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Representations of a p-adic group in characteristic p

G. Henniart and M.-F. Vignéras

For Joseph Bernstein

Abstract. Let F be a locally compact non-archimedean field of residue characteristic p, $\mathbf{G}$ a connected reductive group over F, and R a field of characteristic p. When R is algebraically closed, the irreducible admissible R-representations of $G = \mathbf{G}(F)$ are classified in [J. Amer. Math. Soc. **30** (2017), no. 2, 495–559] in term of supersingular R-representations of the Levi subgroups of G and parabolic induction; there is a similar classification for the simple modules of the pro-p Iwahori Hecke R-algebra $H(G)_R$ in [N. Abe, DOI:10.1515/crelle-2016-0043]. In this paper, we show that both classifications hold true when R is not algebraically closed.

Contents

I. Introduction

I.1 In this paper, p is a prime number, F is a locally compact non-archimedean field of residual characteristic p, $\mathbf{G}$ is a connected reductive group over F and finally R is a field.

Recent applications of automorphic forms to number theory have imposed the study of smooth representations of $G = \mathbf{G}(F)$ on R-vector spaces; indeed one expects a strong relation, à la Langlands, with R-representations of the Galois group of F. The most intriguing case is when the characteristic of R is p - the only established case, however, is that of $GL(2, \mathbb{Q}_p)$.

The first focus is on irreducible representations. When R is algebraically closed of characteristic p, the irreducible admissible R-representations of G have been classified in terms of parabolic induction of supersingular R-representations of Levi subgroups of G [**AHHV**]. But the restriction to algebraically closed R is undesirable: for example, in the work of Breuil and Colmez on $GL(2, \mathbb{Q}_p)$, R is often finite. Here we extend to any R the classification of [**AHHV**] and its consequences.

Let B be a minimal parabolic subgroup of G and I a compatible pro-p Iwahori subgroup of G. If W is a smooth R-representation of G, the space W^I of I-fixed elements is a right module over the Hecke ring $H(G)$ of I in G; it is non-zero if W is, and finite dimensional if W is admissible. Even though W^I might not be simple over $H(G)$ when W is irreducible, it is important to study simple $R \otimes H(G)$-modules. When R is algebraically closed of characteristic p, they have been classified ([**Abe**], see also [**AHenV2**, Cor:4.30]) in terms of supersingular $R \otimes H(M)$-modules, where M is a Levi subgroup of G and $H(M)$ the Hecke ring of $I \cap M$ in M. The classification uses a parabolic induction process from $H(M)$-modules to $H(G)$-modules. Again we extend that classification to any R of characteristic p.

I.2 Before we state our main results more precisely, let us describe our principal tools for reducing them to the known case where R is algebraically closed - those tools are developed in section II.

The idea is to introduce an algebraic closure R^{alg} of R, and study scalar extension from R-representations of G to R^{alg}-representations of G, or from $R \otimes H(G)$-modules to $R^{alg} \otimes H(G)$-modules. The important remark is that when W is an irreducible admissible R-representation of G, or a simple $R \otimes H(G)$-module, its commutant has finite dimension over R. The following result examines what happens for more general extensions R' of R.

THEOREM I.1. [Decomposition theorem] *Let R be a field, A an R-algebra*[1]*, and V a simple A-module with commutant $D = \operatorname{End}_A V$ of finite dimension over R. Let E denote the center of the skew field D, δ the reduced degree of D over E, E_{sep}/R the maximal separable subextension of E/R.*

(1) Let E' be a finite separable extension of E splitting D, L/R the normal closure of E'/R and R' an extension of L. Then the scalar extension $V_{R'}$ of V to R' has length $\delta[E : R]$ and is a direct sum

$$V_{R'} \simeq \oplus^{\delta} \oplus_{j \in \operatorname{Hom}_R(E_{sep}, R')} W'_j$$

of δ copies of a direct sum of $[E_{sep} : R]$ modules W'_j with commutant the local artinian ring $R' \otimes_{j, E_{sep}} E$ which has residue field R'. For each j, the $A_{R'}$-module

[1] all our algebras are associative with unit

W'_j is indecomposable of length $[E : E_{sep}]$, its simple subquotients are all isomorphic to the $A_{R'}$-module $V'_j = R' \otimes_{(R' \otimes_{E_{sep}} E)} W'_j$ which has commutant R', and descend to the finite extension L/R.

If R'/R is normal, the isomorphism classes of the $A_{R'}$-modules W'_j, resp. V'_j, for $j \in \mathrm{Hom}_R(E_{sep}, R')$ form an $\mathrm{Aut}_R(R')$-orbit of cardinality $[E_{sep} : R]$.

(2) Let R^{alg}/R be an algebraic closure. The map which to V as above associates the $\mathrm{Aut}_R(R^{alg})$-orbit of a simple subquotient V' of $V_{R^{alg}}$ induces a bijection

- from the set of isomorphism classes $[V]$ of simple A-modules V with commutant of finite dimension over R,

- onto the set of $\mathrm{Aut}_R(R^{alg})$-orbits of isomorphism classes $[V']$ of absolutely simple $A_{R^{alg}}$-modules V' descending to a finite extension of R.

We note that the $\mathrm{Aut}_R(R^{alg})$-orbit of $[V']$ is finite when V' descends to a finite extension of R. Part (1) of the theorem implies:

Corollary I.2. *For any extension R'/R, the length of the $A_{R'}$-module $V_{R'}$ is $\leq \delta[E : R]$, and the dimension over R' of the commutant of any subquotient of $V_{R'}$ is finite.*

When the field R is perfect (example: R finite or of characteristic 0), every algebraic extension of R is separable [**Lang**, VII §7 Cor. 7.8]. In that simple case, the $A_{R'}$-modules W'_j, are absolutely simple in Thm. I.1; in fact, for any extension R'/R, $V_{R'}$ is semi-simple [**BkiA8**, §12 n^o 1 Prop.1].

The second theorem is a criterion, inspired by [**AHenV1**, Lemma 3.11], for a functor to preserve the lattice of submodules of a module W. If W is an object in an abelian category, we write $\mathcal{L}_W$ for the lattice of its subobjects; if W has finite length, that length is written $\lg(W)$.

Theorem I.3. [Lattice isomorphism] *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between abelian categories having a right adjoint G; write $\epsilon : \mathrm{id} \to GF$ for the unit of the adjunction, and $\eta : FG \to \mathrm{id}$ for the counit.*

(a) Let W be a finite length object in $\mathcal{C}$ such that

(i) $F(Y)$ and $GF(Y)$ are simple for any simple subquotient Y of W;

(ii) $F(W)$ and $GF(W)$ have finite length $\lg(F(W)) = \lg(GF(W)) = \lg(W)$. Then for any subquotient Y of W, $F(Y)$ and $GF(Y)$ have finite length $\lg(F(Y)) = \lg(GF(Y)) = \lg(Y)$, and $\epsilon_Y : Y \to GF(Y)$ is an isomorphism; in addition the maps $Y \mapsto F(Y) : \mathcal{L}_W \to \mathcal{L}_{F(W)}$ and $X \mapsto \epsilon_W^{-1}(G(X)) : \mathcal{L}_{F(W)} \to \mathcal{L}_W$ are lattice isomorphisms, inverse to each other.

(b) Let V be a finite length object in $\mathcal{D}$ such that

(i) $G(X)$ and $FG(X)$ are simple for any simple subquotient X of V;

(ii) $G(V)$ and $FG(V)$ have finite length $\lg(G(V)) = \lg(FG(V)) = \lg(V)$. Then for any subquotient X of V, $G(X)$ and $FG(X)$ have finite length $\lg(G(X)) = \lg(FG(X)) = \lg(X)$, and $\eta_X : FG(X) \to X$ is an isomorphism. In addition, the maps $X \mapsto G(X) : \mathcal{L}_V \to \mathcal{L}_{G(V)}$ and $Y \mapsto \eta_V(F(Y)) : \mathcal{L}_{G(V)} \to \mathcal{L}_V$ are lattice isomorphisms, inverse to each other.

The present formulation and its proof in §II.4 owe much to the referee. We get (b) from (a) by reversing the arrows.

Corollary I.4. *Assume that F is fully faithful. Let W be a finite length object in $\mathcal{C}$ such that*

(i) $F(Y)$ is simple for any simple subquotient Y of W;

(ii) $F(W)$ has finite length $\lg(F(W)) = \lg(W)$.
Then $Y \mapsto F(Y) : \mathcal{L}_W \to \mathcal{L}_{F(W)}$ is a lattice isomorphism.

We end §II with another lattice isomorphism inspired by [**Abe**, Lemma 5.3]. Let R be a field, A an R-algebra, and V a simple A-module with commutant R. We have the tensor product $-\otimes_R V : \mathcal{C} \to \mathcal{D}$ from the abelian category $\mathcal{C}$ of R-vector spaces to the abelian category $\mathcal{D}$ of A-modules; it has a right adjoint $\mathrm{Hom}_A(V, -)$.

Theorem I.5. [Lattice isomorphism and tensor product]

(i) For any R-vector space W, $W \otimes_R V$ is an isotypic A-module of type V and the map $Y \mapsto Y \otimes_R V : \mathcal{L}_W \to \mathcal{L}_{W\otimes_R V}$ is a lattice isomorphism. Moreover, the natural map

$$W \xrightarrow{\epsilon_W} \mathrm{Hom}_A(V, W \otimes_R V) \quad \epsilon_W(w) : v \mapsto w \otimes v$$

is an isomorphism of R-vector spaces.

(ii) For $b_W \in \mathrm{End}_R(W)$, $b_V \in \mathrm{End}_R(V)$ and an R-subspace Y of W, we have: $b_W(Y) \subset Y$ implies $b_W(Y) \otimes_R b_V(V) \subset Y \otimes_R V$ and the converse is true provided that $b_V \neq 0$.

In our applications, the action of A on V extends to an R-algebra A' containing A, and there is an R-basis B of A contained in an R-basis B' of A' such that no element of $B' \setminus B$ acts by 0 on V, the action of B on W by the identity extends to an action of A' and the diagonal action of B' on $W \otimes_R V$ yields an A'-module structure. On $X = V, W$ or $W \otimes_R V$, $b \in B'$ acts via an endomorphism written b_X.

Corollary I.6. *In the above situation, in Theorem* I.5*:*

The map $Y \mapsto Y \otimes_R V$ yields a lattice isomorphism $\mathcal{L}'_W \to \mathcal{L}'_{W\otimes_R V}$ between the lattices of A'-submodules of W and of $W \otimes_R V$. The A'-module structure on $\mathrm{Hom}_A(V, W \otimes_R V)$ such that the isomorphism $W \xrightarrow{\epsilon_W} \mathrm{Hom}_A(V, W \otimes_R V)$ is A'-equivariant, satisfies for all $f \in \mathrm{Hom}_A(V, W \otimes_R V)$,

if $b \in B$ then $b(f) = f$, and if $b \in B' \setminus B$ acts invertibly on V then $b(f) = b_{W\otimes_R V} \circ f \circ b_V^{-1}$.

In that situation the natural map $\mathrm{Hom}_A(V, W \otimes_R V) \otimes_R V \to W \otimes_R V$ is also an isomorphism of A'-modules for $b \in B'$ acting diagonally.

I.3 In §III, for a field R of characteristic p, we prove the classification of the irreducible admissible R-representations of G in terms of supersingular irreducible admissible R-representations of Levi subgroups of G.

We always take our parabolic subgroups to contain a minimal one $B = ZU$ in good position with respect to I. An R-triple (P, σ, Q) of G consists of a parabolic subgroup $P = MN$ of G, a smooth R-representation σ of M, and a parabolic subgroup Q of G satisfying $P \subset Q \subset P(\sigma)$, where $P(\sigma) = M(\sigma)N(\sigma)$ is the maximal parabolic subgroup of G to which σ extends trivially on N; the restriction to M_Q of that extension is denoted by $e_Q(\sigma)$. By definition

(0.1) $$I_G(P, \sigma, Q) = \mathrm{Ind}_{P(\sigma)}^G(\mathrm{St}_Q^{M(\sigma)}(\sigma)) \quad \text{where}$$

(0.2) $$\mathrm{St}_Q^{M(\sigma)}(\sigma) = \mathrm{Ind}_Q^{M(\sigma)}(e_Q(\sigma)) / \sum_{Q \subsetneq Q' \subset P(\sigma)} \mathrm{Ind}_{Q'}^{M(\sigma)}(e_{Q'}(\sigma)),$$

is the generalized Steinberg R-representation of $M(\sigma)$ and $\mathrm{Ind}_Q^{M(\sigma)}$ stands for the parabolic smooth induction functor $\mathrm{Ind}_{Q\cap M(\sigma)}^{M(\sigma)}$. In §III.3 we show that $I_G(P, -, Q)$

and scalar extension are compatible: for any R-triple (P,σ,Q) of G, we have $R'\otimes_R I_G(P,\sigma,Q)\simeq I_G(P,R'\otimes_R\sigma,Q)$ for any field extension R'/R and $I_G(P,\sigma,Q)$ descends to a subfield of R if and only if σ does (Prop.III.13).

What does supersingular mean for an irreducible admissible R-representation π of G ? We know what it means to be a supersingular $H(G)_R=R\otimes_{\mathbb{Z}}H(G)$-module: for all $P\neq G$ containing B, a certain central element T_P of the pro-p Iwahori Hecke ring $H(G)$ should act locally nilpotently [**Vig17**]. We say that π is **supersingular** if the I-invariant module π^I is supersingular as a right $H(G)_R$-module (Definition III.14 in §III.4). In §III.4, we show that supersingularity is compatible with scalar extension (Lemma III.16) and that π^I is supersingular if and only if π^I contains a non-zero supersingular element (Theorem III.17). When R is algebraically closed, this definition is equivalent to the one in [**AHHV**], by [**OV**].

THEOREM I.7. [Classification theorem for G]

For any R-triple (P,σ,Q) of G with σ irreducible admissible supersingular, $I_G(P,\sigma,Q)$ is an irreducible admissible R-representation of G.

If $I_G(P,\sigma,Q)\simeq I_G(P_1,\sigma_1,Q_1)$ for two R-triples (P,σ,Q) and (P_1,σ_1,Q_1) of G with σ,σ_1 irreducible admissible supersingular and P,P_1 containing B, then $P=P_1,Q=Q_1$ and $\sigma\simeq\sigma_1$.

For any irreducible admissible R-representation π of G, there exists an R-triple (P,σ,Q) of G with σ irreducible admissible supersingular and P containing B, such that $\pi\simeq I_G(P,\sigma,Q)$.

When R is algebraically closed, this is the classification theorem of [**AHHV**]. In §III.5 we descend the classification theorem from R^{alg} to R by a formal proof using the decomposition theorem (Thm.I.1) and a lattice isomorphism $\mathcal{L}_{\sigma_{R^{alg}}}\simeq\mathcal{L}_{I_G(P,\sigma_{R^{alg}},Q)}$ when σ is irreducible admissible supersingular and $\sigma_{R^{alg}}$ its scalar extension to R^{alg} (Prop.III.10 in §III.3, Remark III.18 in §III.4).

I.4 In §IV, for a field R of characteristic p we prove a similar classification for the simple right $H(G)_R$-modules. As in [**AHenV2**] when R is algebraically closed, this classification uses for a parabolic subgroup $P=MN$ of G containing B, the parabolic induction functor

$$\mathrm{Ind}_P^{H(G)}:\mathrm{Mod}_R(H(M))\to\mathrm{Mod}_R(H(G))$$

from right $H(M)_R$-modules to right $H(G)_R$-modules, analogue of the parabolic smooth induction: indeed $(\mathrm{Ind}_P^G\sigma)^I$ is naturally isomorphic to $\mathrm{Ind}_P^{H(G)}(\sigma^{I\cap M})$ for a smooth R-representation σ of G [**OV**]. An R-triple $(P,\mathcal{V},Q)$ of $H(G)$ consists of parabolic subgroups $P=MN\subset Q$ of G containing B and of a right $H(M)_R$-module $\mathcal{V}$ with $Q\subset P(\mathcal{V})$ (Definition IV.8); for an R-triple $(P,\mathcal{V},Q)$ of $H(G)$ we define a right $H(G)_R$-module $I_{H(G)}(P,\mathcal{V},Q)$ as for the group.

In Proposition IV.12, we prove that $I_{H(G)}(P,-,Q)$ and scalar extension are compatible, as in the group case (Prop. III.13).

THEOREM I.8. [Classification theorem for $H(G)$]

For any R-triple $(P,\mathcal{V},Q)$ of $H(G)$ with $\mathcal{V}$ simple supersingular, $I_{H(G)}(P,\mathcal{V},Q)$ is a simple $H(G)_R$-module.

If $I_{H(G)}(P,\mathcal{V},Q)\simeq I_{H(G)}(P_1,\mathcal{V}_1,Q_1)$ for R-triples $(P,\mathcal{V},Q)$ and $(P_1,\mathcal{V}_1,Q_1)$ of $H(G)$ with $\mathcal{V}$ and $\mathcal{V}_1$ simple supersingular, then $P=P_1,Q=Q_1$ and $\mathcal{V}\simeq\mathcal{V}_1$.

Any simple right $H(G)_R$-module $\mathcal{X}$ is isomorphic to $I_{H(G)}(P,\mathcal{V},Q)$ for some R-triple $(P,\mathcal{V},Q)$ of $H(G)$ with $\mathcal{V}$ simple supersingular.

The proof follows the same pattern as for the group G, by a descent to R of the classification over R^{alg} [**AHenV2**].

Assuming that R contains a root of unity of order the exponent of the quotient Z_k of the parahoric subgroup of Z by its pro-p Sylow subgroup, the simple supersingular $H(G)_R$-modules are classified [**Oss**], [**Vig17**, Thm.1.6]; in particular when G is semisimple and simply connected, they have dimension 1. With Thm. I.8, we have a complete classification of the $H(G)_R$-modules.

The ring $H(M)$ does not embed in the ring $H(G)$ and different inductions from $\mathrm{Mod}_R(H(M))$ to $\mathrm{Mod}_R(H(G))$ are possible. When R is algebraically closed, Abe proved the classification theorem (Thm.I.8) using one of them, the parabolic coinduction[2] [**Abe**]. In the appendix we define and compare 8 natural inductions following [**Abeparind**]; the classification theorem can be expressed with any these 8 inductions instead of the parabolic induction (for the parabolic coinduction [**AHenV2**, Cor. 4.24]).

I.5 In §V, still with R of characteristic p we give applications (Thm. I.9, I.10, I.12, I.13) of the classification for G (Thm. I.7) and for $H(G)$ (Thm. I.8); they were already known when R is algebraically closed, except for the parts (ii),(iii) of Theorem I.10 below.

THEOREM I.9. [Vanishing of the smooth dual] *The smooth dual of an infinite dimensional irreducible admissible R-representation of G is* 0.

This was proved by different methods when the characteristic of F is 0 in [**Kohl**] and when R is algebraically closed in [**AHenV2**, Thm.6.4]. In §V.1 we deduce easily the theorem from the theorem over R^{alg} using scalar extension to R^{alg} (Thm. I.1).

[Description of $\mathrm{Ind}_P^G \sigma$ for an irreducible admissible R-representation σ of M, and of $\mathrm{Ind}_P^{H(G)} \mathcal{V}$ for a simple $H(M)_R$-module $\mathcal{V}$] Here $P = MN$ is a fixed parabolic subgroup of G.

We write $\mathcal{L}_\pi$ for the lattice of subrepresentations of an R-representation π of G, and $\mathcal{L}_\mathcal{X}$ for the lattice of submodules of an $H(G)_R$-module $\mathcal{X}$.

Recall that for a set X, an upper set in $\mathcal{P}(X)$ is a set $\mathcal{Q}$ of subsets of X, such that if $X_1 \subset X_2 \subset X$ and $X_1 \in \mathcal{Q}$ then $X_2 \in \mathcal{Q}$. Write $\mathcal{L}_{\mathcal{P}(X),\geq}$ for the lattice of upper sets in $\mathcal{P}(X)$. For two subsets X_1, X_2 of X write $X_1 \setminus X_2$ for the complementary set of $X_1 \cap X_2$ in X_1.

By the classification theorems, $\sigma \simeq I_M(P_1 \cap M, \sigma_1, Q \cap M)$ with (P_1, σ_1, Q) an R-triple of G, $Q \subset P$ and σ_1 irreducible admissible supersingular and $\mathcal{V} \simeq I_{H(M)}(P_1 \cap M, \mathcal{V}_1, Q \cap M)$ with $(P_1, \mathcal{V}_1, Q)$ an R-triple of $H(G)$, $Q \subset P$, and $\mathcal{V}_1$ simple supersingular.

With these notations we have:

THEOREM I.10. [Lattices $\mathcal{L}_{\mathrm{Ind}_P^G \sigma}$ and $\mathcal{L}_{\mathrm{Ind}_P^{H(G)} \mathcal{V}}$]

(i) The R-representation $\mathrm{Ind}_P^G \sigma$ of G is multiplicity free of irreducible subquotients $I_G(P_1, \sigma_1, Q')$ for R-triples (P_1, σ_1, Q') of G with $Q' \cap P = Q$.

Sending $I_G(P_1, \sigma_1, Q')$ to $\Delta_{Q'} \cap (\Delta_{P(\sigma_1)} \setminus \Delta_P)$ gives a lattice isomorphism[3]

$$\mathcal{L}_{\mathrm{Ind}_P^G \sigma} \to \mathcal{L}_{\mathcal{P}(\Delta_{P(\sigma_1)} \setminus \Delta_P),\geq}.$$

[2]The parabolic coinduction is the induction used by Abe

[3]see the discussion in [**He**]§11 on the lattice of submodules of a multiplicity free module

(ii) The $H(G)_R$-module $\mathrm{Ind}_P^{H(G)}\mathcal{V}$ *satisfies the analogue of (i).*

(iii) If $\sigma^{I\cap M}$ is simple and the natural surjective map $\sigma^{I\cap M}\otimes_{H(M)}\mathbb{Z}[(I\cap M)\backslash M]\to\sigma$ is bijective, then the I-invariant functor $(-)^I$ and its left adjoint $-\otimes_{H(G)}\mathbb{Z}[I\backslash G]$ give lattice isomorphisms between $\mathcal{L}_{\mathrm{Ind}_P^G(\sigma)}$ and $\mathcal{L}_{\mathrm{Ind}_P^{H(G)}(\sigma^{I\cap M})}$ inverse of each other.

When R is algebraically closed (i) is proved in [**AHenV1**, §3.2]. In §V.2 we prove (i) and (ii); (iii) follows from (i), (ii), Corollary I.4 and the commutativity of the parabolic inductions with $(-)^I$ and $-\otimes_{H(G)}\mathbb{Z}[I\backslash G]$ [**OV**].

Corollary I.11. *1. The socle and the cosocle of $\mathrm{Ind}_P^G\sigma$ are irreducible; $\mathrm{Ind}_P^G\sigma$ is irreducible if and only if P contains $P(\sigma_1)$. The same is true for $\mathrm{Ind}_P^{H(G)}\mathcal{V}$.*

2. Let π be an irreducible admissible R-representation of G; we write $\pi\simeq I_G(P,\sigma,Q)$ with σ irreducible admissible supersingular.

If $\sigma^{I\cap M}$ is simple and the natural map $\sigma^{I\cap M}\otimes_{H(M)}\mathbb{Z}[(I\cap M)\backslash M]\to\sigma$ is bijective, then π^I is simple and $\pi\simeq\pi^I\otimes_{H(G)}\mathbb{Z}[I\backslash G]$.

