The Bang-Bang theorem via Baire category. A Dual Approach

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Abstract

The paper develops a new approach to the classical bang-bang theorem in linear control theory, based on Baire category. Among all controls which steer the system from the origin to a given point $\bar{x}$, we consider those which minimize an auxiliary linear functional ϕ. For all ϕ in a residual set, we show that the minimizing control is unique and takes values within a set of extreme points.

1 Introduction

For $t \in [0,T]$, consider the Cauchy problem

$$\dot{x}(t) \in F\big(x(t)\big)\,, \tag{1.1}$$

$$x(0) \ = \ 0\,, \tag{1.2}$$

where $x \mapsto F(x) \subset I\!R^n$ is a bounded, Hausdorff continuous multifunction with compact convex values. Call $\mathcal{F} \subset \mathcal{C}\big([0,T];\, I\!R^n\big)$ the set of Carathéodory solutions of (1.1). Moreover, call $\mathcal{F}^{ext}$ the set of trajectories of

$$\dot{x}(t) \ \in \ ext\, F(x(t))\,, \tag{1.3}$$

where the time derivative takes values within the set of extreme points of $F(x)$.

If F is Lipschitz continuous w.r.t. the Hausdorff distance, starting with the seminal paper by Cellina [7], it is now well known that the set of extremal solutions $\mathcal{F}^{ext}$ is a *residual* subset of $\mathcal{F}$, i.e. it contains the intersection of countably many open dense subsets [9, 10]. By an application of Baire's theorem, this implies that the set $\mathcal{F}^{ext}$ is nonempty and every solution of (1.1) can be uniformly approximated by solutions of (1.3).

In [3] an alternative approach was developed, still based on Baire category but from a dual point of view. For every $w \in I\!R^n$, consider the compact, convex subset of vectors in $F(x)$

which maximize the inner product with w, namely

$$F^w(x) \doteq \Big\{ y \in F(x)\,; \quad \langle y, w\rangle = \max_{y' \in F(x)} \langle y', w\rangle \Big\}. \tag{1.4}$$

For each continuous path $t \mapsto w(t)$, the multifunction $F^w(t,x) \doteq F^{w(t)}(x)$ is upper semicontinuous with compact, convex values. Hence the Cauchy problem

$$\dot x(t) \in F^{w(t)}\big(x(t)\big)\,, \qquad\qquad x(0) = 0\,, \tag{1.5}$$

has a non-empty, compact set of solutions $\mathcal{F}^w \subset \mathcal{C}([0,T];\, I\!R^n)$. The main result in [3] shows that for "almost all" functions $w \in \mathcal{C}([0,T];\, I\!R^n)$, in the Baire category sense, all solutions of (1.5) are also solutions of (1.3).

Theorem 1. *Let F be a bounded, Hausdorff continuous multifunction on $I\!R^n$, with compact convex values. Then the set*

$$W \doteq \Big\{ w(\cdot)\,; \quad \mathcal{F}^w \subseteq \mathcal{F}^{ext} \Big\} \tag{1.6}$$

is a residual subset of $\mathcal{C}([0,T];, I\!R^n)$.

Notice that here the Lipschitz continuity of F is not required. Incidentally, this yields yet another proof of the classical theorem of Filippov [11], on the existence of solutions to differential inclusions with continuous, non-convex valued right hand side.

The purpose of the present paper is to explore whether a "dual" Baire category approach can be applied also to some boundary value problems (or variational problems) without convexity assumptions. The basic setting is as follows.

Consider a non-convex problem P, and let $\mathcal{S}$ be the set of solutions to a suitable convexified problem $\widehat{P}$. Under natural assumptions, $\mathcal{S}$ will be a nonempty, closed subset of a Banach space, hence a complete metric space. Moreover, one can identify a set $\mathcal{S}^{ext} \subset \mathcal{S}$ of "extremal solutions" which solve the original non-convex problem P.

