\mathbf{w}}^{-1} = \sum_{t \in \mathbb{Z}} c_{n_{s'}}(t) t$, and $\mathbf{n}_{\mathbf{w}}^2 \in \mathbb{Z}$.

This concludes the proof of $\delta(\iota(\tau_v) \tau_u) = \delta(\tau_u \tau_v)$.

We now prove part (2) of Prop. 2.27. Here R is a ring of characteristic different from p and $\delta' : \mathrm{H} \to R$ is the map sending τ_1 to 1 and τ_w to zero for all $w \in \mathscr{N}_{\mathrm{G}} - \{1\}$. It is proved in [47, Lemma 2.3] that δ' is symmetric and non degenerate when R is a field of characteristic different from p. The arguments go through in the case of a ring of characteristic different from p. We recall first the argument of the non degeneracy of δ'. The support of an element in H of the form $a = \sum_{w \in \mathscr{N}_{\mathrm{G}}} \lambda_w \tau_w$, $\lambda_w \in R$, is defined to be the set of all w such that $\lambda_w \neq 0$ and its height, if $a \neq 0$, the minimal length of the elements in its support. If a has height 0, then there is $t \in \mathbb{Z}$ in its support and $\delta'(a \tau_t^{-1}) \neq 0$ so the right ideal generated by a is not contained in the kernel of δ. If a has height $k \geq 1$, then there is $s \in S$ such that $a \tau_{\mathbf{n}_s}$ has height $k - 1$. This is because there is $w_0 \in \mathscr{N}_{\mathrm{G}}$ in the support of a with length k and we choose $s \in S$ such that $\ell(w_0 \mathbf{n}_s) = \ell(w_0) - 1$. Then $a \tau_{\mathbf{n}_s}$ is the sum

- of $\sum_{w, \ell(w) \geq k+1} \lambda_w \tau_w \tau_{\mathbf{n}_s}$ which has height $\geq k$ if it is not zero (use (2.17) and (2.18)),
- of $\sum_{w, \ell(w) = k, \ell(w \mathbf{n}_s) = \ell(w) + 1} \lambda_w \tau_{w \mathbf{n}_s}$ which has height $k + 1$ if it is not zero,
- and of $\sum_{w, \ell(w) = k, \ell(w \mathbf{n}_s) = k - 1} \lambda_w (q_s \tau_{w \mathbf{n}_s} + \tau_w c_{n_s})$ which has height $k - 1$ since $\lambda_{w_0} q_s \neq 0$ in R.

By induction on the height of a we obtain that the right ideal generated by a is not contained in the kernel of δ'. One would proceed the same way for left ideals.

It remains to check that δ' is symmetric that is to say that $\delta'(\tau_w \tau_x) = \delta'(\tau_x \tau_w)$ for any $x, w \in \mathscr{N}_{\mathrm{G}}$. The identity is clear when x or w has length zero. Now suppose $x = \mathbf{n}_s$ for some $s \in S$ and w has length ≥ 1. By (2.18) both $\tau_w \tau_{\mathbf{n}_s}$ and $\tau_{\mathbf{n}_s} \tau_w$ are linear combinations of elements τ_y with $\ell(y) \geq \ell(w) - 1$. So

if $\ell(w) \geq 2$, we have $\delta'(\tau_w \tau_{\mathbf{n}_s}) = \delta'(\tau_{\mathbf{n}_s} \tau_w) = 0$. Now suppose $w = \mathbf{n}_{s'}$ for some $s' \in S$: if $s \neq s'$, then $\delta'(\tau_w \tau_{\mathbf{n}_s}) = \delta'(\tau_{\mathbf{n}_s} \tau_w) = 0$; otherwise, $\tau_w \tau_{\mathbf{n}_s} = \tau_{\mathbf{n}_s}^2 = \tau_{\mathbf{n}_s} \tau_w$ so these elements have the same image by δ'. We conclude that $\delta'(\tau_w \tau_x) = \delta'(\tau_x \tau_w)$ for any $x, w \in \mathscr{N}_{\mathrm{G}}$ by induction on the length of x.

Note that that δ' is symmetric for any ring R (in a general context [54, Prop. I.3.6]). It is only for its non degeneracy that we need R to have characteristic different from p.

$\square$

Notation 2.29. *Let A, B, C be R-algebras, $\alpha : A \to B$ and $\beta : B \to C$ morphisms of algebras. For any left, resp. right, B-module $\mathfrak{m}$ we let $\alpha \mathfrak{m}$, resp. $\mathfrak{m}\alpha$, denote the space $\mathfrak{m}$ with a left, resp. right, A-action through the endomorphism α namely $(a, m) \mapsto \alpha(a)m$, resp. $(m, a) \mapsto m\alpha(a)$. Note that, for a right C-module $\mathfrak{m}$,*

$$(\mathfrak{m}\beta)\alpha \cong \mathfrak{m}\,\beta \circ \alpha \quad \text{as right } A\text{-modules}$$

and for a left C-module $\mathfrak{m}$,

$$\alpha(\beta\mathfrak{m}) \cong \beta \circ \alpha\,\mathfrak{m} \quad \text{as left } A\text{-modules}.$$

If D is an R-algebra and $\gamma : C \to D$ a morphism of algebras, let $\mathfrak{m}$ be a (B, D)-bimodule. Then we denote simply by $\alpha\mathfrak{m}\gamma$ the (A, C)-bimodule $\alpha(\mathfrak{m}\gamma) \cong (\alpha\mathfrak{m})\gamma$.

We attach to $\mathbb{M}$ the choice of a pair $(\delta_{\mathbb{M}}, \iota_{\mathbb{M}})$ as in Proposition 2.27(1). Let $j = \iota_{\mathbb{M}}$ or $\iota_{\mathbb{M}}^{-1}$. Let $\mathfrak{m}$ be a left, resp. right, $\mathrm{H}_{\mathbb{M}}$-module $\mathfrak{m}$; by Remark 2.28(1), the structure of left, resp. right, $\mathrm{H}_{\mathbb{M}}$-module of $j\mathfrak{m}$, resp. $\mathfrak{m}j$, does not depend on the choice of $(\delta_{\mathbb{M}}, \iota_{\mathbb{M}})$.

Proposition 2.30. *There is an isomorphism of $(\mathrm{H}, \mathrm{H}_{\mathbb{M}})$-bimodules*

(2.31) $$\mathrm{Hom}_{\mathrm{H}_{\mathbb{M}}}(\mathrm{H}, \mathrm{H}_{\mathbb{M}}) \cong \mathrm{H}\iota^{-1}\iota_{\mathbb{M}}$$

where the right action of $\mathrm{H}_{\mathbb{M}}$ on the right hand side is via the embedding

$$\mathrm{H}_{\mathbb{M}} \xrightarrow{\iota^{-1} \circ \iota_{\mathbb{M}}} \mathrm{H}.$$

Proof. First recall that given a $(\mathrm{H}_{\mathbb{M}}, \mathrm{H})$-bimodule $\mathfrak{m}$, there is a natural structure of $(\mathrm{H}, \mathrm{H}_{\mathbb{M}})$-bimodule on

- $\mathrm{Hom}_{\mathrm{H}_{\mathbb{M}}}(\mathfrak{m}, \mathrm{H}_{\mathbb{M}})$ given by $((h, h_{\mathbb{M}}), f) \mapsto [m \mapsto f(m.h)h_{\mathbb{M}}]$,
- $\mathrm{Hom}_{\mathrm{H}}(\mathfrak{m}, \mathrm{H})$ given by $((h, h_{\mathbb{M}}), f) \mapsto [m \mapsto h\,f(h_{\mathbb{M}}.m)]$,
- $\mathrm{Hom}_R(\mathfrak{m}, R)$ given by $((h, h_{\mathbb{M}}), f) \mapsto [m \mapsto f(h_{\mathbb{M}}.m.h)]$.

We will check below the following properties. The map $f \mapsto \delta_{\mathbb{M}} \circ f$ induces

(1) an isomorphism of right, resp. left, $\mathrm{H}_{\mathbb{M}}$-modules between $\mathrm{H}_{\mathbb{M}} \cong \mathrm{Hom}_{\mathrm{H}_{\mathbb{M}}}(\mathrm{H}_{\mathbb{M}}, \mathrm{H}_{\mathbb{M}})$ and $\mathrm{Hom}_R(\iota_{\mathbb{M}}\mathrm{H}_{\mathbb{M}}, R)$ (resp. $\mathrm{Hom}_R(\mathrm{H}_{\mathbb{M}}\iota_{\mathbb{M}}^{-1}, R)$),

(2) an isomorphism of $(\mathrm{H}, \mathrm{H}_{\mathbb{M}})$-bimodules between $\mathrm{Hom}_{\mathrm{H}_{\mathbb{M}}}(\mathrm{H}, \mathrm{H}_{\mathbb{M}})$ and $\mathrm{Hom}_R(\iota_{\mathbb{M}}\mathrm{H}, R)$.

and the map $f \mapsto \delta \circ f$ induces

(3) an isomorphism of $(\mathrm{H}, \mathrm{H}_{\mathbb{M}})$-bimodules between $\mathrm{Hom}_{\mathrm{H}}(\iota_{\mathbb{M}}\mathrm{H}\iota, \mathrm{H})$ and $\mathrm{Hom}_R(\iota_{\mathbb{M}}\mathrm{H}, R)$.

Combining (2) and (3) gives an isomorphism of $(H, H_{\mathbb{M}})$-bimodules $\mathrm{Hom}_H(\iota_{\mathbb{M}}H\iota, H) \cong \mathrm{Hom}_{H_{\mathbb{M}}}(H, H_{\mathbb{M}})$. Now consider the natural bijection

$$\mathrm{Hom}_H(\iota_{\mathbb{M}}H\iota, H) \longrightarrow H\iota^{-1}\iota_{\mathbb{M}}, \ f \longmapsto f(1).$$

It is a morphism of $(H, H_{\mathbb{M}})$-bimodules since $(hfh_{\mathbb{M}})(1) = h(f(h_{\mathbb{M}}.1)) = h(f(\iota_{\mathbb{M}}(h_{\mathbb{M}}))) = h(f(1.\iota^{-1}(\iota_{\mathbb{M}}(h_{\mathbb{M}})))) = hf(1)\iota^{-1} \circ \iota_{\mathbb{M}}(h_{\mathbb{M}})$ for $(h, h_{\mathbb{M}}) \in H \times H_{\mathbb{M}}$. This proves the Proposition provided we verify (1), (2) and (3).

We prove (1). We write the proof in the case $\mathbb{M} = \mathbb{G}$ so as to avoid the multiple subscripts. The map $H \to \mathrm{Hom}_R(\iota H, R)$ that we are studying is denoted by F in this proof. It sends $h \in H$ onto $x \mapsto \delta(hx)$. It is easy to see that it is right H-equivariant. It is injective since the kernel of δ does not contain any non zero left ideal (Prop. 2.27(1)). We need to prove that it is surjective. For any $w \in \mathscr{N}_{\mathbb{G}}$, denote by $e_w \in \mathrm{Hom}_R(\iota H, R)$ the map sending τ_w onto $1 \in R$ and τ_n onto 0 for all $n \in \mathscr{N}_{\mathbb{G}} - \{w\}$. These elements form a R-basis of $\mathrm{Hom}_R(\iota H, R)$ and we need to prove that they all lie in the image of F. If $w \in \mathscr{N}_{\mathbb{G}}$ has maximal length, then there is $t \in \mathbb{Z}$ such that $wt = \mathbf{n_w}$ and $e_w = F(\tau_t)$. Now let k with $1 \leq k \leq \ell(\mathbf{w})$ and suppose that e_w lies in the image of F for any $w \in \mathscr{N}_{\mathbb{G}}$ such that $\ell(w) \geq k$. Let $v \in \mathscr{N}_{\mathbb{G}}$ with length $k - 1$ and $s \in S$ such that $\ell(\mathbf{n}_s v) = \ell(v) + 1 = k$. By definition, the image of $e_{\mathbf{n}_s v}$ under the right action of $\tau_{\mathbf{n}_s}$ is the R-linear form $e := e_{\mathbf{n}_s v}(\tau_{\mathbf{n}_s} -)$ which lies in the image of F by hypothesis and since F is right H-equivariant. We have $e(\tau_v) = 1$ and we now prove that for $x \in \mathscr{N}_{\mathbb{G}} - \{v\}$, if τ_x is in the support of e then x has length $\geq k$. Let $x \in \mathscr{N}_{\mathbb{G}} - \{v\}$ such that τ_x is in the support of e. First notice that we have $\ell(\mathbf{n}_s x) = \ell(x) - 1$. Otherwise we would have $\tau_{\mathbf{n}_s}\tau_x = \tau_{\mathbf{n}_s x}$ and $v = x$ since x is in the support of e, contradiction. Therefore, $\tau_{\mathbf{n}_s}\tau_x = \tau_{\mathbf{n}_s}^2 \tau_{\mathbf{n}_s^{-1} x} = (q_s \tau_{\mathbf{n}_s^2} + \sum_{z \in \mathbb{Z}'_s} c_{n_s}(z)\tau_z \tau_{\mathbf{n}_s})\tau_{\mathbf{n}_s^{-1} x} = q_s \tau_{\mathbf{n}_s x} + \sum_{z \in \mathbb{Z}'_s} c_{n_s}(z)\tau_{zx}$. Since τ_x is in the support of e, we have either $\mathbf{n}_s x = \mathbf{n}_s v$ and $x = v$ which is not true, or $zx = \mathbf{n}_s v$ for some $z \in \mathbb{Z}'_s$ and x has length k. It proves the claim. This shows that e is the sum of e_v and of e_x for $x \in \mathscr{N}_{\mathbb{G}}$ with length $\geq k$. By induction hypothesis and since e lies in the image of F, it implies that e_v lies in the image of F. We have proved that F is surjective. The proof of the isomorphism between H and $\mathrm{Hom}_R(H\iota^{-1}, R)$ is similar.

We now prove (2). By (1), the map $f \mapsto \delta_{\mathbb{M}} \circ f$ gives a linear isomorphism $\mathrm{Hom}_{H_{\mathbb{M}}}(H_{\mathbb{M}}, H_{\mathbb{M}}) \xrightarrow{\cong} \mathrm{Hom}_R(H_{\mathbb{M}}, R)$. Recall that H is a free left $H_{\mathbb{M}}$-module (Prop. 2.26). This implies that the map $\phi \mapsto \delta_{\mathbb{M}} \circ \phi$ defines a linear isomorphism $\mathrm{Hom}_{H_{\mathbb{M}}}(H, H_{\mathbb{M}}) \cong \mathrm{Hom}_R(H, R)$. It remains to verify that it induces a morphism of $(H, H_{\mathbb{M}})$-bimodules $\mathrm{Hom}_{H_{\mathbb{M}}}(H, H_{\mathbb{M}}) \cong \mathrm{Hom}_R(\iota_{\mathbb{M}}H, R)$ which is immediate.

To prove (3), first notice that the map $f \mapsto \delta \circ f$ induces a morphism of $(H, H_{\mathbb{M}})$-bimodules from $\mathrm{Hom}_H(\iota_{\mathbb{M}}H\iota, H) \to \mathrm{Hom}_R(\iota_{\mathbb{M}}H, R)$. Using (1) with $\mathbb{M} = \mathbb{G}$, we know that for any element $\varphi \in \mathrm{Hom}_R(H, R)$ there is a unique $f \in \mathrm{Hom}_H(H, H)$ such that $\varphi = \delta \circ f$. But precomposing by ι^{-1} induces a linear isomorphism $\mathrm{Hom}_H(H, H) \xrightarrow{\cong} \mathrm{Hom}_H(\iota_{\mathbb{M}}H\iota, H)$ so $\varphi = \delta \circ f \circ \iota^{-1}$ where $f \circ \iota^{-1} \in \mathrm{Hom}_H(\iota_{\mathbb{M}}H\iota, H)$. $\square$

Remark 2.31. If R is a ring where p is invertible, we may attach to $\mathbb{M}$ the non degenerate symmetric map $\delta'_{\mathbb{M}} : H_{\mathbb{M}} \to R$ as in Proposition 2.27(2) and the corresponding automorphism $\iota'_{\mathbb{M}}$ of $H_{\mathbb{M}}$ is the identity. We then have an isomorphism of $(H, H_{\mathbb{M}})$-bimodules $\mathrm{Hom}_{H_{\mathbb{M}}}(H, H_{\mathbb{M}}) \cong H$.

The strategy to prove this is the same as the one in the proof of Prop. 2.30 with δ', $\delta'_{\mathbb{M}}$, ι' and $\iota'_{\mathbb{M}}$ instead of, respectively, δ, $\delta_{\mathbb{M}}$, ι and $\iota_{\mathbb{M}}$. All arguments go through immediately except for the analogous of point (1) of that proof: we verify here that the map $F' : \mathrm{H} \to \mathrm{Hom}_R(\mathrm{H}, R)$ sending $h \in \mathrm{H}$ onto $x \mapsto \delta'(hx)$ is surjective. We need to prove that e_w (same notation as in the above mentioned proof) lies in the image of F' for any $w \in \mathscr{N}_{\mathbb{G}}$. If w has length zero, then $e_w = F'(\tau_{w^{-1}})$. Now suppose that e_w lies in the image of F' for all w with $\ell(w) \leq k$. Let $v \in \mathscr{N}_{\mathbb{G}}$ with length $k+1$ and $s \in S$ such that $\ell(\mathbf{n}_s v) = \ell(v) - 1$. The R-linear form $e := e_{\mathbf{n}_s v}.\tau_{\mathbf{n}_s} = e_{\mathbf{n}_s v}(\tau_{\mathbf{n}_s} -)$ lies in the image of F' by hypothesis and since F' is right H-equivariant. Because p is invertible in R and using (2.18), we have $e(\tau_v) = q_s \neq 0$. In fact, still using (2.18) one easily checks that $e = q_s e_v$. So e_v lies in the image of F'.

Remark 2.32. Suppose that R is a field. Then the above results follow from the theory of Frobenius algebras. We mentioned in (2.13) that $(\mathbb{M}, \mathbb{B}_{\mathbb{M}}, \mathscr{N}_{\mathbb{M}}, S_J)$ together with the decomposition $\mathbb{B}_{\mathbb{M}} = \mathbb{Z}\mathbb{U}_{\mathbb{M}}$ is a (strongly) split BN-pair of characteristic p [20, Def. 2.20]. The results of [52, Prop. 3.7] [47, Thm. 2.4],[20, Prop. 6.11] apply to the finite Hecke algebras $\mathrm{H}_{\mathbb{M}}$. In particular, they are Frobenius. If R is a field, then the proof of Prop. 2.30 simplifies greatly (by argument of dimension) and one can prove in fact more generally that for $\mathfrak{m}$ a $(\mathrm{H}_{\mathbb{M}}, \mathrm{H})$-bimodule there is an isomorphism of $(\mathrm{H}, \mathrm{H}_{\mathbb{M}})$-bimodules $\mathrm{Hom}_{\mathrm{H}}(\iota_{\mathbb{M}}\mathfrak{m}\iota, \mathrm{H}) \cong \mathrm{Hom}_{\mathrm{H}_{\mathbb{M}}}(\mathfrak{m}, \mathrm{H}_{\mathbb{M}})$.

Remark 2.33. Prop. 2.27 (2) implies that if R is a field of characteristic different from p, then $\mathrm{H}_{\mathbb{M}}$ is not only Frobenius but also symmetric. If R is a field of characteristic p, then $\mathrm{H}_{\mathbb{M}}$ is not symmetric in general [47, Addendum to Theorem 24].

3. Parabolic induction for the finite group and unipotent finite Hecke algebra

Let R be a commutative ring. For $\mathbb{P}$ a standard parabolic subgroup of $\mathbb{G}$ with Levi decomposition $\mathbb{P} = \mathbb{M}\mathbb{N}$, we consider the category $\mathrm{Rep}(\mathbb{M})$ (resp. $\mathrm{Rep}(\mathbb{P})$) of R-representations of $\mathbb{M}$ (resp. $\mathbb{P}$), that is, the category of left $R[\mathbb{M}]$-modules (resp. $R[\mathbb{P}]$-modules), and the category $\mathrm{Mod}(\mathrm{H}_{\mathbb{M}})$ of right $\mathrm{H}_{\mathbb{M}}$-modules.

3.1. Parabolic induction and restriction for the representations of the finite group.
For $\mathrm{V} \in \mathrm{Rep}(\mathbb{M})$, we consider the representation of $\mathbb{G}$ on the space $\mathrm{Ind}_{\mathbb{P}}^{\mathbb{G}}(\mathrm{V})$ of functions $f : \mathbb{G} \to \mathrm{V}$ such that $f(mng) = m.f(g)$ for all $m \in \mathbb{M}$, $n \in \mathbb{N}$ and $g \in \mathbb{G}$. The action of $\mathbb{G}$ is by right translations, $(g, f) \mapsto f(\,.\,g)$. This define the functor of parabolic induction

$$\mathrm{Ind}_{\mathbb{P}}^{\mathbb{G}} : \mathrm{Rep}(\mathbb{M}) \longrightarrow \mathrm{Rep}(\mathbb{G}).$$

For $\mathrm{V} \in \mathrm{Rep}(\mathbb{G})$, we consider the representation of $\mathbb{M}$ on the $\mathbb{N}$-invariant subspace $\mathrm{V}^{\mathbb{N}}$ which defines a functor denoted by

$$(-)^{\mathbb{N}} : \mathrm{Rep}(\mathbb{G}) \longrightarrow \mathrm{Rep}(\mathbb{M})$$

and the representation of $\mathbb{M}$ on the $\mathbb{N}$-coinvariant space $\mathrm{V}_{\mathbb{N}}$ which defines a functor denoted by

$$(-)_{\mathbb{N}} : \mathrm{Rep}(\mathbb{G}) \longrightarrow \mathrm{Rep}(\mathbb{M}).$$

Proposition 3.1. *The functor* $\mathrm{Ind}_{\mathbb{P}}^{\mathbb{G}}$ *is faithful, exact, of right adjoint* $(-)^{\mathbb{N}}$ *and left adjoint* $(-)_{\mathbb{N}}$.

Remark 3.2. The $\mathbb{N}$-invariant and $\mathbb{N}$-coinvariants functors are more naturally defined on the category $\mathrm{Rep}(\mathbb{P})$ of the R-representations of $\mathbb{P}$, where they are respectively the right and left adjoint functors of the inflation $\mathrm{Rep}(\mathbb{M}) \to \mathrm{Rep}(\mathbb{P})$.

Proof. Although this property is well known we do not know a reference and we give a proof. By Frobenius reciprocity, the restriction $\mathrm{Rep}(\mathbb{G}) \to \mathrm{Rep}(\mathbb{P})$ from $\mathbb{G}$ to $\mathbb{P}$ admits a left and right adjoint equal respectively to the induction $R[\mathbb{G}] \otimes_{R[\mathbb{P}]} -$ and to the coinduction $\mathrm{Hom}_{R[\mathbb{P}]}(R[\mathbb{G}], -)$ [8, 2.8.2]. Here, $R[\mathbb{G}]$ is seen as a right $R[\mathbb{P}]$-module for the induction and as a left $R[\mathbb{P}]$-module for the coinduction. It is well known that the coinduction and induction functors $\mathrm{Rep}(\mathbb{P}) \to \mathrm{Rep}(\mathbb{G})$ coincide ([8, §3.3]).