The first assertion for σ and R is algebraically closed is proved in [**AHenV1**, Cor. 3.3 and 4.4].

[Computation of the left adjoint and the right adjoint of the parabolic induction, of π^I for an irreducible admissible R-representation π of G, and of $\mathcal{X}\otimes_{H(G)}\mathbb{Z}[I\backslash G]$ for a simple $H(G)_R$-module $\mathcal{X}$]

For a parabolic subgroup P_1 of G, write $L_{P_1}^G$ for the left adjoint of $\mathrm{Ind}_{P_1}^G$, $R_{P_1}^G$ for its right adjoint [**Vigadjoint**], and $L_{P_1}^{H(G)}$ for the left adjoint of $\mathrm{Ind}_{P_1}^{H(G)}$, $R_{P_1}^{H(G)}$ for its right adjoint [**VigpIwst**].

Theorem I.12. [Adjoint functors of the parabolic induction and of the I-invariant]

(i) $L_{P_1}^G(\pi)$ and $R_{P_1}^G(\pi)$ are 0 or irreducible admissible.

$L_{P_1}^G(\pi)\neq 0\ \Leftrightarrow\ P_1\supset P,\langle P_1,Q\rangle\supset P(\sigma)\ \Leftrightarrow\ L_{P_1}^G(\pi)\simeq I_{M_1}(P\cap M_1,\sigma,Q\cap M_1)$.

$R_{P_1}^G(\pi)\neq 0\ \Leftrightarrow\ P_1\supset Q\ \Leftrightarrow\ R_{P_1}^G(\pi)\simeq I_{M_1}(P\cap M_1,\sigma,Q\cap M_1)$.

(ii) $L_{P_1}^{H(G)}(\mathcal{X})$ and $R_{P_1}^{H(G)}(\mathcal{X})$ satisfy (i) with the obvious modifications.

(iii) We have natural isomorphisms $\pi^I\simeq I_{H(G)}(P,\sigma^{I\cap M},Q)$ and $\mathcal{X}\otimes_{H(G)_R}R[I\backslash G]$

$\simeq I_G(P,\mathcal{V}\otimes_{H(M)_R}R[(I\cap M)\backslash M],Q)$.

Example: $L_{P(\sigma)}^G(I_G(P,\sigma,Q))\simeq R_{P(\sigma)}^G(I_G(P,\sigma,Q))\simeq \mathrm{St}_Q^{M(\sigma)}(\sigma)$ and the analogous for $I_{H(G)}(P,\mathcal{V},Q)$.

Proving Theorem I.12 from the classification theorem needs no new techniques (§V.3).

[Equivalence between supersingularity, supercuspidality and cuspidality]

An irreducible admissible R-representation π of G is said to be

- **supercuspidal** if it is not a subquotient of $\mathrm{Ind}_P^G\tau$ with $\tau\in\mathrm{Mod}_R^\infty(M)$ irreducible admissible for any parabolic subgroups $P=MN\subsetneq G$.

- **cuspidal** if $L_P^G(\pi)=R_P^G(\pi)=0$ for all parabolic subgroups $P\subsetneq G$.

Theorem I.13. *Let π be an irreducible admissible R-representation of G. Then π is supersingular if and only if its is supercuspidal if and only if it is cuspidal.*

The equivalence of supersingular with supercuspidal, resp. cuspidal, follows from Thm. I.10, resp. Thm. I.12. When R is algebraically closed, the first equivalence was proved in [**AHHV**, Thm. VI.2] and the second one in [**AHenV1**, Cor.6.9].

An irreducible admissible R-representation π admits a supercuspidal support: the parabolic subgroup $P = MN$ containing B and the isomorphism class of the irreducible admissible supercuspidal R-representation of σ of M such that π is a subquotient of $\mathrm{Ind}_P^G(\sigma)$ are unique; this follows from Thm. I.7 and Thm. I.13.

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II. Some general algebra

II.1. Review on scalar extension. We consider a field R and an R-algebra A (always associative with unit).

For an extension R' of R (which we see as a field R' containing R), the scalar extension functor $R' \otimes_R - : \mathrm{Mod}_R \to \mathrm{Mod}_{R'}$ from R to R', also denoted $(-)_{R'}$, is faithful exact and left adjoint to the restriction functor from R' to R.

The scalar extension $A_{R'}$ of A is an R'-algebra and if W a (left or right) A-module, $W_{R'}$ is an $A_{R'}$-module. An $A_{R'}$-module W' isomorphic to such a $W_{R'}$ is said to descend to R or to be defined over R, and W is called an R-structure for W' (more precisely the isomorphism $W' \simeq W_{R'}$ is an R-structure for W').

Remark II.1. Let R^{alg} be an algebraic closure of R. If A is a finitely generated R-algebra, an $A_{R^{alg}}$-module W of finite dimension over R^{alg} descends to a finite extension of R. Indeed, if (w_i) is an R^{alg}-basis of W, (a_j) a finite set of generators of A, and $a_j w_i = \sum_k r_{j,i,k} w_k$, the extension R'/R generated by the coefficients $r_{j,i,k} \in R^{alg}$ is finite and the $A_{R'}$-module $\oplus_i R' w_i$ gives an R'-structure for W.

Remark II.2. If V, W are A-modules, the natural map

$$(\mathrm{Hom}_A(V,W))_{R'} \to \mathrm{Hom}_{A_{R'}}(V_{R'}, W_{R'}) \tag{1.1}$$

is injective [**BkiA2**, II §7 n^{o}7 Prop.16] and bijective if R'/R is finite [**BkiA8**, §12, n^{o}2 Lemme 1], or if V is a finitely generated A-module (proof as in [**Pask**, Lemma 5.1])[4].

Let V be a simple A-module; we write D for the commutant $\mathrm{End}_A(V)$, so that D is a division algebra, and E for the center of D. Since V is finitely generated, the commutant of $V_{R'}$ is $D_{R'}$ and its center is $E_{R'}$, by Remark II.~~2.~~ That V is simple has further consequences:

(P1) As an A-module, $V_{R'}$ is a direct sum of A-modules isomorphic to V, i.e. V-isotypic of type V [**BkiA8**, §4, n^{o}4, Prop.1].

(P2) The map $\mathfrak{A} \mapsto \mathfrak{A} V_{R'}$ is a lattice isomorphism of the lattice of right ideals $\mathfrak{A}$ of $D_{R'}$ onto the lattice of $A_{R'}$-submodules W of $V_{R'}$, with inverse $W \mapsto \{d \in D_{R'}, dV_{R'} \subset W\}$ [**BkiA8**, §12, n^{o}2, Thm.2b)].

(P3) For any right ideal $\mathfrak{A}$ of $D_{R'}$, via the isomorphism $V_{R'} \simeq D_{R'} \otimes_D V$, $\mathfrak{A} V_{R'}$ corresponds to $\mathfrak{A} \otimes_D V$. As the functor $X \mapsto X \otimes_D V$ from right D-vector spaces to A-modules is exact, if $\mathfrak{B} \subset \mathfrak{A}$ are right ideals of $D_{R'}$, then $\mathfrak{A} V_{R'}/\mathfrak{B} V_{R'}$ is isomorphic to $(\mathfrak{A}/\mathfrak{B}) \otimes_D V_{R'}$.

[4] We are grateful to the referee for that reference

(P4) If the extension R'/R is finite, $V_{R'}$ has finite length as an A-module, so also as an $A_{R'}$-module; then $D_{R'}$ is left and right artinian and $E_{R'}$ is artinian [**BkiA8**, §12, n^{o}5, Prop.5 a)]. If moreover R'/R is separable, $V_{R'}$ is semisimple [**BkiA8**, §12, n^{o}5, Cor.].

(P5) If $\dim_R D$ is finite, then $\dim_{R'} D_{R'} = \dim_R D$ and the length of the $A_{R'}$-module $V_{R'}$ is $\leq [D : R]$ by (P2); the best bound is given in Thm. I.1.

Remark II.3. A non-zero A-module W is called absolutely simple if $W_{R'}$ is simple for any extension R'/R.

A simple A-module V is absolutely simple if and only if $\mathrm{End}_A V = R$. For $\Rightarrow$ [**BkiA8**, §3,n^{o}2,Cor.2, p.44]. For $\Leftarrow$ follows from (P5). If R is algebraically closed of cardinal $> \dim_R V$, then $D = R$ [**BkiA8**, §3,n^{o}2,Thm.1, p.43].

II.2. A bit of ring and module theory. We examine the tensor product $L \otimes_R E$ of two field extensions L/R and E/R. Seeing the commutative ring $L \otimes_R E$ as a module over itself, its simple subquotients are isomorphic to simple $L \otimes_R E$-modules, that is to simple quotients.

Lemma II.4. *Let E/R be a finite extension and L/E an extension.*

(1) If E/R is purely inseparable, then $L \otimes_R E$ is a commutative artinian local ring with residue field L.

(2) If E/R is separable and L contains a Galois closure of E/R, then

$$L \otimes_R E \simeq \prod_{j \in \mathrm{Hom}_R(E,L)} L \otimes_{j,E} E \simeq L^{[E:R]},$$

and if F/R is a subextension of E/R, the restriction $\mathrm{Hom}_R(E, L) \to \mathrm{Hom}_R(F, L)$ is surjective.

(3) If L/R is normal, then $\mathrm{Aut}_R(L)$ acts transitively on $\mathrm{Hom}_R(E, L)$.

(4) If E/R is normal, the ring homomorphism

$$x \otimes y \mapsto (xj(y))_j \ : \ E \otimes_R E \to \prod_{j \in \mathrm{Aut}_R(E)} E$$

is surjective of kernel the Jacobson radical of $E \otimes_R E$.

Proof. As E/R is finite, the commutative ring $L \otimes_R E$ has finite dimension over L, hence is Artinian. Let R' be a field quotient of $L \otimes_R E$. The quotient map $\varphi : L \otimes_R E \to R', \varphi(x \otimes y) = \varphi_1(x)\varphi_2(y)$, is given by non zero R-homomorphisms $\varphi_1 : L \to R', \varphi_1(x) = \varphi(x \otimes 1)$, and $\varphi_2 : E \to R', \varphi_2(y) = \varphi(1 \otimes y)$.

If E/R is purely inseparable, φ_2 is the restriction of φ_1 to E thus we have (1).

Let $J = \mathrm{Hom}_R(E, L)$ and for $j \in J$, let f_j the surjective map $L \otimes_R E \to L \otimes_{j,E} E \xrightarrow{\simeq} L$. If $j \neq j'$ are distinct in J, and $x \in E$ with $j(x) \neq j'(x)$, we have

$$f_j(j(x) \otimes 1 - 1 \otimes x) = 0, \quad f_{j'}(j(x) \otimes 1 - 1 \otimes x) = j(x) - j'(x) \neq 0.$$

Hence $\mathrm{Ker}\, f_j \neq \mathrm{Ker}\, f_{j'}$. By the Chinese Remainder Theorem,

$$\prod f_j : L \otimes_R E \to \prod_{j \in J} L \otimes_{j,E} E \xrightarrow{\simeq} L^J \tag{2.2}$$

is surjective. It is injective if and only if $[E : R] = |J|$.

If E/R is separable and L contains a Galois closure of E/R, then $[E : R] = |J|$ (and conversely), and for any subextension F/R of E/R, F/R and E/F are separable and L contains a Galois closure of F/R and of E/F, thus the restriction

$\mathrm{Hom}_R(E,L) \to \mathrm{Hom}_R(F,L)$ of kernel $\mathrm{Hom}_F(E,L)$ is surjective by a counting argument since $[E:R]=[E:F][F:R]$. This gives (2).

Let E_{sep}/R be the maximal separable subextension of E/R. The extension E/E_{sep} is purely inseparable and the restriction $\mathrm{Hom}_R(E,L) \to \mathrm{Hom}_R(E_{sep},L)$ is injective.

If L/R is normal, (3) is true as $\mathrm{Hom}_R(E,L) \to \mathrm{Hom}_R(E_{sep},L)$ is injective and (3) is true when E/R is separable by Galois theory. If $L=E$, for $j \in \mathrm{Hom}_R(E,E) = \mathrm{Aut}_R(E)$ and $x,y \in E$, we have $f_j(x \otimes y) = xj(y)$. If R' is a field quotient of $E \otimes_R E$, the quotient map satisfies $\varphi(x \otimes y) = \varphi_1(x)\varphi_2(y)$ for φ_1, φ_2 in $\mathrm{Hom}_R(E,R')$. If moreover E/R is normal, then $R'=E$ and $\varphi = \varphi_1 \circ f_j$ where $\varphi_2 = \varphi_1 \circ j$ in $\mathrm{Aut}_R(E)$. This gives (4). □

Lemma II.5. *Let R'/R be a normal field extension, A an~~d~~ R-algebra and V' a simple $A_{R'}$-module descending to a finite extension of R. Then V' is isomorphic to a submodule of the scalar extension $V_{R'}$ from R to R' of a simple A-module V. For any such V, $\dim_R V$ is finite if $\dim_{R'} V'$ is, and $\dim_R \mathrm{End}_A V$ is finite if $\dim_{R'} \mathrm{End}_{A_{R'}} V'$ is.*

Proof. a) Assume first that the normal extension R'/R is finite. Then $A_{R'}$ is a (free) finitely generated A-module, so V' as an A-module is finitely generated, and in particular has a simple quotient V: $\mathrm{Hom}_A(V',V) \neq 0$. By Remark II.2, $\mathrm{Hom}_{A_{R'}}(V'_{R'}, V_{R'}) \neq 0$.

The $A_{R'}$-module $V'_{R'}$ admits a finite filtration of quotients V'_j for $j \in \mathrm{Aut}_R(R')$, where V'_j is isomorphic to V' with the j-twisted action $(y \otimes a)v' = j(y)av'$ for $y \in R', a \in A, v \in V'$. Indeed, $V'_{R'} = R' \otimes_R V' \simeq (R' \otimes_R R') \otimes_{R'} V'$, the artinian commutative ring $R' \otimes_R R'$ admits a finite filtration with quotients isomorphic to simple $R' \otimes_R R'$-modules, and the simple $R' \otimes_R R'$-modules are R'_j for $j \in \mathrm{Aut}_R(R')$, where R'_j is isomorphic to R' with $x \otimes y \in R' \otimes_R R'$ acting by multiplication by $xj(y)$ by Lemma II.4 (4).

We deduce that $\mathrm{Hom}_{A_{R'}}(V'_j, V_{R'}) \neq 0$ for some $j \in \mathrm{Aut}_R(R')$. But $V_{R'}$ is isomorphic to its j-twists $(V_{R'})_j$ for all $j \in \mathrm{Aut}_R(R')$, so we have $\mathrm{Hom}_{A_{R'}}(V', V_{R'}) \neq 0$.

Let V be any simple A-module with $\mathrm{Hom}_{A_{R'}}(V', V_{R'}) \neq 0$. Then $\mathrm{Hom}_A(V',V) \neq 0$ as $V_{R'}$ as an A-module is V-isotypic, so $\dim_R V$ is finite if $\dim_{R'} V'$ is. Put $D = \mathrm{End}_A(V)$ and $D' = \mathrm{End}_{A_{R'}}(V')$ and let W be the maximal V'-isotypic submodule of $V_{R'}$. Then W is $D_{R'}$-stable and we get a homomorphism $D_{R'} \to \mathrm{End}_{A_{R'}} W$ which is necessarily injective on D, since D is a division algebra. By (P4), $V_{R'}$ has finite length, so W also has finite length and $\mathrm{End}_{A_{R'}} W$ is a matrix algebra over D'; it follows that if $\dim_{R'} D'$ is finite, so is $\dim_{R'}(\mathrm{End}_{A_{R'}} W)$ hence also $\dim_R(\mathrm{End}_{A_{R'}} W)$, $\dim_{R'}(D_{R'})$ and $\dim_R D$.

b) Let us treat the general case. By assumption there is a finite normal subextension L of R in R' and an A_L-module U such that $V' = R' \otimes_L U$ - then U is necessarily simple. By a) $\mathrm{Hom}_{A_L}(U, V_L) \neq 0$ for some simple A-module V and by Remark II.2, $\mathrm{Hom}_{A_{R'}}(V', V_{R'}) \neq 0$.

Conversely, if V is some simple A-module with $\mathrm{Hom}_{A_{R'}}(V', V_{R'}) \neq 0$ then by Remark II.2 again $\mathrm{Hom}_{A_L}(U, V_L) \neq 0$, so the other assertions follow from a). □

We pursue with an easy application of Morita theory in the special case of a matrix ring.

LEMMA II.6. *Let A, B be two rings and n a positive integer.*

1) Let W be an A-module. A ring isomorphism $\mathrm{End}_A W \simeq M(n, B)$ induces an A-module isomorphism $W \simeq \oplus^n V$ for some A-module V with commutant B.

2) If B is a commutative artinian local ring of residue field R, then $M(n, B)$ is left Artinian, and as a left module over itself, all its simple subquotients are isomorphic to R^n.

PROOF. 1) If V is a B-module, then V^n is naturally an $M(n, B)$-module, and the functor $V \mapsto V^n$ is an equivalence from the category of B-modules to the category of $M(n, B)$-modules; that is the elementary case of Morita theory. By that equivalence, if V is a left (A, B) bimodule, then V^n is left $(A, M(n, B))$ bimodule, and any left $(A, M(n, B))$ bimodule structure of V^n comes in that way from a left (A, B) bimodule structure on V. As $\mathrm{End}_A(V^n)$ identifies with $M(n, \mathrm{End}_A(V))$, the condition $\mathrm{End}_A(V^n) = M(n, B)$ is the same as $\mathrm{End}_A(V) = B$, and 1) follows.

2) As a left module over itself, $M(n, B)$ is isomorphic to the direct sum of n copies of B^n (let $M(n, B)$ act on the column vectors). By the equivalence recalled in the proof of 1), the $M(n, B)$-module B^n has the same length as B over itself and its simple subquotients are isomorphic to R^n, hence 2). $\square$

II.3. Proof of the decomposition theorem (Thm.I.1 and Cor.I.2). Let V be a simple A-module with commutant $D = \mathrm{End}_A V$ of finite dimension $\dim_R D$ over R. Let E denote the center of the skew field D, δ the reduced degree of D over E, E_{sep}/R the maximal separable subextension of E/R.

Two well-known properties will be used in the proof:

(P6) A finite extension E'/E splits D, i.e. $E' \otimes_E D \simeq M(\delta, E')$, if and only if E' is isomorphic to a maximal subfield of a matrix algebra over D [**BkiA8**, §15, n^{o}3, Prop.5]. The field D contains a maximal subfield, which a separable extension E'/E of degree δ [**CR**, 7.24 Prop] or [**BkiA8**, loc.cit. and §14, n^{o}7].

(P7) For a finite separable extension E'/E and E'_{sep}/R the maximal separable subextension of E'/R, the natural map $x \otimes y \mapsto xy : E'_{sep} \otimes_{E_{sep}} E \to E'$ is an isomorphism (because always surjective and the dimension over E'_{sep} of both sides is the same $[E : E_{sep}]$ by [**Lang**, VII §7, Cor. 7.5] applied to the finite extensions E'_{sep}/E_{sep} separable and E/E_{sep} purely inseparable).

Proof of Thm.I.1 (1).

Let R'/R be an extension containing a normal closure of a finite separable extension E'/E splitting D. For example, R' can be an algebraic closure R^{alg}/R. Let $J = \mathrm{Hom}_R(E_{sep}, R')$. By Lemma II.4 (1), we have a ring isomorphism

$$R' \otimes_R E_{sep} \simeq \prod_{j \in J} R' \otimes_{j, E_{sep}} E_{sep} \simeq R'^{[E_{sep}:R]}.$$

We denote by e_j the idempotents of $(E_{sep})_{R'}$ associated to this decomposition. Tensoring on the right by E, D, or V over E_{sep} and we get product decompositions

$$E_{R'} = \prod_{j \in J} e_j E_{R'},\ D_{R'} = \prod_{j \in J} e_j D_{R'},\ V_{R'} = \oplus_{j \in J}\, e_j V_{R'}$$

where $e_j E_{R'} \simeq R' \otimes_{j, E_{sep}} E$, $e_j D_{R'} \simeq R' \otimes_{j, E_{sep}} D$, $e_j V_{R'} \simeq R' \otimes_{j, E_{sep}} V$. By Lemma II.4 and (P7), for $j \in J$ there exists $j' \in \mathrm{Hom}_R(E'_{sep}, R')$ of restriction $j'|_{E_{sep}} = j$, and

$$R' \otimes_{j, E_{sep}} E \simeq R' \otimes_{j', E'_{sep}} E'_{sep} \otimes_{E_{sep}} E \simeq R' \otimes_{j', E'_{sep}} E'$$

is a commutative artinian local ring of residue field R'. We obtain ring isomorphisms

$$\begin{aligned} R' \otimes_{j,E_{sep}} D &\simeq R' \otimes_{j,E_{sep}} E \otimes_E D \\ &\simeq R' \otimes_{j',E'_{sep}} E' \otimes_E D \simeq R' \otimes_{j',E'_{sep}} M(\delta, E') \simeq M(\delta, R' \otimes_{j,E_{sep}} E). \end{aligned}$$

By Lemma II.6, there exists an $A_{R'}$-module W'_j such that

$$R' \otimes_{j,E_{sep}} V \simeq \oplus^\delta W'_j, \quad \mathrm{End}_{A_{R'}} W'_j \simeq R' \otimes_{j,E_{sep}} E.$$

By Remark II.2, for $j \in J$, the commutant of the $A_{R'}$-module $e_j V_{R'} = R' \otimes_{j,E_{sep}} V$ is $e_j D = R' \otimes_{j,E_{sep}} D$. Applying (P2) and (P3), the map $\mathfrak{A} \mapsto \mathfrak{A} e_j V_{R'}$ is a lattice isomorphism of the lattice of right ideals $\mathfrak{A}$ of $e_j D_{R'}$ onto the lattice of $A_{R'}$-submodules of $e_j V_{R'}$, and for two right ideals $\mathfrak{A} \subset \mathfrak{B}$ of $e_j D_{R'}$, the $A_{R'}$-module $\mathfrak{B} e_j V_{R'} / \mathfrak{A} e_j V_{R'}$ is isomorphic to $(\mathfrak{B}/\mathfrak{A}) \otimes_{e_j D} e_j V_{R'}$. As $e_j D_{R'} \simeq M(\delta, e_j E)$ and $e_j E$ is a commutative artinian local ring of residue field R, by Lemma II.6, the $A_{R'}$-module W'_j is indecomposable of length $[E : E_{sep}]$ and its simple subquotients are all isomorphic to the $A_{R'}$-module

$$V'_j = R' \otimes_{(R' \otimes_{j,E_{sep}} E)} W'_j$$

with commutant R', hence absolutely simple by Remark II.3.

The group $\mathrm{Aut}_R(R')$ of R-automorphisms of R' acts on the $A_{R'}$-modules, fixing the isomorphism class of the scalar extension from R to R' of an A-module. If R'/R is normal, it acts transitively on the set J by Lemma II.4 (3), and for $g \in \mathrm{Aut}_R(R')$ we have $g(e_j) = e_{g\circ j}$. By Krull-Remak-Schmidt's theorem, $g(W'_j) \simeq W'_{g\circ j}$. The same is true for the simple subquotients: $g(V'_j) \simeq V'_{g\circ j}$.