- **Direct approach:** *Show that the set $\mathcal{S}^{ext}$ of extremal solutions is residual in $\mathcal{S}$.*
- **Dual approach:** *Consider a family of constrained optimization problems*

$$\min_{u(\cdot)\in\mathcal{S}} \; J^w(u), \tag{1.7}$$

where the functional J^w depends on an auxiliary function w, ranging in a Banach space W. For each $w \in W$, call $\mathcal{S}^w$ the set of minimizers. Show that the set $\{w \in W\,; \;\; \mathcal{S}^w \subseteq \mathcal{S}^{ext}\}$ is residual in W.

As a first step, we apply these ideas to derive an alternative proof of the classical bang-bang principle. Namely, consider the linear control system in $I\!R^n$

$$\dot x(t) = A(t)x(t) + B(t)u(t)\,, \qquad\qquad u(t) \in \Omega. \tag{1.8}$$

Here $A(t)$ and $B(t)$ are $n \times n$ and $n \times m$ matrices respectively, while $\Omega \subset I\!R^m$ is a compact convex set. Together with (1.8) we consider a system where the control takes values in the extreme points:

$$\dot{x}(t) \;=\; A(t)x(t) + B(t)u(t)\,, \qquad\qquad u(t) \;\in\; ext\,\Omega. \tag{1.9}$$

Given initial and terminal conditions

$$x(0) = 0, \qquad x(T) = \bar{x}, \tag{1.10}$$

assume that the boundary value problem (1.8)-(1.10) has a solution. Then by the bang-bang theorem [6, 8, 12] the problem (1.9)-(1.10) has a solution as well.

The standard proof of the bang-bang theorem relies on Lyapunov's theorem, providing the convexity of the range of a non-atomic vector measure. An alternative approach, based on Baire category, was developed in [5]. Call

$$\mathcal{S} \;\doteq\; \Big\{x : [0,T] \mapsto I\!R^n\,; \qquad x(0) = 0, \quad x(T) = \bar{x}, \qquad \dot{x}(t) = A(t)x(t) + B(t)u(t)$$
$$\text{for some measurable control } u : [0,T] \mapsto \Omega\Big\}.$$

$$\mathcal{S}^{ext} \;\doteq\; \Big\{x : [0,T] \mapsto I\!R^n\,; \qquad x(0) = 0, \quad x(T) = \bar{x}, \qquad \dot{x}(t) = A(t)x(t) + B(t)u(t)$$
$$\text{for some measurable control } u : [0,T] \mapsto ext\,\Omega\Big\}.$$

As proved in [5], one has

Theorem 2. *Let A, B be bounded, measurable, matrix-valued functions, and let $\Omega \subset I\!R^m$ be a compact convex set. Assume that $\mathcal{S} \neq \emptyset$. Then $\mathcal{S}$ is compact in $\mathcal{C}([0,T];\, I\!R^n)$ and S^{ext} is a residual subset of $\mathcal{S}$. In particular, S^{ext} is nonempty.*

In this paper we develop a "dual" approach, also based on Baire category. Let $\mathcal{U}$ be the set of all measurable functions $u : [0,T] \mapsto \Omega$ such that the solution to the Cauchy problem

$$\dot{x}(t) \;=\; A(t)x(t) + B(t)u(t), \qquad\qquad x(0) = 0 \tag{1.11}$$

satisfies the terminal condition

$$x(T) \;=\; \bar{x}\,. \tag{1.12}$$

Throughout the following, we shall assume that $\mathcal{U}$ is non-empty. Given any continuous function $w \in \mathcal{C}([0,T];\, I\!R^m)$, consider the constrained optimization problem

$$\min_{u \in \mathcal{U}} \int_0^T \Big\langle w(t)\,,\, u(t)\Big\rangle\, dt\,. \tag{1.13}$$

We show that, for "almost all" continuous functions $w \in \mathcal{C}([0,T]; I\!R^m)$, in the sense of Baire category, the problem (1.13) has a unique minimizer. This minimizer is a bang-bang control, i.e. it takes values within the set of extreme points of Ω.