The parabolic induction $\mathrm{Ind}_{\mathbb{P}}^{\mathbb{G}}$ is the composite of the inflation $\mathrm{inf}_{\mathbb{P}} : \mathrm{Rep}(\mathbb{M}) \to \mathrm{Rep}(\mathbb{P})$ from $\mathbb{M}$ to $\mathbb{P}$ and of the coinduction $\mathrm{Rep}(\mathbb{P}) \to \mathrm{Rep}(\mathbb{G})$. As the $\mathbb{N}$-coinvariant functor $\mathrm{Rep}(\mathbb{P}) \to \mathrm{Rep}(\mathbb{M})$ and the $\mathbb{N}$-invariant functor $\mathrm{Rep}(\mathbb{P}) \to \mathrm{Rep}(\mathbb{M})$ are respectively the left adjoint and the right adjoint of the inflation, the functor $(-)_{\mathbb{N}}$ is a left adjoint of $\mathrm{Ind}_{\mathbb{P}}^{\mathbb{G}}$ and the functor $(-)^{\mathbb{N}}$ is a right adjoint of $\mathrm{Ind}_{\mathbb{P}}^{\mathbb{G}}$. As a functor with a left and a right adjoint, $\mathrm{Ind}_{\mathbb{P}}^{\mathbb{G}}$ is exact.

Lastly, it is easy to see that $\mathrm{Ind}_{\mathbb{P}}^{\mathbb{G}}$ is faithful since $R[\mathbb{G}]$ is a free right $R[\mathbb{P}]$-module. $\square$

Remark 3.3. Suppose that R is a field and consider the contravariant endofunctor $\mathrm{V} \mapsto \mathrm{V}^{\vee}$ of $\mathrm{Rep}(\mathbb{G})$ attaching to a representation V its contragredient representation $\mathrm{V}^{\vee} = \mathrm{Hom}_R(\mathrm{V}, R)$. We check that it commutes with parabolic induction, *i.e.* for any representation V of $\mathbb{M}$ we have a natural isomorphism of representations of $\mathbb{G}$:

$$\mathrm{Ind}_{\mathbb{P}}^{\mathbb{G}}(\mathrm{V}^{\vee}) \longrightarrow \mathrm{Ind}_{\mathbb{P}}^{\mathbb{G}}(\mathrm{V})^{\vee}.$$

Let $\langle\, .\, ,\, .\, \rangle : \mathrm{V}^{\vee} \times \mathrm{V} \to k$ denote the duality between V and $\mathrm{V}^{\vee}$. To an element $\xi \in \mathrm{Ind}_{\mathbb{P}}^{\mathbb{G}}(\mathrm{V}^{\vee})$ we associate the linear map

$$\mathrm{Ind}_{\mathbb{P}}^{\mathbb{G}}(\mathrm{V}) \to R, \ f \mapsto \sum_{g \in \mathbb{P} \backslash \mathbb{G}} \langle \xi(g), f(g) \rangle.$$

This clearly defines an injective linear map $\mathrm{Ind}_{\mathbb{P}}^{\mathbb{G}}(\mathrm{V}^{\vee}) \to (\mathrm{Ind}_{\mathbb{P}}^{\mathbb{G}}(\mathrm{V}))^{\vee}$. It is surjective: $\Phi \in (\mathrm{Ind}_{\mathbb{P}}^{\mathbb{G}}(\mathrm{V}))^{\vee}$ is the image of the element $\xi \in \mathrm{Ind}_{\mathbb{P}}^{\mathbb{G}}(\mathrm{V}^{\vee})$ such that $\langle \xi(g), v \rangle = \Phi(f_{\mathbb{P}g,v})$, where $f_{\mathbb{P}g,v} \in \mathrm{Ind}_{\mathbb{P}}^{\mathbb{G}}(\mathrm{V})$ is the function with support in $\mathbb{P}g$ and value v at g.

Now to a representation $\mathrm{V} \in \mathrm{Rep}(\mathbb{P})$, we can also attach its contragredient representation $\mathrm{V}^{\vee} = \mathrm{Hom}_R(\mathrm{V}, R)$. The contragredient of the $\mathbb{N}$-coinvariants of V and the $\mathbb{N}$-invariants of the contragredient of V are functorially isomorphic representations of $\mathbb{M}$: there is a natural isomorphism of representations of $\mathbb{M}$:

$$(3.1) \qquad\qquad\qquad\qquad (\mathrm{V}_{\mathbb{N}})^{\vee} \cong (\mathrm{V}^{\vee})^{\mathbb{N}}.$$

Denote by $\mathrm{V}(\mathbb{N})$ the subspace of V generated by $nv - v$ for all $n \in \mathbb{N}, v \in \mathrm{V}$. By definition, $\mathrm{V}_{\mathbb{N}} = \mathrm{V}/\mathrm{V}(\mathbb{N})$. An element of $\mathrm{V}^{\vee}$ is $\mathbb{N}$-invariant if and only it vanishes on $\mathrm{V}(\mathbb{N})$. Therefore $(\mathrm{V}_{\mathbb{N}})^{\vee} \simeq (\mathrm{V}^{\vee})^{\mathbb{N}}$. It is obvious that the isomorphism is $\mathbb{M}$-equivariant and functorial.

3.2. **Induction, coinduction and restriction for finite Hecke modules.** We recall that $\mathrm{H}_{\mathbb{M}}$ is isomorphic to the subalgebra of H of basis $(\tau_w)_{w \in \mathscr{N}_{\mathbb{M}}}$. We consider the natural restriction functor

$$\mathrm{Res}_{\mathrm{H}_{\mathbb{M}}}^{\mathrm{H}} = - \otimes_{\mathrm{H}} \mathrm{H} : \mathrm{Mod}(\mathrm{H}) \longrightarrow \mathrm{Mod}(\mathrm{H}_{\mathbb{M}})$$

where H is seen as a $(H, H_{\mathbb{M}})$-bimodule. For $\mathfrak{m}$ a right $H_{\mathbb{M}}$-module, we consider the induced H-module $\mathfrak{m} \otimes_{H_{\mathbb{M}}} H$ where H is seen as a $(H_{\mathbb{M}}, H)$-bimodule. This defines the functor of induction for Hecke modules, which we will sometimes call parabolic induction by analogy with the definitions in §3.1:

$$\mathrm{Ind}_{H_{\mathbb{M}}}^{H} = - \otimes_{H_{\mathbb{M}}} H : \mathrm{Mod}(H_{\mathbb{M}}) \longrightarrow \mathrm{Mod}(H).$$

It is left adjoint to $\mathrm{Res}_{H_{\mathbb{M}}}^{H}$. The space $\mathrm{Hom}_{H_{\mathbb{M}}}(H, \mathfrak{m})$ has a structure of right H-module given by $(f, h) \mapsto [f.h : x \mapsto f(hx)]$ and this defines the functor of coinduction for Hecke modules

$$\mathrm{Coind}_{H_{\mathbb{M}}}^{H} = \mathrm{Hom}_{H_{\mathbb{M}}}(H, -) : \mathrm{Mod}(H_{\mathbb{M}}) \longrightarrow \mathrm{Mod}(H).$$

It is right adjoint to $\mathrm{Res}_{H_{\mathbb{M}}}$.

The functor $\mathrm{Ind}_{H_{\mathbb{M}}}^{H}$ has a left adjoint because it commutes with small projective limits as the left $H_{\mathbb{M}}$-module H is free (Prop. 2.26). We denote it by

$$\mathrm{L}_{H_{\mathbb{M}}}^{H} : \mathrm{Mod}(H) \longrightarrow \mathrm{Mod}(H_{\mathbb{M}})$$

and describe it explicitly in the following proposition.

Recall that we attach to $\mathbb{M}$ the choice of a pair $(\delta_{\mathbb{M}}, \iota_{\mathbb{M}})$ as in Proposition 2.27(1). We refer to the comments before Prop. 2.31.

Proposition 3.4. *Let R be a ring. Let $\mathbb{M}' := \mathbf{w} \mathbb{M} \mathbf{w}^{-1}$.*
The isomorphism of algebras $H_{\mathbb{M}'} \xrightarrow{\iota_{\mathbb{M}}^{-1} \circ \iota} H_{\mathbb{M}}$ induces an equivalence of categories $- \iota_{\mathbb{M}}^{-1} \iota : \mathrm{Mod}(H_{\mathbb{M}}) \to \mathrm{Mod}(H_{\mathbb{M}'})$ with quasi-inverse $- \iota^{-1} \iota_{\mathbb{M}} : \mathrm{Mod}(H_{\mathbb{M}'}) \to \mathrm{Mod}(H_{\mathbb{M}})$.
The functors

$$- \otimes_{H_{\mathbb{M}}} H \text{ and } \mathrm{Hom}_{H_{\mathbb{M}'}}(H, - \iota_{\mathbb{M}}^{-1} \iota) \ : \ \mathrm{Mod}(H_{\mathbb{M}}) \to \mathrm{Mod}(H)$$

are naturally isomorphic that is to say

$$\mathrm{Ind}_{H_{\mathbb{M}}}^{H} \cong \mathrm{Coind}_{H_{\mathbb{M}'}}^{H}(- \iota_{\mathbb{M}}^{-1} \iota).$$

The left adjoint of the functor $\mathrm{Ind}_{H_{\mathbb{M}}}^{H}$ is

$$\mathrm{L}_{H_{\mathbb{M}}}^{H} = (\mathrm{Res}_{H_{\mathbb{M}'}}^{H}(-)) \iota^{-1} \iota_{\mathbb{M}} : \mathrm{Mod}(H) \to \mathrm{Mod}(H_{\mathbb{M}}).$$

In particular, $\mathrm{L}_{H_{\mathbb{M}}}^{H}$ is an exact functor.

Proof. For A and B a right and respectively left $H_{\mathbb{M}}$-module, we have a natural transformation

$$A \otimes_{H_{\mathbb{M}}} B \longrightarrow \mathrm{Hom}_{H_{\mathbb{M}}}(\mathrm{Hom}_{H_{\mathbb{M}}}(B, H_{\mathbb{M}}), A), \ a \otimes b \mapsto \varphi : f \mapsto af(b).$$

When $B = H_{\mathbb{M}}$ and therefore when B is a free left $H_{\mathbb{M}}$-module, it is clearly an isomorphism.

By Proposition 2.26, the left $H_{\mathbb{M}}$-module H is free (of finite rank). It follows that the natural transformation above when $B = H$ yields an isomorphism of functors:

$$- \otimes_{H_{\mathbb{M}}} H \cong \mathrm{Hom}_{H_{\mathbb{M}}}(\mathrm{Hom}_{H_{\mathbb{M}}}(H, H_{\mathbb{M}}), -) : \mathrm{Mod}(H_{\mathbb{M}}) \longrightarrow \mathrm{Mod}(H).$$

Combining this with (2.31), we obtain an isomorphism of functors $\mathrm{Mod}(H_{\mathbb{M}}) \to \mathrm{Mod}(H)$

$$- \otimes_{H_{\mathbb{M}}} H \cong \mathrm{Hom}_{H_{\mathbb{M}}}(H \iota^{-1} \iota_{\mathbb{M}}, -)$$

where we recall that the right action of $H_{\mathbb{M}}$ on $H\iota^{-1}\iota_{\mathbb{M}}$ is via the embedding $H_{\mathbb{M}} \xrightarrow{\iota^{-1}\circ\iota_{\mathbb{M}}} H$. The automorphisms ι and $\iota_{\mathbb{M}}$ are described explicitly in Prop. 2.27(1). Let $\widetilde{\mathbf{w}}$ (resp. $\widetilde{\mathbf{w}_{\mathbb{M}}}$) be a lift in $\mathscr{N}_{G}$ (resp. $\mathscr{N}_{\mathbb{M}}$) of the longest element $\mathbf{w}$ of $\mathbb{W}$ (resp. $\mathbf{w}_{\mathbb{M}}$ of $\mathbb{W}_{M}$). We pick $\widetilde{\mathbf{w}_{\mathbb{M}'}} := \widetilde{\mathbf{w}}^{-1}\widetilde{\mathbf{w}_{\mathbb{M}}}\widetilde{\mathbf{w}}$ as a lift in $\mathscr{N}_{\mathbb{M}'}$ of the longest element $\mathbf{w}_{\mathbb{M}'}$ of $\mathbb{W}_{\mathbb{M}'}$. The corresponding automorphism $\iota_{\mathbb{M}'}$ of $H_{\mathbb{M}'}$ as defined in Prop. 2.27(1) then coincides with the restriction to $H_{\mathbb{M}'}$ of $\iota^{-1}\iota_{\mathbb{M}}\iota$.

For any right $H_{\mathbb{M}}$-module $\mathfrak{m}$, have the following isomorphisms of right H-modules (see Notation 2.29):

$$\operatorname{Hom}_{H_{\mathbb{M}}}(H\iota^{-1}\iota_{\mathbb{M}}, \mathfrak{m}) \cong \operatorname{Hom}_{\iota^{-1}(H_{\mathbb{M}})}(H\iota^{-1}\iota_{\mathbb{M}}\iota, \mathfrak{m}\iota) = \operatorname{Hom}_{H_{\mathbb{M}'}}(H\iota^{-1}\iota_{\mathbb{M}}\iota, \mathfrak{m}\iota) \cong \operatorname{Hom}_{H_{\mathbb{M}'}}(H\iota_{\mathbb{M}'}, \mathfrak{m}\iota)$$

$$\cong \operatorname{Hom}_{H_{\mathbb{M}'}}(H, \mathfrak{m}\iota\iota_{\mathbb{M}'}^{-1}) \cong \operatorname{Hom}_{H_{\mathbb{M}'}}(H, \mathfrak{m}\iota_{\mathbb{M}}^{-1}\iota).$$

So the functors $-\otimes_{H_{\mathbb{M}}} H \cong \operatorname{Hom}_{H_{\mathbb{M}'}}(H, -\iota_{\mathbb{M}}^{-1}\iota)$ are isomorphic.   □

Recalling that H is free as a left $H_{\mathbb{M}}$-module (Prop. 2.26), with the notations of Prop. 3.4 we have:

Corollary 3.5. *The induction* $\operatorname{Ind}_{H_{\mathbb{M}}}^{H}$ *is faithful, exact, of exact right adjoint* $\operatorname{Res}_{H_{\mathbb{M}}}^{H}$ *and exact left adjoint* $(\operatorname{Res}_{H_{\mathbb{M}'}}^{H}(-))\iota^{-1}\iota_{\mathbb{M}}$.

Compare with the parabolic functor $\operatorname{Ind}_{P}^{G}$ for the group (§3.1) where the right adjoint $(-)^{\mathbb{N}}$ and the left adjoint $(-)_{\mathbb{N}}$ are not always exact (see Prop. 3.11).

Remark 3.6. If R is a ring of characteristic different from p, then by Remark 2.31, we have an isomorphism of $(H, H_{\mathbb{M}})$-bimodules $\operatorname{Hom}_{H_{\mathbb{M}}}(H, H_{\mathbb{M}}) \cong H$, so following the arguments of the proof of Prop. 3.4 we would prove in this case that the functors $\operatorname{Ind}_{H_{\mathbb{M}}}^{H}$ and $\operatorname{Coind}_{H_{\mathbb{M}}}^{H}$ coincide. In particular, $\operatorname{Res}_{H_{\mathbb{M}}}^{H}$ is not only the right adjoint but also the left adjoint of $\operatorname{Ind}_{H_{\mathbb{M}}}^{H}$, and we have $L_{H_{\mathbb{M}}}^{H} \cong \operatorname{Res}_{H_{\mathbb{M}}}^{H}$. This means that the pair $(H_{\mathbb{M}}, H)$ is a Frobenius pair. Note that by Corollary 3.5, we also have $L_{H_{\mathbb{M}}}^{H} \cong \operatorname{Res}_{H_{\mathbb{M}}}^{H} \cong (\operatorname{Res}_{H_{\mathbb{M}'}}^{H}(-))\iota^{-1}\iota_{\mathbb{M}}$.

Remark 3.7. Let $\mathbb{M}$ and $\mathbb{M}'$ as in Prop. 3.4. From the proposition, we also get the following isomorphism of functors

$$\operatorname{Coind}_{H_{\mathbb{M}}}^{H} \cong \operatorname{Ind}_{H_{\mathbb{M}'}}^{H}(-\iota_{\mathbb{M}}^{-1}\iota).$$

Denote by $\operatorname{Triv}_{H_{\mathbb{M}}}$, respectively $\operatorname{Triv}_{H_{\mathbb{M}'}}$, the restriction to $H_{\mathbb{M}}$, respectively $H_{\mathbb{M}'}$, of the trivial character of $\mathcal{H}$ as defined in Remark 2.24. Then one easily checks that $\operatorname{Triv}_{H_{\mathbb{M}}}\iota_{\mathbb{M}}^{-1}\iota \cong \operatorname{Triv}_{H_{\mathbb{M}'}}$ and therefore

$$\operatorname{Coind}_{H_{\mathbb{M}}}^{H}(\operatorname{Triv}_{H_{\mathbb{M}}}) \cong \operatorname{Ind}_{H_{\mathbb{M}'}}^{H}(\operatorname{Triv}_{H_{\mathbb{M}'}}).$$

Suppose that R has characteristic p. Then one checks that, in general, the right H-modules $\operatorname{Ind}_{H_{\mathbb{M}'}}^{H}(\operatorname{Triv}_{H_{\mathbb{M}'}})$ and $\operatorname{Ind}_{H_{\mathbb{M}}}^{H}(\operatorname{Triv}_{H_{\mathbb{M}}})$ are not isomorphic. (For example, when $\mathbb{G} = \operatorname{GL}_3(\mathbb{F}_q)$ and R is a field of characteristic p, we have $\Pi = \{\alpha_1, \alpha_2\}$. For $\{i, i'\} = \{1, 2\}$ let $\mathbb{M}_i$ the standard Levi subgroup corresponding to $\{\alpha_i\}$ and $s_i \in \mathbb{W}$ the corresponding reflection. We have $\mathbb{M}_{i'} = \mathbf{w}\mathbb{M}_i\mathbf{w}^{-1}$. Then $\operatorname{Ind}_{H_{\mathbb{M}_i}}^{H}(\operatorname{Triv}_{H_{\mathbb{M}_i}})$ is a 3-dimensional vector space. As a right H-module, it has a 2-dimensional socle and the quotient of $\operatorname{Ind}_{H_{\mathbb{M}_i}}^{H}(\operatorname{Triv}_{H_{\mathbb{M}_i}})$ by its socle is the character $\varepsilon_1 \mapsto 1$, $\varepsilon_1\tau_{n_{s_i}} \mapsto 0$, $\varepsilon_1\tau_{n_{s_{i'}}} \mapsto -1$ for H.) So if R has characteristic p, then the functors $\operatorname{Ind}_{H_{\mathbb{M}}}^{H}$ and $\operatorname{Coind}_{H_{\mathbb{M}}}^{H}$ are not isomorphic in general.

3.3. **Commutative diagrams: parabolic induction and unipotent-invariants functor.** Let $\mathbb{P}$ be a standard parabolic subgroup of $\mathbb{G}$ with Levi decomposition $\mathbb{P} = \mathbb{M}\,\mathbb{N}$.

3.3.1. *The $\mathbb{U}$-invariant functor and its left adjoint.* Let R be a commutative ring. We consider the universal module $\mathrm{X}_{\mathbb{M}} = \mathrm{ind}_{\mathbb{U}_{\mathbb{M}}}^{\mathbb{M}}(1)$. Recall from 2.6 that it is naturally a left $\mathrm{H}_{\mathbb{M}}$-module and a representation in $\mathrm{Rep}(\mathbb{M})$. When $\mathbb{M} = \mathbb{G}$, we write simply X instead of $\mathrm{X}_{\mathbb{G}}$. The functor

$$(3.2) \qquad\qquad - \otimes_{\mathrm{H}} \mathrm{X} : \mathrm{Mod}(\mathrm{H}) \to \mathrm{Rep}(\mathbb{G})$$

is the left adjoint of the $\mathbb{U}$-invariant functor

$$(3.3) \qquad\qquad \mathrm{Hom}_{\mathbb{G}}(\mathrm{X}, -) = (-)^{\mathbb{U}} : \mathrm{Rep}(\mathbb{G}) \to \mathrm{Mod}(\mathrm{H}).$$

It is therefore right exact and the functor $(-)^{\mathbb{U}}$ is left exact. The analogous definitions and statements hold for $(-)^{\mathbb{U}_{\mathbb{M}}}$ and $- \otimes_{\mathrm{H}_{\mathbb{M}}} \mathrm{X}_{\mathbb{M}}$.

The functor $- \otimes_{\mathrm{H}} \mathrm{X}$ is left exact if and only if the H-module X is flat. In general, X is not flat over H: if $\mathbb{G} = \mathrm{GL}(n, \mathbb{F}_q)$ and R is a field of characteristic p, then X is a flat H-module if $n = 2$ and $q = p$ or if $n = 3$ and $q = 2$. It is not flat if $n = 2$ and $q \neq p$, if $n = 3$ and $q \neq 2$, and if $n \geq 4$ and $q \neq 2$ ([44, Thm B]).

Remark 3.8. Let R be a field of characteristic different from p. Then, the $\mathbb{U}$-invariant functor and its left adjoint have properties coming from the existence of an idempotent e of $R[\mathbb{U}]$ such that $V^{\mathbb{U}} = eV$ for $V \in \mathrm{Rep}(\mathbb{G})$. In particular, the functor $(-)^{\mathbb{U}}$ is exact. Moreover, it admits a right adjoint because it commutes with small direct sums ([61, Prop. 2.9]). We have $\mathrm{X} = eR[\mathbb{G}]$ and $\mathrm{H} = eR[\mathbb{G}]e$. The functor $- \otimes_{\mathrm{H}} \mathrm{X}$ is fully faithful because it has a right adjoint and the map $\mathrm{M} \mapsto (\mathrm{M} \otimes_{\mathrm{H}} \mathrm{X})^{\mathbb{U}} = \mathrm{M} \otimes_{eR[\mathbb{G}]e} eR[\mathbb{G}]e$ is a functorial isomorphism in $\mathrm{Mod}(\mathrm{H})$ ([34, Prop. 1.5.6]). If $- \otimes_{\mathrm{H}} \mathrm{X}$ admits a left adjoint, then it is exact. Let $\mathrm{Rep}(\mathbb{G})^{\dagger}$ be the full subcategory of $\mathrm{Rep}(\mathbb{G})$ of representations generated by their $\mathbb{U}$-fixed vectors. The following properties are equivalent:

   (1) $(-)^{\mathbb{U}}$ is fully faithful,
   (2) $(-)^{\mathbb{U}} : \mathrm{Rep}(\mathbb{G})^{\dagger} \to \mathrm{Mod}(\mathrm{H})$ is an equivalence of categories of quasi-inverse $- \otimes_{\mathrm{H}} \mathrm{X}$,
   (3) $- \otimes_{\mathrm{H}} \mathrm{X} : \mathrm{Mod}(\mathrm{H}) \to \mathrm{Rep}(\mathbb{G})^{\dagger}$ is an equivalence of categories of quasi-inverse $(-)^{\mathbb{U}}$,
   (4) the category $\mathrm{Rep}(\mathbb{G})^{\dagger}$ is abelian.

The equivalences between (1), (2), (3) follow from [61, Prop. 2.4], the equivalence with (4) follows from [62, §I.6.6]. The properties (1), (2), (3) and (4) are not satisfied already for $\mathbb{G} = \mathrm{GL}(2, \mathbb{F}_q)$ if $q^2 - 1 = 0$ in R, because the representation $\mathrm{Ind}_{\mathbb{B}}^{\mathbb{G}} 1$ generated by its $\mathbb{U}$-invariants has length 3 [33] and $\dim_R(\mathrm{Ind}_{\mathbb{B}}^{\mathbb{G}} 1)^{\mathbb{U}} = 2$. When R is an algebraically closed field of characteristic p, it is proved [44, Prop. 2.13] (based on work by Cabanes in [19]) that the above conditions are equivalent to X being flat over H.

When R is a field, let $\mathrm{Rep}^f(\mathbb{G})$ denote the full subcategory of $\mathrm{Rep}(\mathbb{G})$ of all finite dimensional representations.

Remark 3.9. Suppose that R is a field. We recall that a finite dimensional representation of $\mathrm{Rep}(\mathbb{G})$ is projective in $\mathrm{Rep}(\mathbb{G})$ if and only if it is projective in $\mathrm{Rep}^f(\mathbb{G})$.