The dimension over R' of the commutant of any subquotient of the $A_{R'}$-module

$$R' \otimes_R V = \oplus^\delta \oplus_{j\in J} W'_j$$

is finite (because the length of $R' \otimes_R V$ is finite and R' is the commutant of any of its simple subquotients).

Let L be the normal closure of E'/R in R'/R. These results applied to R'/R and to L/R, show that scalar extension from L to R' induces a lattice isomorphism $\mathcal{L}_{V_L} \to \mathcal{L}_{V_{R'}}$. This ends the proof of Thm.I.1 (1).

Proof of Thm.I.1 (2).

Thm.I.1 1) applies to $R' = R^{alg}$ an algebraic closure of R. It shows that for any simple A-module V with $\dim_R V$ finite, the simple subquotients of $V_{R^{alg}}$ are absolutely simple, descend to a finite subextension of R^{alg}/R and their isomorphism classes form a finite $\mathrm{Aut}_R(R^{alg})$-orbit.

Conversely, let V' be an absolutely simple $A_{R^{alg}}$-module descending to a finite extension L of R. We prove that the $\mathrm{Aut}_R(R^{alg})$-orbit $\mathrm{Aut}_R(R^{alg})[V']$ of the isomorphism class $[V']$ of V' is finite. Let W' denote an A_L-module with scalar extension $W_{R^{alg}} = V'$ to R^{alg}. Necessarily, W is absolutely simple. By Lemma II.5, W' is contained in the scalar extension V_L from R to L of a simple A-module V with $\dim_R V$ finite. We proved that $V_{R^{alg}}$ has finite length and that the isomorphism classes of its simple subquotients form an $\mathrm{Aut}_R(R^{alg})$-orbit. Hence $\mathrm{Aut}_R(R^{alg})[V']$ is finite, and the map $[V] \to \mathrm{Aut}_R(R^{alg})[V']$ in Thm.I.1 (2) is surjective. It is also injective because $V_{R^{alg}}$ is V-isotypic as an A-module (by P1), so the same is true for V'. This ends the proof of Thm.I.1 (2).

Proof of Corollary I.2.

Let L/R be any extension and L^{alg} an algebraic closure of L. The scalar extension from R to L^{alg} is the scalar extension of R to L followed by the scalar extension from L to L^{alg}.

(i) The length of the $A_{L^{alg}}$-module $V_{L^{alg}}$ is $\delta[E:R]$ by part 1) of Thm.I.1, hence the length of the A_L-module V_L is $\leq \delta[E:R]$.

(ii) Let W be a subquotient of V_L. We show that the commutant of W has finite dimension over L. As $W_{L^{alg}}$ is a subquotient of $V_{L^{alg}}$, by part 2) of Thm.I.1, the dimension over L^{alg} of the commutant of $W_{L^{alg}}$ is finite. By (i) the A_L-module W has finite length hence is finitely generated and by Remark II.2, $\dim_{L^{alg}}(\mathrm{End}_{A_{L_{alg}}} W_{L^{alg}}) = \dim_L(\mathrm{End}_{A_L} W)$. This ends the proof of Corollary I.2.

II.4. Proof of the lattice theorems (Thm. I.3, I.5 and Cor. I.4, I.6). Our overall reference for abelian categories is [**KS**, Chapter 8].

Let $\mathcal{C}$ be an abelian category and W an object in $\mathcal{C}$. A subobject of W is an isomorphism class of monomorphisms $f: Y \to W$ [**KS**, Def. 1.2.18]. The ordered set $\mathcal{L}_W$ of subobjects of W is a bounded lattice: the meet of two subobjects $f: Y \to W$ and $f': Y' \to W$ is the kernel of $(f, -f'): Y \oplus Y' \to W$ and their join is its image. As in module categories[5], we write $Y \cap Y'$ for the meet, $Y + Y'$ for the join [**KS**, 8.3.10]; we note the exact sequence

$$0 \to (Y \cap Y') \to (Y \oplus Y') \to Y + Y' \to 0.$$

We define the lattice $\overline{\mathcal{L}}_W$ of quotients of W: it is the lattice of subobjects of W in the opposite category of $\mathcal{C}$. The map which to a subobject Y of W associates its cokernel (written W/Y) yields a lattice isomorphism $\mathcal{L}_W \to \overline{\mathcal{L}}_W$.

If $\mathcal{D}$ is an abelian category and $F: \mathcal{C} \to \mathcal{D}$ a left exact functor, then $Y \mapsto F(Y): \mathcal{L}_W \to \mathcal{L}_{F(W)}$ is an ordered preserving map; if F is not left exact, $F(Y)$ might not be a subobject of $F(W)$ if Y is a subobject of W.

Lemma II.7. *Let $F: \mathcal{C} \to \mathcal{D}$ be a functor between abelian categories which is left or right exact, and let W be a finite length object of $\mathcal{C}$* [**KS**, Ex. 8.20, p. 205].

(i) Assume that $F(Y)$ is 0 or simple (that is, $\lg(F(Y)) \leq 1$) for any simple subquotient Y of W. Then, $F(Y)$ has finite length $\lg(F(Y)) \leq \lg(Y)$ for any subquotient Y of W.

(ii) If moreover $\lg(F(W)) = \lg(W)$, then for any subquotient Y of W, $\lg(F(Y)) = \lg(Y)$ and an exact sequence $0 \to Y' \to Y \to Y'' \to 0$ in $\mathcal{C}$ yields via F an exact sequence $0 \to F(Y') \to F(Y) \to F(Y'') \to 0$ in $\mathcal{D}$; in addition $Y \mapsto F(Y)$ gives an injective morphism of bounded lattices $\mathcal{L}_W \to \mathcal{L}_{F(W)}$.

Proof. (i) Our proof proceeds by induction on the length of $\lg(Y)$ of a subquotient Y of W. By assumption $\lg(F(Y)) \leq \lg(Y)$ if $\lg(Y) \leq 1$. If $\lg(Y) \geq 2$, we choose an exact sequence $0 \to Y' \to Y \to Y'' \to 0$ in $\mathcal{C}$ with non-zero Y', Y''. If F is left exact, $0 \to F(Y') \to F(Y) \to F(Y'')$ is exact, if F is right exact, $F(Y') \to F(Y) \to F(Y'') \to 0$ is exact; in either case we get $\lg(F(Y)) \leq \lg(F(Y')) + \lg(F(Y''))$ which by induction is $\leq \lg(Y') + \lg(Y'') = \lg(Y)$.

(ii) For any subobject Y of W, the exact sequence $0 \to Y \to W \to W/Y \to 0$ gives $\lg(F(W)) \leq \lg(F(Y)) + \lg(F(W/Y))$ as above; applying (i), $\lg(F(Y)) \leq \lg(Y)$, $\lg(F(W/Y)) \leq \lg(W/Y)$. By assumption $\lg(F(W)) = \lg(W) = \lg(Y) + \lg(W/Y)$ so we get equalities throughout: $\lg(F(Y)) = \lg(Y)$ and $\lg(F(W/Y)) = \lg(W/Y)$.

[5] In any case, all our applications are to module categories

For any subquotient Y of W we repeat the argument to get $\lg(F(Y)) = \lg(Y)$. An exact sequence $0 \to Y' \to Y \to Y'' \to 0$ in $\mathcal{C}$ yields a sequence in $\mathcal{D}$

$$0 \to F(Y') \to F(Y) \to F(Y'') \to 0$$

which is exact on one side by the exactness property of F, and on the other side by length count.

It remains to prove the last assertion; if Y is a subobject of W we already know that $F(Y)$ is a subobject of $F(W)$ and that the map $Y \mapsto F(Y) : \mathcal{L}_W \to \mathcal{L}_{F(W)}$ is order preserving. It certainly sends the largest element W of $\mathcal{L}_W$ to the largest element $F(W)$ of $\mathcal{L}_{F(W)}$ and similarly for the smallest elements (the 0 elements). Let us verify that it preserves meets and joins. So let Y, Y' be two objects in $\mathcal{C}$. The two natural monomorphisms $Y \to Y \oplus Y', Y' \to Y \oplus Y'$, upon applying F, give a map $F(Y) \oplus F(Y') \to F(Y \oplus Y')$. If F is right exact, it is an isomorphism [**KS**, line after Prop.3.3.3]. If F is left exact, the map $F(Y \times Y') \to F(Y) \times F(Y')$ coming from the two maps $Y \times Y' \to Y$ and $Y \times Y' \to Y'$, is also an isomorphism [**KS**, Prop.3.3.3]; using the natural isomorphisms $Y \oplus Y' \to Y \times Y'$ and $F(Y) \oplus F(Y') \to F(Y) \times F(Y')$ in the abelian categories $\mathcal{C}$ and $\mathcal{D}$, we see that $F(Y) \oplus F(Y') \to F(Y \oplus Y')$ is an isomorphism too. Applying this to W and W, we see that $\lg(F(W \oplus W)) = 2\lg(F(W)) = 2\lg(W) = \lg(W \oplus W)$. Now let $f : Y \to W, f' : Y' \to W$ be subobjects of W; then applying the results obtained so far to the subobject $(f, -f') : Y \oplus Y' \to W \oplus W$ of $W \oplus W$, we see that the sequence in $\mathcal{D}$

$$0 \to F(Y \cap Y') \to F(Y \oplus Y') \to F(Y + Y') \to 0$$

is exact. But the composite $F(Y) \oplus F(Y') \to F(Y \oplus Y') \to F(Y + Y') \to F(W)$ is obtained from f, f' via F, and we see that $F(Y \cap Y') = F(Y) \cap F(Y')$ and $F(Y + Y') = F(Y) + F(Y')$. If Y, Y' satisfy $F(Y) = F(Y')$ then $F(Y + Y') = F(Y) = F(Y')$ so $\lg(Y + Y') = \lg(Y) = \lg(Y')$, which implies $Y = Y'$, hence the injectivity. ☐

Remark II.8. [**KS**, Prop. 1.5.6]:
For any adjunction (F, G, η, ϵ) between two categories,
- F is fully faithful if and only if the unit ϵ is an isomorphism,
- G is fully faithful if and only if the counit η is an isomorphism,
- the following equivalent properties imply that F, G are quasi-inverses of each other:
  - F and G are fully faithful,
  - F is an equivalence,
  - G is an equivalence.

We are now ready to prove Theorem I.3 and Corollary I.4.

We prove Thm. I.3 (a). We can apply Lemma II.7 to F and W by the assumptions. As above any simple subquotient X of $F(W)$ is isomorphic to $F(Y)$ for some simple subquotient Y of W; thus we can apply Lemma II.7 to G and $F(W)$. Let Y be a subquotient of W; by induction on $\lg(Y)$ we prove now that ϵ_Y is an isomorphism. Through adjunction ϵ_Y corresponds to the identity map $F(Y) \to F(Y)$, in particular ϵ_Y is not 0 if $F(Y)$ is not 0. If Y is simple then $GF(Y)$ is simple and the non-zero map $\epsilon_Y : Y \mapsto GF(Y)$ is an isomorphism. If $\lg(Y) \geq 2$, we choose an exact sequence $0 \to Y' \to Y \to Y'' \to 0$ in $\mathcal{C}$ with non-zero Y', Y''. Applying F

then G gives a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & Y' \arrow[r] \arrow[d, "\epsilon_{Y'}"'] & Y \arrow[r] \arrow[d, "\epsilon_Y"'] & Y'' \arrow[r] \arrow[d, "\epsilon_{Y''}"'] & 0 \\ 0 \arrow[r] & (G\circ F)(Y') \arrow[r] & (G\circ F)(Y) \arrow[r] & (G\circ F)(Y'') \arrow[r] & 0 \end{tikzcd}$$

where the lines are exact. By induction $\epsilon_{Y'}, \epsilon_{Y''}$ are isomorphisms, and so is ϵ_Y. From Lemma II.7 we obtain injective lattice morphisms $\mathcal{L}_W \to \mathcal{L}_{F(W)}$ and $\mathcal{L}_{F(W)} \to \mathcal{L}_{GF(W)}$ whose composite coincides with $Y \mapsto \epsilon_W(Y)$, so they are both bijective and consequently lattice isomorphisms. Hence Thm. I.3 (a).

To prove Theorem I.3 (b) we "reverse the arrows" i.e. consider F and G as functors between the opposite categories to $\mathcal{C}$ and $\mathcal{D}$. Applying (a) we get a lattice isomorphism $U \mapsto G(U) : \overline{\mathcal{L}}_V \to \overline{\mathcal{L}}_{G(V)}$; then $X \mapsto G(X) : \mathcal{L}_V \to \mathcal{L}_{G(V)}$ is an isomorphism because $G(V/X)$ is isomorphic to $G(V)/G(X)$ for a subobject X of V.

By Remark II.8, if F is fully faithful then $\epsilon_Y : Y \to GF(Y)$ is an isomorphism for any object Y of $\mathcal{C}$. Thus Corollary I.4 is an immediate consequence of Theorem I.3 (a).

Remark II.9. The referee noted that if we assume, for W of finite length in $\mathcal{C}$

(i) $F(Y)$ is simple for any simple subquotient $F(Y)$ of W,

(ii) $\lg(F(W)) = \lg(W)$ and ϵ_W is an isomorphism, then ϵ_Y is an isomorphism for any **subobject** Y of W, and $X \mapsto \epsilon_W^{-1}(G(X))$ provides a left inverse to $Y \mapsto F(Y) : \mathcal{L}_W \to \mathcal{L}_{F(W)}$.

Remark II.10. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between abelian categories and W a finite length object of $\mathcal{C}$ satisfying:

$$X \mapsto F(X) : \mathcal{L}_W \to \mathcal{L}_{F(W)} \quad \text{is a lattice isomorphism.}$$

Then any subquotient of W satisfies the same property. Indeed, this is clear for a subobject W' of W. For any exact sequence $0 \to W_1 \to W_2 \to W_3 \to 0$ in $\mathcal{C}$ with W_2 a subobject of W, the sequence $0 \to F(W_1) \to F(W_2) \to F(W_3) \to 0$ in $\mathcal{D}$ is exact by length count. Let $\mathcal{L}_{W_2}(W_1)$ denote the lattice of subobjects Y of W_2 containing W_1. The map $Y \mapsto F(Y) : \mathcal{L}_{W_2}(W_1) \to \mathcal{L}_{F(W_2)}(F(W_1))$ is a lattice isomorphism. Taking the cokernels, it corresponds to a lattice isomorphism $Z \mapsto F(Z) : \mathcal{L}_{W_3} \to \mathcal{L}_{F(W_3)}$.

We now prove the second lattice theorem I.5.

(i) This is classical. See [**BkiA8**, §4 n^{o}4 Prop. 3 b) and n^{o}5 Def. 3 and Thm. 2 a)].

(ii) The first statement is obvious. Assume $b(Y) \otimes_R b(V) \subset Y \otimes_R V$ and let $y \in Y$ and $v \in V$. Any R-linear form λ on V defines a linear map $Y \otimes_R V \to Y$ sending $b(y) \otimes b(v)$ to $\lambda(b(v))b(y)$. If $b_V \neq 0$ we can choose $v \in V$ and λ such that $\lambda(b(v)) \neq 0$ and then $b(y) \in Y$.

We finally prove Corollary I.6. Clearly the lattice isomorphism $Y \mapsto Y \otimes_R V$ in Thm. I.5 (i) sends an A'-submodule of W to an A'-submodule of $W \otimes_R V$. If an A-submodule $Y \otimes_R V$ of $W \otimes_R V$ is A'-stable, then Thm. I.5 (ii) implies that Y is an A'-submodule of W because no element in $B' \setminus B$ acts by 0 on V, as every element of $B' \setminus B$ acts invertibly on V.

The structure of A'-module on W induces a structure of A'-module on $\mathrm{Hom}_A(V, W \otimes_R V)$ such that the isomorphism ϵ_W of Thm. I.5 (i) is A'-equivariant. For $f \in \mathrm{Hom}_A(V, W \otimes_R V)$, we have $b(f) = f$ if $b \in B$ as B acts by the identity on W. If $b \in B \setminus B'$, for all $w \in W$ we have $b(\epsilon_W(w)) = \epsilon_W(b(w))$, meaning that for all $v \in V$, $b(\epsilon_W(w))(v) = b(w) \otimes v = (b_{W\otimes_R V} \circ \epsilon_W(w) \circ b_V^{-1})(v)$ as b_V is invertible. Therefore $b(f) = b_{W\otimes_R V} \circ f \circ b_V^{-1}$ for all $f \in \mathrm{Hom}_A(V, W \otimes_R V)$, if $b \in B \setminus B'$.

III. Classification theorem for G

III.1. Admissibility, K-invariants, and scalar extension. In this section III, R is any field and G is a locally profinite group. An $R[G]$-module π is smooth if $\pi = \cup_K \pi^K$ with K running through the open compact subgroups of G, and is admissible if it is smooth and $\dim_R \pi^K$ is finite for all K. If π^K generates π then $\mathrm{End}_{R[G]}\,\pi \subset \mathrm{End}_R\,\pi^K$. Fix such a K for the rest of §III.1.

The category $\mathrm{Mod}_R(G)$ of $R[G]$-modules and the subcategory $\mathrm{Mod}_R^\infty(G)$ of smooth $R[G]$-modules are abelian, but not the additive subcategory $\mathrm{Mod}_R(G)^a$ of admissible $R[G]$-modules in general (when F has characteristic p). The subcategory $\mathrm{Mod}_R^K(G)$ of $R[G]$-modules π generated by π^K is additive with a generator $R[K\backslash G]$ but is not abelian in general [6]. The commutant of $R[K\backslash G]$ is the Hecke R-algebra

$$\mathrm{End}_{R[G]}\,R[K\backslash G] \simeq R \otimes_{\mathbb{Z}} H(G,K) = H(G,K)_R$$

(the Hecke ring $H(G,K)$ is $\mathrm{End}_{\mathbb{Z}[G]}\,\mathbb{Z}[K\backslash G]$). We have the abelian category $\mathrm{Mod}_R(H(G,K))$ of right $H(G,K)_R$-modules (which we also call $H(G,K)$-modules over R). The functor

$$\mathcal{T} := - \otimes_{H(G,K)} \mathbb{Z}[K\backslash G] : \mathrm{Mod}_R(H(G,K)) \to \mathrm{Mod}_R(G)$$

with image $\mathrm{Mod}_R^K(G)$, is left adjoint to the K-invariant functor $(-)^K : \mathrm{Mod}_R(G) \to \mathrm{Mod}_R(H(G,K))$.

The unit $\epsilon : \mathrm{id}_{\mathrm{Mod}_R(H(G,K))} \to (-)^K \circ \mathcal{T}$ and the counit $\eta : \mathcal{T} \circ (-)^K \to \mathrm{id}_{\mathrm{Mod}_R^K(G)}$ of the adjunction correspond to the natural maps $\mathcal{X} \xrightarrow{\epsilon_{\mathcal{X}}} \mathcal{T}(\mathcal{X})^K, \epsilon_{\mathcal{X}}(x) = x \otimes 1$ for $x \in \mathcal{X} \in \mathrm{Mod}_R(H(G,K))$ and $\mathcal{T}(\pi^K) \xrightarrow{\eta_\pi} \pi, \eta_\pi(v \otimes Kg) = gv$ for $g \in G, v \in \pi^K, \pi \in \mathrm{Mod}_R(G)$.

Lemma III.1. *(i) If π is generated by π^K and $\dim_R \pi^K < \infty$ (in particular if π is irreducible admissible and $\pi^K \neq 0$), then $\dim_R \mathrm{End}_{R[G]}\,\pi$ is finite.*

(ii) Let R'/R be an extension. The adjoint functors $\mathcal{T}$, $(-)^K$, the unit ϵ and the counit η commute with scalar extension: there are natural isomorphisms

$$\mathcal{T}(\mathcal{X})_{R'} \simeq \mathcal{T}(\mathcal{X}_{R'}),\ (\pi^K)_{R'} \simeq (\pi_{R'})^K,\ (\epsilon_{\mathcal{X}})_{R'} \simeq \epsilon_{\mathcal{X}_{R'}},\ (\eta_\pi)_{R'} \simeq \eta_{\pi_{R'}}.$$

In particular, π is admissible if and only if $\pi_{R'}$ is admissible.

(iii) Let R'/R be an extension and π a smooth irreducible R-representation of G generated by π^K. Then, any subquotient π' of $\pi_{R'}$ is generated by π'^K.

Proof. (i) and (ii) are clear. We prove (iii).

Assume that π is generated by π^K. It is clear that $\pi_{R'}$ is generated by $(\pi^K)_{R'}$, hence by $(\pi_{R'})^K = (\pi^K)_{R'}$ (Lemma III.1). Let π' be a subquotient of $\pi_{R'}$ and $\mathfrak{A} \subset \mathfrak{B}$ be right ideals of $D_{R'} = \mathrm{End}_{R'[G]}(\pi_{R'})$ such that

$$\pi' \simeq (\mathfrak{B}/\mathfrak{A}) \otimes_D \pi_{R'}$$

[6] If $\mathrm{Mod}_R^K(G)$ is abelian and G second countable, $\mathrm{Mod}_R^K(G)$ is a Grothendieck category (same proof than for $\mathrm{Mod}_R(G)$ [**Vigadjoint**, lemma 3.2])

(apply (P2) and (P3) in §II.1 to π seen as a simple $R[G]$-module). If $v \in \pi'$, then v is a finite sum $v = \sum_{x \in \mathfrak{B}/\mathfrak{A}, w \in \pi_{R'}} x \otimes w$ and each w is a finite sum $w = \sum_{g \in G, u \in \pi_{R'}^K} gu$; as $x \otimes gu = g(x \otimes u)$ and $x \otimes u \in \pi'^K$, the representation π' is generated by π'^K. $\square$

We deduce that if ϵ or η is an isomorphism of functors, then it is also true if we replace R by a subfield. Recalling Remark II.8:

LEMMA III.2. *If the K-invariant functor $(-)^K : \mathrm{Mod}_R^K(G) \to \mathrm{Mod}_R(H(G,K))$ over R is an equivalence, then it is an equivalence over any subfield R' of R. If $\pi \in \mathrm{Mod}_R^K(G)$ and π^K is defined over R', then π is defined over R'.*

REMARK III.3. Assume that R is a field of characteristic p and K is a pro-p-Iwahori subgroup. The functor $(-)^K$ of Lemma III.2 is an equivalence if $G = GL(2, \mathbb{Q}_p)$ and $p \neq 2$, or if $G = SL(2, \mathbb{Q}_p)$.

Indeed, for $GL(2, \mathbb{Q}_p)$ this is proved under the extra-hypothesis that R contains a $(p-1)$-th root of 1 ([**O**] plus [**K**]), that we can remove with Lemma III.2. For $G = SL(2, \mathbb{Q}_p)$, see [**OS**, Prop. 3.25].

III.2. Decomposition Theorem for G. Let G be a locally profinite group, R'/R a field extension and R^{alg}/R an algebraic closure. We apply Lemma II.5, Theorem I.1 and Corollary I.2 to the group ring $A = R[G]$ and to a smooth R-representation π of G, seen as an A-module V.

We keep the same notations as in §II.1. If π is a smooth irreducible R-representation of G, the scalar extension of π to R' is a smooth R'-representation $\pi_{R'}$ of G. When the commutant $D = \mathrm{End}_{R[G]} \pi$ of π has finite dimension over R, we denote E the center of D, δ the reduced degree of D over E, E'/E a finite separable field extension splitting D, L/R a normal closure of E'/R.

