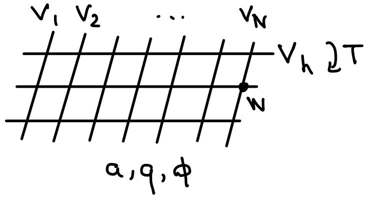
I. History & Backgrounds

1) 1970s



Baxter : integrable lattice model

to solve $\sum_{\text{state}} \prod_{\text{particle}} W$, $W = e^{-\beta E}$

⇝ to solve eigenvalues of T

Q : explicit , $T_a Q_a = e^{-i\phi} Q_{aq} + e^{i\phi} Q_{aq^{-1}}$

Sibuya, Voros : TQ & QQ-systems in ODE.

2) Quantum Affine Algebras (QAA) ?

W : solution of Yang-Baxter eqn. $W = R$

$U_q(\hat{\mathfrak{g}})$: $RTT = TTR$

e.g. 6-vertex model $\to U_q(\hat{sl}_2)$, $\dim(V_i) = \dim(V_h) = 2$

$$T = \mathrm{tr}_{V_h}(R_{V_1 V_h} \cdots R_{V_N V_h}) : V_1 \otimes \cdots \otimes V_N \circlearrowleft$$

3) Bazhanov-Lukyanov-Zamolodchikov, 1994~

$Q = \mathrm{tr}_V(R_{V_1 V} \cdots R_{V_N V})$, some space V

CFT / $\hat{sl}_2$ ⟷ 1-d Schrödinger operator

4) q-character & Grothendieck ring of $U_q(\hat{\mathfrak{g}})$

Frenkel-Reshetikhin, 1999 : q-character χ_q

$$\chi_q(V) = \sum_\gamma \mathrm{mult}(\gamma) \cdot [\gamma] \quad , \gamma \text{ generalized eigenvalue of } U_q\tilde{\mathfrak{h}}$$

$$= \pi_{H\text{-}C} \circ \mathrm{tr}_{V(z)\otimes \cdot}(R) \quad (= \text{eigenvalue of } T.)$$

$R \in U_q\mathfrak{b} \hat{\otimes} U_q\mathfrak{b}^-$, $\pi_{H\text{-}C} : U_q\mathfrak{b}^- \longrightarrow U_q\tilde{\mathfrak{h}}$

$V_h = V_a$ "fundamental rep."

$\chi_q : K_0(\mathcal{C}) \longrightarrow \mathcal{E}$ injective $\Rightarrow$ relations in $K_0(\mathcal{C})$

Kuniba-Nakanishi-Suzuki , Frenkel-Hernandez, 2015

Q-operator : $K_0(\mathcal{O})$ of $U_q(\mathfrak{b})$, $V = L_a^+$ "prefundamental rep"

Hernandez-Jimbo

Frenkel-Hernandez, 2018 $[\frac{\alpha_i}{2}] Q_{i,aq^{-1}} \tilde{Q}_{i,aq} - [-\frac{\alpha_i}{2}] Q_{i,aq} \tilde{Q}_{i,aq^{-1}} = \prod_{j \sim i} Q_{j,a}$

5) Langlands duality

BLZ, Dorey-Dunning-Tateo : $\hat{sl}_2$-IM ⟷ $\hat{sl}_2$-ODE

general $\mathfrak{g}$: $U_q(\hat{\mathfrak{g}})$ ⟷ ${}^L\hat{\mathfrak{g}}$-opers

Feigin, Frenkel, Hernandez, Masoero, Raimondo, Valeri, etc.

· when $\mathfrak{g}$ simply-laced, $\hat{\mathfrak{g}}^L = \hat{\mathfrak{g}} = A^{(1)}, D^{(1)}, E^{(1)}$

· when $\mathfrak{g}$ not simply-Laced, $\hat{\mathfrak{g}}$ non-twisted $B^{(1)}, C^{(1)}, F^{(1)}, G_2^{(1)}$

$\hat{\mathfrak{g}}^L$ twisted affine $A^{(2)}, D^{(2)}, E^{(2)}, D_4^{(3)}$

Hope. $U_q(\hat{\mathfrak{g}}^L) \leftrightsquigarrow \hat{\mathfrak{g}}$-opers.

↳ Masoero, etc ?

↳ twisted quantum affine alg.

II. Maths

$\mathfrak{g}$ finite dim^l simple Lie alg. $/\mathbb{C}$, simply laced, q generic

$U_q(\hat{\mathfrak{g}})$ non-twisted QAA: Drinfeld's generators $x_i^\pm(z)$, $h_i(z), k_i^\pm$

$U_q(\hat{\mathfrak{g}}) \simeq U_q^- \otimes U_q^0 \otimes U_q^+$, $\phi_i^\pm(z) = k_i^\pm \exp(\pm(q-q^{-1}) h_i^\pm(z)) \in U_q^0$

"ℓ-highest weight modules" $L(\gamma) = U_q^- . v_0$, $\gamma = (\gamma_i(z))_{i\in I}$

$$\phi_i^\pm(z).v_0 = \gamma_i(z).v_0$$

$U_q(\mathfrak{b}) \subset U_q(\hat{\mathfrak{g}})$, $\mathcal{O}$: cat. of $U_q(\mathfrak{b})$-modules that are in $\mathcal{O}$ when seen as $U_q(\mathfrak{h})$-modules.