Proof. This is a general and straightforward property. It is obvious that a projective objet of $\mathrm{Rep}(\mathbb{G})$ which is finite dimensional is projective in $\mathrm{Rep}^f(\mathbb{G})$. Conversely, let P be an object in $\mathrm{Rep}^f(\mathbb{G})$ and a surjective $R[\mathbb{G}]$-equivariant morphism $\pi : R[\mathbb{G}]^s \twoheadrightarrow P$ for some $s \geq 0$. If P is projective in $\mathrm{Rep}^f(\mathbb{G})$, then this surjective morphisms splits and P is a direct summand of the free module $R[\mathbb{G}]^s$. So P is projective in $\mathrm{Rep}(\mathbb{G})$. $\hfill\square$

Lemma 3.10. *If R is a field of characteristic p and $\mathbb{U} \neq \{1\}$, then* X *is not a projective object in* $\mathrm{Rep}^f(\mathbb{G})$.

Proof. We use the results of modular representation theory of finite groups recalled in [46, §4]. We suppose that R is a field of characteristic p. Recall that $\mathbb{U}$ being a p-group, the trivial representation is its unique irreducible representation and $R[\mathbb{U}]$ is the only principal indecomposable module in the category $\mathrm{Rep}(\mathbb{U})$ of all $R[\mathbb{U}]$-modules. This implies that for any finite dimensional projective object P in $\mathrm{Rep}(\mathbb{G})$, the dimension of P is equal to the product of $|\mathbb{U}|$ by the dimension of the $\mathbb{U}$-invariant subspace $P^{\mathbb{U}}$ ([46, Cor. 4.6]). If X were projective, its dimension would be $|\mathbb{U}| |\mathscr{N}_{\mathbb{G}}|$. But the dimension of X is strictly smaller than $|\mathbb{U}| |\mathscr{N}_{\mathbb{G}}|$ because it is the sum over $w \in \mathscr{N}_{\mathbb{G}}$ of all $|\mathbb{U}\backslash\mathbb{U}w\mathbb{U}|$, we have $|\mathbb{U}\backslash\mathbb{U}w\mathbb{U}| \leq |\mathbb{U}|$ the integer $|\mathbb{U}\backslash\mathbb{U}w\mathbb{U}|$ is equal to 1 when $w = 1$, and $\mathbb{U} \neq \{1\}$. $\hfill\square$

Proposition 3.11. *If R be a field of characteristic p and $\mathbb{U} \neq \{1\}$, then*

- *The $\mathbb{U}$-invariant functor $(-)^{\mathbb{U}}$ (resp. the restriction of $(-)^{\mathbb{U}}$ to $\mathrm{Rep}^f(\mathbb{G})$) is not right exact.*
- *The $\mathbb{U}$-coinvariant functor $(-)_{\mathbb{U}}$ (resp. the restriction of $(-)_{\mathbb{U}}$ to $\mathrm{Rep}^f(\mathbb{G})$) is not left exact.*

Proof. That the restriction to $\mathrm{Rep}^f(\mathbb{G})$ of the $\mathbb{U}$-invariant functor is not right exact follows immediately from Lemma 3.10. This implies that the $\mathbb{U}$-invariant functor is not right exact. For the second point, note first that the contragredient functor is a contravariant involution of $\mathrm{Rep}^f(\mathbb{G})$ exchanging the $\mathbb{U}$-invariant functor and the $\mathbb{U}$-coinvariant functor in the sense of (3.1). Therefore, the restriction of the $\mathbb{U}$-coinvariant functor to $\mathrm{Rep}^f(\mathbb{G})$ is exact if and only if the restriction of the $\mathbb{U}$-coinvariant functor to $\mathrm{Rep}^f(\mathbb{G})$ is exact. We deduce that the restriction of the $\mathbb{U}$-coinvariant functor to $\mathrm{Rep}^f(\mathbb{G})$ is not left exact which implies that of the $\mathbb{U}$-coinvariant functor is not left exact. $\hfill\square$

Remark 3.12. By Prop. 3.11, if R is a field of characteristic p and $\mathbb{U} \neq \{1\}$, then $(-)^{\mathbb{U}}$ does not have a right adjoint.

3.3.2. *On the $\mathbb{U}$-invariants functor in characteristic p.* In this paragraph, R is a field of characteristic p which we suppose algebraically closed. This ensures that the simple H-modules are 1-dimensional ([52, Lem. 3.13]). By work of Carter and Lusztig [22], the functor (3.3) induces a bijection between irreducible R-representations of $\mathbb{G}$ and simple H-modules.

Since R has characteristic p, every non zero representation $\mathrm{V} \in \mathrm{Rep}(\mathbb{G})$ has a non zero $\mathbb{U}$-fixed vector. Furthermore, the irreducible representations of $\mathbb{G}$ are finite dimensional. Therefore, considering $\mathrm{Mod}^f(\mathrm{H})$ the category of finite dimensional H-modules and $\mathrm{Rep}^{f,\mathbb{U}}(\mathbb{G})$ the full subcategory of $\mathrm{Rep}(\mathbb{G})$ of the finite dimensional representations generated by their $\mathbb{U}$-invariant subspace, it is natural to ask if (3.2) and (3.3) are quasi-inverse equivalences of categories between $\mathrm{Mod}^f(\mathrm{H})$ and $\mathrm{Rep}^{f,\mathbb{U}}(\mathbb{G})$.

The answer is no in general. However, Cabanes described in [19] an additive subcategory $\mathcal{B}(\mathbb{G})$ of $\mathrm{Rep}^f(\mathbb{G})$ which is equivalent to $\mathrm{Mod}^f(\mathrm{H})$ via the functor (3.3). In [44, Prop. 2.13] it is proved (in the setting of $\mathrm{GL}(n, \mathbb{F}_q)$ but the proof is general) that $\mathcal{B}(\mathbb{G})$ and $\mathrm{Rep}^{f,\mathbb{U}}(\mathbb{G})$ coincide if and only if X is a flat H-module. Compare with Remark 3.8. We recalled in §3.3.1 flatness conditions for X when $\mathbb{G} = \mathrm{GL}(n, \mathbb{F}_q)$.

3.3.3. *Commutative diagrams: questions.* We study the connection between the functors of induction for Hecke modules and for representations via the functors (3.2) and (3.3) and their analogs for $\mathbb{M}$. We examine the following questions.

Question 1. *Does the $\mathbb{U}$-invariant functor commute with parabolic induction: is the following diagram commutative?*

(3.4)
$$
\begin{array}{ccc}
\mathrm{Rep}(\mathbb{M}) & \xrightarrow{(-)^{\mathbb{U}_{\mathbb{M}}}} & \mathrm{Mod}(\mathrm{H}_{\mathbb{M}}) \\
\mathrm{Ind}_{\mathbb{P}}^{\mathbb{G}} \downarrow & & \mathrm{Ind}_{\mathrm{H}_{\mathbb{M}}}^{\mathrm{H}} \downarrow \\
\mathrm{Rep}(\mathbb{G}) & \xrightarrow{(-)^{\mathbb{U}}} & \mathrm{Mod}(\mathrm{H})
\end{array}
$$

Passing to left adjoints, this is equivalent to asking whether the following diagram is commutative:

(3.5)
$$
\begin{array}{ccc}
\mathrm{Mod}(\mathrm{H}) & \xrightarrow{-\otimes_{\mathrm{H}} \mathrm{X}} & \mathrm{Rep}(\mathbb{G}) \\
\mathrm{L}_{\mathrm{H}_{\mathbb{M}}}^{\mathrm{H}} \downarrow & & (-)_{\mathbb{N}} \downarrow \\
\mathrm{Mod}(\mathrm{H}_{\mathbb{M}}) & \xrightarrow{-\otimes_{\mathrm{H}_{\mathbb{M}}} \mathrm{X}_{\mathbb{M}}} & \mathrm{Rep}(\mathbb{M})
\end{array}
$$

We answer this question positively in §3.3.4 (Proposition 3.13) for an arbitrary ring R. In passing, we will consider the functor $\dagger : \mathrm{Rep}(\mathbb{M}) \to \mathrm{Rep}(\mathbb{M})$ sending a representation V onto the subrepresentation generated by the $\mathbb{U}_{\mathbb{M}}$-fixed subspace $\mathrm{V}^{\mathbb{U}_{\mathbb{M}}}$ (and likewise for $\dagger : \mathrm{Rep}(\mathbb{G}) \to \mathrm{Rep}(\mathbb{G})$). We show (Proposition 3.16) that $\mathrm{Ind}_{\mathbb{P}}^{\mathbb{G}}$ and $\dagger$ commute.

Question 2. *Does the $\mathbb{U}$-invariant functor commute with the right adjoints of parabolic induction: is the following diagram commutative?*

(3.6)
$$
\begin{array}{ccc}
\mathrm{Rep}(\mathbb{G}) & \xrightarrow{(-)^{\mathbb{N}}} & \mathrm{Rep}(\mathbb{M}) \\
(-)^{\mathbb{U}} \downarrow & & (-)^{\mathbb{U}_{\mathbb{M}}} \downarrow \\
\mathrm{Mod}(\mathrm{H}) & \xrightarrow{\mathrm{Res}_{\mathrm{H}_{\mathbb{M}}}^{\mathrm{H}}} & \mathrm{Mod}(\mathrm{H}_{\mathbb{M}})
\end{array}
$$

Passing to left adjoints, this is equivalent to asking if the following diagram is commutative:

(3.7)
$$
\begin{array}{ccc}
\mathrm{Mod}(\mathrm{H}_{\mathbb{M}}) & \xrightarrow{-\otimes_{\mathrm{H}_{\mathbb{M}}} \mathrm{X}_{\mathbb{M}}} & \mathrm{Rep}(\mathbb{M}) \\
\mathrm{Ind}_{\mathrm{H}_{\mathbb{M}}}^{\mathrm{H}} \downarrow & & \mathrm{Ind}_{\mathbb{P}}^{\mathbb{G}} \downarrow \\
\mathrm{Mod}(\mathrm{H}) & \xrightarrow{-\otimes_{\mathrm{H}} \mathrm{X}} & \mathrm{Rep}(\mathbb{G})
\end{array}
$$

We answer this question positively in §3.3.4 for an arbitrary ring and describe the natural transformations corresponding to (3.7) *in* (3.10) *and* (3.11).

Question 3. *Does the $\mathbb{U}$-invariant functor commute with the left adjoints of parabolic induction: is the following diagram commutative?*

$$
(3.8) \qquad
\begin{array}{ccc}
\mathrm{Rep}(\mathbb{G}) & \xrightarrow{\;(-)_{\mathbb{N}}\;} & \mathrm{Rep}(\mathbb{M}) \\
{\scriptstyle (-)^{\mathbb{U}}}\Big\downarrow & & \Big\downarrow{\scriptstyle (-)^{\mathbb{U}_{\mathbb{M}}}} \\
\mathrm{Mod}(\mathrm{H}) & \xrightarrow{\;\mathrm{L}^{\mathrm{H}}_{\mathrm{H}_{\mathbb{M}}}\;} & \mathrm{Mod}(\mathrm{H}_{\mathbb{M}})
\end{array}
$$

The answer is negative in general, but positive if R is a ring in which p is invertible (Proposition 3.17 and comments following it).

3.3.4. *Commutative diagrams: answers.* The **answer to Question** 1 is positive when R is an arbitrary ring:

Proposition 3.13. *For any representation $V \in \mathrm{Rep}(\mathbb{M})$, there is a natural isomorphism of right H-modules*

$$
V^{\mathbb{U}_{\mathbb{M}}} \otimes_{\mathrm{H}_{\mathbb{M}}} \mathrm{H} \cong (\mathrm{Ind}_{\mathbb{P}}^{\mathbb{G}}(V))^{\mathbb{U}}.
$$

Corollary 3.14. *The diagrams* (3.4) *and* (3.5) *commute.*

The isomorphism of Proposition 3.13 is given by the map (3.9) below. We prove the proposition after the following lemma.

Lemma 3.15.   (1) *The R-module $(\mathrm{Ind}_{\mathbb{P}}^{\mathbb{G}}(V))^{\mathbb{U}}$ is generated by all*

$$
f_{\mathbb{P}\hat{d}\mathbb{U},v}
$$

where v (resp. d) ranges over $V^{\mathbb{U}_{\mathbb{M}}}$ (resp. over $^{\mathbb{M}}W$) and $f_{\mathbb{P}\hat{d}\mathbb{U},v}$ is the unique $\mathbb{U}$-invariant function in $\mathrm{Ind}_{\mathbb{P}}^{\mathbb{G}}(V)$ with support $\mathbb{P}\hat{d}\mathbb{U}$ and value v at a chosen lift $\hat{d}$ of d in $\mathcal{N}_{\mathbb{G}}$.

(2) *For $v \in V^{\mathbb{U}_{\mathbb{M}}}$ and $d \in {}^{\mathbb{M}}W$, we have $f_{\mathbb{P}\hat{d}\mathbb{U},v} = f_{\mathbb{P},v}\tau_{\hat{d}}$.*

(3) *For $v \in V^{\mathbb{U}_{\mathbb{M}}}$ and $w \in \mathcal{N}_{\mathbb{M}}$, we have $f_{\mathbb{P},v}\tau_w = f_{\mathbb{P},v\tau_w^{\mathbb{M}}}$.*

Recall that for W a $\mathbb{G}$-representation, $f \in W^{\mathbb{U}}$, $w \in \mathcal{N}_{\mathbb{G}}$, and $\mathbb{U}w\mathbb{U} = \sqcup_x \mathbb{U}wx$ a decomposition in simple cosets with x ranging over $\mathbb{U} \cap w^{-1}\mathbb{U}w\backslash\mathbb{U}$, we have $f\tau_w = \sum_x (wx)^{-1}f$.

Proof. (1) Let $\tilde{V}$ denote in this proof the representation V trivially inflated to a representation of $\mathbb{P}$. By Lemma 2.15(1), a function $(\mathrm{Ind}_{\mathbb{P}}^{\mathbb{G}}(V))^{\mathbb{U}}$ is a unique linear combination of $\mathbb{U}$-invariant functions with support in $\mathbb{P}\hat{d}\mathbb{U}$, for $d \in {}^{\mathbb{M}}W$. A $\mathbb{U}$-invariant function with support in $\mathbb{P}\hat{d}\mathbb{U}$ is determined by its value at $\hat{d}$ which is an element in $\tilde{V}^{\mathbb{P}\cap\hat{d}\mathbb{U}\hat{d}^{-1}} = V^{\mathbb{U}_{\mathbb{M}}}$ by Lemma 2.15(1)(b).

(2) For $v \in V^{\mathbb{U}_{\mathbb{M}}}$ and $d \in {}^{\mathbb{M}}W$, the function $f_{\mathbb{P},v}\tau_{\hat{d}}$ is $\mathbb{U}$-invariant with support in $\mathbb{P}\hat{d}\mathbb{U}$ and value at $\hat{d}$ equal to v because $\hat{d}\mathbb{U}\hat{d}^{-1}\cap\mathbb{P} \subset \mathbb{U}$ by Lemma 2.15(1)(b). Therefore $f_{\mathbb{P},v}\tau_{\hat{d}}$ is equal to the function $f_{\mathbb{P}\hat{d}\mathbb{U},v}$ defined in (1).

(3) Let $w \in \mathscr{N}_{\mathbb{M}}$. Note that $\mathbb{U}w\mathbb{U} = \mathbb{U}w\mathbb{N}\mathbb{U}_{\mathbb{M}} = \mathbb{U}w\mathbb{U}_{\mathbb{M}}$ since $w \in \mathbb{M}$ normalizes $\mathbb{N}$. Let $\mathbb{U}_{\mathbb{M}}w\mathbb{U}_{\mathbb{M}} = \sqcup_u \mathbb{U}_{\mathbb{M}}wu$ be a disjoint decomposition in simple cosets with u ranging over $\mathbb{U}_{\mathbb{M}} \cap w^{-1}\mathbb{U}_{\mathbb{M}}w \backslash \mathbb{U}_{\mathbb{M}}$. Then $\mathbb{U}w\mathbb{U} = \mathbb{U}w\mathbb{U}_{\mathbb{M}} = \cup_u \mathbb{U}wu$ and the union is disjoint because $(w^{-1}\mathbb{U}w) \cap \mathbb{U}_{\mathbb{M}} = (w^{-1}\mathbb{U}_{\mathbb{M}}ww^{-1}\mathbb{N}w) \cap \mathbb{U}_{\mathbb{M}} = (w^{-1}\mathbb{U}_{\mathbb{M}}w\mathbb{N}) \cap \mathbb{U}_{\mathbb{M}} = (w^{-1}\mathbb{U}_{\mathbb{M}}w) \cap \mathbb{U}_{\mathbb{M}}$. It proves that for $v \in V^{\mathbb{U}_{\mathbb{M}}}$, the $\mathbb{U}$-invariant function $f_{\mathbb{P},v}\tau_w$ has support in $\mathbb{P}$ and value at 1 equal to $v\tau_w^{\mathbb{M}}$.

$\square$

Proof of Proposition 3.13. The map

(3.9) $$V^{\mathbb{U}_{\mathbb{M}}} \otimes_{H_{\mathbb{M}}} H \to (\mathrm{Ind}_{\mathbb{P}}^{\mathbb{G}}(V))^{\mathbb{U}}, v \otimes h \mapsto f_{\mathbb{P},v}h$$

is a well defined morphism of right H-modules. By Proposition 2.26, the vector space $V^{\mathbb{U}_{\mathbb{M}}} \otimes_{H_{\mathbb{M}}} H$ decomposes as the direct sum of all $V^{\mathbb{U}_{\mathbb{M}}} \otimes \tau_d$ for $d \in {}^{\mathbb{M}}W$. For $v \in V^{\mathbb{U}_{\mathbb{M}}}$ and $d \in {}^{\mathbb{M}}W$, the image of $v \otimes \tau_{\hat{d}}$ is equal to $f_{\mathbb{P}\hat{d}\mathbb{U},v}$ by Lemma 3.15(2). This implies that the map is injective since $\mathbb{G} = \sqcup$. The map is surjective by Lemma 3.15(1). $\square$

Proposition 3.16. *For* $V \in \mathrm{Rep}(\mathbb{M})$*, there is a natural isomorphism of representations of* $\mathbb{G}$*:*

$$(\mathrm{Ind}_{\mathbb{P}}^{\mathbb{G}}(V))^{\dagger} \cong \mathrm{Ind}_{\mathbb{P}}^{\mathbb{G}}(V^{\dagger}).$$

Proof. It is easy to see that the representation $\mathrm{Ind}_{\mathbb{P}}^{\mathbb{G}}(V^{\dagger})$ is generated by the functions of the form $f_{\mathbb{P},v}$ for $v \in V^{\mathbb{U}_{\mathbb{M}}}$ so in particular it is generated its $\mathbb{U}$-invariant subspace. The natural $\mathbb{M}$-equivariant injection $V^{\dagger} \to V$ induces, by exactness of the functor $\mathrm{Ind}_{\mathbb{P}}^{\mathbb{G}}$, a $\mathbb{G}$-equivariant injection $\mathrm{Ind}_{\mathbb{P}}^{\mathbb{G}}(V^{\dagger}) \to \mathrm{Ind}_{\mathbb{P}}^{\mathbb{G}}(V)$ whose image is contained in $(\mathrm{Ind}_{\mathbb{P}}^{\mathbb{G}}(V))^{\dagger}$. Its image is exactly $(\mathrm{Ind}_{\mathbb{P}}^{\mathbb{G}}(V))^{\dagger}$ by points (1) and (2) of Lemma 3.15. $\square$

The **answer to Question** 2 is positive when R is an arbitrary ring:

The composition

$$\mathrm{Rep}(\mathbb{G}) \to \mathrm{Rep}(\mathbb{M}) \to \mathrm{Mod}(R), \ V \mapsto V^{\mathbb{N}} \mapsto (V^{\mathbb{N}})^{\mathbb{U}_{\mathbb{M}}}$$

coincides with

$$\mathrm{Rep}(\mathbb{G}) \to \mathrm{Mod}(H) \to \mathrm{Mod}(R), \ V \mapsto V^{\mathbb{U}} \mapsto \mathrm{Res}_{H_{\mathbb{M}}}^{H}(V^{\mathbb{U}})$$

because $\mathbb{U}$ is the semidirect product $\mathbb{U}_{\mathbb{M}} \ltimes \mathbb{N}$. For $v \in V^{\mathbb{U}}$, denote by $\phi_v : X \to V$ (resp. $\phi_v^{\mathbb{M}} : X_{\mathbb{M}} \to V^{\mathbb{N}}$) the $\mathbb{G}$-equivariant (resp. $\mathbb{M}$-equivariant) map sending $\mathbf{1}_{\mathbb{U}}$ (resp. $\mathbf{1}_{\mathbb{U}_{\mathbb{M}}}$) to v. If $\xi_{\mathbb{M}}$ denotes the $\mathbb{M}$-equivariant inclusion $X_{\mathbb{M}} \to X$ described in (2.29), the map $\phi_v^{\mathbb{M}}$ coincides with the restriction of $\phi_v \circ \xi_{\mathbb{M}}$ to $X_{\mathbb{M}} \to V^{\mathbb{N}}$. The right action of $h \in H_{\mathbb{M}}$ on v seen in $(V^{\mathbb{N}})^{\mathbb{U}_{\mathbb{M}}}$ is given by $\phi_v^{\mathbb{M}}(h)$. The right action of $h \in H_{\mathbb{M}}$ on v seen in $\mathrm{Res}_{H_{\mathbb{M}}}^{H}(V^{\mathbb{U}})$ is $\phi_v(\xi_{\mathbb{M}}(h))$. Therefore the identity map is an isomorphism of right $H_{\mathbb{M}}$-modules $(V^{\mathbb{N}})^{\mathbb{U}_{\mathbb{M}}} \cong \mathrm{Res}_{H_{\mathbb{M}}}^{H}(V^{\mathbb{U}})$.