THEOREM III.4. *1) If $\dim_R \mathrm{End}_{R[G]} \pi$ is finite and R'/R is normal and contains L, then*

$$\pi_{R'} \simeq \oplus^{\delta} \oplus_{i \in \mathrm{Hom}_R(E_{sep}, R')} W'_i$$

has length $\delta[E:R]$, W'_i is an indecomposable smooth R'-representation of G. All irreducible subquotients of W'_i have commutant R' and have the same isomorphism class $[V'_i]$; the $[V'_i]$ form a single orbit under $\mathrm{Aut}_R(R')$.

The map $[\pi] \to \mathrm{Aut}_R(R^{alg})[\pi']$ where π' is an irreducible subquotient of $\pi_{R^{alg}}$, is a bijection from the set of isomorphism classes $[\pi]$ of smooth irreducible R-representations π of G with $\dim_R \mathrm{End}_{R[G]} \pi < \infty$ onto the set of $\mathrm{Aut}_R(R^{alg})$-orbit of isomorphism classes $[\pi']$ of smooth absolutely irreducible R^{alg}-representations π' of G descending to some finite extension of R.

2) If $\dim_R \mathrm{End}_{R[G]} \pi$ is finite, $\pi_{R'}$ has length $\leq \delta[E:R]$. For any non-zero subquotient π' of $\pi_{R'}$ we have $\dim_{R'} \mathrm{End}_{R'[G]} \pi' < \infty$ and π' admissible is equivalent to π admissible.

3) If R'/R is normal, a smooth irreducible R'-representation π' of G descending to a finite extension of R is isomorphic to a subrepresentation of $\pi_{R'}$ for some smooth irreducible R-representation π of G. For any such π, $\dim_R \pi$, resp. $\dim_R \mathrm{End}_{R[G]} \pi$, is finite if $\dim_{R'} \pi'$, resp. $\dim_{R'} \mathrm{End}_{R'[G]} \pi'$, is.

PROOF. 1), 3) and the first assertion of 2) follow from Lemma II.5, Theorem I.1 and Corollary I.2. Let us prove the claims about admissibility in 2). Take an algebraic closure R'^{alg} of R' containing R^{alg}. Then $\pi_{R'^{alg}} \simeq (\pi_{R'})_{R'^{alg}} \simeq (\pi_{R^{alg}})_{R'^{alg}}$

and one of the representations $\pi, \pi_{R'}, \pi_{R^{alg}}, \pi_{R'^{alg}}$ is admissible if and only if the other ones are (Lemma III.1 (ii)).

Applying 1), $\pi_{R^{alg}}$ has finite length, its irreducible subquotients are $\mathrm{Aut}_R(R^{alg})$-conjugate, isomorphic to subrepresentations and scalar extension induces a bijection from the isomorphism classes of irreducible subquotients of $\pi_{R^{alg}}$ onto those of $\pi_{R'^{alg}}$. So some irreducible subquotient of $\pi_{R^{alg}}$ is admissible if and only if π is admissible if and only if all irreducible subquotients of $\pi_{R^{alg}}$ are admissible, if and only if all irreducible subquotients of $\pi_{R'^{alg}}$ are admissible.

In a finite length representation, if all irreducible subquotients are admissible, then all subquotients are admissible. So π is admissible if and only if some non-zero subquotient of $\pi_{R'^{alg}}$ is admissible if and only if all subquotients of $\pi_{R'^{alg}}$ are admissible.

Let π' be a non-zero subquotient of $\pi_{R'}$. Then $\pi'_{R'^{alg}}$ is a non-zero subquotient of $\pi_{R'^{alg}}$. As π' is admissible if and only if $\pi'_{R'^{alg}}$ is, we deduce that π' is admissible if and only if π is admissible. $\square$

Let K be an open compact subgroup of G, $R \subset R'$ a field extension, R^{alg} an algebraic closure of R and π an irreducible admissible R'-representation of G with $\pi^K \neq 0$. The rationality field $R[\pi]$ of π is the subfield of R' fixed by the $\mathrm{Aut}(R')$-stabilizer $H[\pi] = \{\sigma \in \mathrm{Aut}(R') \mid R' \otimes_\sigma \pi \simeq \pi\}$ of the isomorphism class $[\pi]$ of π.

Proposition III.5. *(i) Any finite dimensional R^{alg}-representation of $H(G,K)$ descends to a finite extension of R, when the Hecke ring $H(G,K)$ is finitely generated (see Lemma* III.7 *below).*

(ii) If the $H(G,K)_{R'}$-module π^K descends to R, then

a) π descends to R if the pro-order of K is invertible in R.

b) π descends to the subfield of R' fixed by $\mathrm{Aut}_{R[\pi]}(R')$ if the commutant of π^K is R'.

c) π descends to a finite extension of $R[\pi]$ if R is finite and $R' = R^{alg}$.

Proof. (i) follows from [**Viglivre**, II.4.7]: Let (e_i) be a basis of $\mathcal{M}$ and (T_j) a finite set of generators of the ring $H(G,K)$. There are finitely many elements $c_{i,j,k} \in R^{alg}$ such that $e_i T_j = \sum_k c_{i,j,k} e_k$. Let L/R be the finite extension generated by all the $c_{i,j,k}$ and $\mathcal{M}_L$ the L-vector subspace of basis (e_i). Then $\mathcal{M}_L$ is $H(G,K)$-stable and the natural map $R^{alg} \otimes_L \mathcal{M}_L \to \mathcal{M}$ is an $R^{alg}[H(G,K)]$-isomorphism.

(ii) We suppose that $\pi^K \neq 0$ descends to R; we choose an $H(G,K)_R$-stable submodule $(\pi^K)_R \subset \pi^K$ generated over R by an R'-basis of π^K; put π_R for the R-subrepresentation of π generated by $(\pi^K)_R$.

a) By assumption the pro-order of K is invertible in R, By [**Viglivre**] one can put on the space $H(G)_R$ of locally constant compactly supported functions from G to R a structure of convolution algebra such that the characteristic function $e = e_K$ of K is an idempotent; then $H(G,K)_R$ appears as $eH(G)_R e$. A smooth R-representation π of G is naturally a $H(G)_R$-module and $H(G,K)_R$ acts on $\pi^K = e\pi$ via the inclusion $eH(G)_R e \subset H(G)_R$. Since π is an irreducible admissible R'-representation of G with $\pi^K \neq 0$, π^K is a simple $H(G,K)_{R'}$-module [**Viglivre**] and π can be recovered from π^K. Indeed, following [**BK**, 4.2.3 Prop.], if $\mathcal{X}$ is a simple $H(G,K)_{R'}$-module, then $\mathcal{X} \otimes_{\mathbb{Z}} \mathbb{Z}[K\backslash G]$ has a maximal subrepresentation killed by

e, the corresponding quotient X is irreducible and the quotient map induces an $H(G,K)_{R'}$-isomorphism $\mathcal{X} \simeq Xe$. If $\mathcal{X} = \pi^K$ then $X = \pi$.

Since π^K is a simple $H(G,K)_{R'}$-module, $(\pi^K)_R$ is a simple $H(G,K)_R$-module. Applying the above procedure over R, we consider the quotient ρ of $(\pi^K)_R \otimes_{\mathbb{Z}} \mathbb{Z}[K\backslash G]$ by its maximal subrepresentation W killed by e; it is an irreducible and admissible R-representation of G. We have the exact sequence

$$0 \to R' \otimes_R W \to R' \otimes_R (\pi^K)_R \otimes_{\mathbb{Z}} \mathbb{Z}[K\backslash G] \to R' \otimes_R \rho \to 0.$$

Clearly $(R' \otimes_R W)e = 0$ and $R' \otimes_R \rho$ isomorphic to a direct sum of copies of ρ as an $R[G]$-module has no non-zero subrepresentation killed by e. It follows that $R' \otimes_R W$ is the maximal subrepresentation of $R' \otimes_R (\pi^K)_R \otimes_{\mathbb{Z}} \mathbb{Z}[K\backslash G]$ killed by e, hence $\pi \simeq R' \otimes_R \rho$ descends to R.

b) and c) The set $\{gv \mid g \in G\}$ certainly generates π as an R'-vector space, so we can extract a basis $\{g_i v \mid i \in I\}$. For $g \in G$ we express $gv = \sum_{i \in I} \lambda_i g_i v$ with unique $\lambda_i \in R'$, almost all 0. We will show:

(*) $\sigma(\lambda_i) = \lambda_i$ for all $i \in I$ and

- for all $\sigma \in \mathrm{Aut}_{R[\pi]}(R')$ if $\mathrm{End}_{R'[G]} \pi^K = R'$,

- for all $\sigma \in \mathrm{Aut}_L(R')$ for some finite extension $L/R[\pi]$ if R is finite and $R' = R^{alg}$.

This will imply that for all $i \in I$, λ_i lies in the subfield L of R' fixed by $\mathrm{Aut}_{R[\pi]}(R')$ if $\mathrm{End}_{R'[G]} \pi^K = R'$, and in a finite extension $L/R[\pi]$ if R is finite and $R' = R^{alg}$. Thus, the L-vector subspace V of π of basis $(g_i v)_{i \in I}$ is stable by G, it is an L-subrepresentation π_L of π such that the natural isomorphism $R' \otimes_L \pi_L \to \pi$ is an $R'[G]$-isomorphism.

To prove (*) it suffices to find for all σ in (*) an intertwining operator $A_\sigma : \pi \to R' \otimes_\sigma \pi$ such that $A_\sigma(v) = 1 \otimes v$. Indeed, for such an operator $A = A_\sigma$,

$$\begin{aligned} 1 \otimes gv = A(gv) = A(\sum_{i\in I} \lambda_i g_i v) &= \sum_{i \in I} \lambda_i A(g_i v) \\ = \sum_{i \in I} \lambda_i (1 \otimes g_i v) &= \sum_{i\in I} 1 \otimes \sigma(\lambda_i) g_i v = 1 \otimes \sum_{i \in I} \sigma(\lambda_i) g_i v; \end{aligned}$$

so $\sum_{i \in I} \lambda_i g_i v = \sum_{i \in I} \sigma(\lambda_i) g_i v$, that is, $\sigma(\lambda_i) = \lambda_i$ for all $i \in I$.

To find A_σ, we note that for $\sigma \in \mathrm{Aut}_{R[\pi]}(R')$, the natural map $f : (\pi^K)_R \to R' \otimes_\sigma \pi^K$ sending x to $1 \otimes x$ extends to an intertwining operator $\pi^K \to R' \otimes_\sigma \pi^K$.

- If $\mathrm{End}_{R'[G]} \pi^K = R'$, then any intertwining operator $\pi \to R' \otimes_\sigma \pi$ restricts on $(\pi^K)_R$ to a multiple of f, hence we can find A_σ. This ends the proof of (iii) in the case b).

- If R is finite and $R' = R^{alg}$, we choose a (topological) generator τ of the (pro)cyclic group $\mathrm{Aut}_{R[\pi]}(R^{alg})$ and an R'-basis of π^K contained in $(\pi^K)_R$; the restriction $A_\tau^K : \pi^K \to R^{alg} \otimes_\tau \pi^K$ of A_τ to π^K has a matrix $\mathrm{Mat}(A_\tau^K)$ on this basis. The coefficients $\mathrm{Mat}(A_\tau^K)$ are fixed by τ^m for some positive integer m. For any positive integer k, we have the intertwining operator $A_{\tau^{mk}} = (\tau^{m-1}(A_\tau) \dots \tau(A_\tau) A_\tau)^k : \pi \to R^{alg} \otimes_{\tau^{mk}} \pi$ with restriction $A^K_{\tau^{mk}} = (\tau^{m-1}(A^K_\tau) \dots \tau(A^K_\tau) A^K_\tau)^k$ to π^K of matrix $\mathrm{Mat}(A^K_{\tau^m})^k$. As the order of $\mathrm{Mat}(A^K_{\tau^m})$ is finite, we can choose k_o such that $\mathrm{Mat}(A^K_{\tau^{mk_o}})$ is the identity. Then $A_{\tau^{mk_0}}(v) = 1 \otimes v$. Therefore the subfield of R^{alg} fixed by τ^{mk_0} is a finite extension $R'/R[\pi]$ such that $A_\sigma(v) = 1 \otimes v$ for all $\sigma \in \mathrm{Aut}_{R'}(R^{alg})$. This ends the proof of (iii) in the case c). $\square$

REMARK III.6. If the K-invariant functor $(-)^K : \mathrm{Mod}_{R'}^K(G) \to \mathrm{Mod}_{R'}(H(G,K))$ over R' is an equivalence (Lemma III.2), then π^K descends to R if and only if π does.

III.3. The representations $I_G(P,\sigma,Q)$. Until the end of the article G is a p-adic reductive group (in the following sense).

The base field F is locally compact non-archimedean of residue characteristic p. A linear algebraic group over F is written with a boldface letter like $\mathbf{H}$, and its group of F-points by the corresponding ordinary letter $H = \mathbf{H}(F)$. We fix an arbitrary connected reductive F-group $\mathbf{G}$, a maximal F-split torus $\mathbf{T}$ in $\mathbf{G}$ and a minimal F-parabolic subgroup $\mathbf{B}$ of $\mathbf{G}$ containing $\mathbf{T}$; we write $\mathbf{Z}$ for the centralizer of $\mathbf{T}$ in $\mathbf{G}$ and $\mathbf{U}$ for the unipotent radical of $\mathbf{B}$. We denote by $\mathbf{G}^{is}$ the product of the isotropic simple components of the simply connected cover of the derived group of $\mathbf{G}$.

Let Φ^+ denote the set of roots of $\mathbf{T}$ in $\mathbf{U}$, $\Delta \subset \Phi^+$ the set of simple roots. We say that P is a **parabolic subgroup** of G and write $P = MN$ to mean that $\mathbf{P}$ is an F-parabolic subgroup of $\mathbf{G}$ containing $\mathbf{B}$, $\mathbf{M}$ the Levi subgroup containing $\mathbf{Z}$ and $\mathbf{N}$ the unipotent radical; the parabolic subgroups P of G are in bijection $P \mapsto \Delta_P = \Delta_M$ with the subsets Δ. For $J \subset \Delta$ we write $P_J = M_J N_J$ for the corresponding parabolic subgroup; for a singleton $J = \{\alpha\}$ we rather write $P_\alpha = M_\alpha N_\alpha$. We have $G = M\langle {}^G N\rangle$ for the normal subgroup $\langle {}^G N\rangle$ of G generated by N.

The image of G^{is} in G is the normal subgroup G' of G generated by U, and $G = ZG'$. Set P^{is} for the parabolic subgroup of G^{is} of image $P \cap G'$ in G.

LEMMA III.7. *Let K be an open compact subgroup of G. The Hecke ring $H(G,K) = \mathrm{End}_{\mathbb{Z}[G]} \mathbb{Z}[K\backslash G]$ is finitely generated, if K is a normal subgroup of a special parahoric subgroup of G and admits an Iwahori decomposition*[7].

PROOF. It is only proved that $\mathbb{Z}[1/p] \otimes_{\mathbb{Z}} H(G,K)$ is finitely generated in [**Viglivre**, II.2.13 Prop.].

When G is compact, the lemma is obvious as the set $K\backslash G/K$ is finite.

When G is compact modulo its centre Z_G, this is also clear as the set $K\backslash G/KZ_G$ is finite and the group $Z_G/(Z_G \cap K)$ is commutative and finitely generated. One can choose a finite set of representatives g_i such that all the double classes of G modulo K are of the form $Kg_i zK$ for $z \in Z_G$ and representatives z_j of a finite set of generators of $Z_G/(Z_G \cap K)$. The product of Kg_iK and of $Kz_jK = Kz_j = z_jK$ is Kg_iz_jK, and the ring $H(G,K)$ is generated by the Kg_iz_jK.

For G general, the same arguments imply that the ring $H(Z^+, K\cap Z)$ is finitely generated (Z^+ is the positive monoid cf.§IV.1). When K has an Iwahori decomposition and is a normal subgroup of a special parahoric subgroup K_0 of G, the map $(K \cap Z)z(K \cap Z) \mapsto KzK : H(Z^+, K \cap Z) \to H(Z^+, K)$ is a ring embedding of image the subring of $H(G,K)$ generated by the elements KzK for $z \in Z^+$ [**VigSelecta**, II.4], and moreover the Cartan decomposition [**HV1**, 6.3 Prop.] implies $H(G,K) = H(K_0,K)H(Z^+,K)H(K_0,K)$ [**Viglivre**, II.2.13 Prop.]. Thus, the ring $H(G,K)$ is finitely generated. $\square$

[7] K is called " bien placé par rapport à (B,Z,U)" in [**Viglivre**, II.1.3 (vi)]

REMARK III.8. If K is an Iwahori or a pro-p Iwahori subgroup of G, then $H(G,K)$ is a finite module over its centre and the centre is finitely generated [**VigpIwc**].

Until the end of the article R is a field of characteristic p. We are interested in irreducible admissible R-representations of G.

For a parabolic group $P = MN$ of G, the smooth parabolic induction functor $\mathrm{Ind}_P^G : \mathrm{Mod}_R^\infty(M) \to \mathrm{Mod}_R^\infty(G)$ is fully faithful, and admits a left adjoint L_P^G and a right adjoint R_P^G [**Vigadjoint**]. The right adjoint R_P^G respects admissibility[**AHenV1**, Cor. 4.13] hence is equal on admissible representation to the Emerton's $\overline{P}$-ordinary part functor $\mathrm{Ord}_{\overline{P}}^G$ where $\overline{\mathbf{P}}$ is the opposite of $\mathbf{P}$ with respect to $\mathbf{B}$ [**Eme**, 3.1.9 Definition].

For a pair of parabolic subgroups $Q \subset P$ of G, write Ind_Q^M for $\mathrm{Ind}_{Q\cap M}^M$ and consider the Steinberg R-representation $\mathrm{St}_Q^M(R)$ of M, quotient of $\mathrm{Ind}_Q^M(R)$ (R stands for the trivial R-representation $\mathrm{Triv}_{Q\cap M}$ of $Q\cap M$) by the sum $\sum_{Q'} \mathrm{Ind}_{Q'}^M(R)$, Q' running through the parabolic subgroups of G with $Q \subsetneq Q' \subset P$. The R-representation $\mathrm{St}_Q^M(R)$ of M is absolutely irreducible and admissible [**Ly**], and $\mathrm{St}_Q^M(R) \simeq R \otimes_{\mathbb{Z}} \mathrm{St}_Q^M$ where $\mathrm{St}_Q^M = \mathrm{St}_Q^M(\mathbb{Z})$.

Writing $P_2 = M_2N_2$ for the parabolic subgroup corresponding to $\Delta_P \setminus \Delta_Q$, the inflation to M_2^{is} of the restriction of St_Q^M to M_2' is $\mathrm{St}_{(Q\cap M_2)^{is}}^{M_2^{is}}(R)$ ([**AHHV**, II.8 Proof of Proposition and Remark] when R is algebraically closed, but the proofs do not use this hypothesis). Therefore the action of M_2' on $\mathrm{St}_Q^M(R)$ is absolutely irreducible.

To an R-representation σ of M are associated the following parabolic subgroups of G:

a) $P_\sigma = M_\sigma N_\sigma$ corresponding to the set Δ_σ of $\alpha \in \Delta \setminus \Delta_M$ such that $Z \cap M'_\alpha$ acts trivially on σ.

b) $P(\sigma) = M(\sigma)N(\sigma)$ corresponding to $\Delta(\sigma) = \Delta_P \cup \Delta_\sigma$. By [**AHHV**, II.7 Proposition and Remark 2] which remain valid when R is not algebraically closed, there exists an extension $e(\sigma)$ to $P(\sigma)$ of σ trivial on N; we write also $e(\sigma)$ for its restriction to $M(\sigma)$. For $P \subset Q \subset P(\sigma)$, the generalized Steinberg representation $\mathrm{St}_Q^{M(\sigma)}(\sigma)$ of $M(\sigma)$ defined in §I (0.2), is admissible and isomorphic to $e(\sigma) \otimes_{\mathbb{Z}} \mathrm{St}_Q^{M(\sigma)}$.

c) $P_{min} = M_{min}N_{min} \subset P$ the smallest parabolic subgroup of G such that σ is extended from an R-representation σ_{min} of M_{min} trivially on $N_{min} \cap M$ [**AHenV1**, Lemma 2.9]. Then $\Delta(\sigma_{min}) = \Delta(\sigma)$, $e_Q(\sigma) = e_Q(\sigma_{min})$, and $\Delta_{\sigma_{min}}$, $\Delta_{\sigma_{min}} \setminus \Delta_{P_{min}}$ are orthogonal [**AHenV1**, Lemma 2.10]. This implies that $M(\sigma) = M_{min}M'_{\sigma_{min}}$, M_{min} normalizes $M'_{\sigma_{min}}$, and that $e(\sigma)$ is trivial on $M'_{\sigma_{min}}$.

DEFINITION III.9. An R-triple (P,σ,Q) of G consists of a parabolic subgroup $P = MN$ of G, a smooth R-representation σ of M, and a parabolic subgroup Q of G with $P \subset Q \subset P(\sigma)$. The smooth R-representation of G defined by an R-triple (P,σ,Q) of G is

$$I_G(P,\sigma,Q) = \mathrm{Ind}_{P(\sigma)}^G(\mathrm{St}_Q^{M(\sigma)}(\sigma)).$$

The representation $I_G(P,\sigma,Q)$ is equal to $I_G(P_{min},\sigma_{min},Q)$ [**AHenV1**, Lemma 2.11]; it is admissible when σ is admissible [**AHenV1**, Thm.4.21].

PROPOSITION III.10. *Let (P,σ,Q) be an R-triple of G such that, σ is admissible of finite length, $P(\sigma)=P(\tau)$ and $I_G(P,\tau,Q)$ is irreducible for each irreducible subquotient τ of σ. Then $P(\sigma)=P(\sigma')$ for any non-zero subrepresentation σ' of σ, and the map $\sigma'\mapsto I_G(P,\sigma',Q):\mathcal{L}_\sigma\to\mathcal{L}_{I_G(P,\sigma,Q)}$ is a lattice isomorphism.*

PROOF. Clearly $P(\sigma)\subset P(\sigma')$. As σ' has finite length, it contains an irreducible subrepresentation τ. From $P(\sigma)\subset P(\sigma')\subset P(\tau)$ and $P(\sigma)=P(\tau)$, we get $P(\sigma)=P(\sigma')$.

