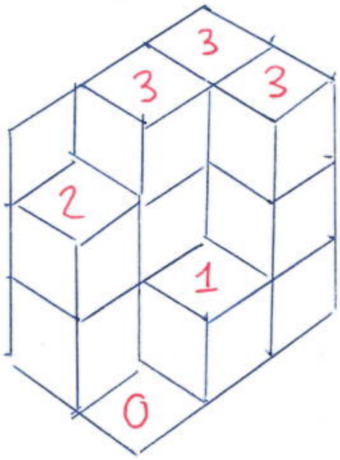
Lozenge tilings, plane partitions and the RSK algorithm.

Let us consider again lozenge tilings of an hexagon:
alias "piles of cubes"



In red we record the heights of the piles of cubes.
View these numbers from the top, with
the highest pile placed in the top left corner:

$$\begin{matrix} 3 & 3 \\ 3 & 1 \\ 2 & 0 \end{matrix}$$

This is a <u>plane partition</u>.

Let's give a formal definition:

a plane partition is an array of numbers with finite support $\pi = (\pi_{ij})_{i,j\geq 1}$
with $\pi_{ij} \in \mathbb{N} = \{0,1,2,3,\dots\}$
such that $\pi_{ij} \geq \pi_{i+1,j}$, $\pi_{ij} \geq \pi_{i,j+1}$ for all $i,j \geq 1$.

To make the connection with lozenge tilings of an $a \times b \times c$ hexagon:

- set $\pi_{ij} = 0$ for $i > a$ or $j > b$.
- assume $\pi_{11} \leq c$.

these are "boxed" plane partitions.

But let us here consider general plane partitions.

<u>Volume</u> $|\pi| := \sum_{i,j \geq 1} \pi_{ij} < \infty$ (by the finite support assumption).

<u>Theorem</u> (MacMahon): the "q-volume" generating function of plane partitions reads:

$$M_q = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^n} \cdot \quad \Big(= \sum_{\substack{\pi \\ \text{plane} \\ \text{partition}}} q^{|\pi|} \Big)$$

This is actually a consequence of the MacMahon formula for the q^{volume} g.f. of lozenge tilings (boxed plane partitions) we have seen before:

$$M_q(a,b,c) = \prod_{i=1}^{a} \prod_{j=1}^{b} \prod_{k=1}^{c} \frac{1 - q^{i+j-1}}{1 - q^{i+j-2}}.$$

just take $a, b, c \to \infty$ (makes sense formally and analytically for $|q| < 1$).

It is interesting to compare the MacMahon generating function to that of (integer, 1D) partitions.

Def: a partition is a sequence of numbers $(\lambda_i)_{i\geq 1}$ with finite support, $\lambda_i \in \mathbb{N}$, such that $\lambda_i \geq \lambda_{i+1}$ for all $i \geq 1$

Theorem (Euler?)? Let the size of a partition λ be $|\lambda| := \sum_{i\geq 1} \lambda_i$ then

$$\sum_{\lambda \text{ partition}} q^{|\lambda|} = \prod_{n\geq 1} \frac{1}{1-q^n}.$$

Remarks: • integer partitions = 1D partitions = 2D Diagrams (Young)
plane partitions = 2D partitions. = 3D diagrams. (cubes.)

It is tempting to conjecture that 4D diagrams have a generating function like $\prod_{n\geq 1} \frac{1}{(1-q^n)^{n^2}}$
but this is too good to be true.
I think Nekrasov found some identities for 4D diagrams but never looked at the details.

• The inverse of the partition g.f. is the Euler function

$$\phi(q) = \prod_{n\geq 1} (1-q^n)$$

which has many interesting number-theoretical properties (connection with modular forms, etc.)

- Asymptotics (for the analytic combinatorics fans)

$$[q^N] \prod_{n\geq 1} \frac{1}{1-q^n} \underset{N\to\infty}{\sim} \frac{1}{4N\sqrt{3}} e^{\pi\sqrt{2N/3}}.$$

(Ramanujan)

$$[q^N) \, \mathrm{TM}_q \sim C_1 N^{-25/36} e^{C_2 N^{2/3}}. \qquad (?)$$

a general method for handling asymptotics of coefficients of series of the form

$$f(q) = \prod_{n\geq 1} \frac{1}{(1-q^n)^n}$$

is Meinardus' method, see the Analytic combinatorics textbook by Flajolet and Sedgewick.