Therefore, the diagram (3.6) commutes and so does (3.7) by adjunction. We now describe a natural transformation between $\mathrm{Ind}_{H_{\mathbb{M}}}^{H}(-) \otimes_H X$ and $\mathrm{Ind}_{\mathbb{P}}^{\mathbb{G}}(- \otimes_{H_{\mathbb{M}}} X_{\mathbb{M}})$. Let $\mathfrak{m}$ be a right $H_{\mathbb{M}}$-module. There is a natural $H_{\mathbb{M}}$-equivariant map

$$\mathfrak{m} \longrightarrow (\mathfrak{m} \otimes_{H_{\mathbb{M}}} X_{\mathbb{M}})^{\mathbb{U}_{\mathbb{M}}}, \ m \mapsto m \otimes \mathbf{1}_{\mathbb{U}_{\mathbb{M}}}.$$

where, for Y a set, $\mathbf{1}_Y$ denotes the characteristic function of Y. Using Proposition 3.13 and adjunction, we have a morphism of right H-modules

$$\mathfrak{m} \otimes_{\mathrm{H}_{\mathbb{M}}} \mathrm{H} \longrightarrow (\mathfrak{m} \otimes_{\mathrm{H}_{\mathbb{M}}} \mathrm{X}_{\mathbb{M}})^{\mathbb{U}_{\mathbb{M}}} \otimes_{\mathrm{H}_{\mathbb{M}}} \mathrm{H} \cong (\mathrm{Ind}_{\mathbb{P}}^{\mathbb{G}}(\mathfrak{m} \otimes_{\mathrm{H}_{\mathbb{M}}} \mathrm{X}_{\mathbb{M}}))^{\mathbb{U}}$$

and a morphism of representations of $\mathbb{G}$

$$(3.10) \qquad\qquad \mathrm{Ind}_{\mathrm{H}_{\mathbb{M}}}^{\mathrm{H}}(\mathfrak{m}) \otimes_{\mathrm{H}} \mathrm{X} = \mathfrak{m} \otimes_{\mathrm{H}_{\mathbb{M}}} \mathrm{X} \longrightarrow \mathrm{Ind}_{\mathbb{P}}^{\mathbb{G}}(\mathfrak{m} \otimes_{\mathrm{H}_{\mathbb{M}}} \mathrm{X}_{\mathbb{M}})$$

sending $m \otimes \mathbf{1}_{\mathbb{U}}$ onto the function $f_{\mathbb{P}, m \otimes \mathbf{1}_{\mathbb{U}_{\mathbb{M}}}}$ with support in $\mathbb{P}$ and value $m \otimes \mathbf{1}_{\mathbb{U}_{\mathbb{M}}}$ at 1. On the other hand, the natural $\mathrm{H}_{\mathbb{M}}$-equivariant map

$$\mathfrak{m} \to \mathrm{Res}_{\mathrm{H}_{\mathbb{M}}}^{\mathrm{H}}((\mathfrak{m} \otimes_{\mathrm{H}_{\mathbb{M}}} \mathrm{X})^{\mathbb{U}}) = ((\mathfrak{m} \otimes_{\mathrm{H}_{\mathbb{M}}} \mathrm{X})^{\mathbb{N}})^{\mathbb{U}_{\mathbb{M}}}$$

induces a morphism of representations of $\mathbb{M}$

$$\mathfrak{m} \otimes_{\mathrm{H}_{\mathbb{M}}} \mathrm{X}_{\mathbb{M}} \longrightarrow (\mathfrak{m} \otimes_{\mathrm{H}_{\mathbb{M}}} \mathrm{X})^{\mathbb{N}}$$

which in turn by adjunction (Prop. 3.1) induces a morphism of representations of $\mathbb{G}$

$$(3.11) \qquad\qquad \mathrm{Ind}_{\mathbb{P}}^{\mathbb{G}}(\mathfrak{m} \otimes_{\mathrm{H}_{\mathbb{M}}} \mathrm{X}_{\mathbb{M}}) \longrightarrow \mathfrak{m} \otimes_{\mathrm{H}_{\mathbb{M}}} \mathrm{X} = \mathrm{Ind}_{\mathrm{H}_{\mathbb{M}}}^{\mathrm{H}}(\mathfrak{m}) \otimes_{\mathrm{H}} \mathrm{X}$$

and one checks that (3.10) and (3.11) are inverse to each other.

We study **Question 3**. We will show the following: if R is a field of characteristic different from p, then the answer is positive. (recall that in that case $\mathrm{L}_{\mathrm{H}_{\mathbb{M}}} = \mathrm{Res}_{\mathrm{H}_{\mathbb{M}}}^{\mathrm{H}}$ by Remark 3.6); if R is a field of characteristic p, then the answer is negative.

Proposition 3.17. *Suppose that R is a ring in which p is invertible. Let V be a representation of $\mathbb{G}$. Then we have an isomorphism of $\mathrm{H}_{\mathbb{M}}$-modules*

$$(\mathrm{V}_{\mathbb{N}})^{\mathbb{U}_{\mathbb{M}}} \simeq \mathrm{Res}_{\mathrm{H}_{\mathbb{M}}}^{\mathrm{H}}(\mathrm{V}^{\mathbb{U}}).$$

Proof. The map $\mathrm{V} \to \mathrm{V}_{\mathbb{N}}, v \mapsto v \mod \mathrm{V}(\mathbb{N})$ restricts into an isomorphism $\mathrm{V}^{\mathbb{N}} \to \mathrm{V}_{\mathbb{N}}$ of R-representations of $\mathbb{M}$ [54, I.4.9 Prop.]. It gives by restriction an isomorphism of right $\mathrm{H}_{\mathbb{M}}$-modules $(\mathrm{V}^{\mathbb{N}})^{\mathbb{U}_{\mathbb{M}}} \to (\mathrm{V}_{\mathbb{N}})^{\mathbb{U}_{\mathbb{M}}}$. To conclude, recall that as a right $\mathrm{H}_{\mathbb{M}}$-module, $(\mathrm{V}^{\mathbb{N}})^{\mathbb{U}_{\mathbb{M}}}$ is isomorphic to $\mathrm{Res}_{\mathrm{H}_{\mathbb{M}}}^{\mathrm{H}}(\mathrm{V}^{\mathbb{U}})$ by the commutativity of (3.6). $\qquad\qquad \square$

Suppose that R is a field of characteristic p. When $\mathbb{P} = \mathbb{B}$, we have $\mathbb{M} = \mathbb{Z}$, $\mathbb{U}_{\mathbb{M}} = \{1\}$, $\mathbb{N} = \mathbb{U}$ and the Hecke algebra $\mathrm{H}_{\mathbb{Z}}$ is naturally isomorphic to the group algebra $R[\mathbb{Z}]$. Via this identification, the second vertical map of the diagram (4.13) is the natural equivalence between $\mathrm{Rep}(\mathbb{Z})$ and $\mathrm{Mod}(R[\mathbb{Z}])$. Since $\mathrm{L}_{\mathrm{H}_{\mathbb{Z}}}^{\mathrm{H}}$ and the $\mathbb{U}$-invariants functor are left exact, the commutativity of the diagram (4.13) would imply the left exactness of $(-)_{\mathbb{U}}$. This contradicts Prop. 3.11 when $\mathbb{U} \neq \{1\}$.

4. Parabolic induction for the p-adic group and the pro-p Iwahori Hecke algebra

Let R be a commutative ring. For P a standard parabolic subgroup of G with standard Levi decomposition P = M N and corresponding to the subset $J \subseteq \Pi$, we consider the category $\mathrm{Mod}(\mathcal{H})$ (resp. $\mathrm{Mod}(\mathcal{H}_{\mathrm{M}})$) of right $\mathcal{H}$-modules (resp. $\mathcal{H}_{\mathrm{M}}$-modules) and the category $\mathrm{Rep}(\mathrm{G})$ (resp. $\mathrm{Rep}(\mathrm{M})$) of smooth R-representations of G (resp. M): a smooth R-representation of G is a R-module endowed with a R-linear action of G such that the stabilizers of the points are open subgroups of G.

4.1. **Parabolic induction and restriction for the representations of the p-adic group.** For V $\in \mathrm{Rep}(\mathrm{M})$, we consider the representation of G on the space of functions $f : \mathrm{G} \to \mathrm{V}$ such that $f(mng) = m.f(g)$ for all $m \in \mathrm{M}$, $n \in \mathrm{N}$ and $g \in \mathrm{G}$, the action of G being by right translation $(g, f) \mapsto f(\,.\,g)$. Its smooth part is denoted by $\mathrm{Ind}_{\mathrm{P}}^{\mathrm{G}}(\mathrm{V})$. This defines the functor of *parabolic induction*

$$\mathrm{Ind}_{\mathrm{P}}^{\mathrm{G}} : \mathrm{Rep}(\mathrm{M}) \longrightarrow \mathrm{Rep}(\mathrm{G}).$$

It is exact and faithful [61, this follows from (5) and (6)]. It has a left adjoint: the N-coinvariant functor

$$(4.1) \qquad\qquad (-)_{\mathrm{N}} : \mathrm{Rep}(\mathrm{G}) \longrightarrow \mathrm{Rep}(\mathrm{M})$$

which associates to V the representation of M on the coinvariant space V_{N} [54, 2.1 Remarque]. It has a right adjoint [61, Prop 4.2] which we denote by

$$\mathbf{R}_{\mathrm{P}}^{\mathrm{G}} : \mathrm{Rep}(\mathrm{M}) \longrightarrow \mathrm{Rep}(\mathrm{G}).$$

4.2. **Induction, coinduction and restriction for pro-p Iwahori Hecke modules.** We identify $\mathcal{H}_{\mathrm{M}+}$ and $\mathcal{H}_{\mathrm{M}-}$ with their respective natural images in $\mathcal{H}$ via the linear map θ and θ^* respectively (§2.5.2). For $\mathfrak{m}$ a right $\mathcal{H}_{\mathrm{M}}$-module, we consider the induced $\mathcal{H}$-module

$$\mathfrak{m} \otimes_{\mathcal{H}_{\mathrm{M}+,\theta}} \mathcal{H}$$

where the notation $\mathcal{H}_{\mathrm{M}+,\theta}$ is a reminder that we identify $\mathcal{H}_{\mathrm{M}+}$ with its image in $\mathcal{H}$ via θ. We will also denote this module by $\mathrm{Ind}_{\mathcal{H}_{\mathrm{M}}}^{\mathcal{H}}(\mathfrak{m})$. This defines a functor called *(parabolic) induction* for Hecke modules

$$(4.2) \qquad\qquad \mathrm{Ind}_{\mathcal{H}_{\mathrm{M}}}^{\mathcal{H}} := - \otimes_{\mathcal{H}_{\mathrm{M}+,\theta}} \mathcal{H} : \mathrm{Mod}(\mathcal{H}_{\mathrm{M}}) \longrightarrow \mathrm{Mod}(\mathcal{H}).$$

The space

$$\mathrm{Hom}_{\mathcal{H}_{\mathrm{M}-,\theta^*}}(\mathcal{H}, \mathfrak{m}),$$

where the notation $\mathcal{H}_{\mathrm{M}-,\theta^*}$ is a reminder that we identify $\mathcal{H}_{\mathrm{M}-}$ with its image in $\mathcal{H}$ via θ^*, has a structure of right $\mathcal{H}$-module given by $(f, h) \mapsto [f.h : x \mapsto f(hx)]$. We will also denote this module by $\mathrm{Coind}_{\mathcal{H}_{\mathrm{M}}}^{\mathcal{H}}(\mathfrak{m})$. This defines the functor of *(parabolic) coinduction* for Hecke modules

$$(4.3) \qquad\qquad \mathrm{Coind}_{\mathcal{H}_{\mathrm{M}}}^{\mathcal{H}} := \mathrm{Hom}_{\mathcal{H}_{\mathrm{M}-,\theta^*}}(\mathcal{H}, -) : \mathrm{Mod}(\mathcal{H}_{\mathrm{M}}) \longrightarrow \mathrm{Mod}(\mathcal{H}).$$

Let $\mathbf{w}$ denote as before the longest element of W and $\mathbf{w}_{\mathbb{M}}$ the longest element in $\mathrm{W}_{\mathbb{M}}$. As in Proposition 2.27, we pick lifts $\widetilde{\mathbf{w}}$ and $\widetilde{\mathbf{w}_{\mathbb{M}}}$ for these elements in $\mathscr{N}_{\mathrm{G}}$ and $\mathscr{N}_{\mathbb{M}}$ respectively. Recall (see (2.10)) that $\mathscr{N}_{\mathrm{G}}$ (resp. $\mathscr{N}_{\mathbb{M}}$) identifies with a subgroup of W(1) (resp. $\mathrm{W}_{\mathrm{M}}(1)$) and we may therefore consider $\widetilde{\mathbf{w}}$ (resp. $\widetilde{\mathbf{w}_{\mathbb{M}}}$) as a element in W(1) (resp. $\mathrm{W}_{\mathrm{M}}(1) \subseteq \mathrm{W}(1)$). We have automorphisms of R-algebras

$$\iota : \mathcal{H} \to \mathcal{H}, \ \tau_w \mapsto \tau_{\widetilde{\mathbf{w}} w \widetilde{\mathbf{w}}^{-1}} \quad \text{for } w \in \mathrm{W}(1) \quad \text{and} \quad \iota_{\mathrm{M}} : \mathcal{H}_{\mathrm{M}} \to \mathcal{H}_{\mathrm{M}}, \ \tau_w \mapsto \tau_{\widetilde{\mathbf{w}_{\mathbb{M}}} w \widetilde{\mathbf{w}_{\mathbb{M}}}^{-1}} \quad \text{for } w \in \mathrm{W}_{\mathrm{M}}(1)$$

whose restrictions to the finite Hecke algebras, H and $H_\mathbb{M}$ respectively, coincide with ι and $\iota_\mathbb{M}$ defined in Proposition 2.27. Let $M' := \mathbf{w}M\mathbf{w}^{-1}$. It is the Levi subgroup of the standard parabolic subgroup P' corresponding to $\mathbf{w}.J \subseteq \Pi$. Note that $\widetilde{\mathbf{w}}^{-1}\widetilde{\mathbf{w}}_\mathrm{M}$ is a lift for $\mathbf{ww}_\mathrm{M} \in W$ (the latter is denoted by w_0^M in [66] above Definition 1.7). We have an R-algebra isomorphism $\iota_\mathrm{M}^{-1} \circ \iota : \mathcal{H}_{\mathrm{M}'} \xrightarrow{\simeq} \mathcal{H}_\mathrm{M}$ (compare with [66, Prop. 2.20]). It induces an equivalence of categories $- \iota_\mathrm{M}^{-1}\iota : \mathrm{Mod}(\mathcal{H}_\mathrm{M}) \to \mathrm{Mod}(\mathcal{H}_{\mathrm{M}'})$ with inverse $-\iota^{-1}\iota_\mathrm{M} : \mathrm{Mod}(\mathcal{H}_{\mathrm{M}'}) \to \mathrm{Mod}(\mathcal{H}_\mathrm{M})$. By [66, Theorem 1.8], we have an isomorphism of functors (compare with Prop. 3.4):

$$(4.4) \qquad\qquad \mathrm{Ind}_{\mathcal{H}_\mathrm{M}}^\mathcal{H} \cong \mathrm{Coind}_{\mathcal{H}_{\mathrm{M}'}}^\mathcal{H}(-\iota_\mathrm{M}^{-1}\iota).$$

Using (4.4) and the classical properties of induction and coinduction ([66, §3]), one obtains ([66, Thm 1.6 and 1.9]):

(1) The induction functor $\mathrm{Ind}_{\mathcal{H}_\mathrm{M}}^\mathcal{H}$ is faithful and exact. It has an exact left adjoint denoted by $\mathrm{L}_{\mathcal{H}_\mathrm{M}}^\mathcal{H}$ and the functor

$$(4.5) \qquad\qquad \mathrm{R}_{\mathcal{H}_\mathrm{M}}^\mathcal{H} := \mathrm{Hom}_\mathcal{H}(\mathcal{H}_\mathrm{M} \otimes_{\mathcal{H}_{\mathrm{M}^+,\theta}} \mathcal{H}, -)$$

as a right adjoint.

(2) If $\mathfrak{m}$ is a $\mathcal{H}_\mathrm{M}$-module which is finitely generated over R, then $\mathrm{Ind}_{\mathcal{H}_\mathrm{M}}^\mathcal{H}(\mathfrak{m})$ is finitely generated over R.

(3) Suppose that R is a field, and let $\mathfrak{m}$ be a $\mathcal{H}$-module which is finite dimensional over R. The image of $\mathfrak{m}$ by the left (resp. right) adjoint to $\mathrm{Ind}_{\mathcal{H}_\mathrm{M}}^\mathcal{H}$ is finite dimensional over R.

Note that the left adjoint of the induction is exact because it is given by a localization at a central element [66, Theorem 1.9]. Analogous results about the coinduction functor hold ([66, Cor. 1.10]).

Let $\mathfrak{m}$ be a $\mathcal{H}$-module. We have $\mathrm{R}_{\mathcal{H}_\mathrm{M}}^\mathcal{H}(\mathfrak{m}) = \mathrm{Hom}_{\mathcal{H}_{\mathrm{M}^+,\theta}}(\mathcal{H}_\mathrm{M}, \mathfrak{m})$ and we recall that $\mathcal{H}_\mathrm{M} = \mathcal{H}_{\mathrm{M}^+}[\tau^{-1}]$ is a localization of $\mathcal{H}_{\mathrm{M}^+}$ at a central element τ (§2.5.2). The map $f \mapsto f(1)$ is an homomorphism from $\mathrm{Hom}_{\mathcal{H}_{\mathrm{M}^+,\theta}}(\mathcal{H}_\mathrm{M}, \mathfrak{m})$ to the submodule $\cap_{n\in\mathrm{N}}\mathfrak{m}\,\theta(\tau)^n$ of infinitely $\theta(\tau)$-divisible elements of $\mathfrak{m}$. When $\mathfrak{m}$ a noetherian R-module, this map is bijective ([5, Lemma 4.10]). Therefore (3) for the right adjoint to $\mathrm{Ind}_{\mathcal{H}_\mathrm{M}}^\mathcal{H}$ is a particular case of:

(4) Let $\mathfrak{m}$ be a $\mathcal{H}$-module which is noetherian over R. Then $\mathrm{R}_{\mathcal{H}_\mathrm{M}}^\mathcal{H}(\mathfrak{m})$ is finitely generated over R.

Remark 4.1. We have an isomorphism of R-modules

$$\bigoplus_{d\in{}^\mathbb{M}\mathbb{W}} \mathfrak{m} \cong \mathrm{Ind}_{\mathcal{H}_\mathrm{M}}^\mathcal{H}(\mathfrak{m})$$

given by $(m_d)_{d\in{}^\mathbb{M}\mathbb{W}} \mapsto \sum_{d\in{}^\mathbb{M}\mathbb{W}} m_d \otimes \tau_{\tilde{d}}$ ([66, Rmk. 3.7], see also [41, Prop. 5.2] when $G = \mathrm{GL}_n(F)$ and R an algebraically closed field of characteristic p).

Remark 4.2.   (1) The functor $\mathrm{L}_{\mathcal{H}_\mathrm{M}}^\mathcal{H}$ is explicitly described in [66, Thm. 1.9]: from (4.3) and (4.4) we get an isomorphism

$$(4.6) \qquad\qquad \mathrm{L}_{\mathcal{H}_\mathrm{M}}^\mathcal{H} \cong (- \otimes_{\mathcal{H}_{(\mathrm{M}')^-,\theta*}} \mathcal{H}_{\mathrm{M}'})\iota^{-1}\iota_\mathrm{M}.$$

(2) Abe has recently proved [3, Prop. 4.13 (2)] that the coinduction functor $\mathrm{Coind}_{\mathcal{H}_M}^{\mathcal{H}}$ is isomorphic to $\mathrm{Hom}_{\mathcal{H}_{(M')^+,\theta^*}}(\mathcal{H}, -\iota_M^{-1}\iota)$. Applying (4.4), this means that $\mathrm{Hom}_{\mathcal{H}_{M^+,\theta^*}}(\mathcal{H}, -)$ is isomorphic to the induction functor $\mathrm{Ind}_{\mathcal{H}_M}^{\mathcal{H}} = - \otimes_{\mathcal{H}_{M^+,\theta}} \mathcal{H} \simeq \mathrm{Hom}_{\mathcal{H}_{(M')^-,\theta^*}}(\mathcal{H}, -\iota_M^{-1}\iota)$ where the notation $\mathcal{H}_{M^+,\theta^*}$ is a reminder that we identify $\mathcal{H}_{M^+}$ with its image in $\mathcal{H}$ via θ^*. This implies that we have an isomorphism of functors

$$\text{(4.7)} \qquad \mathrm{L}_{\mathcal{H}_M}^{\mathcal{H}} \cong - \otimes_{\mathcal{H}_{M^+,\theta^*}} \mathcal{H}_M.$$

4.3. Commutative diagrams: parabolic induction and the pro-p Iwahori invariants functor.

4.3.1. *The functor of $\mathcal{U}$-invariants and its left adjoint.* We consider the universal module $\mathbf{X}_M = \mathrm{ind}_{\mathcal{U}_M}^M(1)$ introduced in §2.5.1. It is naturally a left $\mathcal{H}_M$-module and a representation in $\mathrm{Rep}(M)$. Recall that when $M = G$, we write simply $\mathbf{X} = \mathrm{ind}_{\mathcal{U}}^G(1)$ instead of $\mathbf{X}_G$. The functor

$$\text{(4.8)} \qquad - \otimes_{\mathcal{H}} \mathbf{X} : \mathrm{Mod}(\mathcal{H}) \to \mathrm{Rep}(G)$$

is the left adjoint of the $\mathcal{U}$-invariant functor

$$\text{(4.9)} \qquad \mathrm{Hom}_G(\mathbf{X}, -) = (-)^{\mathcal{U}} : \mathrm{Rep}(G) \to \mathrm{Mod}(\mathcal{H}).$$

It is therefore right exact and the functor $(-)^{\mathcal{U}}$ is left exact.

4.3.2. *On the functor of $\mathcal{U}$-invariants in characteristic p.* Suppose that R is an algebraic closure of the residue field of F. The functor (4.9) has been studied in the case of $\mathbf{G} = \mathrm{GL}_2$ and in the case of $\mathrm{G} = \mathrm{SL}_2$ ([40], [37]). It yields an equivalence between the category of representations of $\mathbf{G} = \mathrm{GL}_2(\mathbb{Q}_p)$ (resp. $\mathrm{G} = \mathrm{SL}_2(\mathbb{Q}_p)$) generated by their $\mathcal{U}$-fixed vectors and the category of $\mathcal{H}$-modules. This equivalence fails when $\mathrm{G} = \mathrm{GL}_2(\mathrm{F})$ for $\mathrm{F} \neq \mathbb{Q}_p$. Although it has not been studied in general, it is expected to fail for groups of higher rank: for example, the functor (4.8) is not exact when $\mathbf{G} = \mathrm{GL}_3$ and $p \neq 2$ ([45]).

Remark 4.3. For $\pi \in \mathrm{Rep}(G)$ admissible, π irreducible does not imply $\pi^{\mathcal{U}}$ irreducible. Counter examples are constructed in [14] in the case of $\mathrm{G} = \mathrm{GL}_2(\mathrm{F})$ where F is a strict unramified extension of $\mathbb{Q}_p$.

4.3.3. *Commutative diagrams: questions.* We study the connection between the functors of induction for Hecke modules and for representations via the functors (4.8) and (4.9) and their analogs for M. We examine the following questions.

Question 4. *Does the $\mathcal{U}$-invariant functor commute with parabolic induction: is the following diagram commutative?*

$$\text{(4.10)} \qquad \begin{array}{ccc} \mathrm{Rep}(M) & \xrightarrow{(-)^{\mathcal{U}_M}} & \mathrm{Mod}(\mathcal{H}_M) \\ {\scriptstyle \mathrm{Ind}_P^G} \downarrow & & \downarrow {\scriptstyle \mathrm{Ind}_{\mathcal{H}_M}^{\mathcal{H}}} \\ \mathrm{Rep}(G) & \xrightarrow{(-)^{\mathcal{U}}} & \mathrm{Mod}(\mathcal{H}) \end{array}$$

We answer this question positively in §4.3.4 (Proposition 4.4).

Question 5. *Does the $\mathcal{U}$-invariant functor commute with the right adjoints of parabolic induction: is the following diagram commutative?*

(4.11)
$$
\begin{array}{ccc}
\mathrm{Rep}(\mathrm{G}) & \xrightarrow{\quad \mathbf{R}_{\mathrm{P}}^{\mathrm{G}} \quad} & \mathrm{Rep}(\mathrm{M}) \\
(-)^{\mathcal{U}} \downarrow & & (-)^{\mathcal{U}_{\mathrm{M}}} \downarrow \\
\mathrm{Mod}(\mathcal{H}) & \xrightarrow{\mathrm{R}_{\mathcal{H}_{\mathrm{M}}}^{\mathcal{H}}=\mathrm{Hom}_{\mathcal{H}}(\mathcal{H}_{\mathrm{M}}\otimes_{\mathcal{H}_{\mathrm{M}+,\theta}}\mathcal{H},-)} & \mathrm{Mod}(\mathcal{H}_{\mathrm{M}})
\end{array}
$$

Passing to left adjoints, this is equivalent to asking whether the following diagram is commutative:

(4.12)
$$
\begin{array}{ccc}
\mathrm{Mod}(\mathcal{H}_{\mathrm{M}}) & \xrightarrow{-\otimes_{\mathcal{H}_{\mathrm{M}}}\mathbf{X}_{\mathrm{M}}} & \mathrm{Rep}(\mathrm{M}) \\
\mathrm{Ind}_{\mathcal{H}_{\mathrm{M}}}^{\mathcal{H}} \downarrow & & \mathrm{Ind}_{\mathrm{P}}^{\mathrm{G}} \downarrow \\
\mathrm{Mod}(\mathcal{H}) & \xrightarrow{-\otimes_{\mathcal{H}}\mathbf{X}} & \mathrm{Rep}(\mathrm{G})
\end{array}
$$

We answer this question positively in §4.3.4 (Corollary 4.7). In passing, we will consider the functor $\dagger : \mathrm{Rep}(\mathrm{M}) \to \mathrm{Rep}(\mathrm{M})$ *sending a representation* V *onto the subrepresentation generated by the* $\mathcal{U}_{\mathrm{M}}$-*fixed subspace* $\mathrm{V}^{\mathcal{U}_{\mathrm{M}}}$ *(and likewise* $\dagger : \mathrm{Rep}(\mathrm{G}) \to \mathrm{Rep}(\mathrm{G})$). We show (Corollary 4.8) that $\mathrm{Ind}_{\mathrm{P}}^{\mathrm{G}}$ and $\dagger$ commute.