We are in the situation of Corollary I.6 for $A=R[M'_\sigma]\subset A'=R[M(\sigma)]$ and the $R[M(\sigma)]$-modules $W=e(\sigma)$ and $V=\mathrm{St}_Q^{M(\sigma)}(R)$, with the basis $B=M'_\sigma$ of A acting by the identity on W and the basis $B'=M(\sigma)$ of A' acting invertibly on V. Applying Cor.I.6, the natural maps

$$e(\sigma)\to\mathrm{Hom}_{R[M'_\sigma]}(\mathrm{St}_Q^{M(\sigma)}(R),\mathrm{St}_Q^{M(\sigma)}(\sigma)),$$
$$\mathrm{Hom}_{R[M'_\sigma]}(\mathrm{St}_Q^{M(\sigma)}(R),\mathrm{St}_Q^{M(\sigma)}(\sigma))\otimes_R\mathrm{St}_Q^{M(\sigma)}(R)\to\mathrm{St}_Q^{M(\sigma)}(\sigma)$$

are $R[M(\sigma)]$-isomorphisms and $\sigma'\mapsto\mathrm{St}_Q^{M(\sigma)}(\sigma'):\mathcal{L}_\sigma\to\mathcal{L}_{\mathrm{St}_Q^{M(\sigma)}(\sigma)}$ is a lattice isomorphism. In particular, $\mathrm{St}_Q^{M(\sigma)}(\sigma)$ has finite length, $\lg(\mathrm{St}_Q^{M(\sigma)}(\sigma))=\lg(\sigma)$, and the irreducible subquotients $\mathrm{St}_Q^{M(\sigma)}(\sigma)$ are $\mathrm{St}_Q^{M(\sigma)}(\tau)$ for the irreducible subquotients τ of σ. As $I_G(P,\tau,Q)$ is irreducible and equal to $\mathrm{Ind}_{P(\sigma)}^G\mathrm{St}_Q^{M(\sigma)}(\tau))$ for each τ, we are in the situation of Corollary I.4 for the fully faithful exact functor $F=\mathrm{Ind}_{P(\sigma)}^G:\mathrm{Mod}_R(M(\sigma))\to\mathrm{Mod}_R(G)$ having a right adjoint $G=R_P^G$, and $W=\mathrm{St}_Q^{M(\sigma)}(\sigma)$. We deduce that the map $\sigma'\mapsto I_G(P,\sigma',Q):\mathcal{L}_\sigma\to\mathcal{L}_{I_G(P,\sigma,Q)}$ is a lattice isomorphism. □

REMARK III.11. $I_G(P,\sigma,Q)$ determines the isomorphism class of $e(\sigma)$ because

$$e(\sigma)\simeq\mathrm{Hom}_{R[M'_\sigma]}(\mathrm{St}_Q^{P(\sigma)}(R),R_{P(\sigma)}^G(I_G(P,\sigma,Q)))$$

(proof of Prop. III.10 and $R_{P(\sigma)}^G(I_G(P,\sigma,Q))\simeq\mathrm{St}_Q^{P(\sigma)}(\sigma)$).

Let R' be a field containing R. Scalar extension from R to R' commutes with the different steps in the construction of $I_G(P,\sigma,Q)$:

PROPOSITION III.12. *(i) The parabolic induction functor Ind_P^G commutes with the scalar restriction from R' to R and with the scalar extension from R to R'. The left adjoint L_P^G (resp. right adjoint R_P^G) of the parabolic induction commutes with scalar extension (resp. restriction).*

(ii) If $\pi\in\mathrm{Mod}_R^\infty(G)$ is such that $\pi_{R'}\simeq\mathrm{Ind}_P^G(\sigma')$ with $\sigma'\in\mathrm{Mod}_{R'}^\infty(M)$, then σ' is isomorphic to $(L_P^G\pi)_{R'}$.

PROOF. (i) Choosing a continuous section $P\backslash G\to G$, $\mathrm{Ind}_P^G\sigma$ identifies with $\sigma\otimes_{\mathbb{Z}}C_c^\infty(P\backslash G,\mathbb{Z})$ as an R-module [**AHenV1**]; this implies the first assertions, and the next sentence follows by adjunction. Part (ii) follows because Ind_P^G is fully faithful. □

PROPOSITION III.13. [Strong compatibility of $I_G(P,-,Q)$ with scalar extension]

(i) Let (P,σ,Q) be an R-triple of G. Then

$P(\sigma)=P(\sigma_{R'})$, $(P,\sigma_{R'},Q)$ is an R'-triple of G, and if σ is irreducible and σ' a non-zero subquotient of $\sigma_{R'}$, then $P(\sigma)=P(\sigma')$. Moreover,

$(e(\sigma))_{R'}=e(\sigma_{R'})$, $(\mathrm{St}_Q^{P(\sigma)}(\sigma))_{R'}\simeq\mathrm{St}_Q^{P(\sigma_{R'})}(\sigma_{R'})$ and $I_G(P,\sigma,Q)_{R'}\simeq I_G(P,\sigma_{R'},Q)$.

(ii) Let (P,σ,Q) be an R'-triple of G. If $e(\sigma)$ or $\mathrm{St}_Q^{P(\sigma)}(\sigma)$ or $I_G(P,\sigma,Q)$ descends to R, then σ descends to R.

Precisely, if $e(\sigma)=\tau_{R'}$ or $\mathrm{St}_Q^{P(\sigma)}(\sigma)=\rho_{R'}$ or $I_G(P,\sigma,Q)=\pi_{R'}$ for R-representations τ of $M(\sigma)$ or ρ of $M(\sigma)$ or π of G,

then σ is the scalar extension of the natural R-representation of M on τ, or $\mathrm{Hom}_{R[M'_\sigma]}(\mathrm{St}_Q^{P(\sigma)}(R),\rho)$, or $\mathrm{Hom}_{R[M'_\sigma]}(\mathrm{St}_Q^{P(\sigma)}(R),L^G_{P(\sigma)}\pi)$.

PROOF. (i) $\sigma_{R'}$ is a direct sum of $R[M]$-modules isomorphic to σ. If σ is irreducible, any subquotient σ' of $\sigma_{R'}$ is σ-isotypic. For $\alpha\in\Delta-\Delta_P$, $Z\cap M'_\alpha$ acts trivially on an $R'[M]$-module τ if and only if it acts trivially on τ seen as an $R[M]$-module. So $P(\sigma)=P(\sigma_{R'})$ (hence $(P,\sigma_{R'},Q)$ is an R'-triple of G), and if σ is irreducible $P(\sigma)=P(\sigma')$. It is clear from the definition that the extension commutes with scalar extension $R'\otimes_R e(\sigma)=e(R'\otimes_R\sigma)$. The scalar extension of $\mathrm{St}_Q^{P(\sigma)}(\sigma)=e(\sigma)\otimes_{\mathbb{Z}}\mathrm{St}_Q^{P(\sigma)}$ from R to R' is $R'\otimes_R\mathrm{St}_Q^{P(\sigma)}(\sigma)=R'\otimes_R e(\sigma)\otimes_{\mathbb{Z}}\mathrm{St}_Q^{P(\sigma)}\simeq e(R'\otimes_R\sigma)\otimes_{\mathbb{Z}}\mathrm{St}_Q^{P(\sigma)}\simeq\mathrm{St}_Q^{P(\sigma)}(\sigma_{R'})=\mathrm{St}_Q^{P(\sigma_{R'})}(\sigma_{R'})$. The scalar extension of $I_G(P,\sigma,Q)=\mathrm{Ind}^G_{P(\sigma)}(\mathrm{St}_Q^{P(\sigma)}(\sigma))$ from R to R' is $R'\otimes_R I_G(P,\sigma,Q)\simeq\mathrm{Ind}^G_{P(\sigma)}(\mathrm{St}_Q^{P(\sigma)}(\sigma_{R'}))=\mathrm{Ind}^G_{P(\sigma_{R'})}(\mathrm{St}_Q^{P(\sigma_{R'})}(\sigma_{R'}))=I_G(P,\sigma_{R'},Q)$.

(ii) If $I_G(P,\sigma,Q)=\pi_{R'}$, we have $\mathrm{St}_Q^{P(\sigma)}(\sigma)\simeq(L^G_{P(\sigma)}\pi)_{R'}$ (Proposition III.12 (ii)).

If $\mathrm{St}_Q^{P(\sigma)}(\sigma)\simeq\rho_{R'}$, then $e(\sigma)\simeq\mathrm{Hom}_{R'[M'_\sigma]}(\mathrm{St}_Q^{P(\sigma)}(R'),\rho_{R'})$ (Remark III.11); as $\mathrm{St}_Q^{P(\sigma)}(R')=\mathrm{St}_Q^{P(\sigma)}(R)_{R'}$, is irreducible, $\mathrm{Hom}_{R'[M'_\sigma]}(\mathrm{St}_Q^{P(\sigma)}(R'),\rho_{R'})\simeq\mathrm{Hom}_{R[M'_\sigma]}(\mathrm{St}_Q^{P(\sigma)}(R),\rho)_{R'}$ (Remark II.2).

If $e(\sigma)\simeq\tau_{R'}$ then $\sigma\simeq(\tau|_M)_{R'}$ because the restriction to M commutes with scalar extension. □

III.4. Supersingular representations.

We keep the notations of §III.3. When R is algebraically closed and π is an irreducible admissible R-representation of G, in [**AHHV**] the definition of supersingularity uses the Hecke algebras defined by the irreducible smooth R-representations of the special parahoric subgroups of G. Two equivalent simpler criterions using the pro-p Iwahori Hecke R-algebra of G are given in [**OV**, Thm. 5.3]. We will use these equivalent criterions to extend the definition of supersingularity to the situation where R is not algebraically closed, and π is a non-zero smooth representation generated by its pro-p Iwahori invariants.

Let I be a pro-p Iwahori subgroup of G **compatible with** B, so that $I\cap M$ is a pro-p Iwahori subgroup of M for any parabolic subgroup $P=MN$ (we recall that P contains $B=ZU$ and M contains Z). Let Z_0 be the unique parahoric subgroup of Z and Z_1 the pro-p Sylow subgroup of Z_0. We defined in §III.1 the pro-p Iwahori Hecke ring $H(G,I)=H(G)$, the pro-p Iwahori Hecke R-algebra $H(G)_R$ and the categories $\mathrm{Mod}_R(H(G))$ and $\mathrm{Mod}_R^\infty(G)$. The elements in $H(G)$ with support in G'

form a subring $H(G')$ normalized by a subring of $H(G)$ isomorphic to $\mathbb{Z}[\Omega]$ for a commutative finitely generated subgroup Ω, $H(G)$ is the product of $H(G')$ by $\mathbb{Z}[\Omega]$ and

$$H(G') \cap \mathbb{Z}[\Omega] \simeq \mathbb{Z}[Z'_k],\ Z'_k = (Z_0 \cap G')/(Z_1 \cap G').$$

To M is associated a certain element T_M in $H(G')$ which is central in $H(G)$ [**Vig17**].

Definition III.14. 1. An non-zero element v in a right $H(G)_R$-module is called supersingular if $vT_M^n = 0$ for all $M \neq G$ and some positive integer n. A non-zero $H(G)_R$-module is called supersingular if its non-zero elements are supersingular.

2. A non-zero smooth R-representation π of G generated by π^I is called supersingular if the right $H(G)_R$-module π^I is supersingular.

Any non-zero R-representation of G has a non-zero I-invariant vector, as the characteristic of R is p, hence any irreducible smooth R-representation π of G is generated by π^I. As explained above, when π is irreducible admissible and R algebraically closed, our definition of supersingularity is equivalent to the definition given in [**AHHV**] by [**OV**, Thm. 5.3].

Remark III.15. 1. Let $0 \to \mathcal{V}' \to \mathcal{V} \to \mathcal{V}'' \to 0$ be an exact sequence of $H(G)_R$-modules. Then $\mathcal{V}$ is supersingular if and only if $\mathcal{V}'$ and $\mathcal{V}''$ are supersingular.

2. When R contains a root of unity of order the exponent of $Z_k = Z_0/(Z_0 \cap I)$, the simple supersingular $H(G)_R$-modules are classified [**Vig17**, Thm. 6.18]; as $H(G')_R$-modules, they are sums of supersingular characters.

3. The group $\mathrm{Aut}(R)$ of automorphisms of R acts on $\mathrm{Mod}_R(G)$ and on $\mathrm{Mod}_R(H(G))$. Clearly, the action of $\mathrm{Aut}(R)$ commutes with the I-invariant functor, and respects supersingularity, irreducibility, and admissibility.

Supersingularity commutes with scalar extension:

Lemma III.16. *Let R'/R an extension.*

1) A (right) $H(G)_R$-module $\mathcal{X}$ is supersingular if and only if $\mathcal{X}_{R'}$ is; a smooth R-representation π of G generated by π^I is supersingular if and only if $\pi_{R'}$ is.

2) Let π be a smooth irreducible R-representation π of G with $\dim_R \mathrm{End}_{R[G]} \pi < \infty$ and π' be a non-zero subquotient of $\pi_{R'}$. Then π is supersingular if and only if π' is supersingular.

Proof. 1) In $\mathcal{X}_{R'} = R' \otimes_R \mathcal{X}$, we have $(r' \otimes x)T_M = r' \otimes xT_M$ for $r' \in R', x \in \mathcal{X}$; clearly the non-zero elements of $\mathcal{X}_{R'}$ are supersingular if and only if the non-zero elements of $\mathcal{X}$ are supersingular.

If π is generated by π^I, then $\pi_{R'}$ is generated by $\pi^I_{R'} = (\pi^I)_{R'}$ (Lemma III.1 (iii)). By the previous case, π is supersingular if and only if $\pi_{R'}$ is.

2) Any non-zero subquotient π' of $\pi_{R'}$ is generated by π'^I because π is (Lemma III.1 (iii)). The proof that π is supersingular if and only if π' is supersingular is the same as for admissible. Applying Thm. III.4 2) and Remark III.15, we can replace "admissible" by "supersingular" in the proof of Thm. III.4 3). $\square$

As an application, supersingularity for an irreducible admissible R-representation of G can be detected on a weaker property, as in the case where R is algebraically closed:

Theorem III.17. *Let π be an irreducible admissible R-representation of G. Then π is supersingular if and only if π^I contains a non-zero supersingular element.*

PROOF. Suppose that π^I contains a non-zero supersingular element. By Lemma III.1, $(\pi^I)_{R^{alg}} = (\pi_{R^{alg}})^I$. By Lemma III.16 and [**OV**, Thm. 5.3], $(\pi^I)_{R^{alg}}$ is supersingular. By Thm III.4, $\pi_{R^{alg}}$ has finite length. The irreducible subrepresentations of $\pi_{R^{alg}}$ are supersingular. By Lemma III.16, π is supersingular. The converse is obvious. □

REMARK III.18. *The scalar extension to R^{alg} of a R-triple (P, σ, Q) of G where σ is irreducible admissible supersingular, is an R^{alg}-triple $(P, \sigma_{R^{alg}}, Q)$ of G satisfying the hypotheses of Proposition* III.10: the irreducible subquotients τ of $\sigma_{R^{alg}}$ are supersingular (Lemma III.16), $P(\tau) = P(\sigma) = P(\sigma_{R^{alg}})$ (Prop.III.13 (i)), and $I_G(P, \tau, Q)$ is irreducible (Classification theorem for G over R^{alg} [**AHHV**]).

III.5. Classification of irreducible admissible R-representations of G. We prove in this section the classification theorem for G (Thm. I.7). The arguments are formal and rely on:

1 The decomposition theorem for G (Thm.III.4).

2 The classification theorem for G (Thm.I.7) over an algebraic closure R^{alg} of R [**AHHV**].

3 The compatibility of scalar extension from R to R^{alg} with supersingularity (Lemma III.16) and the strong compatibility with $I_G(P, -, Q)$ (Prop.III.13).

4 The lattice isomorphism $\mathcal{L}_{\sigma_{R^{alg}}} \to \mathcal{L}_{I_G(P, \sigma_{R^{alg}}, Q)}$ for the scalar extension $\sigma_{R^{alg}}$ to R^{alg} of an irreducible admissible supersingular R-representation σ (Prop.III.10 and Rem.III.18).

We start the proof with an R-triple $(P = MN, \sigma, Q)$ be of G with σ irreducible admissible supersingular. We show that $I_G(P, \sigma, Q)$ is irreducible. By the decomposition theorem for M, $\sigma_{R^{alg}}$ has finite length, $I_G(P, \sigma_{R^{alg}}, Q)$ also by the lattice isomorphism $\mathcal{L}_{\sigma_{R^{alg}}} \to \mathcal{L}_{I_G(P, \sigma_{R^{alg}}, Q)}$, and $I_G(P, \sigma, Q)_{R^{alg}} \simeq I_G(P, \sigma_{R^{alg}}, Q)$ by compatibility of the scalar extension with $I_G(P, -, Q)$; as the scalar extension is faithful and exact, $I_G(P, \sigma, Q)$ has also finite length. Let π be an irreducible R-subrepresentation of $I_G(P, \sigma, Q)$. As $I_G(P, \sigma, Q)$ is admissible, π is admissible. The scalar extension $\pi_{R^{alg}}$ is isomorphic to a subrepresentation of $I_G(P, \sigma, Q)_{R^{alg}} \simeq I_G(P, \sigma_{R^{alg}}, Q)$. By the lattice isomorphism $\mathcal{L}_{\sigma_{R^{alg}}} \to \mathcal{L}_{I_G(P, \sigma_{R^{alg}}, Q)}$, $\pi_{R^{alg}} \simeq I_G(P, \rho, Q)$ for a subrepresentation ρ of $\sigma_{R^{alg}}$. The representation ρ descends to R because $I_G(P, \rho, Q)$ does, by the strong compatibility of $I_G(P, -, Q)$ with scalar extension. But $\sigma_{R^{alg}}$ has no proper subrepresentation descending to R by the decomposition theorem for G, so $\rho = \sigma_{R^{alg}}$ and $\pi_{R^{alg}} = I_G(P, \sigma_{R^{alg}}, Q) \simeq I_G(P, \sigma, Q)_{R^{alg}}$, or equivalently, $\pi \simeq I_G(P, \rho, Q)$.

Next, let (P, σ, Q) and (P_1, σ_1, Q_1) be two R-triples of G with σ, σ_1 irreducible admissible supersingular and $I_G(P, \sigma, Q) \simeq I_G(P_1, \sigma_1, Q_1)$. By scalar extension $I_G(P, \sigma_{R^{alg}}, Q) \simeq I_G(P_1, (\sigma_1)_{R^{alg}}, Q_1)$. The classification theorem over R^{alg} implies $P = P_1, Q = Q_1$ and some irreducible subquotient σ^{alg} of $\sigma_{R^{alg}}$ is isomorphic to some irreducible subquotient σ_1^{alg} of $(\sigma_1)_{R^{alg}}$. As R-representations of G, σ^{alg} is σ-isotypic and σ_1^{alg} is σ_1-isotypic, hence σ, σ_1 are isomorphic.

Finally, let π be an arbitrary irreducible admissible R-representation of G. By the decomposition theorem for G, its scalar extension $\pi_{R^{alg}}$ has finite length; we choose an irreducible subrepresentation π^{alg} of $\pi_{R^{alg}}$. By the decomposition theorem for G, π^{alg} is admissible, descends to a finite extension of R. By the classification theorem over R^{alg},

$$\pi^{alg} \simeq I_G(P, \sigma^{alg}, Q)$$

for an R^{alg}-triple $(P = MN, \sigma^{alg}, Q)$ of G with σ^{alg} irreducible admissible supersingular. By the strong compatibility of $I_G(P, -, Q)$ with scalar extension, σ^{alg} descends to a finite extension of R. By the decomposition theorem for M, σ^{alg} is contained in the scalar extension $\sigma_{R^{alg}}$ of an irreducible admissible R-representation σ. By compatibility of scalar extension with supersingularity and $I_G(P, -, Q)$, (P, σ, Q) is an R-triple of G, σ is supersingular and $I_G(P, \sigma_{R^{alg}}, Q) \simeq I_G(P, \sigma, Q)_{R^{alg}}$. By the lattice isomorphism $\mathcal{L}_{\sigma_{R^{alg}}} \to \mathcal{L}_{I_G(P,\sigma_{R^{alg}},Q)}$, $I_G(P, \sigma^{alg}, Q)$ is contained in $I_G(P, \sigma_{R^{alg}}, Q)$. The irreducible representation π^{alg} is isomorphic to an irreducible subrepresentation of $I_G(P, \sigma, Q)_{R^{alg}}$. The decomposition theorem for G implies that

$$\pi \simeq I_G(P, \sigma, Q).$$

This ends the proof of the classification theorem for G (Theorem I.7).

IV. Classification theorem for $H(G)$

Let R be a field of characteristic p and G a p-adic reductive group, as in §III.3. Let I be a pro-p Iwahori subgroup of G compatible with B, $H(G)$ the pro-p Iwahori Hecke ring, $H(G)_R = R \otimes_{\mathbb{Z}} H(G)$, Z_1 the pro-p Sylow of the unique parahoric subgroup Z_0 of Z and $Z_k = Z_0/Z_1$, as in §III.4.

In this section we prove results analogous to those of Section §III but for right $H(G)_R$-modules. Although the I-invariant functor and its left adjoint relate R-representations of $H(G)$ and G, the relation in characteristic p is weaker than in the complex case and does not permit to deduce the case of the pro-p Iwahori Hecke algebra from the case of the group: similar results for $H(G)$ and G have to be proved separately.

IV.1. Pro-p Iwahori Hecke ring. The center $Z(H(G))$ of the pro-p Iwahori Hecke ring $H(G)$ is a finitely generated subring and $H(G)$ is a finitely generated module over its center; the same is true for the center of $H(G)_R$ [**VigpIwc**]. This implies that the dimension over R of a simple $H(G)_R$-module is finite [**Hn**, 2.8 Prop.].

Let $P = MN$ be a parabolic subgroup of G. The pro-p Iwahori Hecke ring $H(M)$ of M for the pro-p Iwahori subgroup $I \cap M$ does not embed in the ring $H(G)$. However we are in the good situation where $H(M)$ is a localization of a subring $H(M^+)$ (of elements supported in the positive monoid $M^+ := \{m \in M \mid m(I \cap N)m^{-1} \subset I \cap N\}$) which embeds in $H(G)$. We explain this in more detail after introducing more notations than in §III.3 and §III.4; our main reference is [**VigpIw**].

An upper or lower index M indicates an object defined for M; for G we suppress the index. We write $\mathcal{N}_M$ for the F-points of the normalizer of $\mathbf{T}$ in $\mathbf{M}$, $\mathbb{W}_M = \mathcal{N}_M/Z$, $W_M = \mathcal{N}_M/Z_1$, $W_{M'}$ for the image of $M' \cap \mathcal{N}_M$ in W_M, $\Lambda = Z/Z_1$, lg_M for the length of W_M, Ω_M for the image in W_M of the $\mathcal{N}_M$-normalizer of $(I \cap M)$; Ω_M is also the set of $u \in W_M$ of length $\mathrm{lg}_M(u) = 0$ (the group $\Omega = \Omega_G$ was introduced in §III.4).

The natural map $W_M \to (I \cap M)\backslash M/(I \cap M)$ is bijective, $W_{M'}$ is a normal subgroup W_M and a quotient of $W_{M^{is}}$ (via the quotient map $M^{is} \to M'$), and we have $W_M = W_{M'}\Omega_M, W_{M'} \cap \Omega_M = W_{M'} \cap Z_k$.