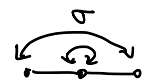
$U_q(\mathfrak{b}) = \langle k_i^\pm, e_i \rangle$

<u>Thm</u> (Hernandez-Jimbo): simples in $\mathcal{O}$ are ℓ-highest weight modules $L(\gamma)$ s.t. $\gamma_i(z)$ rational.

<u>Ex</u>. $\gamma(z) = (1, \cdots, 1, 1-az, 1, \cdots, 1) =: \psi_{i,a}$

$L_{i,a}^+ := L(\psi_{i,a})$ positive prefundamental.

$\chi_q(L_{i,a}^+)$ is simple $\Rightarrow$ Baxter's explicit construction.



A_3

—(twisted)

When $\mathfrak{g}$ is simply-laced, its Dynkin diagram has non-trivial autom. σ

$U_q(\hat{\mathfrak{g}}^\sigma)$ twisted QAA: $x^\pm_{\sigma(i)}(z) = x_i^\pm(\omega z)$, $h_{\sigma(i)}(z) = h_i(z)$

w/ modified commuting relations. ω: root of 1 of order $|\sigma|$

$U_q(\mathfrak{b}^\sigma) \subset U_q(\hat{\mathfrak{g}}^\sigma)$ Borel subalg.

$\chi_q^\sigma : K_0(\mathcal{O}) \to \mathcal{E}^\sigma$ comm.

<u>Thm</u> (W.) simples in $\mathcal{O}(U_q \mathfrak{b}^\sigma)$ are ℓ-highest weight modules $L(\gamma)$ s.t. $\gamma_i(z)$ rational, $i \in I$, and $\gamma_{\sigma(i)}(z) = \gamma_i(\omega z)$

· From non-twisted to twisted?

folding of characters: $\gamma_i(z) \longmapsto \prod_{k=1}^{|\sigma|} \gamma_{\sigma^k(i)}(\omega^k z) =: \gamma_i^\sigma$

<u>ex</u>: A_3. $\gamma = (\gamma_1(z), \gamma_2(z), \gamma_3(z))$

$$\downarrow$$

$$\gamma^\sigma = (\gamma_1(z)\gamma_3(-z), \gamma_2(z)\gamma_2(-z), \gamma_3(z)\gamma_1(-z))$$

$[\gamma] \mapsto [\gamma^\sigma]$ induces folding on characters.

Thm (Hernandez) If S is an element in $\mathrm{Im}(\chi_q(K_0(\mathcal{C}(U_q\hat{\mathfrak{g}}))))$,

then S^σ is an element in $\mathrm{Im}(\chi_q^\sigma(K_0(\mathcal{C}(U_q\hat{\mathfrak{g}}^\sigma))))$.

Non-thm: When $S=\chi_q(V)$, we don't know if $S^\sigma = \chi_q^\sigma(W)$ for some representation W of $U_q\hat{\mathfrak{g}}^\sigma$.

↳ Partial response: **Thm** (Hernandez) True for KR-modules $L(\gamma)$

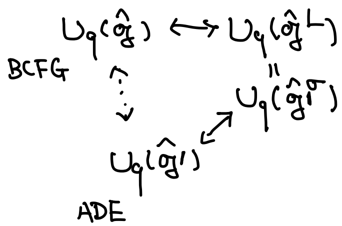
i.e. $\gamma = \left(1,\cdots,1,q^N\frac{1-q^{-1}az}{1-q^{2N-1}az},1,\cdots,1\right)$. Moreover,

$W = L(\gamma^\sigma)$.

Conj. When $\gamma_i(z)$ has its poles & zeros in $q^{\mathbb{Z}}$, then the folding of $\frac{\chi_q(L(\gamma))}{\chi(L(\gamma))}$ equals to $\frac{\chi_q^\sigma(L(\gamma^\sigma))}{\chi^\sigma(L(\gamma^\sigma))}$.

Here χ, χ^σ are usual charaters w.r.t. $U_q\mathfrak{h}$.

Importance:
- to establish results for twisted QAA.
- to study Frenkel–Hernandez's interpolating duality
- to study type BCFG QAA.



Partial response (W):

Thm: true for $L(\gamma)$ prefundamental & for $\gamma = \psi_{i,a}^{-1}\prod_{j\sim i}\psi_{j,aq}$.