Question 6. *Does the $\mathcal{U}$-invariant functor commute with the left adjoints of parabolic induction: is the following diagram commutative?*

(4.13)
$$
\begin{array}{ccc}
\mathrm{Rep}(\mathrm{G}) & \xrightarrow{\quad (-)_{\mathrm{N}} \quad} & \mathrm{Rep}(\mathrm{M}) \\
(-)^{\mathcal{U}} \downarrow & & (-)^{\mathcal{U}_{\mathrm{M}}} \downarrow \\
\mathrm{Mod}(\mathcal{H}) & \xrightarrow{\quad \mathrm{L}_{\mathcal{H}_{\mathrm{M}}}^{\mathcal{H}} \quad} & \mathrm{Mod}(\mathcal{H}_{\mathrm{M}})
\end{array}
$$

We recall that the left adjoint to $\mathrm{Ind}_{\mathcal{H}_{\mathrm{M}}}^{\mathcal{H}}$ is described in Remark 4.2. The answer is negative in general as we show in Corollary 4.13 where we work with $\mathrm{G} = \mathrm{GL}_2(\mathbb{Q}_p)$, $\mathrm{M} = \mathrm{T}$ and R *is a field of characteristic* p. But when R *is a field of characteristic different from* p, *we prove in Proposition 4.11 that the answer is positive.*

4.3.4. *Commutative diagrams: answers.* The **answer to Question** 4 is positive. We prove that the diagrams (3.4) and (3.6) commute by proving the following result.

Proposition 4.4. *For any representation* $\mathrm{V} \in \mathrm{Rep}(\mathrm{M})$, *there is a natural isomorphism of right* $\mathcal{H}$-*modules*
$$
\mathrm{Ind}_{\mathcal{H}_{\mathrm{M}}}^{\mathcal{H}}(\mathrm{V}^{\mathcal{U}_{\mathrm{M}}}) = \mathrm{V}^{\mathcal{U}_{\mathrm{M}}} \otimes_{\mathcal{H}_{\mathrm{M}+}} \mathcal{H} \cong (\mathrm{Ind}_{\mathrm{P}}^{\mathrm{G}}(\mathrm{V}))^{\mathcal{U}}.
$$

The isomorphism of Proposition 4.4 is given by the map (4.14) below. We prove the proposition after the following lemma. Here we consider $^{\mathbb{M}}\mathbb{W}$ as a subset of $\mathbb{W}$ and fix a lift $\hat{d} \in \mathcal{N}_{\mathrm{G}} \cap \mathcal{K}$ for each $d \in {}^{\mathbb{M}}\mathbb{W}$.

Lemma 4.5. *Let* V *be a representation of* M.

(1) *The R-module $(\mathrm{Ind}_{\mathrm{P}}^{\mathrm{G}}(\mathrm{V}))^{\mathcal{U}}$ is generated by all*

$$f_{\mathrm{P}\hat{d}\mathcal{U},v}$$

where d (resp. v) ranges over ${}^{\mathbb{M}}\mathbb{W}$ (resp. $\mathrm{V}^{\mathcal{U}_{\mathrm{M}}}$) and $f_{\mathrm{P}\hat{d}\mathcal{U},v}$ denotes the unique $\mathcal{U}$-invariant function in $\mathrm{Ind}_{\mathrm{P}}^{\mathrm{G}}(\mathrm{V})$ with support $\mathrm{P}\hat{d}\mathcal{U}$ and value v at $\hat{d}$.

(2) *For $v \in \mathrm{V}^{\mathcal{U}_{\mathrm{M}}}$ and $d \in {}^{\mathbb{M}}\mathbb{W}$, we have $f_{\mathrm{P}\hat{d}\mathcal{U},v} = f_{\mathrm{P}\mathcal{U},v}\tau_{\hat{d}}$.*

(3) *For $v \in \mathrm{V}^{\mathcal{U}_{\mathrm{M}}}$ and $w \in \mathrm{W}_{\mathrm{M}^+}(1)$, we have $f_{\mathrm{P}\mathcal{U},v}\,\tau_w = f_{\mathrm{P}\mathcal{U},v\tau_w^{\mathrm{M}}}$.*

Proof. These results are proved for $\mathrm{G} = \mathrm{GL}(n,\mathrm{F})$ in [41, 6A 1,2 and 3] and for G split in [42, §5.5.1 and 5.5.2]. We recall the arguments here and check that they are valid for general G. They are similar to the ones in the proof of Lemma 3.15.

(1) By Lemma 2.15(2), a function $(\mathrm{Ind}_{\mathrm{P}}^{\mathrm{G}}(\mathrm{V}))^{\mathcal{U}}$ is a unique linear combination of $\mathcal{U}$-invariant functions with support in $\mathrm{P}\hat{d}\mathcal{U}$, for $d \in {}^{\mathbb{M}}\mathbb{W}$. Denote in this proof by $\tilde{\mathrm{V}}$ the representation V trivially inflated to a representation of P. A $\mathcal{U}$-invariant function with support in $\mathrm{P}\hat{d}\mathcal{U}$ is determined by its value at $\hat{d}$ which is an element in $\tilde{\mathrm{V}}^{\mathrm{P}\cap d\mathcal{U}d^{-1}} = \tilde{\mathrm{V}}^{\mathcal{U}_{\mathrm{M}}\mathrm{N}} = \mathrm{V}^{\mathcal{U}_{\mathrm{M}}}$ by Lemma 2.15(2)(c).

Now recall that for X a representation of G, and $f \in X^{\mathcal{U}}$, $w \in \mathrm{W}(1)$, if $\mathcal{U}\hat{w}\mathcal{U} = \sqcup_x \mathcal{U}\hat{w}x$ is a decomposition in simple cosets with x ranging over $(\mathcal{U} \cap \hat{w}^{-1}\mathcal{U}\hat{w})\backslash\mathcal{U}$, then we have $f\tau_w = \sum_x (\hat{w}x)^{-1}f$. Using this we check (2): for $v \in \mathrm{V}^{\mathcal{U}_{\mathrm{M}}}$ and $d \in {}^{\mathbb{M}}\mathbb{W}$, the function $f_{\mathrm{P}\mathcal{U},v}\tau_{\hat{d}}$ is $\mathcal{U}$-invariant with support in $\mathrm{P}\hat{d}\mathcal{U}$ and value at $\hat{d}$ equal to v because $\hat{d}\mathcal{U}\hat{d}^{-1} \cap \mathrm{P}\mathcal{U} \subset \mathcal{U}$ by Lemma 2.15(2)(b).

Point (3) is a direct consequence of Lemma 2.16(1). $\square$

Proof of Proposition 4.4. Analogously to the proof of Proposition 3.13, consider the morphism of right $\mathcal{H}$-modules

$$(4.14) \qquad \mathrm{V}^{\mathcal{U}_{\mathrm{M}}} \otimes_{\mathcal{H}_{\mathrm{M}^+}} \mathcal{H} \to (\mathrm{Ind}_{\mathrm{P}}^{\mathrm{G}}(\mathrm{V}))^{\mathcal{U}}, \quad v \otimes h \mapsto f_{\mathrm{P}\mathcal{U},v}\,h.$$

It is well defined by Lemma 4.5(3). By Remark 4.1, the vector space $\mathrm{V}^{\mathcal{U}_{\mathrm{M}}} \otimes_{\mathcal{H}_{\mathrm{M}^+}} \mathcal{H}$ decomposes as the direct sum of all $\mathrm{V}^{\mathcal{U}_{\mathrm{M}}} \otimes \tau_{\hat{d}}$ for $d \in {}^{\mathbb{M}}\mathbb{W}$. For $v \in \mathrm{V}^{\mathrm{U}_{\mathrm{M}}}$ and $d \in {}^{\mathbb{M}}\mathbb{W}$, the image of $v \otimes \tau_{\hat{d}}$ is equal to $f_{\mathrm{P}\hat{d}\mathrm{U},v}$ by Lemma 4.5(2). This proves that the map is bijective by Lemma 4.5(1) and its proof. $\square$

We now **answer Question** 5 positively by proving that the diagram (4.12) commutes. Let $\mathfrak{m}$ be a right $\mathcal{H}_{\mathrm{M}}$-module. The natural morphism of right $\mathcal{H}_{\mathrm{M}}$-modules $\mathfrak{m} \to (\mathfrak{m} \otimes_{\mathcal{H}_{\mathrm{M}}} \mathbf{X}_{\mathrm{M}})^{\mathcal{U}_{\mathrm{M}}}$, $m \mapsto m \otimes 1_{\mathcal{U}_{\mathrm{M}}}$ induces a morphism of right $\mathcal{H}$-modules

$$\mathrm{Ind}_{\mathcal{H}_{\mathrm{M}}}^{\mathcal{H}}(\mathfrak{m}) = \mathfrak{m} \otimes_{\mathcal{H}_{\mathrm{M}^+,\theta}} \mathcal{H} \to (\mathfrak{m} \otimes_{\mathcal{H}_{\mathrm{M}}} \mathbf{X}_{\mathrm{M}})^{\mathcal{U}_{\mathrm{M}}} \otimes_{\mathcal{H}_{\mathrm{M}^+,\theta}} \mathcal{H} \cong (\mathrm{Ind}_{\mathrm{P}}^{\mathrm{G}}(\mathfrak{m} \otimes_{\mathcal{H}_{\mathrm{M}}} \mathbf{X}_{\mathrm{M}}))^{\mathcal{U}}.$$

(the isomorphism comes from Proposition 4.4). By adjunction, this in turn induces a G-equivariant map

$$(4.15) \qquad \mathrm{Ind}_{\mathcal{H}_{\mathrm{M}}}^{\mathcal{H}}(\mathfrak{m}) \otimes_{\mathcal{H}} \mathbf{X} \longrightarrow \mathrm{Ind}_{\mathrm{P}}^{\mathrm{G}}(\mathfrak{m} \otimes_{\mathcal{H}_{\mathrm{M}}} \mathbf{X}_{\mathrm{M}})$$

sending $m \otimes 1_{\mathcal{U}} \in \mathfrak{m} \otimes_{\mathcal{H}_{\mathrm{M}^+,\theta}} \mathbf{X}$ to $f_{\mathrm{P}\mathcal{U},m\otimes 1_{\mathcal{U}_{\mathrm{M}}}}$ with the notation in Lemma 4.5. Our goal is to prove that (4.15) is an isomorphism which will prove that (4.12) commutes.

In the case when $\mathfrak{m} = \mathcal{H}_M$, note that the space $\mathrm{Ind}_P^G(\mathbf{X}_M)$ is naturally endowed with a structure of left $\mathcal{H}_M$-module which commutes with the action of G: the image hf of $f \in \mathrm{Ind}_P^G(\mathbf{X}_M)$ by $h \in \mathcal{H}_M$ is given by $(hf)(g) = hf(g)$ for $g \in G$. In this case, (4.15) is the G-equivariant and left $\mathcal{H}_M$-equivariant map

$$(4.16) \qquad\qquad \mathcal{H}_M \otimes_{\mathcal{H}_{M^+,\theta}} \mathbf{X} \longrightarrow \mathrm{Ind}_P^G(\mathbf{X}_M)$$

sending $1 \otimes 1_{\mathcal{U}} \in \mathcal{H}_M \otimes_{\mathcal{H}_{M^+,\theta}} \mathbf{X}$ to $f_{P\mathcal{U},1_{\mathcal{U}_M}}$.

Proposition 4.6. *The map* (4.16) *is an isomorphism of representations of* G *and of left* $\mathcal{H}_M$-*modules.*

Before proving the proposition, we draw two corollaries.

Corollary 4.7. *For any* $\mathfrak{m} \in \mathrm{Mod}(\mathcal{H}_M)$, *the* G-*equivariant map* (4.15) *is an isomorphism.*

Proof of Corollary 4.7. By [30, Lemma 7.7], the canonical G-equivariant map

$$(4.17) \qquad\qquad \mathfrak{m} \otimes_{\mathcal{H}_M} \mathrm{Ind}_P^G(\mathbf{X}_M) \longrightarrow \mathrm{Ind}_P^G(\mathfrak{m} \otimes_{\mathcal{H}_M} \mathbf{X}_M)$$

is an isomorphism. Tensoring the isomorphism (4.16) by $\mathfrak{m}$ and composing with (4.17) gives the composite isomorphism

$$\mathfrak{m} \otimes_{\mathcal{H}_{M^+,\theta}} \mathbf{X} \xrightarrow{\simeq} \mathfrak{m} \otimes_{\mathcal{H}_M} \mathrm{Ind}_P^G(\mathbf{X}_M) \xrightarrow{\simeq} \mathrm{Ind}_P^G(\mathfrak{m} \otimes_{\mathcal{H}_M} \mathbf{X}_M)$$

which coincides with (4.15). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

The notation † was introduced after (4.12). We have:

Corollary 4.8. *For* $V \in \mathrm{Rep}(M)$, *there is an isomorphism of representations of* G:

$$(\mathrm{Ind}_P^G(V))^\dagger \cong \mathrm{Ind}_P^G(V^\dagger).$$

The representation $V \in \mathrm{Rep}(M)$ *is generated by its* $\mathcal{U}_M$-*invariant subspace if and only if the representation* $\mathrm{Ind}_P^G(V) \in \mathrm{Rep}(G)$ *is generated by its* $\mathcal{U}$-*invariant subspace.*

Proof. Let $V \in \mathrm{Rep}(M)$.
• The representation $(\mathrm{Ind}_P^G(V))^\dagger$ is the image of the natural G-equivariant morphism

$$(4.18) \qquad\qquad (\mathrm{Ind}_P^G(V))^{\mathcal{U}} \otimes_{\mathcal{H}} \mathbf{X} \to \mathrm{Ind}_P^G(V), \quad f \otimes 1_{\mathcal{U}} \mapsto f,$$

corresponding by adjunction to the identity map of $(\mathrm{Ind}_P^G(V))^{\mathcal{U}}$. By Proposition 4.4, the $\mathcal{H}$-equivariant map

$$V^{\mathcal{U}_M} \otimes_{\mathcal{H}_{M^+,\theta}} \mathcal{H} \to (\mathrm{Ind}_P^G(V))^{\mathcal{U}}$$

sending $v \otimes 1$ onto $f_{P\mathcal{U},v}$ is an isomorphism. It induces an isomorphism $V^{\mathcal{U}_M} \otimes_{\mathcal{H}_{M^+,\theta}} \mathbf{X} \to (\mathrm{Ind}_P^G(V))^{\mathcal{U}} \otimes_{\mathcal{H}}$ $\mathbf{X}$. Composing the latter with (4.18) gives the G-equivariant map

$$(4.19) \qquad\qquad V^{\mathcal{U}_M} \otimes_{\mathcal{H}_{M^+,\theta}} \mathbf{X} \to \mathrm{Ind}_P^G(V), \quad v \otimes 1_{\mathcal{U}} \mapsto f_{P\mathcal{U},v}.$$

which has image $(\mathrm{Ind}_P^G(V))^\dagger$.
• The representation $V^\dagger$ is the image of the natural M-equivariant morphism

$$(4.20) \qquad\qquad \varphi_V : V^{\mathcal{U}_M} \otimes_{\mathcal{H}_M} \mathbf{X}_M \to V, \quad v \otimes 1_{\mathcal{U}_M} \mapsto v$$

corresponding by adjunction to the identity map of $V^{\mathcal{U}_M}$. By right exactness of the functor Ind_P^G, the map (4.20) induces a G-equivariant map

$$(4.21) \qquad \mathrm{Ind}_P^G(\varphi_V) : \mathrm{Ind}_P^G(V^{\mathcal{U}_M} \otimes_{\mathcal{H}_M} \mathbf{X}_M) \to \mathrm{Ind}_P^G(V), \quad f \mapsto \varphi_V \circ f.$$

which has image $\mathrm{Ind}_P^G(V^\dagger)$. By Corollary 4.7, we have a G-equivariant isomorphism $V^{\mathcal{U}_M} \otimes_{\mathcal{H}_{M^+,\theta}} \mathbf{X} \cong \mathrm{Ind}_P^G(V^{\mathcal{U}_M} \otimes_{\mathcal{H}_M} \mathbf{X}_M)$ that sends $v \otimes \mathbf{1}_{\mathcal{U}}$ onto $f_{P\mathcal{U}, v \otimes \mathbf{1}_{u_M}}$. Composing the latter with $\mathrm{Ind}_P^G(\varphi_V)$ gives the G-equivariant map

$$(4.22) \qquad V^{\mathcal{U}_M} \otimes_{\mathcal{H}_{M^+,\theta}} \mathbf{X} \to \mathrm{Ind}_P^G(V), v \otimes 1_{\mathcal{U}} \mapsto f_{P\mathcal{U},v}.$$

which has image $\mathrm{Ind}_P^G(V^\dagger)$.

• Since (4.19) and (4.22) coincide, we proved that the two subrepresentations $\mathrm{Ind}_P^G(V^\dagger)$ and $(\mathrm{Ind}_P^G(V))^\dagger$ of $\mathrm{Ind}_P^G(V)$ coincide. This will implies the second statement $\mathrm{Ind}_P^G(V^\dagger) = \mathrm{Ind}_P^G(V)$ is equivalent to $V^\dagger = V$, as the functor Ind_P^G is exact and faithful [61, this follows from (5) and (6)]. $\qquad \square$

We now prove Proposition 4.6 via the following lemmas. First we prove the surjectivity of

$$(4.16) \qquad \mathcal{H}_M \otimes_{\mathcal{H}_{M^+,\theta}} \mathbf{X} \longrightarrow \mathrm{Ind}_P^G (\mathbf{X}_M), \ 1 \otimes 1_{\mathcal{U}} \mapsto f_{P\mathcal{U}, 1_{u_M}}$$

in Lemma 4.9. Its injectivity is proved in Lemma 4.10.

Lemma 4.9. *The map* (4.16) *is surjective.*

Proof. This is a direct consequence of [30, Lemma 6.3] (which is valid for smooth representations over a commutative ring) as we explain now. Let $t \in T$ be a central element in M satisfying $-\alpha(\tilde{\nu}(t)) > 0$ for all $\alpha \in \Sigma^+ - \Sigma_M^+$ (see Remark 2.11). We denote its by μ_{M^+} its image in $\Lambda_{M^+}(1) \subset W_{M^+}(1)$. Then $\mathrm{Ind}_P^G(\mathbf{X}_M)$ is generated as a representation of G by the functions f_n, $n \in \mathbb{Z}$ with support $P\mathcal{U} = P\mathcal{U}_{N^{op}}$ and respective value $t^{-n} \mathbf{1}_{\mathcal{U}_M}$ on $\mathcal{U}_{N^{op}}$. Note that such a function is $\mathcal{U}$-invariant: for $k = k^+ k_0 k^- \in \mathcal{U} = \mathcal{U}_N \mathcal{U}_M \mathcal{U}_{N^{op}}$, we have $f_n(k) = k_0.t^{-n} \mathbf{1}_{\mathcal{U}_M} = t^{-n} \mathbf{1}_{\mathcal{U}_M} = f_n(1)$ since t in central in M. Since $t^{-n} \mathbf{1}_{\mathcal{U}_M} = (\tau_{\mu_{M^+}}^M)^n$ seen as an element of $\mathbf{X}_M$, we have in fact $f_n = f_{P\mathcal{U},(\tau_{\mu_{M^+}}^M)^n} = (\tau_{\mu_{M^+}}^M)^n f_{P\mathcal{U}, \mathbf{1}_{u_M}}$. By definition, this element is in the image of (4.16). $\qquad \square$

Lemma 4.10. (1) *The set* $\mathcal{U}M^+\mathcal{K}$ *is the disjoint union of all* $\mathcal{U}M^+\hat{d}\mathcal{U}$ *for* $d \in {}^{\mathbb{M}}W$.

    (2) *Let* $\mathbf{Y}$ *be the set the functions in* $\mathbf{X}$ *with support in* $\mathcal{U}M^+\mathcal{K}$. *The restriction to* $\mathbf{Y}$ *of the G-equivariant map*

$$F : \mathbf{X} \longrightarrow \mathrm{Ind}_P^G(\mathbf{X}_M), \ \mathbf{1}_{\mathcal{U}} \longmapsto f_{P\mathcal{U}, \mathbf{1}_{u_M}}$$

    is injective.

    (3) *For any element* $x \in \mathbf{X}$, *there is* $r \in \mathbb{N}$ *such that* $\tau_{\mu_{M^+}}^r x \in \mathbf{Y}$, *where* $\mu_{M^+} \in W_{M^+}(1)$ *was defined in the proof of Lemma 4.9.*

    (4) *The map* (4.16) *is injective.*

Proof of Lemma 4.10. Proof of (1). Compare this proof with [41, Lemma 2.4]. Recall that $W_0'(1) = (\mathcal{N}_G \cap \mathcal{K})/Z^1 \subset W(1)$ and we denote by $W_{M,0}'(1)$ the subgroup $(\mathcal{N}_M \cap \mathcal{K})/Z^1$ of $W(1)$. We may see ${}^{\mathbb{M}}W$ defined in Lemma 2.12 as a system of representatives of the right cosets $W_{M,0}'(1) \backslash W_0'(1)$. We have

$\mathcal{U}\mathrm{M}^+\mathcal{K} = \mathcal{U}\mathrm{W}_{\mathrm{M}+}(1)\mathcal{U}\mathrm{W}_0'(1)\mathcal{U} = \mathcal{U}\mathrm{W}_{\mathrm{M}+}(1)\mathcal{U}_{\mathrm{M}}\mathrm{W}_0'(1)\mathcal{U}$. The last equality follows from $\mathcal{U} = \mathcal{U}_{\mathrm{N}}\mathcal{U}_{\mathrm{M}}\mathcal{U}_{\mathrm{N}^{\mathrm{op}}}$, from $k\mathcal{U}_{\mathrm{N}^{\mathrm{op}}}k^{-1} \subset \mathcal{U}$ for $k \in \mathcal{K}$ and the fact that $\mathrm{W}_{\mathrm{M}+}(1)$ normalizes $\mathcal{U}_{\mathrm{N}}$. Hence

$$\mathcal{U}\mathrm{M}^+\mathcal{K} = \cup_{d\in{}^{\mathbb{M}}\mathbb{W}}\mathcal{U}\mathrm{W}_{\mathrm{M}+}(1)\mathcal{U}_{\mathrm{M}}\mathrm{W}_{\mathrm{M},0}'(1)\hat{d}\mathcal{U} \subset \cup_{d\in{}^{\mathbb{M}}\mathbb{W}}\mathcal{U}\mathrm{M}^+\hat{d}\mathcal{U}.$$

The inclusion in the other direction $\cup_{d\in{}^{\mathbb{M}}\mathbb{W}}\mathcal{U}\mathrm{M}^+\hat{d}\mathcal{U} \subset \mathcal{U}\mathrm{M}^+\mathcal{K}$ is obvious. To prove that the union is disjoint, we first write $\mathcal{U}\mathrm{M}^+\hat{d}\mathcal{U} = \mathcal{U}\mathrm{W}_{\mathrm{M}+}(1)\mathcal{U}\hat{d}\mathcal{U}$ for $d \in {}^{\mathbb{M}}\mathbb{W}$. By Lemma 2.14, the latter is equal to $\mathcal{U}\mathrm{W}_{\mathrm{M}+}(1)\hat{d}\mathcal{U}$. This proves that $\mathcal{U}\mathrm{M}^+\hat{d}\mathcal{U}$ is the disjoint union of all $\mathcal{U}w\hat{d}\mathcal{U}$ for $w \in \mathrm{W}_{\mathrm{M}+}(1)$ and that the sets $\mathcal{U}\mathrm{M}^+\hat{d}\mathcal{U}$ are disjoint when d ranges over ${}^{\mathbb{M}}\mathbb{W}$.