For $m\in M$ and $w=w(m)\in W_M$ image of $m_1\in \mathcal{N}_M$ such that $(I\cap M)m(I\cap M)=(I\cap M)m_1(I\cap M)$ (denoted also $(I\cap M)w(I\cap M)$), the characteristic function of $(I\cap M)m(I\cap M)$ seen as an element of $H(M)$ is written $T^M(m)$ or $T^M(w)$; we have also $T^{M,*}(m)=T^{M,*}(w)$ in $H(M)$ defined by $T^{M,*}(w)T^M(w^{-1})=[(I\cap M)w(I\cap M):(I\cap M)]$ [**VigpIw**, Prop.4.13]. For $u\in\Omega_M$, $T^{M,*}(u)=T^M(u)$ is invertible of inverse $T^M(u^{-1})$. The $\mathbb{Z}$-module $H(M)$ is free with a natural basis $(T^M(w))_{w\in W_M}$, and another basis $(T^{M,*}(w))_{w\in W_M}$, called the $*$-basis. The $\mathbb{Z}$-submodule of basis $(T^M(u)=T^{M,*}(u))_{u\in Z_k}$ is the subring $H(Z_0\cap M)$ of elements supported on Z_0. The relations satisfied by the natural basis and the $*$-basis are the braid relations for $w_1,w_2\in W_M$ such that $\lg_M(w_1w_2)=\lg_M(w_1)+\lg_M(w_2)$:

$$T^M(w_1)T^M(w_2)=T^M(w_1w_2),\quad T^{M,*}(w_1)T^{M,*}(w_2)=T^{M,*}(w_1w_2),$$

and the quadratic relations with a change of sign for $s\in W_{M'},\lg_M(s)=1$:

$$T^M(s)^2=q_s+c_sT^M(s),\quad T^{M,*}(s)^2=q_s-c_sT^{M,*}(s)$$

where $q_s=[(I\cap M)s(I\cap M):(I\cap M)]$ and $c_s\in H(Z_0\cap M')$ the subring of elements supported on $Z_0\cap M'$, satisfy the congruences $q_s\equiv 0$ modulo p and $c_s\equiv -1$ modulo the ideal of $H(Z_0\cap M')$ generated by p and $T(u)-1$ for $u\in Z_k\cap W_{M'}$ [**VigpIw**]. Both q_s and c_s do not depend on M but $\lg_M$ depends on M. The quotient map $W_{M^{is}}\to W_{M'}$ respects the length and the coefficients of the quadratic relations, the surjective natural linear map from $H(M^{is})$ to the subring $H(M')$ of elements supported on M', is a ring homomorphism sending $T^{M^{is}}(w)$ to $T^M(w')$ and $T^{M^{is},*}(w)$ to $T^{M,*}(w')$ if $w'\in W_{M'}$ is the image of $w\in W_{M^{is}}$.

The injective linear maps associated to the bases

$$T^M(m)\mapsto T(m):H(M)\xrightarrow{\theta_M^G}H(G),\quad T^{M,*}(m)\mapsto T^*(m):H(M)\xrightarrow{\theta_M^{G,*}}H(G),$$

generally do not respect the product but their restrictions to the subrings $H(M^+)$ and $H(M^-)$ (of elements supported on the inverse monoid M^- of M^+) do.

Remark IV.1. 1. For $P=MN\subset Q=M_QN_Q$, we have inclusions for $\epsilon\in\{+,-\}$:

$M^\epsilon\subset M_Q^\epsilon,\quad \theta_M^G(H(M^\epsilon))\subset\theta_{M_Q}^G(H(M_Q^\epsilon)),\quad \theta_M^{G,*}(H(M^\epsilon))\subset\theta_{M_Q}^{G,*}(H(M_Q^\epsilon))$.

2. When Δ_M and $\Delta\setminus\Delta_M$ are orthogonal, the situation is simpler. For $P_2=M_2N_2$ the parabolic subgroup of G corresponding to $\Delta\setminus\Delta_M$:

G' is the direct product of M' and of M_2', $G=MM_2'$, $W'=W_{M_2'}W_{M'}$, $W_{M_2'}\cap W_{M'}\Omega=W_{M_2'}\cap Z_k$, $W=W_{M'}W_{M_2'}\Omega$ and for $w\in W_{M'},w_2\in W_{M_2'},u\in\Omega$, $\lg(ww_2u)=\lg_M(w)+\lg_{M_2}(w_2)$. The braid and quadratic relations satisfied by $T(w)=T^G(w)$ for $w\in W_M$ are the same as for $T^M(w)$, the same is true for $T(w)$ and for M_2. Moreover, $\theta_M^G=\theta_M^{G,*}$, $M'\subset M^+\cap M^-$ and $H(M')\times H(M_2')\xrightarrow{\theta_M^G\times\theta_{M_2}^G}H(G')$ is a ring isomorphism.

IV.2. Parabolic induction $\mathrm{Ind}_P^{H(G)}$. For a parabolic subgroup $P=MN$ of G, the parabolic inductions for the pro-p Iwahori Hecke rings and for the groups

$$\mathrm{Ind}_P^{H(G)}:=-\otimes_{H(M^+),\theta_M^G}H(G):\mathrm{Mod}_R(H(M))\to\mathrm{Mod}_R(H(G)),$$
$$\mathrm{Ind}_P^G:\mathrm{Mod}_R^\infty(M)\to\mathrm{Mod}_R^\infty(G)$$

are compatible with the pro-p Iwahori invariant functor and its left adjoint: [**OV**, Prop.4.4, Prop.4.6] gives natural isomorphisms:

$$(2.1)\qquad (-)^I \circ \mathrm{Ind}_P^G \simeq \mathrm{Ind}_P^{H(G)} \circ (-)^{I\cap M},$$

$$(- \otimes_{H(G)} \mathbb{Z}[I\backslash G]) \circ \mathrm{Ind}_P^{H(G)} \simeq \mathrm{Ind}_P^G \circ (- \otimes_{H(M)} \mathbb{Z}[(I\cap M)\backslash M]).$$

The parabolic induction $\mathrm{Ind}_P^{H(G)}$ for the pro-p Iwahori Hecke rings has a right adjoint $R_P^{H(G)}$ and a left adjoint $L_P^{H(G)}$ as for the groups, [**VigpIwst**]. As $-\otimes_{H(M^+),\theta_M^G} H(G) \simeq \mathrm{Hom}_{H(M^+),\theta_M^{G,*}}(H(G),-)$ (Proposition VI.1 in the appendix below):

$$(2.2)\qquad L_P^{H(G)} \simeq - \otimes_{H(M^+),\theta_M^{G,*}} H(M), \quad R_P^{H(G)} = \mathrm{Hom}_{H(M^+),\theta_M^G}(H(M),-).$$

The right adjoint functors R_P^G and $R_P^{H(G)}$ are compatible with the pro-p Iwahori invariant functor but the left adjoint functors are not [**OV**, Cor.4.13].

Remark IV.2. For the pro-p Iwahori Hecke algebra, the left adjoint $L_P^{H(G)}$ being a localization is exact but for the group, the left adjoint L_P^G is not exact.

Proposition IV.3. *Let $P = MN, P_1 = M_1N_1$ be two parabolic subgroups of G. We have:*

(i) $R_{P_1}^{H(G)} \circ \mathrm{Ind}_P^{H(G)} \simeq \mathrm{Ind}_{P\cap P_1}^{H(M_1)} \circ R_{P\cap P_1}^{H(M)}$.

(ii) $L_{P_1}^{H(G)} \circ \mathrm{Ind}_P^{H(G)} \simeq \mathrm{Ind}_{P\cap P_1}^{H(M_1)} \circ L_{P\cap P_1}^{H(M)}$.

(iii) The parabolic induction functor $\mathrm{Ind}_P^{H(G)}$ is fully faithful.

Proof. (i) is proved for the parabolic coinduction and its right adjoint in [**Abeparind**, Prop. 5.1][8]. Using the relation between the parabolic induction and coinduction given in the appendix we get (i).

(ii) follows from (i) by left adjunction and exchanging P, P_1.

(iii) The isomorphism (i) is described in the proof [**Abeparind**, Lemma 5.2]. For $P_1 = P$, one checks that it is given by the unit $\mathrm{id} \to R_P^{H(G)} \circ \mathrm{Ind}_P^{H(G)}$ of the adjunction. Applying Remark II.8, the functor $\mathrm{Ind}_P^{H(G)}$ is fully faithful. □

IV.3. The $H(G)_R$-module $\mathrm{St}_Q^{H(G)}(\mathcal{V})$. The "trivial" representation of $H(G)$ is $\mathrm{Triv}_{H(G)} = (\mathrm{Triv}_G)^I$ where Triv_G is the trivial $\mathbb{Z}$-representation of G. Let $P = MN$ be a parabolic subgroup of G and $\mathrm{St}_P^{H(G)} := (\mathrm{St}_P^G)^I$. Put $\mathrm{Triv}_{H(G)_R} = R\otimes_{\mathbb{Z}} \mathrm{Triv}_{H(G)}$ and $\mathrm{St}_P^{H(G)}(R) := R\otimes_{\mathbb{Z}} \mathrm{St}_P^{H(G)}$; they are $H(G)_R$-modules. The $H(G)_R$-module $\mathrm{Ind}_P^{H(G)}(\mathrm{Triv}_{H(M)_R}) = \mathrm{Ind}_Q^{H(G)}(R)$ is isomorphic to $(\mathrm{Ind}_Q^G(R))^I$ (§IV.2). By [**Ly**], $\mathrm{St}_P^{H(G)}(R)$ is absolutely simple and isomorphic to the cokernel of the natural map

$$(3.3)\qquad \oplus_{P\subsetneq Q\subset G}(\mathrm{Ind}_Q^G(R))^I \to (\mathrm{Ind}_P^G(R))^I.$$

One knows that $T^*(z)$ acts trivially on $\mathrm{Ind}_P^{H(G)}(\mathbb{Z})$ and on $\mathrm{St}_P^{H(G)}$ for $z\in Z\cap M'$ [**AHenV2**, Ex.3.14].

Let $\mathcal{V}$ be a non-zero right $H(M)_R$-module, and $P_{\mathcal{V}} = M_{\mathcal{V}}N_{\mathcal{V}}$, $P(\mathcal{V}) = M(\mathcal{V})N(\mathcal{V})$ the parabolic subgroups of G corresponding to:

$$\Delta_{\mathcal{V}} = \{\alpha\in\Delta \text{ orthogonal to } \Delta_M, v = vT^{M,*}(z) \text{ for all } v\in\mathcal{V}, z\in Z\cap M'_\alpha\},$$

[8] What we call parabolic coinduction is denoted by I_P in [**Abeparind**, §4] and called parabolic induction

$\Delta(\mathcal{V}) = \Delta_M \cup \Delta_{\mathcal{V}}$ [**Abe**] [**AHenV2**, Def.4.12]. Different consequences for $M(\mathcal{V})$ of the orthogonality of Δ_M and $\Delta_{\mathcal{V}}$ are described in Remark IV.1 2.

Definition IV.4. There is a unique right $H(M(\mathcal{V}))_R$-module $e(\mathcal{V})$ equal to $\mathcal{V}$ as an R-vector space, where $T^{M(\mathcal{V}),*}(m)$ acts by $T^{M,*}(m)$ for $m \in M$ and by the identity for $m \in M'_{\mathcal{V}}$ [**AHenV2**, Def.3.8 and remark before Cor. 3.9]; we say that $e(\mathcal{V})$ is the extension of $\mathcal{V}$ to $H(M(\mathcal{V}))$ or that $\mathcal{V}$ is the restriction of $e(\mathcal{V})$ to $H(M)$.

Remark IV.5. Extension to $H(M(\mathcal{V}))$ gives a lattice isomorphism $\mathcal{L}_{\mathcal{V}} \to \mathcal{L}_{e(\mathcal{V})}$.

For $P = MN \subset Q = M_Q N_Q \subset P(\mathcal{V})$, we define similarly the extension $e_Q(\mathcal{V})$ of $\mathcal{V}$ to $H(M_Q)$. When $P \subset Q = M_Q N_Q$, we write $\mathrm{St}_P^{H(Q)} := \mathrm{St}_{P\cap M_Q}^{H(M_Q)}$.

Lemma IV.6. *Assume that Δ_M is orthogonal to $\Delta \setminus \Delta_M$ and that we have right $H(G)_R$-modules $\mathcal{X}$ extending an $H(M)_R$-module and $\mathcal{Y}$ extending an $H(M_2)_R$-module, where $P_2 = M_2 N_2$ is the parabolic subgroup of G corresponding to $\Delta \setminus \Delta_M$.*

Then, there is a structure of right $H(G)_R$-module on $\mathcal{X} \otimes_R \mathcal{Y}$ where $T^*(w)$ and $T(w)$ for $w \in W$ act diagonally, and on $\mathrm{Hom}_{\theta_{M_2}^{G,*}(H(M'_2))}(\mathcal{Y}, \mathcal{X} \otimes_R \mathcal{Y})$, where $T^*(w)$ acts by the identity for $w \in W_{M'_2}$ and by

$$(T^*(w)_{\mathcal{X}} \otimes T^*(w)_{\mathcal{Y}}) \circ - \circ (T^*(w)_{\mathcal{Y}})^{-1} \quad \text{for } w \in W_{M'}\Omega,$$

where $T^*(w)_{\mathcal{X}}$ and $T^*(w)_{\mathcal{Y}}$ are the actions of $T^*(w)$ on $\mathcal{X}$ and $\mathcal{Y}$.

Proof. For $\mathcal{X} \otimes_R \mathcal{Y}$ see [**AHenV2**, Prop.3.15, Cor.3.17].

Put $\mathcal{Z} = \mathrm{Hom}_{\theta_{M_2}^{G,*}(H(M'_2))}(\mathcal{Y}, \mathcal{X} \otimes_R \mathcal{Y})$; we check that the action $T^*(w)_{\mathcal{Z}}$ of $T^*(w)$ on $\mathcal{Z}$ for $w \in W$ defined in the lemma, respects the braid and quadratic relations (§IV.1). The braid relations follow from $W = W_{M'_2} W_{M'} \Omega$ and $T^*(w w_2 u) = T^*(w) T^*(w_2) T^*(u)$ if $w \in W_{M'}, w_2 \in W_{M'_2}, u \in \Omega$ (Remark IV.1 2). For the quadratic relations, let $s_2 \in W_{M'_2}$ and $s \in W_{M'}$ of length 1. Then $T^*(s_2)_{\mathcal{X}}, T^*(s_2)_{\mathcal{Z}}$ and $T^*(s)_{\mathcal{Y}}$ are the identity. As $-c(s_2)_{\mathcal{Z}}$ is the identity and the characteristic of R is p, $T^*(s_2)_{\mathcal{Z}}$ verifies the quadratic relation; $T^*(s)_{\mathcal{Z}}(-) = (T^*(s)_{\mathcal{X}} \otimes \mathrm{id}_{\mathcal{Y}}) \circ -$ satisfies the quadratic relation because $T^*(s)_{\mathcal{X}}$ does (§IV.1). $\square$

Assume $P \subset Q \subset P(\mathcal{V}) = G$, in particular Δ_M and $\Delta \setminus \Delta_M$ are orthogonal. We have $(\mathrm{St}_Q^G)^I = (\mathrm{St}_Q^G)^{I \cap M'_2}$ [**AHenV2**, §4.2, proof of theorem 4.7], the right $H(G)_R$-modules:

$$e(\mathcal{V}) \otimes_R \mathrm{Ind}_Q^{H(G)}(R), \quad \mathrm{St}_Q^{H(G)}(\mathcal{V}) = e(\mathcal{V}) \otimes_R \mathrm{St}_Q^{H(G)}(R),$$
$$\mathrm{Hom}_{H(M'_2)_R}(e(\mathcal{V}), \mathrm{St}_Q^{H(G)}(\mathcal{V}))$$

where $T^*(w)$ acts diagonally for $w \in W$ on the first and second ones, and for the third one, the map $\theta_{M_2}^G = \theta_{M_2}^{G,*}$ embeds $H(M'_2)$ in $H(G)$ (Remark IV.1 2), $T^*(w)$ acts by the identity for $w \in W_{M'_2}$ and by $T^*(w) \circ - \circ T^*(w)^{-1}$ for $w \in W_{M'}\Omega$ (Lemma IV.6).

From the $H(G)_R$-isomorphism

$$\mathrm{Ind}_Q^{H(G)}(e_Q(\mathcal{V})) \simeq e(\mathcal{V}) \otimes (\mathrm{Ind}_Q^G(R))^I$$

explicated in ([**AHenV2**, Prop.4.5], and the inclusion $(\mathrm{Ind}_{Q_1}^G(R))^I \subset (\mathrm{Ind}_Q^G(R))^I$ for $P \subset Q \subset Q_1$, we obtain an injective $H(G)_R$-isomorphism

$\mathrm{Ind}_{Q_1}^{H(G)}(e_{Q_1}(\mathcal{V})) \xrightarrow{\iota^G(Q,Q_1)} \mathrm{Ind}_Q^{H(G)}(e_Q(\mathcal{V}))$ and an $H(G)_R$-map

$$\oplus_{Q\subsetneq Q_1\subset G} \mathrm{Ind}_{Q_1}^{H(G)}(e_{Q_1}(\mathcal{V})) \xrightarrow{\oplus_{Q\subsetneq Q_1\subset G}\iota^G(Q,Q_1)} \mathrm{Ind}_Q^{H(G)}(e_Q(\mathcal{V})) \tag{3.4}$$

of cokernel isomorphic to $\mathrm{St}_Q^{H(G)}(\mathcal{V})$ [**AHenV2**, Cor.4.6].

PROPOSITION IV.7. *Assume $P \subset Q \subset P(\mathcal{V}) = G$.*

(i) The natural maps $e(\mathcal{V}) \to \mathrm{Hom}_{H(M'_2)_R}(\mathrm{St}_Q^{H(G)}(R), e(\mathcal{V})\otimes_R \mathrm{St}_Q^{H(G)}(R))$ and $\mathrm{Hom}_{H(M'_2)_R}(\mathrm{St}_Q^{H(G)}(R), \mathrm{St}_Q^{H(G)}(\mathcal{V}))\otimes_R \mathrm{St}_Q^{H(G)}(R) \to \mathrm{St}_Q^{H(G)}(\mathcal{V})$ are $H(G)_R$-isomorphisms.

(ii) The map $Y \mapsto Y\otimes_R \mathrm{St}_Q^{H(G)}(R) : \mathcal{L}_{e(\mathcal{V})} \to \mathcal{L}_{\mathrm{St}_Q^{H(G)}(\mathcal{V})}$ is a lattice isomorphism of inverse $X \to \{y \in e(\mathcal{V}),\ y\otimes_{\mathbb{Z}} \mathrm{St}_Q^{H(G)} \subset X\}$.

PROOF. We are in the setting of Cor. I.6 for $A = H(M'_2)_R \subset A' = H(G)_R$ (the inclusion is via $\theta_{M_2}^G = \theta_{M_2}^{G,*}$), the bases $B = (T^*(w))_{w\in W_{M'_2}}$ and $B' = (T^*(w))_{w\in W}$, the right A-module $\mathcal{V}$, and the right A'-module $V = \mathrm{St}_Q^{H(G)}(R) = e(\mathrm{St}_Q^{H(M_2)}(R))$, absolutely simple as an A-module where T^*_w for $w \in W\setminus W_{M'_2}$ (contained in $W_{M'}\Omega$) acts invertibly. □

IV.4. The module $I_{H(G)}(P,\mathcal{V},Q)$.

DEFINITION IV.8. An R-triple $(P,\mathcal{V},Q)$ of $H(G)$ consists of a parabolic subgroup $P = MN$ of G, a right $H(M)_R$-module $\mathcal{V}$, a parabolic subgroup Q of G with $P \subset Q \subset P(\mathcal{V})$. To an R-triple $(P,\mathcal{V},Q)$ of $H(G)$ is attached a right $H(G)_R$-module

$$I_{H(G)}(P,\mathcal{V},Q) = \mathrm{Ind}_{P(\mathcal{V})}^{H(G)}(\mathrm{St}_Q^{H(M(\mathcal{V}))}(\mathcal{V}))$$

isomorphic to the cokernel of the $H(G)_R$-homomorphism

$$\oplus_{Q\subsetneq Q_1\subset P(\mathcal{V})} \mathrm{Ind}_{Q_1}^{H(G)}(e_{Q_1}(\mathcal{V})) \xrightarrow{\oplus_{Q\subsetneq Q_1\subset P(\mathcal{V})}\iota^G(Q_1,Q)} \mathrm{Ind}_Q^{H(G)}(e_Q(\mathcal{V}))$$

where $\iota^G(Q_1,Q) = \mathrm{Ind}_{P(\mathcal{V})}^{H(G)}(\iota^{M(\mathcal{V})}(Q\cap M(\mathcal{V}), Q_1\cap M(\mathcal{V})))$.

We can recover $\mathrm{St}_Q^{H(M(\mathcal{V}))}(\mathcal{V})$ and $e(\mathcal{V})$ from $I_{H(G)}(P,\mathcal{V},Q)$ and $P(\mathcal{V})$:

$$\mathrm{St}_Q^{H(M(\mathcal{V}))}(\mathcal{V}) \simeq L_{H(M(\mathcal{V}))}^{H(G)}(I_{H(G)}(P,\mathcal{V},Q))) \tag{4.5}$$

by Proposition IV.3(ii) and

$$e(\mathcal{V}) \simeq \mathrm{Hom}_{H(M'_{\mathcal{V}})}(\mathrm{St}_Q^{H(M(\mathcal{V}))}(R), L_{H(M(\mathcal{V}))}^{H(G)}(I_{H(G)}(P,\mathcal{V},Q))) \tag{4.6}$$

by Proposition IV.7(i).

PROPOSITION IV.9. *Let $(P,\mathcal{V},Q)$ be an R-triple of $H(G)$ with $\mathcal{V}$ of finite length and such that for each irreducible subquotient $\mathcal{X}$ of $\mathcal{V}$, $P(\mathcal{V}) = P(\mathcal{X})$ and $I_{H(G)}(P,\mathcal{X},Q)$ is simple. Then $P(\mathcal{V}) = P(\mathcal{V}')$ for any non-zero $H(M)_R$-submodule $\mathcal{V}'$ of $\mathcal{V}$; moreover the map $\mathcal{V}' \mapsto I_{H(G)}(P,\mathcal{V}',Q) : \mathcal{L}_{\mathcal{V}} \to \mathcal{L}_{I_{H(G)}(P,\mathcal{V},Q)}$ is a lattice isomorphism.*

PROOF. $P(\mathcal{V}) = P(\mathcal{V}')$ is proved as in Proposition III.10. We are in the situation of Corollary I.6 (proof of Prop.IV.7 for $M(\mathcal{V})$ instead of G). So $\mathrm{St}_Q^{H(M(\mathcal{V}))}(\mathcal{V})$

has finite length, and its irreducible subquotients are $\mathrm{St}_Q^{H(M(\mathcal{V}))}(\mathcal{X})$ for the irreducible subquotients $\mathcal{X}$ of $\mathcal{V}$. If $I_G(P,\mathcal{X},Q) = \mathrm{Ind}_{P(\mathcal{V})}^G(\mathrm{St}_Q^{M(\mathcal{V})}(\mathcal{X}))$ is irreducible for all $\mathcal{X}$, we are in the situation of Corollary I.4 for $F = \mathrm{Ind}_{P(\mathcal{V})}^{H(G)}$ and $W = \mathrm{St}_Q^{H(M(\mathcal{V}))}(\mathcal{V})$ because $\mathrm{Ind}_{P(\mathcal{V})}^{H(G)}$ has a right adjoint and is exact fully faithful (Proposition IV.3 (iii)) so the map $\mathcal{V}' \mapsto I_G(P,\mathcal{V}',Q) : \mathcal{L}_{\mathcal{V}} \to \mathcal{L}_{I_G(P,\mathcal{V},Q)}$ is a lattice isomorphism. □

REMARK IV.10. *The scalar extension to R^{alg} of a R-triple $(P,\mathcal{V},Q)$ of $H(G)$ where $\mathcal{V}$ is simple supersingular, is an R^{alg}-triple $(P,\sigma_{R^{alg}},Q)$ of $H(G)$ satisfying the hypotheses of Proposition* IV.9, as for the group (Remark III.18). By the decomposition theorem and Lemma III.16, $\mathcal{V}_{R^{alg}}$ has finite length and its irreducible subquotients $\mathcal{X}$ are supersingular, $P(\mathcal{X}) = P(\mathcal{V}) = P(\mathcal{V}_{R^{alg}})$ (Prop.IV.12 (ii)), and $I_{H(G)}(P,\mathcal{X},Q)$ is irreducible by the classification theorem for $H(G)$ over R^{alg} (Thm.I.8 [**AHenV2**]).