Theorem 3. *Let A, B be bounded, measurable, matrix-valued functions, and let $\Omega \subset I\!R^m$ be a compact convex set. Let $\mathcal{W} \subseteq \mathcal{C}([0,T];\, I\!R^m)$ be the set of all continuous functions w*

such that the variational problem (1.13) has a unique minimizer, satisfying $u(t) \in ext\,\Omega$ for a.e. $t \in [0,T]$. Then $\mathcal{W}$ is residual in $\mathcal{C}([0,T];\, I\!\!R^m)$.

A proof of the theorem will be given in the next section. For the basic theory of multifunctions and differential inclusions we refer to the classical monograph [1]. Definitions and basic properties of Young measures, mentioned in the proof, can be found in [14, 15, 16].

2 Proof of Theorem 3

1. Given the compact convex set $\Omega \subset I\!\!R^m$, consider the function $\varphi : I\!\!R^m \mapsto I\!\!R \cup \{-\infty\}$ defined by

$$\varphi(y) \doteq \max\left\{ \int_0^1 |f(s) - y|^2\, ds \ : \quad f : [0,1] \mapsto \Omega\,, \quad \int_0^1 f(s)\, ds = y \right\},$$

with the provision that $\varphi(y) \doteq -\infty$ if $y \notin \Omega$. Otherwise stated, $\varphi(y)$ is the maximum variance among all probability measures supported on Ω, whose barycenter is at y. As proved in [2], φ is upper semicontinuous and concave on Ω. Moreover, we have the equivalence

$$\varphi(y) = 0 \qquad \Longleftrightarrow \qquad y \in ext\,\Omega. \tag{2.1}$$

A measurable control $u : [0,T] \mapsto \Omega$ takes values in $ext\,\Omega$ for a.e. t iff

$$\int_0^T \varphi(u(t))\, dt = 0\,. \tag{2.2}$$

2. The solution to the Cauchy problem (1.11) can be written as

$$x(t) = \int_0^t M(t,s) B(s) u(s)\, ds\,,$$

where $M(t,s)$ is the $n \times n$ matrix fundamental solution to the linear homogeneous system $\dot x = A(t)x$. In other words,

$$\frac{\partial}{\partial t} M(t,s) = A(t) M(t,s)\,, \qquad\qquad M(s,s) = I_n\,,$$

where I_n is the $n \times n$ identity matrix. Setting $G(s) \doteq M(T,s)B(s)$, for any $w \in \mathcal{C}([0,T];\, I\!\!R^m)$ the optimization problem (1.13) can be reformulated as follows.

(OP) *Find $u : [0,T] \mapsto \Omega$ which minimizes the integral*

$$J^w(u) \doteq \int_0^T \Big\langle w(t)\,,\, u(t) \Big\rangle\, dt \tag{2.3}$$

subject to

$$\int_0^T G(t)\, u(t)\, dt = \bar x\,. \tag{2.4}$$

3. For any $\varepsilon > 0$ and $N \geq 1$, consider the set $\mathcal{W}_{\varepsilon,N} \subseteq \mathcal{C}([0,T];\, I\!\!R^m)$ of all functions w with the following property. If

$$u(t) \;\in\; \operatorname*{argmin}_{\omega\in\Omega} \Big\langle \omega\,,\, w(t) - \lambda G(t)\Big\rangle \tag{2.5}$$

for some row vector $\lambda = (\lambda_1, \ldots, \lambda_n) \in [-N, N]^n$ and all $t \in [0,T]$, then

$$\int_0^T \varphi(u(t))\, dt \;<\; \varepsilon\,. \tag{2.6}$$

In the next two steps we will prove that each $\mathcal{W}_{\varepsilon,N}$ is open and dense, hence the intersection

$$\mathcal{W}^\sharp \;\doteq\; \bigcap_{\varepsilon>0,\, N\geq 1} \mathcal{W}_{\varepsilon,N} \tag{2.7}$$

is residual in $\mathcal{C}([0,T];\, I\!\!R^m)$.