Proof of (2). By (1), the space $\mathbf{Y}$ decomposes as the direct sum of the subspaces $\mathbf{Y}_d$ of the functions with support in $\mathcal{U}\mathrm{M}^+\hat{d}\mathcal{U}$, for $d \in {}^{\mathbb{M}}\mathbb{W}$. The image of $\mathbf{Y}_d$ under F is contained in the subspace of the functions in $\mathrm{Ind}_{\mathrm{P}}^{\mathrm{G}}(\mathbf{X}_{\mathrm{M}})$ with support in $\mathrm{P}\mathcal{U}\mathrm{M}^+\hat{d}\mathcal{U}$. The set $\mathrm{P}\mathcal{U}\mathrm{M}^+\hat{d}\mathcal{U} = \mathrm{P}\mathcal{U}\mathrm{W}_{\mathrm{M}+}(1)\mathcal{U}\hat{d}\mathcal{U}$ is $\mathrm{P}\mathcal{U}\hat{d}\mathcal{U}$ by Lemma 2.16(1). Therefore, $F(\mathbf{Y}_d)$ is contained in the subspace $\mathrm{Ind}_{\mathrm{P}}^{\mathrm{P}\hat{d}\mathcal{U}}(\mathbf{X}_{\mathrm{M}})$ of the functions in $\mathrm{Ind}_{\mathrm{P}}^{\mathrm{G}}(\mathbf{X}_{\mathrm{M}})$ with support in $\mathrm{P}\hat{d}\mathcal{U}$. Using Lemma 2.15(2), the restriction to $\mathbf{Y}$ of F is injective if and only if its restriction to each $\mathbf{Y}_d$ is injective.

Let $d \in {}^{\mathbb{M}}\mathbb{W}$. By Lemma 2.16(3) (and the proof of point (1) above), the set $\mathcal{U}\mathrm{M}^+\hat{d}\mathcal{U}$ is the disjoint union $\sqcup_w\mathcal{U}\hat{w}\hat{d}\mathcal{U} = \sqcup_{w,x,u_x,y}\mathcal{U}\hat{w}xu_x\hat{d}y$ for

$$(4.23) \qquad w \in \mathrm{W}_{\mathrm{M}+}(1),\ x \in \hat{w}^{-1}\mathcal{U}_{\mathrm{M}}\hat{w} \cap \mathcal{U}_{\mathrm{M}}\backslash\mathcal{U}_{\mathrm{M}},\ u_x \in (\hat{w}x)^{-1}\mathcal{U}\hat{w}x \cap \mathcal{U}_{\mathrm{N}^{\mathrm{op}}}\backslash\mathcal{U}_{\mathrm{N}^{\mathrm{op}}},\ y \in \mathcal{U} \cap \hat{d}^{-1}\mathcal{U}\hat{d}\backslash\mathcal{U}.$$

We may write an element $g \in \mathbf{Y}_d$ as a R-linear combination

$$g = \sum_y g_y, \quad \text{with } g_y = \sum_{w,x,u_x} \lambda_{w,x,u_x,y}\mathbf{1}_{\mathcal{U}\hat{w}xu_x\hat{d}y}, \quad \lambda_{w,x,u_x,y} \in R$$

with w, x, u_x as in (4.23). By Lemma 2.16(2), the set $\mathrm{P}\hat{d}\mathcal{U}$ is the disjoint union $\sqcup_y\mathrm{P}\mathcal{U}\hat{d}y$ and note that $F(g_y)$ is the component of $F(g)$ with support in $\mathrm{P}\mathcal{U}\hat{d}y$. Suppose that $F(g) = 0$. Then $F(g_y) = 0$ for any y. We admit for a moment that for $u \in \mathcal{U}_{\mathrm{N}^{\mathrm{op}}}$ and $w \in \mathrm{W}_{\mathrm{M}+}(1)$, $x \in \mathcal{U}_{\mathrm{M}}$, $u_x \in \mathcal{U}_{\mathrm{N}^{\mathrm{op}}}$, $y \in \mathcal{U}$ we have the following formula

$$(4.24) \qquad [F(\mathbf{1}_{\mathcal{U}\hat{w}xu_x\hat{d}y})](u\hat{d}y) = \begin{cases} \mathbf{1}_{\mathcal{U}_{\mathrm{M}}\hat{w}x} & \text{if } \mathcal{U}\hat{w}xu_x = \mathcal{U}\hat{w}xu \\ 0 & \text{otherwise.} \end{cases}$$

Then for y as above and $u \in \mathcal{U}_{\mathrm{N}^{\mathrm{op}}}$ we have $[F(g_y)](u\hat{d}y) = \sum_{w,x} \lambda_{w,x,u_x,y}\mathbf{1}_{\mathcal{U}_{\mathrm{M}}\hat{w}x}$ for $w \in \mathrm{W}_{\mathrm{M}+}(1)$, $x \in \hat{w}^{-1}\mathcal{U}_{\mathrm{M}}\hat{w}\cap\mathcal{U}_{\mathrm{M}}\backslash\mathcal{U}_{\mathrm{M}}$, and u_x the only element in $(\hat{w}x)^{-1}\mathcal{U}\hat{w}x\cap\mathcal{U}_{\mathrm{N}^{\mathrm{op}}}\backslash\mathcal{U}_{\mathrm{N}^{\mathrm{op}}}$ such that $\mathcal{U}\hat{w}xu_x = \mathcal{U}\hat{w}xu$. This proves that $\lambda_{w,x,u_x,y} = 0$ for all w, x, u_x, y.

Proof of (4.24): Recall that $f := F(\mathbf{1}_{\mathcal{U}\hat{w}xu_x\hat{d}y}) = f_{\mathrm{P}\mathcal{U},\mathbf{1}_{\mathcal{U}_{\mathrm{M}}}}(.(\hat{w}xu_x\hat{d}y)^{-1})$ has support in $\mathrm{P}\mathcal{U}\hat{w}xu_x\hat{d}y \subseteq \mathrm{P}\mathcal{U}\hat{d}y$ by Lemma 2.16(1), and we have $f(u\hat{d}y) = f_{\mathrm{P}\mathcal{U},\mathbf{1}_{\mathcal{U}_{\mathrm{M}}}}(u(\hat{w}xu_x)^{-1})$. If $\mathcal{U}\hat{w}xu_x = \mathcal{U}\hat{w}xu$, then $\hat{w}xu_xu^{-1} \in \mathcal{U}\hat{w}x$ and $f(u\hat{d}y) = f_{\mathrm{P}\mathcal{U},\mathbf{1}_{\mathcal{U}_{\mathrm{M}}}}((\hat{w}xu_xu^{-1})^{-1}) = f_{\mathrm{P}\mathcal{U},\mathbf{1}_{\mathcal{U}_{\mathrm{M}}}}((\hat{w}x)^{-1}) = (\hat{w}x)^{-1}\mathbf{1}_{\mathcal{U}_{\mathrm{M}}} = \mathbf{1}_{\mathcal{U}_{\mathrm{M}}\hat{w}x}$. If $\mathcal{U}\hat{w}xu_x \neq \mathcal{U}\hat{w}xu$, then $u \notin \mathrm{P}\mathcal{U}\hat{w}xu_x$ by Lemma 2.16(1) and therefore $f(u\hat{d}y) = 0$. This proves (4.24).

Proof of (3). The idea of the argument comes from [48, Lemma 12 p.80] where $\mathrm{G} = \mathrm{GL}(n, F)$. A version of it in the case when G is split can be found in [31, Lemma 2.20] and in the general case in [30, Lemma 6.2]. From the proof of the latter we get that $(\mathrm{G}, \mathcal{B}, \mathscr{N}_{\mathrm{G}})$ is a generalized Tits system. It

follows that $\mathrm{G} = \mathcal{B}Z\mathcal{K} = \mathcal{U}Z\mathcal{K}$ and it is enough to prove the statement for $x = \mathbf{1}_{\mathcal{U}z}$ for $z \in \mathrm{Z}$ (as $\tau^r_{\mu^+_{\mathrm{M}^+}} \mathbf{1}_{\mathcal{U}g} = \mathbf{1}_{\mathcal{U}t^r\mathcal{U}g}$ for $g \in \mathrm{G}$). It also follows that there is a finite subset $A \subset \mathscr{N}_{\mathrm{G}}$ such that for any $z' \in \mathrm{Z}$ we have

$$z'\mathcal{U}z \subset z'\mathcal{B}z \subset \cup_{a \in A}\mathcal{B}z'a\mathcal{B} \subset \cup_{a \in A}\mathcal{B}z'a\mathcal{K}.$$

Since $\mathcal{K}$ is special, for any $a \in \mathscr{N}_{\mathrm{G}}$, there is $z_a \in \mathrm{Z}$ such that $a\mathcal{K} = z_a\mathcal{K}$. Pick $r \geq 1$ such that $t^r z_a \in \mathrm{M}^+$ for any $a \in A$, where $t \in \mathrm{T} \subseteq \mathrm{Z}$ is a lift for μ_{M^+} as in the proof of Lemma 4.9. Then we have

$$t^r\mathcal{U}z \subset \cup_{a \in A}\mathcal{B}t^r z_a\mathcal{K} \subset \mathcal{B}\mathrm{M}^+\mathcal{K} = \mathcal{U}\mathrm{M}^+\mathcal{K}.$$

Now $\tau^r_{\mu_{\mathrm{M}^+}} \mathbf{1}_{\mathcal{U}z} = \mathbf{1}_{\mathcal{U}t^r\mathcal{U}z}$ so $\tau^r_{\mu_{\mathrm{M}^+}} \mathbf{1}_{\mathcal{U}z} \in \mathbf{Y}$.

Proof of (4). By definition, the element μ_{M^+} is central in $\mathrm{W}_{\mathrm{M}}(1)$ and $\tau^{\mathrm{M}}_{\mu_{\mathrm{M}^+}}$ is invertible in $\mathcal{H}_{\mathrm{M}}$. Consider an element in $\mathcal{H}_{\mathrm{M}} \otimes_{\mathcal{H}_{\mathrm{M}^+},\theta} \mathbf{X}$. By (3) it can be written $(\tau^{\mathrm{M}}_{\mu_{\mathrm{M}^+}})^{-r} \otimes y$ for $y \in \mathbf{Y}$ and $r \in \mathrm{N}$ large enough. Its image by the left $\mathcal{H}_{\mathrm{M}}$-equivariant map (4.16) is $(\tau^{\mathrm{M}}_{\mu_{\mathrm{M}^+}})^{-r}F(y)$. Suppose that this image is zero, then $F(y) = 0$ and by (2), we have $y = 0$. This concludes the proof of (4). $\qquad\square$

The **answer to Question** 6 is positive when R is a field of characteristic different from p.

Proposition 4.11. *(i) When p is invertible in R,*

$$\mathrm{L}^{\mathcal{H}}_{\mathcal{H}_{\mathrm{M}}} \cong - \otimes_{\mathcal{H}_{\mathrm{M},\theta - \delta_{\mathrm{P}}}} \mathcal{H}_{\mathrm{M}},$$

where the notation $\mathcal{H}_{\mathrm{M},\theta - \delta_{\mathrm{P}}}$ is a reminder that we identify $\mathcal{H}_{\mathrm{M}}$ with its image in $\mathcal{H}$ via $\theta^-\delta_{\mathrm{P}}$.

(ii) When R is a field of characteristic different from p and $V \in \mathrm{Rep}(G)$ we have $(V_{\mathrm{N}})^{\mathcal{U}_{\mathrm{M}}} \simeq V^{\mathcal{U}} \otimes_{\mathcal{H}_{\mathrm{M},\theta - \delta_{\mathrm{P}}}} \mathcal{H}_M$.

Proof. When p is invertible in R we have $\theta^{*,+} = \theta^-\delta_{\mathrm{P}}$ by Lemma 2.21 and $\mathrm{L}^{\mathcal{H}}_{\mathcal{H}_{\mathrm{M}}} \cong - \otimes_{\mathcal{H}_{\mathrm{M},\theta^{*,+}}} \mathcal{H}_{\mathrm{M}}$ by (4.7). When R is a field of characteristic different from p, the isomorphism $(V_{\mathrm{N}})^{\mathcal{U}_{\mathrm{M}}} \simeq V^{\mathcal{U}} \otimes_{\mathcal{H}_{\mathrm{M},\theta - \delta_{\mathrm{P}}}} \mathcal{H}_{\mathrm{M}}$ is a particular case of [55, II.10.1 3)]. $\qquad\square$

The **answer to Question** 6 is negative when R is a field of characteristic p. The existence of a counter-example was suggested to us by Noriyuki Abe.

Proposition 4.12. *When $\mathrm{G} = \mathrm{GL}(2,\mathbb{Q}_p)$ and R is a field of characteristic p, there exists an extension π of the trivial representation $\mathrm{Triv}_{\mathrm{G}}$ by the Steinberg representation St_{G} which is not generated by its $\mathcal{U}$-invariant subspace.*

Corollary 4.13. *Let $\mathrm{G} = \mathrm{GL}(2,\mathbb{Q}_p)$. Suppose that R is a field of characteristic p. There is no functor $\mathbf{F} : \mathrm{Mod}(\mathcal{H}) \to \mathrm{Mod}(\mathcal{H}_{\mathrm{T}})$ such that $\mathbf{F}(V^{\mathcal{U}}) = (V_{\mathrm{U}})^{\mathcal{U}_{\mathrm{T}}}$ for any representation $V \in \mathrm{Rep}(G)$.*

Proof of the corollary. Here we work with the parabolic subgroup $\mathrm{B} = \mathrm{TU}$. Consider the representation π of the proposition. We are going to check that $\mathrm{St}^{\mathcal{U}}_{\mathrm{G}} \cong \pi^{\mathcal{U}}$. If there was a functor as in the corollary, we would have $((\mathrm{St}_{\mathrm{G}})_{\mathrm{U}})^{\mathcal{U}_{\mathrm{T}}} \cong (\pi_{\mathrm{U}})^{\mathcal{U}_{\mathrm{T}}}$. However, we are going to prove $(\mathrm{St}_{\mathrm{G}})_{\mathrm{U}} = \{0\}$ and $\pi_{\mathrm{U}} \neq \{0\}$ hence $((\mathrm{St}_{\mathrm{G}})_{\mathrm{U}})^{\mathcal{U}_{\mathrm{T}}} = \{0\}$ and $(\pi_{\mathrm{U}})^{\mathcal{U}_{\mathrm{T}}} \neq \{0\}$.

Recall that St_{G} is irreducible [56, Thm. 4.7] (and therefore generated by $\mathrm{St}^{\mathcal{U}}$), so the natural inclusion $\mathrm{St}^{\mathcal{U}}_{\mathrm{G}} \subset \pi^{\mathcal{U}}$ is an equality as $\pi^{\mathcal{U}}$ does not generate π and π/St is one dimensional. Now $(\mathrm{Triv}_{\mathrm{G}})_{\mathrm{U}}$ is one

dimensional and the U-coinvariant functor is right exact so $\pi_U \neq \{0\}$. The restriction of St_G to U is the space $C_c^\infty(U, R)$ of locally constant functions with compact support in U with the natural action of U [57, Lemma 3]. The group U does not have a Haar measure with value in R of characteristic p (since U does not contain a compact open subgroup of pro-order prime to p) hence $(St_G)_U = \{0\}$.   □

Proof of Proposition 4.12. Let R be a field of characteristic p. The one dimensional $\mathcal{H}$-modules $\text{Sign}_{\mathcal{H}}$ and $\text{Triv}_{\mathcal{H}}$ have been described in Remark 2.24. They are respectively isomorphic to $St_G^{\mathcal{U}}$ and $\text{Triv}_G^{\mathcal{U}}$ ([56, 4.4]). We have $\text{Sign}_{\mathcal{H}}(\tau_w) = (-1)^{\ell(w)}$, $\text{Triv}_{\mathcal{H}}(\tau_w) = 1$ if $\ell(w) = 0$ and $\text{Triv}_{\mathcal{H}}(\tau_w) = 0$ if $\ell(w) > 0$ for any $w \in W(1)$ since R has characteristic p,.

First we suppose that R is an algebraic closure of the residue field of F.

The existence of a representation π sitting in an exact sequence of the form $0 \to St_G \to \pi \to \text{Triv}_G \to 0$ and not generated by its $\mathcal{U}$-invariants follows from the following facts:

(1) The $\mathcal{U}$-invariant functor is an equivalence from the subcategory of representations of G generated by their $\mathcal{U}$-invariants to $\text{Mod}(\mathcal{H})$ with inverse $- \otimes \mathbf{X}$ (§4.3.2).
(2) $\dim_R \text{Ext}_G^1(\text{Triv}_G, St_G) > 1$ in the category $\text{Rep}(G)$.
(3) $\dim_R \text{Ext}_{\mathcal{H}}^1(\text{Triv}_{\mathcal{H}}, \text{Sign}_{\mathcal{H}}) = 1$ in the category $\text{Mod}(\mathcal{H})$.

Property (2) follows from [24, Thm. VII.4.18] (there is no symmetry, $\dim_R \text{Ext}_G^1(St_G, \text{Triv}_G) = 1$ by [27, Prop.4.3.30(ii)]). Colmez works over a finite subfield of R, but R being the union of its finite subfields, the results of Colmez are valid over R. A smooth extension E of Triv_G by St_G gives by (non smooth) duality an extension $E^\vee$ of the (non-smooth) dual $St_G^\vee$ by Triv_G. Let $A^+ = \begin{pmatrix} \mathbb{Q}_p^\times & 0 \\ 0 & 1 \end{pmatrix}$. As $\dim_R(St_G^\vee)^{A^+} = 1$ [24, Lemma. VII.4.16], the inverse image of $(St^\vee)^{A^+}$ in $E^\vee$ gives a representation of A^+ which is an extension of Triv_{A^+} by Triv_{A^+}. Such extensions are classified by the group $\text{Hom}(\mathbb{Q}_p^\times, R)$ of smooth (i.e. locally constant) group homomorphisms $\mathbb{Q}_p^\times \to R$. This construction induces a map

$$res_{A^+} : \text{Ext}_G^1(\text{Triv}_G, St_G) \to \text{Hom}(\mathbb{Q}_p^\times, R).$$

For each $\tau \in \text{Hom}(\mathbb{Q}_p^\times, R)$, Colmez constructs a smooth extension E_τ of Triv_G by St_G with $res_{A^+}(E_\tau) = \tau$. This implies $\dim_R \text{Ext}_G^1(\text{Triv}_G, St_G) \geq \dim_R \text{Hom}(\mathbb{Q}_p^\times, R) > 1$ (recall $\mathbb{Q}_p^\times \simeq \mathbb{Z} \times \mathbb{Z}/(p-1)\mathbb{Z} \times \mathbb{Z}_p$ if $p \neq 2$, and $\mathbb{Q}_2^\times \simeq \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}_2$ [50, Thm.2]). Note that Colmez shows that E_τ admits a central character and that res_{A^+} is an isomorphism if we restrict ourselves to smooth representations of G with a central character.

To prove (3), first we verify that as vector spaces we have

(4.25) $$\text{Ext}_{\mathcal{H}}^i(\text{Triv}_{\mathcal{H}}, \text{Sign}_{\mathcal{H}}) \cong \text{Ext}_{\mathcal{H}}^i(\text{Sign}_{\mathcal{H}}, \text{Triv}_{\mathcal{H}})$$

for any $i \geq 0$ and then we show that $\dim_R \text{Ext}_{\mathcal{H}}^1(\text{Sign}_{\mathcal{H}}, \text{Triv}_{\mathcal{H}}) = 1$.

Consider the involutive automorphism η of the algebra $\mathcal{H}$ ([62, Propositions 4.13 and 4.23], [64, (4)], introduced in the split case in [58, Cor. 2] (see also [43, §4.7] and note that in the last two references where τ_w^* is defined differently from the current article) satisfying

(4.26) $$\eta(\tau_w) = (-1)^{\ell(w)}\tau_w^* \quad (w \in W(1)).$$

For M a right $\mathcal{H}$-module, define $M\eta$ to be the right $\mathcal{H}$-module on the vector space M with the action of $\mathcal{H}$ twisted by η, i.e. $(m,h) \mapsto m\eta(h)$. For example, using (2.19), we obtain that the sign character $\mathrm{Sign}_{\mathcal{H}}$ and the trivial character $\mathrm{Triv}_{\mathcal{H}}$ (defined in Remark 2.24) satisfy

$$\mathrm{Triv}_{\mathcal{H}} = \mathrm{Sign}_{\mathcal{H}}\eta \quad \text{and} \quad \mathrm{Sign}_{\mathcal{H}} = \mathrm{Triv}_{\mathcal{H}}\eta.$$

When $\ell(w) = 0$ we have indeed $\tau_w^* = \tau_w$ and when $\ell(w) > 0$ we have $\mathrm{Sign}_{\mathcal{H}}(\tau_w^*) = 0$ because $\tau_{n_s}^* = \tau_{n_s} + |\mathbb{Z}_s'|^{-1} \sum_{z \in \mathbb{Z}_s'} \tau_z$ and therefore $\mathrm{Sign}_{\mathcal{H}}(\tau_{n_s}^*) = -1 + 1 = 0$ for any $s \in S_{aff}$ (in the case when G is split, compare with [42, §5.4.2]).

The map $\mathcal{H} \to \mathcal{H}\eta$, $h \mapsto \eta(h)$ being an isomorphism of right $\mathcal{H}$-modules, M is projective if and only if $M\eta$ is projective. Lastly, given M and N two right $\mathcal{H}$-modules, the identity map yields an isomorphism of vector spaces $\mathrm{Hom}_{\mathcal{H}}(M,N) \cong \mathrm{Hom}_{\mathcal{H}}(M\eta, N\eta)$. Therefore, if $P_\bullet = (P_i)_{i \geq 0}$ is a projective resolution of $\mathrm{Sign}_{\mathcal{H}}$, then $P\eta_\bullet := (P_i\eta)_{i \geq 0}$ is naturally a projective resolution of $\mathrm{Triv}_{\mathcal{H}}$. We apply the functor $\mathrm{Hom}_{\mathcal{H}}(-, \mathrm{Sign}_{\mathcal{H}}) = \mathrm{Hom}_{\mathcal{H}}(-, \mathrm{Triv}_{\mathcal{H}}\eta)$ to the complex $P\eta_\bullet \to \mathrm{Triv}_{\mathcal{H}}$ and obtain that $\mathrm{Ext}_{\mathcal{H}}^i(\mathrm{Triv}_{\mathcal{H}}, \mathrm{Sign}_{\mathcal{H}})$ is the ith cohomology space of the complex $\mathrm{Hom}_{\mathcal{H}}(P\eta_\bullet, \mathrm{Triv}_{\mathcal{H}}\eta)$ which is isomorphic to the complex $\mathrm{Hom}_{\mathcal{H}}(P_\bullet, \mathrm{Triv}_{\mathcal{H}})$. This proves (4.25).