We now check the compatibility of $I_{H(G)}(P,\mathcal{V},Q)$ with scalar extension, as for the group (Propositions III.12 and III.13). Let R'/R be a field extension.

PROPOSITION IV.11. *(i) The parabolic induction commutes with the scalar restriction from R' to R and with the scalar extension from R to R'. Hence the left (resp. right) adjoint of the parabolic induction commutes with scalar extension (resp. restriction).*

(ii) An $H(M)_{R'}$-module $\mathcal{V}'$ and an $H(G)_R$-module $\mathcal{X}$ such that $\mathrm{Ind}_P^{H(G)}\mathcal{V}' \simeq \mathcal{X}_{R'}$, we have $\mathcal{V}' \simeq (L_P^{H(G)}\mathcal{X})_{R'}$.

PROOF. As for the group (Proposition III.12). Note that (i) is valid for commutative rings $R \subset R'$. □

PROPOSITION IV.12. *(i) Let $(P,\mathcal{V},Q)$ be an R-triple of $H(G)$. Then $P(\mathcal{V}) = P(\mathcal{V}_{R'})$; if $\mathcal{V}$ is simple and $\mathcal{V}'$ is a subquotient of $\mathcal{V}_{R'}$, then $P(\mathcal{V}) = P(\mathcal{V}')$ and*

$$(e(\mathcal{V}))_{R'} = e(\mathcal{V}_{R'}),\ \mathrm{St}_Q^{H(M(\mathcal{V}))}(\mathcal{V})_{R'} \simeq \mathrm{St}_Q^{H(M(\mathcal{V}))}(\mathcal{V}_{R'}),$$
$$I_{H(G)}(P,\mathcal{V},Q)_{R'} \simeq I_{H(G)}(P,\mathcal{V}_{R'},Q).$$

(ii) Let $(P,\mathcal{V}',Q)$ be an R'-triple of $H(G)$ such that $e(\mathcal{V}')$, resp. $\mathrm{St}_Q^{H(M(\mathcal{V}'))}(\mathcal{V}')$, resp. $I_{H(G)}(P,\mathcal{V},'Q)$, descend to R. Then $\mathcal{V}'$ descends to R.

Precisely, if $e(\mathcal{V}')$, resp. $\mathrm{St}_Q^{H(M(\mathcal{V}'))}(\mathcal{V}')$, resp. $I_{H(G)}(P,\mathcal{V}',Q)$, is the scalar extension from R to R' of $\mathcal{X}$, resp. $\mathcal{Y}$, resp. $\mathcal{Z}$, then $\mathcal{V}'$ is the scalar extension from R to R' of the natural action of $H(M)_R$ on $\mathcal{X}$, resp. $\mathrm{Hom}_{H(M'_{\mathcal{V}'})_R}(\mathrm{St}_Q^{H(M(\mathcal{V}'))}(R),\mathcal{Y})$, resp. $\mathrm{Hom}_{H(M'_{\mathcal{V}'})_R}(\mathrm{St}_Q^{H(M(\mathcal{V}'))}(R), L_{P(\mathcal{V}')}^{H(G)}\mathcal{Z})$.

PROOF. (i) As for the group (Proposition III.13).

(ii) If $I_{H(G)}(P,\mathcal{V}',Q) = \mathcal{Z}_{R'}$ then $\mathrm{St}_Q^{H(M(\mathcal{V}'))}(\mathcal{V}') = \mathcal{Y}_{R'}$ where $\mathcal{Y} \simeq L_{H(M(\mathcal{V}'))}^{H(G)}(\mathcal{Z})$ by (i) and (4.5).

If $\mathrm{St}_Q^{H(M(\mathcal{V}'))}(\mathcal{V}') = \mathcal{Y}_{R'}$, then $e(\mathcal{V}') = \mathcal{X}_{R'}$ where

$$\mathcal{X} \simeq \mathrm{Hom}_{H(M'_{\mathcal{V}'})_R}(\mathrm{St}_Q^{H(M(\mathcal{V}'))}(R),\mathcal{Y})$$

as $e(\mathcal{V}') \simeq \mathrm{Hom}_{H(M'_{\mathcal{V}'})_{R'}}(\mathrm{St}_Q^{H(M(\mathcal{V}'))}(R'), \mathcal{Y}_{R'})$ (Prop. IV.7) and $\mathrm{St}_Q^{H(M(\mathcal{V}'))}(R') \simeq (\mathrm{St}_Q^{H(M(\mathcal{V}'))}(R))_{R'}$.

If $e(\mathcal{V}') = \mathcal{X}_{R'}$ then $T^{M(\mathcal{V}'),*}(m)$ acts trivially on $\mathcal{X}_{R'}$ for $m \in M'_{\mathcal{V}'}$ hence also on $\mathcal{X}$ and $\mathcal{V}'$ is the scalar extension to R' of $\mathcal{X}$ seen as a $H(M)_R$-module. □

IV.5. Classification of simple modules over the pro-p Iwahori Hecke algebra. As in §III.5 for the group, the classification theorem for $H(G)$ over R^{alg} (Thm.I.8) descends to R by a formal proof relying on:

1 The decomposition theorem for $H(G)$ (Thm.I.1).

2 The classification theorem for $H(G)$ over R^{alg} (Thm.I.8 [**AHenV2**]).

3 The strong compatibility of scalar extension with $I_{H(G)}(P, -, Q)$ (Prop. IV.12) and supersingularity (Lemma III.16).

4 The lattice isomorphism $\mathcal{L}_{\mathcal{V}_{R^{alg}}} \to \mathcal{L}_{I_{H(G)}(P, \mathcal{V}_{R^{alg}}, Q)}$ for the scalar extension $\mathcal{V}_{R^{alg}}$ to R^{alg} of a simple supersingular $H(M)_R$-module $\mathcal{V}$ (Prop.IV.7 and Remark IV.10).

We start the proof with an R-triple $(P, \mathcal{V}, Q)$ of $H(G)$ with $\mathcal{V}$ simple supersingular and we prove that $I_{H(G)}(P, \mathcal{V}, Q)$ is simple. By the decomposition theorem, the $H(G)_{R^{alg}}$-module $\mathcal{V}_{R^{alg}}$ has finite length, and $I_{H(G)}(P, \mathcal{V}_{R^{alg}}, Q)$ also by the lattice isomorphism $\mathcal{L}_{\mathcal{V}_{R^{alg}}} \to \mathcal{L}_{I_{H(G)}(P, \mathcal{V}_{R^{alg}}, Q)}$. Scalar extension is faithful and exact and $I_{H(G)}(P, \mathcal{V}, Q)_{R^{alg}} \simeq I_{H(G)}(P, \mathcal{V}_{R^{alg}}, Q)$ so $I_{H(G)}(P, \mathcal{V}, Q)$ has also finite length. We choose a simple $H(G)_R$-submodule $\mathcal{X}$ of $I_{H(G)}(P, \mathcal{V}, Q)$. The $H(G)_{R^{alg}}$-module $\mathcal{X}_{R^{alg}}$ is contained in $I_{H(G)}(P, \mathcal{V}, Q)_{R^{alg}}$ hence $\mathcal{X}_{R^{alg}} \simeq I_{H(G)}P, \mathcal{V}', Q)$ for an $H(M)_{R^{alg}}$-submodule $\mathcal{V}'$ of $\mathcal{V}_{R^{alg}}$ by (5.8) and the lattice isomorphism $\mathcal{L}_{\mathcal{V}_{R^{alg}}} \to \mathcal{L}_{I_{H(G)}(P, \mathcal{V}_{R^{alg}}, Q)}$. As $I_{H(G)}(P, \mathcal{V}', Q)$ descends to R, $\mathcal{V}'$ is also by the strong compatibility of $I_{H(G)}(P, -, Q)$ with scalar extension. But no proper $H(M)_{R^{alg}}$-submodule of $\mathcal{V}_{R^{alg}}$ descends to R by the decomposition theorem for $H(G)$, so $\mathcal{V}' = \mathcal{V}_{R^{alg}}$, $\mathcal{X}_{R^{alg}} = I_{H(G)}(P, \mathcal{V}_{R^{alg}}, Q)$ and $\mathcal{X}_{R^{alg}} \simeq I_{H(G)}(P, \mathcal{V}, Q)_{R^{alg}}$ by compatibility of scalar extension with $I_{H(G)}(P, -, Q)$. So $\mathcal{X} \simeq I_{H(G)}(P, \mathcal{V}, Q)$ and $I_{H(G)}(P, \mathcal{V}, Q)$ is simple.

Next, let $(P, \mathcal{V}, Q)$ and $(P_1, \mathcal{V}_1, Q_1)$ be two R-triples of $H(G)$ with $\mathcal{V}, \mathcal{V}_1$ simple supersingular and $I_{H(G)}(P, \mathcal{V}, Q) \simeq I_{H(G)}(P_1, \mathcal{V}_1, Q_1)$. The scalar extensions to R^{alg} are isomorphic $(I_{H(G)}(P, \mathcal{V}, Q))_{R^{alg}} \simeq (I_{H(G)}(P_1, (\mathcal{V}_1), Q_1))_{R^{alg}}$. The classification theorem for $H(G)$ over R^{alg} and (5.8) imply $P = P_1, Q = Q_1$ and some simple $H(M)_{R^{alg}}$-subquotient $\mathcal{V}^{alg}$ of $\mathcal{V}_{R^{alg}}$ is isomorphic to some simple $H(M)_{R^{alg}}$-subquotient $\mathcal{V}_1^{alg}$ of $(\mathcal{V}_1)_{R^{alg}}$. As $\mathcal{V}^{alg}$ is $\mathcal{V}$-isotypic and $\mathcal{V}_1^{alg}$ is $\mathcal{V}_1$-isotypic as $H(M)_R$-module, $\mathcal{V}$ and $\mathcal{V}_1$ are isomorphic.

Finally, let $\mathcal{X}$ be an arbitrary simple $H(G)_R$-module. By the decomposition theorem, the $H(G)_{R^{alg}}$-module $\mathcal{X}_{R^{alg}}$ has finite length; we choose a simple submodule $\mathcal{X}^{alg}$ of $\mathcal{X}_{R^{alg}}$. By the classification theorem over R^{alg},

$$\mathcal{X}^{alg} \simeq I_{H(G)}(P, \mathcal{V}^{alg}, Q) \tag{5.7}$$

for an R^{alg}-triple $(P = MN, \mathcal{V}^{alg}, Q)$ of $H(G)$ where $\mathcal{V}^{alg}$ is a simple supersingular $H(M)_{R^{alg}}$-module. By the decomposition theorem, $\mathcal{X}^{alg}$ descends to a finite extension of R, and also $\mathcal{V}^{alg}$ by strong compatibility of scalar extension with $I_{H(G)}(P, -, Q)$. By the decomposition theorem, $\mathcal{V}^{alg}$ is contained in the scalar extension $\mathcal{V}_{R^{alg}}$ to R^{alg} of a simple $H(M)_R$-module $\mathcal{V}$. By compatibility of scalar extension with $I_{H(G)}(P,-,Q)$ and supersingularity, $\mathcal{V}$ is supersingular, $(P,\mathcal{V},Q)$ is an R-triple of G and

$$I_{H(G)}(P,\mathcal{V}_{R^{alg}},Q)\simeq I_{H(G)}(P,\mathcal{V},Q)_{R^{alg}}. \tag{5.8}$$

We have $I_{H(G)}(P,\mathcal{V}^{alg},Q)\subset I_{H(G)}(P,\mathcal{V}_{R^{alg}},Q)$ by the lattice isomorphism $\mathcal{L}_{\mathcal{V}_{R^{alg}}}\to \mathcal{L}_{I_{H(G)}(P,\mathcal{V}_{R^{alg}},Q)}$. The decomposition theorem and $\mathcal{X}^{alg}\subset I_{H(G)}(P,\mathcal{V},Q)_{R^{alg}}$ imply

$$\mathcal{X}\simeq I_{H(G)}(P,\mathcal{V},Q).$$

This ends the proof of the classification theorem for $H(G)$ (Thm.I.8). $\square$

V. Applications

Let R be a field of characteristic p and G a reductive p-adic group as in §III.3.

V.1. Vanishing of the smooth dual. The dual of $\pi\in \mathrm{Mod}_R(G)$ is $\mathrm{Hom}_R(\pi,R)$ with the contragredient action of G, that is, $(gf)(gx)=f(x)$ for $g\in G, f\in \mathrm{Hom}_R(\pi,R), x\in\pi$. The smooth dual of π is $\pi^\vee:=\cup_K \mathrm{Hom}_R(\pi,R)^K$ where K runs through the open compact subgroups of G.

A finite dimensional smooth R-representation of G is fixed by an open compact subgroup, and its smooth dual is equal to its dual.

We prove Theorem I.9. Let R^{alg}/R be an algebraic closure and let π be a non-zero irreducible admissible R-representation π of G. By Remark II.2, $(\pi^\vee)_{R^{alg}}\subset(\pi_{R^{alg}})^\vee$. Assume that $\pi^\vee\neq 0$. Then, $(\pi^\vee)_{R^{alg}}\neq 0$, hence $(\pi_{R^{alg}})^\vee\neq 0$. We know that $\pi_{R^{alg}}$ has finite length (Thm. III.4), so $\rho^\vee\neq 0$ for some irreducible subquotient ρ of $\pi_{R^{alg}}$. By the theorem over R^{alg} [**AHenV2**, Thm.6.4], the R^{alg}-dimension of ρ is finite. The R^{alg}-dimension is constant on the $\mathrm{Aut}_R(R^{alg})$-orbit of ρ. By the decomposition theorem (Thm. III.4), the R^{alg}-dimension of $\pi_{R^{alg}}$ is finite. It is equal to the R-dimension of π. So we proved that $\pi^\vee\neq 0$ implies that the R-dimension of π is finite. $\square$

V.2. Lattice of submodules (Proof of Theorem I.10)**.**

V.2.1. We recall some properties of the I-invariant functor and of its left adjoint. Let σ be a smooth R-representation of M.

1. The parabolic induction commutes with $(-)^I$ and its left adjoint $-\otimes_{H(G)}\mathbb{Z}[I\backslash G]$ (§IV.2 (2.1)).

2. If the natural surjective $R[G]$-map (counit of the adjunction) $\sigma^{I\cap M}\otimes_{H(M,I\cap M)}\mathbb{Z}[(I\cap M)\backslash M]\to\sigma$ is an $R[M]$-isomorphism, it follows from 1 and the full faithfulness of Ind_P^G that $(\mathrm{Ind}_P^G(\sigma))^I\otimes_{H(G,I)}\mathbb{Z}[I\backslash G]$ is isomorphic to $\mathrm{Ind}_P^G(\sigma)$[9].

3. The natural $R[G]$-map $(\mathrm{Triv}_{H(G)}\otimes_{H(G)}\mathbb{Z}[I\backslash G])^I\to\mathrm{Triv}_G$ where Triv_G is the trivial R-representation of G and $\mathrm{Triv}_{H(G)}=(\mathrm{Triv}_G)^I$ [**OV**, end of the proof of Lemma 2.25].

4. $I_G(P,\sigma,Q)^I\simeq I_{H(G)}(P,\sigma^{I\cap M},Q)$ if $\sigma=\sigma_{min}$ (§III.3) and $P(\sigma)=P(\sigma^{I\cap M})$ [**AHenV2**, Thm. 4.17].

Lemma V.1. *Let σ be an irreducible admissible supersingular R-representation of M. Then $\sigma=\sigma_{min}$, $P(\sigma)=P(\sigma^{I\cap M})$, so $I_G(P,\sigma,Q)^I\simeq I_{H(G)}(P,\sigma^{I\cap M},Q)$.*

[9] One can check that the natural surjective map (counit of the adjunction) $(\mathrm{Ind}_P^G(\sigma))^I\otimes_{H(G,I)}\mathbb{Z}[I\backslash G]\to\mathrm{Ind}_P^G(\sigma)$ is an $R[G]$-isomorphism

PROOF. The equality $\sigma = \sigma_{min}$ follows from the classification (Thm.I.7) because σ is supersingular (§III.4). When $\sigma = \sigma_{min}$, then Δ_σ is orthogonal to Δ_M (§III.3). As σ being irreducible is generated by $\sigma^{I\cap M}$, $P(\sigma) = P(\sigma^{I\cap M})$ [**AHenV2**, Thm.3.13]. $\square$

5. $I_{H(G)}(P, \mathcal{V}, Q) \otimes_{H(G)} \mathbb{Z}[I\backslash G] \simeq I_G(P, \mathcal{V} \otimes_{H(M)} \mathbb{Z}[(I \cap M)\backslash M], Q)$ if $\mathcal{V}$ is a simple supersingular $H(M)_R$-module (more generally, if $P(\mathcal{V}) = P(\mathcal{V} \otimes_{H(M)} \mathbb{Z}[(I \cap M)\backslash M])$ when $\mathcal{V} \otimes_{H(M)} \mathbb{Z}[(I \cap M)\backslash M] \neq 0$) [**AHenV2**, Cor. 5.12, 5.13].

PROPOSITION V.2. *Let σ be an irreducible admissible supersingular R-representation of M such that $\sigma^{I\cap M}$ simple and the map $\sigma^{I\cap M} \otimes_{H(M)} \mathbb{Z}[(I \cap M)\backslash M] \to \sigma$ is bijective. Then,*

$\mathrm{Ind}_P^G(\sigma)$ *has multiplicity* 1 *and irreducible subquotients $I_G(P, \sigma, Q)$ for $P \subset Q \subset P(\sigma)$.*

$(\mathrm{Ind}_P^G \sigma)^I \simeq \mathrm{Ind}_P^{H(G)}(\sigma^{I\cap M})$ *has multiplicity* 1 *and simple subquotients $I_G(P, \sigma, Q)^I \simeq I_{H(G)}(P, \sigma^{I\cap M}, Q)$ for $P \subset Q \subset P(\sigma)$.*

$\mathrm{Ind}_P^{H(G)}(\sigma^{I\cap M}) \otimes_{H(G)} \mathbb{Z}[I\backslash G] \simeq \mathrm{Ind}_P^G(\sigma, Q)$ *and* $I_{H(G)}(P, \sigma^{I\cap M}, Q) \otimes_{H(G)} \mathbb{Z}[I\backslash G] \simeq I_G(P, \sigma, Q)$ *for $P \subset Q \subset P(\sigma)$.*

PROOF. This follows from the above properties 1 to 5, Lemma V.1, the classification theorems I.7, I.8 and from [**AHHV**, III.24 Prop., the proof is valid for R not algebraically closed]. $\square$

V.2.2. $\mathrm{Ind}_P^G(R)$ *and* $\mathrm{Ind}_P^{H(G)}(R)$. By [**Ly**, §9], the R-representation $\mathrm{Ind}_P^G(\mathrm{Triv}_M) = \mathrm{Ind}_P^G(R)$ of G is multiplicity free of irreducible subquotients $\mathrm{St}_Q^G(R)$ for $P \subset Q \subset G$. The $H(G)_R$-module $\mathrm{Ind}_P^{H(G)}(R) = (\mathrm{Ind}_P^G R)^I$ has a filtration with subquotients $\mathrm{St}_Q^G(R)^I = \mathrm{St}_Q^{H(G)}(R)$ for $P \subset Q \subset G$. By the classification theorem, the $\mathrm{St}_Q^{H(G)}(R)$ are simple not isomorphic. So $\mathrm{Ind}_P^{H(G)}(R)$ is multiplicity free of simple subquotients $\mathrm{St}_Q^{H(G)}(R)$ for $P \subset Q \subset G$.

Applying 1, 2 and 3 in §V.2.1, we see that $\mathrm{Ind}_P^{H(G)}(R) \otimes_{H(G)} \mathbb{Z}[I\backslash G]$ and $\mathrm{Ind}_P^G(R)$ are isomorphic; this implies that $\mathrm{St}_P^{H(G)}(R) \otimes_{H(G)} \mathbb{Z}[I\backslash G]$ and $\mathrm{St}_P^G(R)$ are also isomorphic.

We can apply Thm. I.3 (b) to the functor $F = - \otimes_{H(G)} \mathbb{Z}[I\backslash G] : \mathrm{Mod}_R(H(G)) \to \mathrm{Mod}_R(G)$ of right adjoint $G = (-)^I$, and the $H(G)_R$-module $V = \mathrm{Ind}_P^{H(G)}(R)$. So $(- \otimes_{H(G)} \mathbb{Z}[I\backslash G], (-)^I)$ give lattice isomorphisms between $\mathcal{L}_{\mathrm{Ind}_P^{H(G)}(R)}$ and $\mathcal{L}_{\mathrm{Ind}_P^G(R)}$.

For $P \subset Q \subset G$, the subrepresentation of $\mathrm{Ind}_P^G(R)$ with cosocle $\mathrm{St}_Q^G(R)$ is $\mathrm{Ind}_Q^G(R)$, and sending $\mathrm{St}_Q^G(R)$ for $P \subset Q$ to $\Delta_Q \setminus \Delta_P$ induces a lattice isomorphism from $\mathcal{L}_{\mathrm{Ind}_P^G(R)}$ onto the set of upper sets in $\mathcal{P}(\Delta \setminus \Delta_P)$; to an upper set in $\mathcal{P}(\Delta \setminus \Delta_P)$ is associated the subrepresentation $\sum_J \mathrm{Ind}_{P_{J\cup\Delta_P}}^G(R)$ for J in the upper set [**AHenV1**, Prop.3.6].

V.2.3. $\mathrm{Ind}_P^G(\mathrm{St}_Q^M(R))$ *and* $\mathrm{Ind}_P^{H(G)}(\mathrm{St}_Q^{H(M)}(R))$ *for $Q \subset P$.* This case is a direct consequence of §V.2.2 because $\mathrm{Ind}_P^G(\mathrm{St}_Q^M(R))$ is a quotient of $\mathrm{Ind}_Q^G(R)$:

$$\mathrm{Ind}_P^G(\mathrm{St}_Q^M(R)) = \mathrm{Ind}_Q^G(R) / \sum_{Q \subsetneq Q_1 \subset P} \mathrm{Ind}_{Q_1}^G(R).$$

We deduce from §V.2.2 that $\mathrm{Ind}_P^G(\mathrm{St}_Q^M(R))$ is multiplicity free of irreducible subquotients $\mathrm{Ind}_P^G(\mathrm{St}_{Q'}^G(R))$ for $Q \subset Q'$ but Q' does not contain any Q_1 such that

$Q \subsetneq Q_1 \subset P$, that is, $Q = Q' \cap P$. The subrepresentation $\mathrm{Ind}_P^G(\mathrm{St}_{Q'}^M(R))$ of $\mathrm{Ind}_P^G(\mathrm{St}_Q^M(R))$ has cosocle $\mathrm{St}_{Q'}^G$. Sending $\mathrm{St}_{Q'}^G(R)$ to $\Delta_{Q'} \cap (\Delta \setminus \Delta_P)$ gives a lattice isomorphism from $\mathcal{L}_{\mathrm{Ind}_P^G(\mathrm{St}_Q^M(R))}$ onto the lattice of upper sets in $\mathcal{P}(\Delta \setminus \Delta_P)$ (which does not depend on Q). We deduce also from §V.2.2 and Remark II.10 that $-\otimes_{H(G)} \mathbb{Z}[I\backslash G]$ and $(-)^I$ give lattice isomorphisms between $\mathcal{L}_{\mathrm{Ind}_P^{H(G)}(\mathrm{St}_Q^{H(M)}(R))}$ and $\mathcal{L}_{\mathrm{Ind}_P^G(\mathrm{St}_Q^M(R))}$.