4. We first show that each set $\mathcal{W}_{\varepsilon,N}$ is open in $\mathcal{C}([0,T];\, I\!\!R^m)$. Indeed, consider a sequence $(w_i)_{i\geq 1}$ converging uniformly to w and such that $w_i \notin \mathcal{W}_{\varepsilon,N}$ for every i. Then there exist sequences $\lambda^i \in [-N,N]^n$ and $u_i : [0,T] \mapsto \Omega$ satisfying

$$u_i(t) \;\in\; \operatorname*{argmin}_{\omega\in\Omega} \Big\langle \omega\,,\, w_i(t) - \lambda^i G(t)\Big\rangle \qquad \text{for all } t \in [0,T] \tag{2.8}$$

and

$$\int_0^T \varphi(u_i(t))\, dt \;\geq\; \varepsilon\,. \tag{2.9}$$

By taking subsequences, we can assume that $\lambda^i \to \lambda \in [-N,N]^n$ and u_i converges weakly in $\mathbf{L}^2([0,T], I\!\!R^m)$ to some function u. By the linearity of the functional in (2.5), and the convexity of the set Ω, this limit function $u(\cdot)$ satisfies (2.5), for a.e. $t \in [0,T]$.

By the upper semicontinuity and the concavity of φ in Ω it follows

$$\int_0^T \varphi(u(t))\, dt \;\geq\; \varepsilon\,. \tag{2.10}$$

Hence $w \notin \mathcal{W}_{\varepsilon,N}$, showing that $\mathcal{W}_{\varepsilon,N}$ is open in $\mathcal{C}([0,T];\, I\!\!R^m)$.

5. In this and the next two steps we will prove that each $\mathcal{W}_{\varepsilon,N}$ is dense in $\mathcal{C}([0,T];\, I\!\!R^m)$. Consider the function $\Phi : I\!\!R^m \times I\!\!R^m \mapsto I\!\!R$ defined by

$$\Phi(v,w) \;\doteq\; \max\left\{\, \varphi(u)\,;\; u \in \operatorname*{argmin}_{\omega\in\Omega}\; \langle \omega\,,\, w - v\rangle \,\right\}. \tag{2.11}$$

Observing that the map Φ is upper semicontinuous, for every $\eta > 0$ we can consider the Lipschitz continuous approximation

$$\Phi^\eta(v,w) \;\doteq\; \max\; \Big\{\Phi\,(v',w') - \eta\,|v'-v| - \eta\,|w'-w|\;;\quad (v',w') \in I\!\!R^m \times I\!\!R^m\Big\}. \tag{2.12}$$

Notice that for any $\eta' \leq \eta$ it holds

$$\Phi^{\eta'}(v,w) \;\geq\; \Phi^\eta(v,w), \qquad \text{for all } (v,w) \in I\!\!R^m \times I\!\!R^m\,. \tag{2.13}$$

For any fixed $v \in I\!R^m$, it is known that $\Phi(v, w) = 0$ for almost every $w \in I\!R^m$. Indeed (see [13]) $\Phi(v, w) = 0$ if and only if the minimum

$$\psi(w) \doteq \min_{\omega \in \Omega} \langle \omega, w - v \rangle$$

is attained at a single point. This is true if and only if the Lipschitz map $z \mapsto \psi(z)$ is differentiable at the point $z = w$. By Rademacher's theorem, this holds for a.e. $w \in I\!R^n$.

We claim that for every $\delta > 0$ and $K, M > 0$ there exists $\eta > 0$ sufficiently large so that

$$\int_{B(0,K)} \Phi^\eta(v, w)\, dw < \delta \qquad \text{for all } v \in I\!R^m,\ |v| \leq M. \tag{2.14}$$

Otherwise, let $(v_k, \eta_k)_{k \geq 1}$ be a sequence of vectors in $I\!R^m \times I\!R$ satisfying that $\lim_{k\to\infty} \eta_k = \infty$, $\lim_{k\to\infty} v_k = \bar v$ and

$$\int_{B(0,K)} \Phi^{\eta_k}(v_k, w)\, dw \geq \delta \qquad \text{for all } k \geq 1. \tag{2.15}$$

From (2.13), for any $\eta > 0$ there is $K_\eta > 0$ such that

$$\int_{B(0,K)} \Phi^\eta(v_k, w)\, dw \geq \delta, \qquad \text{for all } k \geq K_\eta.$$

The Lipschitz continuity of Φ^η implies

$$\int_{B(0,K)} \Phi^\eta(\bar v, w)\, dw = \lim_{k\to\infty} \int_{B(0,K)} \Phi^\eta(v_k, w) \geq \delta.$$

Therefore, letting $\eta \to +\infty$ we obtain

$$\int_{B(0,K)} \Phi(\bar v, w)\, dw \geq \delta. \tag{2.16}$$

This yields a contradiction because $\Phi(\bar v, w) = 0$ for almost every w.