We now prove $\dim_R \mathrm{Ext}_{\mathcal{H}}^1(\mathrm{Sign}_{\mathcal{H}}, \mathrm{Triv}_{\mathcal{H}}) = 1$. In $\mathrm{Mod}(\mathcal{H})$, the exact sequence

(4.27) $$0 \to \mathrm{Triv}_{\mathcal{H}} \to \mathrm{Triv}_{\mathcal{H}_{\mathrm{T}}} \otimes_{\mathcal{H}_{\mathrm{T}+,\theta}} \mathcal{H} = \mathrm{Ind}_{\mathcal{H}_{\mathrm{T}}}^{\mathcal{H}}(\mathrm{Triv}_{\mathcal{H}_{\mathrm{T}}}) \to \mathrm{Sign}_{\mathcal{H}} \to 0$$

does not split: by Prop. 4.4, we know that $\mathrm{Ind}_{\mathcal{H}_{\mathrm{T}}}^{\mathcal{H}}(\mathrm{Triv}_{\mathcal{H}_{\mathrm{T}}})$ is isomorphic to the $\mathcal{U}$-invariant subspace of the parabolic induction $\mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}}(1)$. This $\mathcal{U}$-invariant subspace is described explicitly in [56, Thm. 4.2] and one easily checks that it sits indeed in the nonsplit exact sequence (4.27). Applying $\mathrm{Hom}_{\mathcal{H}}(\mathrm{Sign}_{\mathcal{H}}, -)$, it induces an exact sequence

$$\{0\} \to \mathrm{Hom}_{\mathcal{H}}(\mathrm{Sign}_{\mathcal{H}}, \mathrm{Sign}_{\mathcal{H}}) \to \mathrm{Ext}_{\mathcal{H}}^1(\mathrm{Sign}_{\mathcal{H}}, \mathrm{Triv}_{\mathcal{H}}) \to \mathrm{Ext}_{\mathcal{H}}^1(\mathrm{Sign}_{\mathcal{H}}, \mathrm{Ind}_{\mathcal{H}_{\mathrm{T}+}}^{\mathcal{H}}(\mathrm{Triv}_{\mathcal{H}_{\mathrm{T}}})).$$

The exactness of the functor $\mathrm{Ind}_{\mathcal{H}_{\mathrm{T}}}^{\mathcal{H}}$ and of its left adjoint $\mathrm{L}_{\mathcal{H}_{\mathrm{T}}}^{\mathcal{H}} : \mathrm{Mod}(\mathcal{H}) \to \mathrm{Mod}(\mathcal{H}_{\mathrm{T}})$ (§4.2, property (1)) imply that we have isomorphic functors ([28, p. 163]) :

$$\mathrm{Ext}_{\mathcal{H}}^1(-, \mathrm{Ind}_{\mathcal{H}_{\mathrm{T}}}^{\mathcal{H}}(-)) \cong \mathrm{Ext}_{\mathcal{H}_{\mathrm{T}}}^1(\mathrm{L}_{\mathcal{H}_{\mathrm{T}}}^{\mathcal{H}}(-), -).$$

The claim in (3) then easily follows from $\mathrm{L}_{\mathcal{H}_{\mathrm{T}}}^{\mathcal{H}}(\mathrm{Sign}_{\mathcal{H}}) = \{0\}$ which we prove here. In our context, the Hecke algebra $\mathcal{H}_{\mathrm{T}}$ identifies with the group algebra $R[\mathrm{T}/\mathrm{T}^1]$.

Using (4.6), we have

$$\mathrm{L}_{\mathcal{H}_{\mathrm{T}}}^{\mathcal{H}} = (- \otimes_{\mathcal{H}_{\mathrm{T}-,\theta^*}} \mathcal{H}_{\mathrm{T}})\iota^{-1}\iota_{\mathrm{T}}.$$

We claim that $\mathrm{Sign}_{\mathcal{H}} \otimes_{\mathcal{H}_{\mathrm{T}-,\theta^*}} \mathcal{H}_{\mathrm{T}} = \{0\}$.

Let $t \in \mathrm{T}$ be an element satisfying $-\alpha(\tilde{\nu}(t)) < 0$ for $\alpha \in \Sigma^+$. We denote by μ_- its image in $\Lambda_{\mathrm{T}-}(1) = \mathrm{W}_{\mathrm{T}-}(1)$. Seen in $\mathrm{W}_{\mathrm{T}}(1)$, the element μ_- has length zero and $\tau_{\mu_-}^{\mathrm{T},*} = \tau_{\mu_-}^{\mathrm{T}}$. Seen in $\mathrm{W}(1)$, it has non zero length and therefore

$$\mathrm{Sign}_{\mathcal{H}}(\theta^*(\tau_{\mu_-}^{\mathrm{T}})) = \mathrm{Sign}_{\mathcal{H}}(\theta^*(\tau_{\mu_-}^{\mathrm{T},*})) = \mathrm{Sign}_{\mathcal{H}}(\tau_{\mu_-}^*) = \mathrm{Triv}_{\mathcal{H}}(\eta(\tau_{\mu_-}^*)) = (-1)^{\ell(\mu_-)}\mathrm{Triv}_{\mathcal{H}}(\tau_{\mu_-}) = 0.$$

In $\mathrm{Sign}_{\mathcal{H}} \otimes_{\mathcal{H}_{\mathrm{T}-,\theta^*}} \mathcal{H}_{\mathrm{T}}$ we have $1 \otimes 1 = 1 \otimes \tau_{\mu_-}^{\mathrm{T}}(\tau_{\mu_-}^{\mathrm{T}})^{-1} = \mathrm{Sign}_{\mathcal{H}}(\theta^*(\tau_{\mu_-}^{\mathrm{T}})) \otimes (\tau_{\mu_-}^{\mathrm{T}})^{-1} = 0$. This proves the claim and concludes the proof of $\dim_R \mathrm{Ext}_{\mathcal{H}}^1(\mathrm{Sign}_{\mathcal{H}}, \mathrm{Triv}_{\mathcal{H}}) = 1$.

When R is an algebraic closure of the residue field of F, we deduce from (1), (2) and (3) the existence of an extension E_τ of Triv_G by St_G which is not generated by its $\mathcal{U}$-invariants.

Now, let R be an arbitrary field of characteristic p. We are not aware of a version of (1) over a non algebraically closed field, but we do not need it. We note that (3) and its proof are valid for R, that the representation E_τ constructed by Colmez is defined over the finite field generated by the values of τ, and that we can choose $\tau \in \mathrm{Hom}(\mathbb{Q}_p^*, \mathbb{F}_p)$. Therefore we can choose the extension E_τ to be defined on the prime field $\mathbb{F}_p$. By tensor product with R, we obtain an extension (defined over R) of Triv_G by St_G which is not generated by its $\mathcal{U}$-invariants. $\square$

5. Supersingularity for Hecke modules and representations

Unless otherwise mentioned, R will be an algebraically closed field of characteristic p. The goal of this section is to prove Theorem 5.3.

5.1. **Supersingularity for Hecke modules and representations: definitions.** The classification of all smooth irreducible admissible R-representations of G up to the supercuspidal representations is described in [4]. This generalizes [1] and [31] which treat respectively the case when G split and the case when $G = \mathrm{GL}(n, F)$. A classification of all simple right $\mathcal{H}$-modules up to the supersingular modules is given in [2]. The classification of all supersingular right $\mathcal{H}$-modules is described in [64]. The latter generalizes [42] which treats the case when G is split. We recall the following definitions:

(1) An R-representation of G is *supercuspidal* if it is admissible irreducible, and if furthermore it is not a subquotient of $\mathrm{Ind}_P^G \tau$ for some standard parabolic subgroup P of G with standard Levi decomposition $P = MN$ where $M \neq G$ and $\tau \in \mathrm{Rep}(M)$ is an admissible irreducible representation [4, I.3].

(2) A smooth representation π of G is *supersingular* if it is non zero admissible irreducible and if, furthermore, there is an irreducible smooth representation ρ of $\mathcal{K}$ such that the space

$$\mathrm{Hom}_G(\mathrm{ind}_{\mathcal{K}}^G \rho, \pi) = \mathrm{Hom}_{\mathcal{K}}(\rho, \pi|_{\mathcal{K}}) \neq \{0\}$$

is non zero and contains an eigenvector for the left action of the center of $\mathcal{H}(G, \mathcal{K}, \rho) = \mathrm{End}_G(\mathrm{ind}_{\mathcal{K}}^G(\rho))$ with a supersingular eigenvalue χ. This means that given a standard parabolic subgroup P with Levi decomposition $P = MN$, the character χ extends to the center of the spherical Hecke algebra $\mathcal{H}(M, \mathcal{K} \cap M, \rho_{N \cap \mathcal{K}})$ via the Satake homomorphism $\mathcal{H}(G, \mathcal{K}, \rho) \hookrightarrow \mathcal{H}(M, \mathcal{K} \cap M, \rho_{N \cap \mathcal{K}})$ only when $P = G$ ([4, I.5]). A smooth representation of G which is non zero admissible irreducible and not supersingular will be sometimes called *nonsupersingular*.

Remark 5.1. For $\pi \in \mathrm{Rep}(G)$ irreducible admissible, π is supercuspidal if and only if π is supersingular. This theorem is proved in [4, Theorem 5] which generalizes [31] and [1].

(3) In [64, §1.2 and §1.3], a central subalgebra $\mathcal{Z}_T$ of the pro-p Iwahori Hecke algebra $\mathcal{H}$ is defined (it was first introduced in the split case in [42, Definition 5.10] where it is denoted by $\mathcal{Z}^\circ(\mathcal{H})$). There is an isomorphism $\mathcal{Z}_T \cong R[X_*^+(T)]$ ([42, Prop. 2.10], [64, Thm 1.4]). Denote by $X_*^0(T) \subset X_*^+(T)$

the maximal subgroup of the monoid $X_*^+(T)$. Then $R[X_*(T)^+ - X_*^0(T)]$ identifies with a proper ideal $\mathcal{Z}_{T,\ell>0}$ of $\mathcal{Z}_T$. As a vector space, $\mathcal{Z}_T$ is isomorphic to the direct sum $R[X_*^0(T)] \oplus \mathcal{Z}_{T,\ell>0}$.

A morphism of R-algebras $\mathcal{Z}_T \to R$ is called *supersingular* if its kernel contains $\mathcal{Z}_{T,\ell>0}$.

A non-zero element v of a right $\mathcal{H}$-module σ is called *supersingular* if there is an integer $n \geq 1$ such that $v\mathcal{Z}_{T,\ell>0}^n = \{0\}$. The module σ is called *supersingular* when all its elements are supersingular (this is different from the definition [64, Def 6.10] where we suppose $\sigma \mathcal{Z}_{T,\ell>0}^n = \{0\}$ for some integer $n \geq 1$).

Note that a supersingular module as defined above does not even necessarily have finite length.

A simple right $\mathcal{H}$-module which is not supersingular will be sometimes called *nonsupersingular*.

Remark 5.2. Recall that R is an algebraically closed field of characteristic p. We recall the following facts which, in the case when G is split, are contained in [42, §5.3].

(a) A simple right $\mathcal{H}$-module is finite dimensional. This follows immediately from [59, 5.3] since $\mathcal{H}$ is finitely generated over its center $\mathcal{Z}(\mathcal{H})$ and since $\mathcal{Z}(\mathcal{H})$ is an R-algebra of finite type [63, Thm 1.2]. These properties remain true when the center is replaced by the central subalgebra $\mathcal{Z}_T$ ([42, Prop. 2.5], [64, Thm 1.4]).

(b) The central subalgebra $\mathcal{Z}_T \subset \mathcal{H}$ acts on a simple (hence finite dimensional) $\mathcal{H}$-module $\mathfrak{m}$ by a character.

(c) A non zero finite dimensional $\mathcal{H}$-module $\mathfrak{m}$ is supersingular if and only if all the subquotients of $\mathfrak{m}$ seen as a $\mathcal{Z}_T$-module are supersingular characters of $\mathcal{Z}_T$.

(d) A submodule and a quotient of a supersingular $\mathcal{H}$-module are supersingular.

(e) A non zero finite dimensional $\mathcal{H}$-module is not supersingular if and only if it contains a nonsupersingular simple submodule.

Only the direct implication in Point (e) requires a justification. We first recall the following general facts: let R be a field, A an R-algebra, and C a central subalgebra of A. The functors $\operatorname{Ext}_A^r(-,-)$ are considered in the abelian category $\operatorname{Mod}_A^{fd}$ of right A-modules which are finite dimensional over R. Given an ideal $\mathcal{J}$ of C, a finite dimensional right A-module $\mathfrak{m}$ which is killed by some positive power of $\mathcal{J}$ is called $\mathcal{J}$-torsion. We have:

i. The category $\operatorname{Mod}_A^{fd}$ decomposes into the direct sum, over the maximal ideals $\mathcal{M}$ of C, of the subcategory of $\mathcal{M}$-torsion modules.

ii. An A-module of finite dimension over R is $\mathcal{M}$-torsion if and only if, seen as a C-module, it admits a composition series where all quotients are isomorphic to $C/\mathcal{M}$.

iii. Given an ideal $\mathcal{J}$ of C, the category of finite dimensional $\mathcal{J}$-torsion A-modules decomposes into the direct sum, over the maximal ideals $\mathcal{M}$ of C containing $\mathcal{J}$, of the subcategory of $\mathcal{M}$-torsion modules.

iv. Let $\mathcal{M}$ and $\mathcal{M}'$ be two maximal ideals of C. If there exists a $\mathcal{M}$-torsion module $\mathfrak{m} \in \operatorname{Mod}_A^{fd}$ and a $\mathcal{M}'$-torsion module $\mathfrak{m}' \in \operatorname{Mod}_A^{fd}$ such that $\operatorname{Ext}_A^r(\mathfrak{m}, \mathfrak{m}') \neq \{0\}$ for some integer $r \geq 0$, then $\mathcal{M} = \mathcal{M}'$.

When $C = A$, the first three properties are [26, Theorem 2.13 b, c]. In general, for $\mathfrak{m} \in \mathrm{Mod}_A^{fd}$, the set of $x \in \mathfrak{m}$ which are killed by some positive power of $\mathcal{M}$ is the component of $\mathfrak{m}$ in the subcategory of $\mathcal{M}$-torsion finite dimensional C-modules. To prove iv., note that the C-module $\mathrm{Ext}_A^r(\mathfrak{m}, \mathfrak{m}')$ is killed by a positive power of $\mathcal{M}$ and by a positive power of $\mathcal{M}'$. The localization of $\mathrm{Ext}_A^r(\mathfrak{m}, \mathfrak{m}')$ at any maximal ideal of C different from $\mathcal{M}$ or from $\mathcal{M}'$ is 0. If $\mathcal{M} \neq \mathcal{M}'$ then the localization of $\mathrm{Ext}_A^r(\mathfrak{m}, \mathfrak{m}')$ at any maximal ideal of C is 0. This is equivalent to $\mathrm{Ext}_A^r(\mathfrak{m}, \mathfrak{m}') = \{0\}$ [26, Lemma 2.8].

These general statements apply when R is a field of characteristic p, $A = \mathcal{H}$, $C = \mathcal{Z}_{\mathrm{T}}$ and $\mathcal{J} = \mathcal{Z}_{\mathrm{T}, \ell > 0}$. When the field R is furthermore algebraically closed, we have $\mathcal{Z}_{\mathrm{T}}/\mathcal{M} \simeq R$ for any maximal ideal $\mathcal{M}$ of $\mathcal{Z}_{\mathrm{T}}$ and a character $\chi : \mathcal{Z}_{\mathrm{T}} \to R$ is supersingular if its kernel contains $\mathcal{Z}_{\mathrm{T}, \ell > 0}$. Statements i.-iv. in this context have already been observed in [42, §5.3] (where G was supposed split). The direct implication in Point (e) follows from them.

5.2. **Main theorem.**

Theorem 5.3. *Suppose that R is an algebraically closed field of characteristic p. Let $\pi \in \mathrm{Rep}(\mathrm{G})$ be an irreducible admissible representation. Then the following are equivalent:*

(1) *π is supersingular;*
(2) *the finite dimensional $\mathcal{H}$-module $\pi^{\mathcal{U}}$ is supersingular.*
(3) *the finite dimensional $\mathcal{H}$-module $\pi^{\mathcal{U}}$ admits a supersingular subquotient.*

When $\mathbf{G} = \mathbf{Z}$, the theorem is trivial because all finite dimensional $\mathcal{H}$-modules are supersingular and all irreducible admissible representations of G are supersingular.

It is obvious that (2) implies (3). In this article, we prove that $(3) \Rightarrow (1) \Rightarrow (2)$, that is to say,

(5.1) $\pi^{\mathcal{U}}$ admits a supersingular subquotient $\Rightarrow \pi$ supersingular $\Rightarrow \pi^{\mathcal{U}}$ supersingular.

The proof of (5.1) is given in §5.4. It requires the results of Remark 5.4 and §5.3. Note that several of these preliminary results hold not only when R is an algebraically closed field of characteristic zero but also when it is an arbitrary commutative ring R.

Remark 5.4. That (2) implies (1) is proved in [64, Prop 7.10, Rmk 7.11]. (When $\mathbf{G}$ is F-split, $\mathbf{G_{der}} = \mathbf{G_{sc}}$ and the characteristic of F is 0, the arguments to prove this implication were given in [42] in Equation (58) and Lemma 5.25). The arguments are the following. When π is not supersingular, there is an irreducible representation ρ of $\mathcal{K}$ such that

$$\mathrm{Hom}_{\mathrm{G}}(\mathrm{ind}_{\mathcal{K}}^{\mathrm{G}} \rho, \pi) = \mathrm{Hom}_{\mathcal{K}}(\rho, \pi|_{\mathcal{K}}) \neq \{0\}$$

and this space contains an eigenvector with a nonsupersingular eigenvalue for the left action of the center of $\mathcal{H}(\mathrm{G}, \rho)$ by definition of the supersingularity. Then $\pi^{\mathcal{U}}$ contains a nonsupersingular submodule therefore the $\mathcal{H}$-module $\pi^{\mathcal{U}}$ is not supersingular.

5.3. **Standard triples.** Let R be a commutative ring.

5.3.1. *Standard triples of* G. A standard triple of G is a triple (P, σ, Q) where

-   P is a standard parabolic subgroup of G with standard Levi decomposition $P = MN$,
-   σ is a R-representation of M,
-   Q is a parabolic subgroup such that $P \subset Q \subset P(\sigma)$ where $P(\sigma)$ is the parabolic subgroup of G attached to $\Pi(\sigma) = \Pi_M \sqcup \Pi_\sigma$ where Π_σ is the set of $\alpha \in \Pi$ such that $Z \cap M'_\alpha$ acts trivially on σ.

Standard triples (P, σ, Q) and (P, σ', Q) of G where $\sigma \simeq \sigma'$ are called equivalent. A standard triple (P, σ, Q) of G is called smooth (resp. supersingular, resp. supercuspidal) if σ is smooth (resp. supersingular, resp. supercuspidal).

Given a parabolic subgroup Q as above, we will often denote its standard Levi decomposition by $Q = M_Q N_Q$.

Remark 5.5.   (1) For a standard triple (P, σ, Q) of G, the representation σ of M extends to a unique representation $e_Q(\sigma)$ of Q which is trivial on N [4, II.7 (i)].

(2) Let $M_\sigma \subset G$ denote the standard Levi subgroup of G such that $\Pi_{M_\sigma} = \Pi_\sigma$. The subgroup $Z \cap M'_\sigma$ is generated by all $Z \cap M'_\alpha$ for $\alpha \in \Pi_\sigma$ and $M'_\alpha = G'_\alpha$ (notation 2.1) [4, II.3, II.4]; its image $_1\Lambda_{M'_\sigma}$ in the pro-p Iwahori Weyl group $W(1)$ of G is generated by the images $_1\Lambda_{M'_\alpha}$ of $Z \cap M'_\alpha$ in $W(1)$. If σ is supercuspidal, the roots $\alpha \in \Pi_\sigma$ are orthogonal to Π_M. If the roots $\alpha \in \Pi_\sigma$ are orthogonal to Π_M, the group $_1\Lambda_{M'_\sigma}$ is contained in the group of elements of length 0 of the pro-p Iwahori Weyl group $W_M(1)$ of M. We have $\sigma^{\mathcal{U}_M}(\tau_w^{M,*}) = \mathrm{Id}$ for $w \in {}_1\Lambda_{M'_\sigma}$, as $v\tau_w^{M,*} = v\tau_w^M = \hat{w}^{-1}v = v$ for $v \in \sigma^{\mathcal{U}_M}, \hat{w} \in Z \cap M'_\sigma$.

Let (P, σ, Q) be a standard smooth triple of G. For $Q \subset Q' \subset P(\sigma)$, the R-representation $\mathrm{Ind}_{Q'}^G(e_{Q'}(\sigma))$ of G is smooth and embeds naturally in the smooth R-representation $\mathrm{Ind}_Q^G(e_Q(\sigma))$ because the representation $e_{Q'}(\sigma)$ of Q' extending σ trivially on N is equal to $e_Q(\sigma)$ on Q. We define the following smooth R-representation of G:

$$I_G(P, \sigma, Q) := \mathrm{Ind}_Q^G(e_Q(\sigma)) / \sum_{Q \subsetneq Q' \subset P(\sigma)} \mathrm{Ind}_{Q'}^G(e_{Q'}(\sigma)).$$

5.3.2. *Classification of the irreducible admissible representations of* G. Assume that R is an algebraically closed field of characteristic p. The main result of [4, Thm.s. 1, 2, 3 and 5] says that the map $(P, \sigma, Q) \to I_G(P, \sigma, Q)$ is a bijection from the standard supercuspidal triples of G up to equivalence onto the irreducible admissible smooth R-representations of G modulo isomorphism.

Remark 5.6. Let (P, σ, Q) be a standard supercuspidal triple of G. The representation $I_G(P, \sigma, Q)$ is supersingular if and only if $P = G$ [4, Thm. 5]). It is finite dimensional if and only if $P = B$ and $Q = G$ [4, I.5 Rmk. 2)] in which case we have $I_G(B, \sigma, G) = e_G(\sigma)$. Any irreducible smooth representation of Z is supercuspidal and supersingular.

Remark 5.7. The pro-p Iwahori group $\mathcal{U}$ acts trivially on the irreducible smooth finite dimensional R-representations of G. Such a representation is indeed of the form $e_G(\sigma)$ for an irreducible smooth

representation σ of Z by the classification theorem. Then use the Iwahori decomposition (2.14) which $\mathcal{U}$ admits with respect to B, and U, U^{op}, and the fact that U, U^{op} act trivially on $e_G(\sigma)$, and Z^1 acts trivially on σ hence on $e_G(\sigma)$.

5.3.3. *Standard triples of $\mathcal{H}$.* A standard triple of the pro-p Iwahori Hecke R-algebra $\mathcal{H}$ is a triple (P, σ, Q) where

- P is a standard parabolic subgroup of G with standard Levi decomposition P = MN,
- σ is right $\mathcal{H}_M$-module,
- Q is a parabolic subgroup with $P \subset Q \subset P(\sigma)$ where $P(\sigma)$ is the standard parabolic subgroup corresponding to $\Pi(\sigma) = \Pi_M \sqcup \Pi_\sigma \subset \Pi$ where Π_σ is the set of roots $\alpha \in \Pi$ which are orthogonal to Π_M and such that $\sigma(\tau_w^{M,*}) = \mathrm{Id}$ if $w \in {}_1\Lambda_{M'_\alpha}$ (as defined in Remark 5.5).

Standard triples (P, σ, Q) and (P, σ', Q) of $\mathcal{H}$ with $\sigma \simeq \sigma'$ are called equivalent. When σ is finite dimensional (resp. simple, resp. supersingular), the standard triple (P, σ, Q) of $\mathcal{H}$ is called finite dimensional (resp. simple, resp. supersingular).

By Remark 5.5, if (P, σ, Q) is a superscuspidal standard triple of G then $(P, \sigma^{\mathcal{U}_M}, Q)$ is a standard triple of $\mathcal{H}$.