Cor: $Q_{i,a} = \frac{[L^+_{i,a}]}{\chi(L^+_{i,a})}$, $\tilde{Q}_{i,a} = \frac{[L(\psi_{i,a}^{-1}\prod\psi_{j,aq})]}{\chi(L(\cdots))}$

We can construct TQ & $Q\tilde{Q}$-systems for $U_q(\hat{\mathfrak{g}}^\sigma)$.

Ex: $Q\tilde{Q}$ for $A_3^{(1)} \Rightarrow Q\tilde{Q}$ for $A_3^{(2)}$: at $i=\bar{2}$

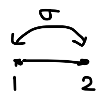
$$[\tfrac{\alpha_{\bar{2}}}{2}]Q_{\bar{2},aq^{-1}}\tilde{Q}_{\bar{2},aq} - [-\tfrac{\alpha_{\bar{2}}}{2}]Q_{\bar{2},aq}\tilde{Q}_{\bar{2},aq^{-1}} = Q_{\bar{1},a}Q_{\bar{1},-a}$$

TQ for $A_2^{(1)} \Rightarrow$ TQ for $A_2^{(2)}$:

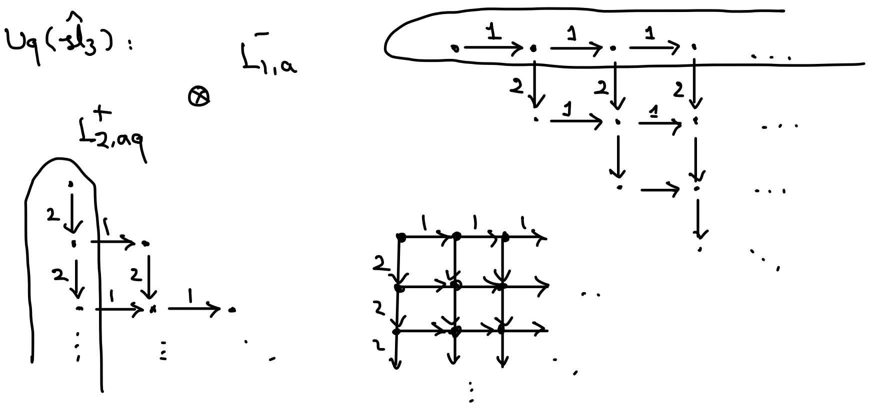
$$T_aQ_{aq}Q_{-aq^2} = [\tfrac{\alpha}{2}]Q_{aq^{-1}}Q_{-aq^2} + Q_{aq^3}Q_{-a} + [-\tfrac{\alpha}{2}]Q_{aq}Q_{-aq^4}$$

Prop:
- The condition on poles & zeros is necessary.
- The denominator & numerator are not respectively equal.

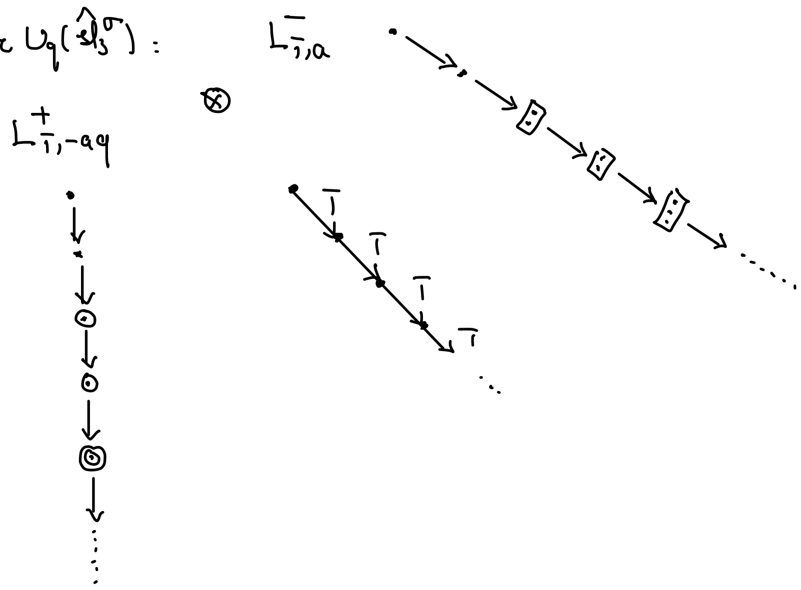
Ex: $\mathfrak{g} = A_2 = sl_3$. $\hat{\mathfrak{g}} = A_2^{(1)} = \hat{sl}_3$, $\hat{\mathfrak{g}}^\sigma = A_2^{(2)} = \hat{sl}_3^\sigma$



For $U_q(\hat{sl}_3)$:



For $U_q(\hat{\mathfrak{sl}}_3^{\sigma})$:



$\chi(L(\gamma)) \neq \chi^{\sigma}(L(\gamma^{\sigma}))$.