6. Choose a constant $L > 0$ such that

$$\varphi(u) \leq L \qquad \text{for all } u \in \Omega. \tag{2.17}$$

Relying on Lusin's theorem, we can replace the measurable matrix-valued function G with a continuous function $\widehat G$ such that

$$\text{meas}\Big(\big\{t;\ \ \widehat G(t) \neq G(t)\big\}\Big) < \frac{\varepsilon}{4L}. \tag{2.18}$$

Let any radius $\rho > 0$ and any $\tilde w \in \mathcal{C}([0,T];\, I\!R^m)$ be given. By the previous step, we can find $\eta > 0$ large enough so that

$$⨍_{B(\tilde w(t),\rho)} \Phi^\eta\left(\lambda \widehat G(t),\, w\right) dw \leq \frac{\varepsilon}{4T}, \tag{2.19}$$

for every $t \in [0,T]$ and $\lambda \in [-N, N]^n$. Here and in the sequel, $⨍_S f\, dx$ denotes the average value of the function f over the set S. From (2.19) we deduce

$$\int_0^T ⨍_{B(\tilde w(t),\rho)} \Phi^\eta\left(\lambda \widehat G(t),\, w\right) dw\, dt \leq \frac{\varepsilon}{4}. \tag{2.20}$$

Consider a sequence of continuous functions $w_\nu \in \mathcal{C}([0,T];\, I\!R^m)$, weakly converging to $\tilde w$, such that

(i) $w_\nu(t) \in B(\tilde w(t), \rho)$, for all $\nu \geq 1$ and $t \in [0,T]$,

(ii) as $\nu \to \infty$, the limit is described by the family of Young measures $\{\mu_t\,;\quad t\in[0,T]\}$, where μ_t is the probability measure uniformly distributed on the ball $B(\tilde w(t), \rho)$.

Since Φ^η is continuous, this yields

$$\lim_{\nu\to\infty} \int_0^T \Phi^\eta\,(\,\lambda\,\widehat G(t)\,,\,w_\nu(t)\,)\,dt \;=\; \int_0^T \fint_{B(\tilde w(t),\rho)} \Phi^\eta\,(\,\lambda\,\widehat G(t)\,,\,w\,)\,dw\,dt\,.$$

Choosing $w = w_\nu$ for some ν sufficiently large, in view of (2.20) we obtain

$$\int_0^T \Phi^\eta\,(\,\lambda\,\widehat G(t)\,,\,w(t)\,)\,dt \;\leq\; \frac{\varepsilon}{3}\,. \tag{2.21}$$

In turn, by (2.17) and (2.18) and the obvious inequality $\Phi \leq \Phi^\eta$, this implies

$$\int_0^T \Phi\,(\,\lambda\,G(t)\,,\,w(t)\,)\,dt \;\leq\; L\,\frac{\varepsilon}{4L} + \int_0^T \Phi^\eta\,(\,\lambda\,\widehat G(t)\,,\,w(t)\,)\,dt \;\leq\; \frac{\varepsilon}{4} + \frac{\varepsilon}{3}\,. \tag{2.22}$$

Since $\|w - \tilde w\|_{\mathcal C} \leq \rho$, and $\rho > 0$ can be chosen arbitrarily small, this proves the density of $\mathcal W_{\varepsilon,N}$.