Let (P, σ, Q) a standard triple of $\mathcal{H}$. There exists a unique $\mathcal{H}_{M_Q}$-module structure $e_{\mathcal{H}_{M_Q}}(\sigma)$ on σ such that $e_{\mathcal{H}_{M_Q}}(\sigma)(\tau_w^{M_Q,*}) = \sigma(\tau_w^{M,*})$ for $w \in W_M(1)$ and $e_{\mathcal{H}_{M_Q}}(\sigma)(\tau_w^{M_Q,*}) = \mathrm{Id}_\sigma$ for $w \in W_{M'_{Q,\sigma}}(1)$ where $M_{Q,\sigma} \subset G$ is the standard Levi subgroup corresponding to $\Pi_{M_Q} \cap \Pi_\sigma$. Let $P \subset Q \subset Q_1 \subset P(\sigma)$; we have $W_{M_Q}(1) \subset W_{M_{Q_1}}(1)$, and ${}^{M_Q}\mathbb{W}_{M_{Q_1}}$ the set of $w \in W_{M_{Q_1}}$ with minimal length in the coset $W_{M_Q}w$; it is proved in [6, Proposition 4.5] that the map

$$x \otimes 1 \mapsto x \otimes \Big(\sum_{d \in {}^{M_Q}\mathbb{W}_{M_{Q_1}}} T_{\tilde{d}} \Big) : e_{\mathcal{H}_{M_{Q_1}}}(\sigma) \otimes_{\mathcal{H}_{M^+_{Q_1}}, \theta} \mathcal{H} \to e_{\mathcal{H}_{M_Q}}(\sigma) \otimes_{\mathcal{H}_{M^+_Q}, \theta} \mathcal{H}$$

is a well defined injective $\mathcal{H}$-equivariant morphism from $\mathrm{Ind}^{\mathcal{H}}_{\mathcal{H}_{M_{Q_1}}}(e_{\mathcal{H}_{M_{Q_1}}}(\sigma))$ into $\mathrm{Ind}^{\mathcal{H}}_{\mathcal{H}_{M_Q}}(e_{\mathcal{H}_{M_Q}}(\sigma))$; one may then define the right $\mathcal{H}$-module

$$(5.2) \qquad I_{\mathcal{H}}(P, \sigma, Q) := \mathrm{Ind}^{\mathcal{H}}_{\mathcal{H}_{M_Q}}(e_{\mathcal{H}_{M_Q}}(\sigma)) / \sum_{Q \subsetneq Q_1 \subset P(\sigma)} \mathrm{Ind}^{\mathcal{H}}_{\mathcal{H}_{M_{Q_1}}}(e_{\mathcal{H}_{M_{Q_1}}}(\sigma)).$$

The smooth parabolic induction Ind^G_P corresponds to the induction $\mathrm{Ind}^{\mathcal{H}}_{\mathcal{H}_M}$ via the pro-p-Iwahori invariant functor and its left adjoint. This is proved in the current paper in Prop 4.4 and Corollary 4.7 (see the commutative diagrams in Questions 4 and 5). It is proved in [6, Thm 4.17 and Cor. 5.13] that this implies:

Theorem 5.8. *(i) Let (P, σ, Q) be a supercuspidal standard triple of G. Then $(P, \sigma^{\mathcal{U}_M}, Q)$ is a standard $\mathcal{H}$-triple and the $\mathcal{H}$-modules*

$$I_G(P, \sigma, Q)^{\mathcal{U}} \cong I_{\mathcal{H}}(P, \sigma^{\mathcal{U}_M}, Q)$$

are isomorphic.

(ii) Let (P, σ, Q) *be a standard triple of* $\mathcal{H}$*. Then* $(P, \sigma \otimes_{\mathcal{H}_M} \mathbf{X}_M, Q)$ *is a standard smooth triple of* G *and the smooth* R*-representations of* G

$$I_{\mathcal{H}}(P, \sigma, Q) \otimes_{\mathcal{H}} \mathbf{X} \cong I_G(P, \sigma \otimes_{\mathcal{H}_M} \mathbf{X}_M, Q)$$

are isomorphic. In particular $e_{\mathcal{H}}(\sigma) \otimes_{\mathcal{H}} \mathbf{X} \cong e_G(\sigma \otimes_{\mathcal{H}_M} \mathbf{X}_M)$ *when* $Q = P(\sigma) = G$.

Note that in the case when $P = B$, $M = Z$ and σ is the trivial character of $\mathcal{H}_Z$, the isomorphism $e_{\mathcal{H}}(\sigma) \otimes_{\mathcal{H}} \mathbf{X} \simeq e_G(\sigma \otimes_{\mathcal{H}_Z} \mathbf{X}_Z)$ is the isomorphism proved in Lemma 2.25.

5.3.4. *Classification of the simple* $\mathcal{H}$*-modules.* Assume that R is an algebraically closed field of characteristic p.

Theorem 5.9. *The map* $(P, \sigma, Q) \to I_{\mathcal{H}}(P, \sigma, Q)$ *is a bijection from the set of standard simple supersingular triples up to equivalence of* $\mathcal{H}$ *onto the set of simple* $\mathcal{H}$*-modules modulo isomorphism.*

This variant of the classification of Abe [2] (Abe's classification uses coinduction instead of induction) is proved in [6, Cor.4.30]; we prefer the induction because it is compatible with the parabolic induction functor and with the $\mathcal{U}$-invariant functor (Theorem 5.8).

Proposition 5.10. *Let* $P = MN$ *be a standard parabolic subgroup of* G *with its standard Levi decomposition and* σ *a simple supersingular right* $\mathcal{H}_M$*-module. Then the composition factors of the induced module* $\mathrm{Ind}_{\mathcal{H}_M}^{\mathcal{H}}(\sigma)$ *are* $I_{\mathcal{H}}(P, \sigma, Q)$ *for* $P \subset Q \subset P(\sigma)$.

This is the variant for an *induced* $\mathcal{H}$-module of a result of Abe [2, Cor. 4.26] which considers the right $\mathcal{H}$-module *coinduced* from a simple supersingular right $\mathcal{H}_M$-module.

Theorem 5.9 and Proposition 5.10 follows from the comparison between induction and coinduction.

Remark 5.11. (i) The $\mathcal{H}$-module $I_{\mathcal{H}}(P, \sigma, Q)$ is simple and supersingular if and only if $P = Q = G$ and σ is simple and supersingular (this is a part of the classification).

(ii) Let τ be a simple right $\mathcal{H}_M$-module and (P^M, σ, Q^M) a standard simple supersingular triple of $\mathcal{H}_M$ such that $\tau = I_{\mathcal{H}_M}(P^M, \sigma, Q^M)$. Let $R \subset P$ be the standard parabolic subgroup of G such that the standard Levi subgroup M_R of R is equal to the standard Levi subgroup of P^M. Then, the composition factors of the $\mathcal{H}$-module $I_{\mathcal{H}}(P, \tau, Q)$ are composition factors of $\mathrm{Ind}_{\mathcal{H}_{M_R}}^{\mathcal{H}}(\sigma)$ (apply Proposition 5.10 and the transitivity of the parabolic induction [66, Thm. 1.4]). Hence they are of the form $I_{\mathcal{H}}(R, \sigma, S)$ for some parabolic subgroups S such that $R \subset S \subset R(\sigma)$ which can be explicitly determined, if necessary. From (i), none of the simple subquotients of the $\mathcal{H}$-module $I_{\mathcal{H}}(P, \tau, Q)$ is supersingular if $R \neq G$. Otherwise $P = Q = R = G$ and $I_{\mathcal{H}}(G, \tau, G) = \tau$ is supersingular.

(iii) When τ is a finite dimensional right $\mathcal{H}_M$-module and $P \neq G$, the $\mathcal{H}$-module $I_{\mathcal{H}}(P, \tau, Q)$ admits no supersingular subquotient (Apply (ii) to the composition factors of the $\mathcal{H}_M$-module τ).

(iv) A simple right $\mathcal{H}_M$-module is supersingular if and only if it is not a subquotient of $\mathrm{Ind}_{\mathcal{H}_M}^{\mathcal{H}}(\tau)$ for some standard Levi subgroup $M \neq G$ and some simple $\mathcal{H}_M$-module τ. Compare with the definition of a supercuspidal representation of G in §5.1 (1).

5.4. **Proof of (5.1).** We suppose that $G \neq Z$. Let $\pi \in \text{Rep}(G)$ be a non zero irreducible admissible representation. Then $\pi^{\mathcal{U}}$ is a non-zero finite dimensional right $\mathcal{H}$-module.

A) We prove the second implication of (5.1) *i.e.* that if π is supersingular, then $\pi^{\mathcal{U}}$ is supersingular.

Suppose that the $\mathcal{H}$-module $\pi^{\mathcal{U}}$ is not supersingular. By Remark 5.2(e), the $\mathcal{H}$-module $\pi^{\mathcal{U}}$ contains a simple nonsupersingular submodule. By §5.3.4 and Remark 5.11, there is a standard simple supersingular triple (P, σ, Q) of $\mathcal{H}$ with $P = MN \neq G$ such that $I_{\mathcal{H}}(P, \sigma, Q)$ is contained in $\pi^{\mathcal{U}}$. We prove that the representation π of G is not supercuspidal.

> a/ Suppose $Q \neq G$. We have by adjunction and by the commutativity of the diagram (4.12):
>
> $$0 \neq \text{Hom}_{\mathcal{H}}(\text{Ind}_{\mathcal{H}_{M_Q}}^{\mathcal{H}}(e_{\mathcal{H}_{M_Q}}(\sigma)), \pi^{\mathcal{U}}) \cong \text{Hom}_{kG}(\text{Ind}_{\mathcal{H}_{M_Q}}^{\mathcal{H}}(e_{\mathcal{H}_{M_Q}}(\sigma)) \otimes_{\mathcal{H}} \mathbf{X}, \pi)$$
>
> $$\cong \text{Hom}_{kG}(\text{Ind}_Q^G(e_{\mathcal{H}_{M_Q}}(\sigma) \otimes_{\mathcal{H}_{M_Q}} \mathbf{X}_{M_Q}), \pi).$$
>
> Since the simple $\mathcal{H}_{M_Q}$-module $e_{\mathcal{H}_{M_Q}}(\sigma)$ has a central character, the representation $e_{\mathcal{H}_{M_Q}}(\sigma) \otimes_{\mathcal{H}_{M_Q}} \mathbf{X}_{M_Q}$ of M_Q has a central character. We deduce that π is not supercuspidal by [30, Prop. 7.9].
>
> b/ Suppose $Q = G$. We have $e_{\mathcal{H}}(\sigma) \subset \pi^{\mathcal{U}}$ and by adjunction, π is a quotient of $e_{\mathcal{H}}(\sigma) \otimes_{\mathcal{H}} \mathbf{X}$. But $e_{\mathcal{H}}(\sigma) \otimes_{\mathcal{H}} \mathbf{X} \cong e_G(\sigma \otimes_{\mathcal{H}_M} \mathbf{X}_M)$ (Theorem 5.8(ii)). The group N acts trivially on π because $e_G(\sigma \otimes_{\mathcal{H}_M} \mathbf{X}_M)$ is the representation of G trivial on N extending the representation $\sigma \otimes_{\mathcal{H}_M} \mathbf{X}_M$ of M. Hence π is not supercuspidal because the space of N-coinvariants of a supercuspidal representation of G is zero (this follows immediately from [4, II.7 Prop.] and from the classification recalled in §5.3.2).

B) We prove the first implication of (5.1) *i.e* that if $\pi^{\mathcal{U}}$ admits a supersingular subquotient then π is supersingular.

Suppose that π is nonsupersingular. Consider a supersingular standard triple (P, σ, Q) of G such that $\pi \cong I_G(P, \sigma, Q)$. As π is not supersingular, we have $P \neq G$. By Theorem 5.8(i) we have

$$\pi^{\mathcal{U}} \cong I_G(P, \sigma, Q)^{\mathcal{U}} = I_{\mathcal{H}}(P, \sigma^{\mathcal{U}_M}, Q).$$

By Remark 5.11 (iii) applied to $I_{\mathcal{H}}(P, \sigma^{\mathcal{U}_M}, Q)$, $\pi^{\mathcal{U}}$ admits no supersingular subquotient because $P \neq G$.

Remark 5.12. When **G** is semisimple and simply connected, we do not need to refer to [6] to prove the second implication of (5.1), that is to say, π is supersingular $\Rightarrow \pi^{\mathcal{U}}$ is supersingular. This reference appears only through Theorem 5.8 in the proof. When **G** is semisimple and simply connected, it suffices to refer to Lemma 2.25 as we explain here: **G** is then the direct product of its almost simple components and we can reduce to an isotropic component. When **G** is almost simple, simply connected and isotropic, we have, in case **A)** b/ of the proof: $P \neq Q = G$ implies $P = B, M = Z$ and σ is the trivial module of $\mathcal{H}_Z$. This is because $\Pi = \Pi_M \sqcup \Pi_\sigma$ (§5.3.3) and Π irreducible implies $\Pi_M = \emptyset$. As $\mathcal{H}_Z = R[Z/Z^1]$, the module σ identifies with an irreducible representation σ_Z of Z. But $\Pi = \Pi(\sigma)$ implies $\Pi = \Pi(\sigma_Z)$, meaning that σ_Z extends to a representation of G which is trivial on G′ (as defined in Notation 2.1). But $G = G'$ because **G** is almost simple, simply connected and isotropic [4, II.4 Prop.]. Hence σ_Z is the trivial representation of Z.

Remark 5.13. Let $\mathbf{G}^{sc} \to \mathbf{G}^{der}$ be the simply connected cover of the derived group $\mathbf{G}^{der}$ of $\mathbf{G}$, and let $\mathbf{C}$ be the connected center of $\mathbf{G}$. We have a natural homomorphism $\iota : \mathrm{G}^{sc} \to \mathrm{G}$. We can prove that π is supersingular $\Rightarrow \pi^{\mathcal{U}}$ is supersingular using Remark 5.12 when:

(5.3)   The restriction to $\iota(\mathrm{G}^{sc})$ of any supercuspidal R-representation π of G has finite length.

Property (5.3) is true when the index of the normal subgroup $\iota(\mathrm{G}^{sc})C$ of G is finite (for instance when the characteristic of F is 0) or when R is the field of complex numbers ([51]) We do not know whether (5.3) is true for any algebraically closed field R.

6. Parabolic induction functors for Hecke modules and their adjoints in the p-adic and the finite case: summary and comparison table

Let R be a commutative ring. We consider P a standard parabolic subgroup of G with standard Levi decomposition $\mathrm{P} = \mathrm{M}\,\mathrm{N}$ and $\mathbb{P}$ the standard parabolic subgroup of $\mathbb{G}$ with standard Levi decomposition $\mathbb{P} = \mathbb{M}\,\mathbb{N}$ image of $\mathrm{P} \cap \mathcal{K}$ in $\mathbb{G}$. We pick lifts $\widetilde{\mathbf{w}}$ and $\widetilde{\mathbf{w}_{\mathbb{M}}}$ in $W(1)$ for the longest elements $\mathbf{w}$ and $\mathbf{w}_{\mathbb{M}}$ of $\mathbb{W}$ and $\mathbb{W}_{\mathbb{M}}$ respectively as in §4.2. Let $\mathrm{M}' = \widetilde{\mathbf{w}}\mathrm{M}\widetilde{\mathbf{w}}^{-1}$. Recall that we defined in §4.2 quasi-inverse equivalences of categories $-\iota_{\mathrm{M}}^{-1}\iota : \mathrm{Mod}(\mathcal{H}_{\mathrm{M}}) \to \mathrm{Mod}(\mathcal{H}_{\mathrm{M}'})$ and $-\iota^{-1}\iota_{\mathrm{M}} : \mathrm{Mod}(\mathcal{H}_{\mathrm{M}'}) \to \mathrm{Mod}(\mathcal{H}_{\mathrm{M}})$. Up to isomorphism, these functors are not affected by the choice of other lifts for $\mathbf{w}$ and $\mathbf{w}_{\mathbb{M}}$. Therefore, the automorphism of R-algebras

(6.1) $\mathcal{H}_{\mathrm{M}'} \to \mathcal{H}_{\mathrm{M}}, \; \tau_x \mapsto \tau_{\widetilde{\mathbf{w}_{\mathbb{M}}}\widetilde{\mathbf{w}}x(\widetilde{\mathbf{w}_{\mathbb{M}}}\widetilde{\mathbf{w}})^{-1}}$ and its inverse $\mathcal{H}_{\mathrm{M}} \to \mathcal{H}_{\mathrm{M}'}$

define quasi-inverse equivalences of categories

(6.2) $\iota_{\mathrm{M}',\mathrm{M}} : \mathrm{Mod}(\mathcal{H}_{\mathrm{M}}) \to \mathrm{Mod}(\mathcal{H}_{\mathrm{M}'})$ and $\iota_{\mathrm{M},\mathrm{M}'} : \mathrm{Mod}(\mathcal{H}_{\mathrm{M}'}) \to \mathrm{Mod}(\mathcal{H}_{\mathrm{M}})$

which are respectively isomorphic to $-\iota_{\mathrm{M}}^{-1}\iota$ and $-\iota^{-1}\iota_{\mathrm{M}}$. They restrict to quasi-inverse equivalences of categories

(6.3) $\iota_{\mathbb{M}',\mathbb{M}} : \mathrm{Mod}(\mathrm{H}_{\mathbb{M}}) \to \mathrm{Mod}(\mathrm{H}_{\mathbb{M}'})$ and $\iota_{\mathbb{M},\mathbb{M}'} : \mathrm{Mod}(\mathrm{H}_{\mathbb{M}'}) \to \mathrm{Mod}(\mathrm{H}_{\mathbb{M}})$.

which are respectively isomorphic to $-\iota_{\mathbb{M}}^{-1}\iota$ and $-\iota^{-1}\iota_{\mathbb{M}}$ (defined in Proposition 3.4).

In §2.5.2, we recalled the definition of the embeddings θ and θ^* of $\mathcal{H}_{\mathrm{M}+}$ and $\mathcal{H}_{\mathrm{M}-}$ into $\mathcal{H}$. In the table below, given an embedding of algebras $\phi : \mathcal{H}_{\mathrm{M}}{}^{\pm} \to \mathcal{H}$, we recall that the index $\mathcal{H}_{\mathrm{M}\pm,\phi}$ means that we see $\mathcal{H}$ as an $\mathcal{H}_{\mathrm{M}\pm}$-module via ϕ. In Remark 2.20 where p is invertible in R, the embeddings θ^+, θ^{*+} and θ^- of $\mathcal{H}_{\mathrm{M}}$ into $\mathcal{H}$ are defined. Again, in the table below, given an embedding of algebras $\phi : \mathcal{H}_{\mathrm{M}} \to \mathcal{H}$, the index $\mathcal{H}_{\mathrm{M},\phi}$ means that we see $\mathcal{H}$ as an $\mathcal{H}_{\mathrm{M}}$-module via ϕ.

When R is an arbitrary ring and when p is invertible in R, the table below summarizes and compares the definitions of the parabolic induction functors $\mathrm{Ind}_{\mathcal{H}_{\mathrm{M}}}^{\mathcal{H}} : \mathrm{Mod}(\mathcal{H}_{\mathrm{M}}) \to \mathrm{Mod}(\mathcal{H})$ and $\mathrm{Ind}_{\mathrm{H}_{\mathbb{M}}}^{\mathrm{H}} : \mathrm{Mod}(\mathrm{H}_{\mathbb{M}}) \to \mathrm{Mod}(\mathrm{H})$ and of their respective adjoints.

The top left quadrant of the table (finite case, arbitrary R) is justified by Proposition 3.4. The lower left quadrant of the table (finite case, p invertible R) is justified by Remark 3.6. The lower right quadrant of the table (p-adic case, p invertible in R) is justified in Remark 4.2(2) based on recent work by Abe [3]

	Finite case	p-Adic case
Arbitrary R	$\mathrm{Ind}_{\mathrm{H}_\mathbb{M}}^\mathrm{H} = - \otimes_{\mathrm{H}_\mathbb{M}} \mathrm{H} \cong \mathrm{Hom}_{\mathrm{H}_{\mathbb{M}'}}(\mathrm{H}, -) \circ \iota_{\mathbb{M}',\mathbb{M}}$ $\mathrm{R}_{\mathrm{H}_\mathbb{M}}^\mathrm{H} \cong \mathrm{Res}_{\mathrm{H}_\mathbb{M}}^\mathrm{H}$ $\mathrm{L}_{\mathrm{H}_\mathbb{M}}^\mathrm{H} \cong \iota_{\mathbb{M},\mathbb{M}'} \circ \mathrm{Res}_{\mathrm{H}_{\mathbb{M}'}}^\mathrm{H}$	$\mathrm{Ind}_{\mathcal{H}_\mathrm{M}}^\mathcal{H} = - \otimes_{\mathcal{H}_{\mathrm{M}+,\theta}} \mathcal{H} \cong \mathrm{Hom}_{\mathcal{H}_{\mathrm{M}'-,\theta*}}(\mathcal{H}, -) \circ \iota_{\mathrm{M}',\mathrm{M}}$ $\mathrm{R}_{\mathcal{H}_\mathrm{M}}^\mathcal{H} \cong \mathrm{Hom}_\mathcal{H}(\mathcal{H}_\mathrm{M} \otimes_{\mathcal{H}_{\mathrm{M}+,\theta}} \mathcal{H}, -)$ $\mathrm{L}_{\mathcal{H}_\mathrm{M}}^\mathcal{H} \cong \mathrm{Res}_{\mathcal{H}_{\mathrm{M},\theta*+}}^\mathcal{H} \cong \iota_{\mathrm{M},\mathrm{M}'} \circ (- \otimes_{\mathcal{H}_{\mathrm{M}'-,\theta*}} \mathcal{H}_{\mathrm{M}'})$
p invertible in R	$\mathrm{Ind}_{\mathrm{H}_\mathbb{M}}^\mathrm{H} - = - \otimes_{\mathrm{H}_\mathbb{M}} \mathrm{H} \cong \mathrm{Hom}_{\mathrm{H}_\mathbb{M}}(\mathrm{H}, -)$ $\mathrm{R}_{\mathrm{H}_\mathbb{M}}^\mathrm{H} \cong \mathrm{Res}_{\mathrm{H}_\mathbb{M}}^\mathrm{H}$ $\mathrm{L}_{\mathrm{H}_\mathbb{M}}^\mathrm{H} \cong \mathrm{Res}_{\mathrm{H}_\mathbb{M}}^\mathrm{H}$	$\mathrm{Ind}_{\mathcal{H}_\mathrm{M}}^\mathcal{H} = - \otimes_{\mathcal{H}_{\mathrm{M},\theta+}} \mathcal{H} \cong \mathrm{Hom}_{\mathcal{H}_{\mathrm{M},\theta*+}}(\mathcal{H}, -)$ $\mathrm{R}_{\mathcal{H}_\mathrm{M}}^\mathcal{H} \cong \mathrm{Res}_{\mathcal{H}_{\mathrm{M},\theta+}}^\mathcal{H}$ $\mathrm{L}_{\mathcal{H}_\mathrm{M}}^\mathcal{H} \cong \mathrm{Res}_{\mathcal{H}_{\mathrm{M},\theta*+}}^\mathcal{H} \cong \mathrm{Res}_{\mathcal{H}_{\mathrm{M},\theta-\delta_P}}^\mathcal{H}$

(see the isomorphism (4.7) for $\mathrm{L}_{\mathcal{H}_\mathrm{M}}^\mathcal{H}$ and recall that it is valid for an arbitrary R) and in Proposition 4.11. The top right quadrant of the table (p-adic case, arbitrary R) is justified in §4.2 (see there the references to [66]) as well as by the isomorphism (4.7).

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University of British Columbia, 1984 Mathematics Road, Vancouver, BC V6T 1Z2, Canada
E-mail address: `ollivier@math.ubc.ca`

Institut de Mathématiques de Jussieu, UMR 7586, 175 Rue du Chevaleret 75013 Paris, France
E-mail address: `marie-france.vigneras@imj-prg.fr`