V.2.4. $\mathrm{Ind}_P^G \sigma$ *for* σ *irreducible admissible supersingular and* $\mathrm{Ind}_P^{H(G)} \mathcal{V}$ *for* $\mathcal{V}$ *simple supersingular.* $\mathrm{Ind}_P^G \sigma$ admits a filtration with quotients $I_G(P, \sigma, Q) = \mathrm{Ind}_{P(\sigma)}^G(\mathrm{St}_Q^{M(\sigma)}(\sigma))$ for $P \subset Q \subset P(\sigma)$, and by the classification theorem the $I_G(P, \sigma, Q)$ are irreducible and not isomorphic; so $\mathrm{Ind}_P^G(\sigma)$ is multiplicity free of irreducible subquotients $I_G(P, \sigma, Q)$ for $P \subset Q \subset P(\sigma)$. The maps

$$X \mapsto e(\sigma) \otimes_R X \mapsto \mathrm{Ind}_{P(\sigma)}^G(e(\sigma) \otimes_R X) : \mathcal{L}_{\mathrm{Ind}_P^{M(\sigma)}(R)} \to \mathcal{L}_{e(\sigma)\otimes_R \mathrm{Ind}_P^{M(\sigma)}(R)} \to \mathcal{L}_{\mathrm{Ind}_P^G(\sigma)}$$

are lattice isomorphisms: this follows from the lattice theorems and the classification theorem (Thm.I.3, Thm.I.5, Thm.I.7), as in Proposition III.10 (for R algebraically closed [**AHenV1**, Prop.3.8]).

The same arguments show that $\mathrm{Ind}_P^{H(G)}(\mathcal{V})$ is multiplicity free of simple subquotients $I_{H(G)}(P, \mathcal{V}, Q)$ for $P \subset Q \subset P(\mathcal{V})$ and that the maps

$$\begin{aligned} Y \mapsto e(\mathcal{V}) \otimes_R Y \mapsto \mathrm{Ind}_{P(\mathcal{V})}^{H(G)}(e(\mathcal{V}) \otimes_R Y) : \mathcal{L}_{\mathrm{Ind}_P^{H(M(\mathcal{V}))}(R)} \\ \to \mathcal{L}_{e(\mathcal{V})\otimes_R \mathrm{Ind}_P^{H(M(\mathcal{V}))}(R)} \to \mathcal{L}_{\mathrm{Ind}_P^{H(G)}(\mathcal{V})} \end{aligned}$$

are lattice isomorphisms, by applying Thm.I.3, Thm.I.5, Thm.I.8, as in Proposition IV.9.

V.2.5. $\mathrm{Ind}_P^G(\mathrm{St}_Q^M(\sigma_1))$ *and* $\mathrm{Ind}_P^{H(G)}(\mathrm{St}_Q^{H(M)}(\mathcal{V}_1))$ *for an R-triple* (P_1, σ_1, P) *of* G, $P_1 \subset Q \subset P$, σ_1 *irreducible admissible supersingular and similarly for* $\mathcal{V}_1$. This is a direct consequence of §V.2.4 because

$$\mathrm{Ind}_P^G(\mathrm{St}_Q^M(\sigma_1)) = (\mathrm{Ind}_Q^G e_Q(\sigma_1))/(\sum_{Q \subsetneq Q_1 \subset P} \mathrm{Ind}_{Q_1}^G e_{Q_1}(\sigma_1))$$

is a subquotient of $\mathrm{Ind}_{P_1}^G(\sigma_1)$ as $e_Q(\sigma_1) \subset \mathrm{Ind}_{P_1}^{M_Q}(\sigma_1)$ and similarly for $\mathcal{V}_1$. We have $\mathrm{Ind}_{Q_1}^G e_{Q_1}(\sigma_1) \simeq \mathrm{Ind}_{P(\sigma_1)}^G(e(\sigma_1) \otimes_R \mathrm{ind}_{Q_1}^{M(\sigma_1)}(R))$, and a lattice isomorphism (§V.2.4):

$$X \mapsto \mathrm{Ind}_{P(\sigma_1)}^G(e(\sigma_1) \otimes_R X) : \mathcal{L}_{\mathrm{Ind}_{P_1}^{M(\sigma_1)}(R)} \to \mathcal{L}_{\mathrm{Ind}_{P_1}^G(\sigma_1)}$$

inducing a lattice isomorphism (Remark II.10):

$$\mathcal{L}_{\mathrm{Ind}_P^{M(\sigma_1)}(\mathrm{St}_Q^M(R))} \to \mathcal{L}_{\mathrm{Ind}_P^G(\mathrm{St}_Q^M(\sigma_1))}.$$

The R-representation $\mathrm{Ind}_P^G(\mathrm{St}_Q^M(\sigma_1))$ is multiplicity free of irreducible subquotients $I_G(P_1, \sigma_1, Q')$ for the R-triples (P_1, σ_1, Q') of G with $Q' \cap P = Q$ (§V.2.3). And similarly for $\mathcal{V}_1$ with the same arguments and references.

V.2.6. $\mathrm{Ind}_P^G \sigma$ *for* σ *irreducible admissible and* $\mathrm{Ind}_P^{H(G)} \mathcal{V}$ *for* $\mathcal{V}$ *simple.* By the classification theorem, there exists an R-triple (P_1, σ_1, Q) of G with $Q \subset P$, σ_1 irreducible admissible supersingular such that

$$\sigma \simeq I_M(P_1 \cap M, \sigma_1, Q \cap M) = \mathrm{Ind}_{P(\sigma_1)\cap M}^M(\mathrm{St}_{Q\cap M}^{M(\sigma_1)\cap M}(\sigma_1)).$$

The transitivity of the induction implies $\mathrm{Ind}_P^G \sigma \simeq \mathrm{Ind}_{P(\sigma_1)\cap P}^G(\mathrm{St}_Q^{M(\sigma_1)\cap M}(\sigma_1))$. This is the case §V.2.5 with $P(\sigma_1)\cap P$. The R-representation $\mathrm{Ind}_P^G \sigma$ of G is multiplicity free of irreducible subquotients $I_G(P_1,\sigma_1,Q')$ for the R-triples (P_1,σ_1,Q') of G with $Q'\cap P = Q$ (note that $Q'\subset P(\sigma_1), Q\subset P$). The map

$$X \mapsto \mathrm{Ind}_{P(\sigma_1)}^G(e(\sigma_1)\otimes_R X) : \mathcal{L}_{\mathrm{Ind}_{P(\sigma_1)\cap P}^{M(\sigma_1)}(\mathrm{St}_Q^{M(\sigma_1)\cap M}(R))} \to \mathcal{L}_{\mathrm{Ind}_P^G(\sigma)}$$

is a lattice isomorphism. And similarly for $\mathcal{V}$ with the same arguments and references.

V.2.7. *Invariants by the pro-p Iwahori subgroup.* We start with an irreducible admissible R-representation σ of M and we keep the notations of §V.2.6. The classification theorem shows that

$$\sigma^{I\cap M} \text{ is simple } \Leftrightarrow \sigma_1^{I\cap M_1} \text{ is simple}$$

because $\sigma^{I\cap M}\simeq I_{H(M)}(P_1\cap M,\sigma_1^{I\cap M_1},Q\cap M)$ (§V.2.1) and $\sigma_1^{I\cap M_1}$ is supersingular of finite length.

Put $\mathcal{V}_1=\sigma_1^{I\cap M_1}$, and assume first that $P(\sigma_1)=P(\mathcal{V}_1)$. In §V.2.3 we saw that the maps

$$(2.1)\qquad X\mapsto X^{I\cap M(\sigma_1)},\quad Y\mapsto Y\otimes_{H(M(\sigma_1))}\mathbb{Z}[I\cap M(\sigma_1)\backslash M(\sigma_1)]$$

between $\mathcal{L}_{\mathrm{Ind}_{P(\sigma_1)\cap P}^{M(\sigma_1)}(\mathrm{St}_Q^{M(\sigma_1)\cap M}(R))}$ and $\mathcal{L}_{\mathrm{Ind}_{P(\sigma_1)\cap P}^{H(M(\sigma_1))}(\mathrm{St}_Q^{H(M(\sigma_1)\cap M)}(R))}$, are lattice isomorphisms. They induce lattice isomorphisms between $\mathcal{L}_{\mathrm{Ind}_P^G(\sigma)}$ and $\mathcal{L}_{\mathrm{Ind}_P^{H(G)}(\mathcal{V})}$:

(2.2)

$$\mathrm{Ind}_{P(\sigma_1)}^G(e(\sigma_1)\otimes_R X)\mapsto \mathrm{Ind}_{P(\mathcal{V}_1)}^{H(G)}(e(\mathcal{V}_1)\otimes_R X^{I\cap M(\sigma_1)}),$$

(2.3)

$$\mathrm{Ind}_{P(\mathcal{V}_1)}^{H(G)}(e(\mathcal{V}_1)\otimes_R Y)\mapsto \mathrm{Ind}_{P(\sigma_1)}^G(e(\sigma_1)\otimes_R (Y\otimes_{H(M(\sigma_1))}\mathbb{Z}[(I\cap M(\sigma_1))\backslash M(\sigma_1)])).$$

by the lattice isomorphisms of §V.2.6 with $\mathcal{L}_{\mathrm{Ind}_P^G(\sigma)}$ and $\mathcal{L}_{\mathrm{Ind}_P^{H(G)}(\mathcal{V})}$.

We assume now that $\sigma^{I\cap M}$ is simple and the natural map $\sigma^{I\cap M}\otimes_{H(M)}\mathbb{Z}[(I\cap M)\backslash M]\to\sigma$ bijective, and we prove that the map $Y\mapsto Y\otimes_{H(G)}\mathbb{Z}[I\backslash G]$: $\mathcal{L}_{\mathrm{Ind}_P^{H(G)}(\sigma^{I\cap M})}\to\mathcal{L}_{\mathrm{Ind}_P^G(\sigma)}$ is a lattice isomorphism. By Lemma V.1, $P(\sigma_1)=P(\mathcal{V}_1)$. By Remark II.10, it is enough to prove it when $\sigma=\sigma_1$, that is, σ is supersingular. For that, we use Thm. I.3 (b) with $F=-\otimes_{H(G)}\mathbb{Z}[I\backslash G]:\mathrm{Mod}_R(H(G))\to\mathrm{Mod}_R(G)$ of right adjoint $(-)^I$ and $V=\mathrm{Ind}_P^G\sigma$ which satisfy the hypotheses by Prop.V.2. This ends the proof of Thm. I.10.

V.3. Proof of Theorem I.12. Proving Theorem I.12 from the classification theorem needs no new techniques. It suffices to quote for $R_{P_1}^G(\pi)$ [**AHenV1**, Corollary 6.5], for $L_{P_1}^G(\pi)$ [**AHenV1**, Cor. 6.2, 6.8], for $L_{P_1}^{H(G)}(\mathcal{X})$ and $R_{P_1}^{H(G)}(\mathcal{X})$ ([**Abeparind**, Thm. 5.20] when R is algebraically closed, but this hypothesis is not used), for π^I and $\mathcal{X}\otimes_{H(G)}\mathbb{Z}[I\backslash G]$ [**AHenV2**, Thm.4.17, Thm.5.11].

VI. Appendix: Eight inductions $\mathrm{Mod}_R(H(M))\to\mathrm{Mod}_R(H(G))$

For a commutative ring R and a parabolic subgroup $P=MN$ of G, eight different inductions $\mathrm{Mod}_R(H(M))\to\mathrm{Mod}_R(H(G))$

$$-\otimes_{H(M^\epsilon),\theta^\eta}H(G)\quad \textit{and}\quad \mathrm{Hom}_{H(M^\epsilon),\theta^\eta}(H(G),-)\quad \textit{for } \epsilon\in\{+,-\},\ \eta\in\{\ ,*\}$$

are associated to the elements of $\{\otimes, \mathrm{Hom}\} \times \{+,-\} \times \{\theta, \theta^*\}$ where $\theta := \theta_M^G$ and $\{\theta^\eta, \theta^{*\eta}\} = \{\theta, \theta^*\}$ as sets (see IV.1). The triple $(\otimes, +, \theta)$ corresponds to the parabolic induction $\mathrm{Ind}_P^{H(G)}(-) = - \otimes_{H(M^+),\theta} H(G)$ and the triple $(\mathrm{Hom}, -, \theta^*)$ corresponds to $\mathrm{Hom}_{H(M^-),\theta^*}(H(G), -)$ that we call parabolic coinduction. Before comparing these eight inductions, we define the " twist by $n_{w_G w_M}$" and the involution $\iota^M_{\mathrm{lg}-\mathrm{lg}_M}$.

Twist by $n_{w_G w_M}$. We choose an injective homomorphism $w \mapsto n_w : \mathbb{W} \to W$ from the Weyl group $\mathbb{W}$ of Δ to W satisfying the braid relations (there is no canonical choice).

Put $w_M = w_P$ for the longest element of the finite Weyl group $\mathbb{W}_M$ of M (see §IV.1), and $P^{op} = M^{op} N^{op}$ for the parabolic subgroup of G corresponding to $\Delta_{M^{op}} = \Delta_{P^{op}} = w_G w_P(\Delta_P) = w_G(-\Delta_P)$ (it is contained in Δ and is the image of Δ_P by the opposition involution $\alpha \mapsto w_G(-\alpha)$ [**T**, 1.5.1]). The conjugation $w \mapsto n_{w_G w_M} w n^{-1}_{w_G w_M} : W_M \to W_{M^{op}}$ by $n_{w_G w_M}$ is a group isomorphism inducing the ring isomorphism "twist by $n_{w_G w_M}$":

$$H(M) \to H(M^{op}), \quad T_w^M \to T^{M^{op}}_{n_{w_G w_M} w n^{-1}_{w_G w_M}} \qquad (w \in W_M)$$

sending also $T_w^{M,*})$ to $T^{M^{op},*}_{n_{w_G w_M} w n^{-1}_{w_G w_M}}$ [**Abe**, §4.3]. It restricts to an isomorphism $H(M^\epsilon) \to H(M^{op,-\epsilon})$ [**VigpIwst**, Prop.2.20], and its inverse is the twist by $n_{w_G w_{M^{op}}}$, because $n_{w_G w_{P^{op}}} = n_{w_P w_G} = n^{-1}_{w_G w_P}$.

We have the functor "twist by $n_{w_G w_M}$":

$$\mathrm{Mod}_R(H(M)) \xrightarrow{n_{w_G w_M}(-)} \mathrm{Mod}_R(H(M^{op})),$$

where the spaces of $\mathcal{V} \in \mathrm{Mod}_R(H(M))$ and $n_{w_G w_M}(\mathcal{V}) \in \mathrm{Mod}_R(H(M^{op}))$ are the same and $vT_w^M = vT^{M^{op}}_{n_{w_G w_M} w n^{-1}_{w_G w_M}}$ for $v \in \mathcal{V}, w \in W_M$.

Involution $\iota^M_{\mathrm{lg}-\mathrm{lg}_M}$ [**Abeparind**, §4.1]. The two commuting involutions ι^M and $\iota_{\mathrm{lg}-\mathrm{lg}_M}$ of the ring $H(M)$:

$(T_w^M, T_w^{M,*}) \xrightarrow{\iota^M} (-1)^{\mathrm{lg}_M(w)}(T_w^{M,*}, T_w^M)$ [**VigpIw**, Prop. 4.23],

$(T_w^M, T_w^{M,*}) \xrightarrow{\iota_{\mathrm{lg}-\mathrm{lg}_M}} (-1)^{\mathrm{lg}(w)-\mathrm{lg}_M(w)}(T_w^M, T_w^{M,*})$ [**Abeparind**, Lemmas 4.2, 4.3, 4.4, 4.5].

give by composition an involution $\iota^M_{\mathrm{lg}-\mathrm{lg}_M}$ of $H(M)$

$(T_w^M, T_w^{M,*}) \xrightarrow{\iota^M_{\mathrm{lg}-\mathrm{lg}_M}} (-1)^{\mathrm{lg}(w)}(T_w^{M,*}, T_w^M)$.

The twist by $n_{w_G w_M}$ and the involution $\iota^M_{\mathrm{lg}-\mathrm{lg}_M}$ commute, and the image of T_w^M for $w \in W_M$ by

$$n_{w_G w_M}(-) \circ \iota^M_{\mathrm{lg}-\mathrm{lg}_M} = \iota^{M^{op}}_{\mathrm{lg}-\mathrm{lg}_{M^{op}}} \circ n_{w_G w_M}(-) : H(M) \to H(M^{op})$$

is

$$(-1)^{\mathrm{lg}(n_{w_G w_M} w n^{-1}_{w_G w_M})} T^{M^{op},*}_{n_{w_G w_M} w n^{-1}_{w_G w_M}} = (-1)^{\mathrm{lg}(w)} T^{M^{op},*}_{n_{w_G w_M} w n^{-1}_{w_G w_M}}$$

(the length lg_M of W_M is invariant by conjugation by w_M, and $\mathrm{lg}(n_{w_G w_M} w n^{-1}_{w_G w_M})$ $= \mathrm{lg}(n_{w_G} n_{w_M} w n^{-1}_{w_M} n^{-1}_{w_G}) = \mathrm{lg}(n_{w_M} w n^{-1}_{w_M}) = \mathrm{lg}(w)$). By functoriality, we get a functor

$$\mathrm{Mod}_R(H(M)) \xrightarrow{(-)^{\iota^M_{\mathrm{lg}-\mathrm{lg}_M}}} \mathrm{Mod}_R(H(M)).$$

When $M = G$, we write simply ι^G.

We are now ready for the comparison of the eight inductions, which follows from different propositions in [**Abeparind**] and [**Abeinv**]. Let $\mathcal{V}$ be any right $H(M^\epsilon)_R$-module. .

PROPOSITION VI.1. *Exchanging* $+,-$ *corresponds to the twist by* $n_{w_G w_M}$*,*

$$\mathcal{V} \otimes_{H(M^\epsilon),\theta^\eta} H(G) \simeq n_{w_G w_M}(\mathcal{V}) \otimes_{H(M^{op,-\epsilon}),\theta^\eta} H(G), \tag{0.1}$$

$$\mathrm{Hom}_{H(M^\epsilon),\theta^\eta}(H(G),\mathcal{V}) \simeq \mathrm{Hom}_{H(M^{op,-\epsilon}),\theta^\eta}(H(G), n_{w_G w_M}(\mathcal{V})). \tag{0.2}$$

Exchanging θ, θ^* *corresponds to the involutions* $\iota^M_{\mathrm{lg}-\mathrm{lg}_M}$ *and* ι^G*.*

$$(\mathcal{V} \otimes_{H(M^\epsilon),\theta^\eta} H(G))^{\iota^G} \simeq \mathcal{V}^{\iota^M_{\mathrm{lg}-\mathrm{lg}_M}} \otimes_{H(M^\epsilon),\theta^{*\eta}} H(G), \tag{0.3}$$

$$\mathrm{Hom}_{H(M^\epsilon),\theta^\eta}(H(G),\mathcal{V})^{\iota^G} \simeq \mathrm{Hom}_{H(M^\epsilon),\theta^{*\eta}}(H(G),\mathcal{V}^{\iota^M_{\mathrm{lg}-\mathrm{lg}_M}}). \tag{0.4}$$

Exchanging $\otimes, \mathrm{Hom}$ *corresponds to the involutions* $\iota^M_{\mathrm{lg}-\mathrm{lg}_M}$ *and* ι^G*,*

$$(\mathcal{V} \otimes_{H(M^\epsilon),\theta^\eta} H(G))^{\iota^G} \simeq \mathrm{Hom}_{H(M^\epsilon),\theta^\eta}(H(G),\mathcal{V}^{\iota^M_{\mathrm{lg}-\mathrm{lg}_M}}). \tag{0.5}$$

REMARK VI.2. By (0.3) and (0.5), exchanging $(\otimes, \theta^\eta)$ and $(\mathrm{Hom}, \theta^{*\eta})$ respects the isomorphism class:

$$\mathcal{V} \otimes_{H(M^\epsilon),\theta^\eta} H(G) \simeq \mathrm{Hom}_{H(M^\epsilon),\theta^{*\eta}}(H(G),\mathcal{V}). \tag{0.6}$$

In Propositions $\simeq$ mean that there are natural isomorphisms described in [**Abeparind**] and [**Abeinv**]

Duality Put ζ for the anti-involution of $H(G)$ defined by $\zeta(T_w) = T_{w^{-1}}$ for $w \in W$; we have also $\zeta(T^*_w) = T^*_{w^{-1}}$ [**VigpIwst**, Remark 2.12]. The dual of a right $H(G)_R$-module $\mathcal{X}$ is $\mathcal{X}^* = \mathrm{Hom}_R(\mathcal{X}, R)$ where $h \in H(G)_R$ acts on $f \in \mathcal{X}^*$ by $(fh)(x) = f(x\zeta(h))$ [**Abeinv**, Introduction].

PROPOSITION VI.3. *The dual exchanges* $(\otimes, +)$ *and* $(\mathrm{Hom}, -)$*:*

$$(\mathcal{V} \otimes_{H(M^\epsilon),\theta^\eta} H(G))^* \simeq \mathrm{Hom}_{H(M^{-\epsilon}),\theta^\eta}(H(G),\mathcal{V}^*), \tag{0.7}$$

$$\mathcal{V}^* \otimes_{H(M^\epsilon),\theta^\eta} H(G) \simeq (\mathrm{Hom}_{H(M^{-\epsilon}),\theta^\eta}(H(G),\mathcal{V}))^*. \tag{0.8}$$

PROOF. Applying (0.6), an upper isomorphism (0.7) for any $(\epsilon, \theta^\eta, \mathcal{V})$ is equivalent to a lower isomorphism (0.8) for any $(\epsilon, \theta^\eta, \mathcal{V})$. It suffices to prove (0.7).

An isomorphism (0.7) for $(+, \theta)$ and any $\mathcal{V}$ is implicit in [**Abeinv**, §4.1]. Using (0.1) (0.2), we get an isomorphism (0.7) for $(-, \theta)$ and any $\mathcal{V}$; so we proved (0.7) for θ and any $\epsilon, \mathcal{V}$. The image by ι^G of an isomorphism (0.7) for $(\theta, \epsilon, \mathcal{V}^{\iota^M_{\mathrm{lg}-\mathrm{lg}_M}})$ is an isomorphism (0.7) for $(\epsilon, \theta^*, \mathcal{V})$, because the anti-involution ζ_M of $H(M)$ commutes with the involution $\iota^M_{\mathrm{lg}-\mathrm{lg}_M}$, and their composite sends $(T^M_w, T^{M,*}_w)$ to $(-1)^{\mathrm{lg}(w)}(T^{M,*}_{w^{-1}}, T^M_{w^{-1}})$ for $w \in W_M$, as $\mathrm{lg}(w) = \mathrm{lg}(w^{-1})$. This ends the proof of (0.7). □

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Laboratoire de Mathématiques d'Orsay, Université de Paris-Sud, CNRS, Université Paris-Saclay, Orsay cedex F-91405 France

Email address: Guy.Henniart@math.u-psud.fr

Institut de Mathématiques de Jussieu-Paris Rive Gauche, 4 Place Jussieu, Paris 75005 France

Email address: marie-france.vigneras@imj-prg.fr