7. The existence of a sequence $(w_\nu)_{\nu\geq 1}$ satisfying the properties (i)-(ii) in the previous step follows from a standard construction. Consider a sequence of points $y_j \in B(0,\rho)$ uniformly distributed on the ball centered at the origin with radius ρ. That means

$$\lim_{\nu\to\infty} \frac{1}{\nu} \sum_{j=1}^{\nu} f(y_j) \;=\; \fint_{B(0,\rho)} f\,dx\,, \tag{2.23}$$

for every continuous function $f : I\!R^m \mapsto I\!R$. For a given $\nu \geq 1$, divide the interval $[0,T]$ into ν^2 subintervals, inserting the times $t_\ell = h\,\ell$, with $h = T/\nu^2$. Consider the piecewise continuous function

$$z_\nu(t) \;\doteq\; \tilde w(t) + y_j \qquad \text{if} \quad t \in [t_{\ell-1},\, t_\ell] \quad \text{with } \ell = m\,\nu + j \text{ for some integer } m.$$

Then choose a continuous function w_ν such that

$$\text{meas}\Big(\{t\,;\quad w_\nu(t) \neq z_\nu(t)\}\Big) \;<\; \frac{1}{\nu}\,.$$

This sequence satisfies the required properties.

8. Now assume that $w \in \mathcal W^\sharp$. We claim that the optimization problem **(OP)** has a unique solution $u : [0,T] \mapsto ext\,\Omega$. Indeed, assume that u_1, u_2 are two distinct solutions. Then $u(t) = \frac{u_1(t)+u_2(t)}{2}$ is also a solution. Set $v(t) \doteq \frac{u_1(t)-u_2(t)}{2}$ and consider the reachable subspace

$$Y \;\doteq\; \left\{ \int_0^T G(t)v(t)\theta(t)\,dt\,; \qquad \theta : [0,T] \mapsto I\!R \quad \text{measurable} \right\} \subseteq I\!R^n. \tag{2.24}$$

Consider the control system on the product space $I\!R \times Y$, with variables (y_0, y).

$$\begin{cases} \dot y_0(t) &= \langle w(t)\,,\, v(t)\rangle\,\theta(t)\,, \\ \dot y(t) &= G(t)v(t)\theta(t)\,. \end{cases} \qquad\qquad \begin{cases} y_0(0) &= 0\,, \\ y(0) &= 0\,. \end{cases}$$

Then the control $\theta^*(t) \equiv 0$ provides an optimal solution to the problem

$$\text{minimize:}\ \ y_0(T)$$

$$\text{subject to} \qquad \theta(t) \in [-1,1]\,, \qquad y(T) = 0\,.$$

The necessary conditions for optimality yield the existence of a nonzero vector $\lambda = (\lambda_0, \lambda) \in I\!R \times Y$ such that

$$\theta^*(t) \;=\; \arg\min_{\theta\in[-1,1]} \Big(\lambda_0\langle w(t), v(t)\rangle - \langle \lambda,\, G(t)v(t)\rangle\Big)\theta$$

for a.e. $t \in [0,T]$. If $\lambda_0 = 0$, then the non-zero vector $\lambda \in Y$ has the property

$$\Big\langle \lambda,\, G(t)v(t)\Big\rangle \;=\; 0 \qquad\qquad \text{for a.e. } t \in [0,T]\,.$$

But this contradicts the definition (2.24) of Y.

If $\lambda_0 \neq 0$, by a normalization we can assume $\lambda_0 = 1$. Then there exists a vector $\lambda \in Y$ such that

$$\Big\langle w(t),\, v(t)\Big\rangle - \Big\langle \lambda,\, G(t)v(t)\Big\rangle \;=\; 0 \qquad\qquad \text{for a.e. } t \in [0,T]\,.$$

This implies

$$u(t) \;\in\; \operatorname*{argmin}_{\omega\in\Omega} \Big\langle \omega\,,\, w(t) - \lambda G(t)\Big\rangle$$

while

$$\int_0^T \varphi(u(t))\,dt \;\geq\; \int_0^T |v(t)|^2\,dt \;>\; 0.$$

This contradicts the assumption $w \in \mathcal{W}^\sharp$. We thus conclude that $\mathcal{W}^\sharp \subseteq \mathcal{W}$. By (2.7), $\mathcal{W}^\sharp$ is residual, hence the same is true for $\mathcal{W}$. □

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