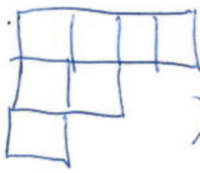
<u>NB:</u> the exponents $N^{1/2}$, $N^{2/3}$ appearing in the exponentials can also be predicted by physical arguments. (entropy...)

But ~~let~~ me now present another route to the MacMahon g.f. M_q that uses the RSK (Robinson-Schensted-Knuth) algorithm.

First, we need some definitions.

Let λ, μ be two partitions. (representable by Young diagrams) aka "shapes"

Write $\mu \subset \lambda$ if $\mu_i \leq \lambda_i$ for all i.



$\lambda = (4,3,1,0,\ldots)$

λ/μ is then called a "skew" shape.

We say that λ, μ are <u>interlaced</u> if

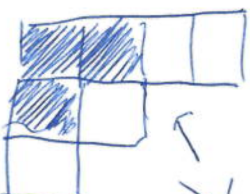
$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots$$

and ~~we~~ then write $\lambda \succ \mu$.

In terms of Young diagrams, this says that λ/μ is a "horizontal strip": it does not contain two boxes on top of each other.

<u>Ex:</u> $\lambda = (4,2,1)$

$\mu = (2,1,0)$



λ/μ (white boxes).

Plane partitions and sequences of interlaced partitions

Given a plane partition π, define a sequence of integer partitions $(\lambda^{(k)})_{k \in \mathbb{Z}}$ by

$$\lambda^{(k)}_i = \begin{cases} \pi_{i,\, i+k} & \text{if } k \geq 0 \\ \pi_{i+|k|,\, i} & \text{if } k \leq 0 . \end{cases}$$

<u>Observation</u>: we have:

$$\cdots \preceq \lambda^{(-2)} \preceq \lambda^{(-1)} \preceq \lambda^{(0)} \succeq \lambda^{(1)} \succeq \lambda^{(2)} \succeq \cdots$$

with $\lambda^{(k)} = \emptyset := (0,0,\ldots)$ (empty partition) for $|k|$ large enough.

<u>Semi standard Young tableaux</u>:

Let λ be a partition. A semistandard Young tableau of shape λ is a filling of the boxes of the Young diagram of λ by positive integer numbers, that is weakly increasing along rows and strictly increasing along columns.

<u>Ex</u>: $\lambda = (4,2,1)$
$$\begin{array}{|c|c|c|c|} \hline 1 & 1 & 2 & 3 \\ \hline 3 & 3 \\ \cline{1-2} 4 \\ \cline{1-1} \end{array}$$

Observation: a SSYT of shape λ corresponds to a sequence of interlaced partitions

$$\phi = \mu^{(0)} \prec \mu^{(1)} \prec \mu^{(2)} \prec \cdots \quad \text{such that } \mu^{(k)} = \lambda$$

for k large enough.

($\mu^{(k)}/\mu^{(k-1)}$ are the boxes filled by the integer k, they must form a horizontal strip).

NB: a standard Young tableau (SYT) of shape λ is a SSYT in which we assume the entries to be all distinct and equal to $1, 2, 3, \ldots, |\lambda|$.

Ex:

1	2	4	6
3	5		
7			

Schur functions (combinatorial definition)

Let $x_1, x_2, x_3 \ldots$ be formal variables.

Define the weight of a SSYT T

$$x^T := \prod_{i \geq 1} x_i^{\#\{\text{entries equal to } i \text{ in } T\}}.$$

The Schur function $s_\lambda(x_1, \ldots x_n)$ is then defined as

$$s_\lambda(x_1, x_2 \ldots) = \sum_{\substack{T \text{ SSYT of} \\ \text{shape } \lambda}} x^T.$$

Ex: $s_{(1)} = x_1 + x_2 + x_3 + \ldots$

Exercise show that $s_\lambda(x_1, x_2, \ldots)$ is symmetric in its variables

Hint: it is sufficient to prove the symmetry in x_k and x_{k+1} for all $k \geq 1$. This may be done by the Bender-Knuth involution ...

Back to plane partition plane partitions are in bijection with pairs of SSYT having the same shape (combine the previous observations, shape is $\lambda^{(0)}$ ...).

Corollary we have $M_q = \sum_{\lambda \text{ partition}} s_\lambda(q^{\frac{1}{2}}, q^{\frac{3}{2}}, \ldots) s_\lambda(q^{\frac{1}{2}}, q^{\frac{3}{2}}, \ldots)$

(needs a slight argument to relate volume of plane partitions with the entries of the SSYT's ..).

The MacMahon is then a corollary of the more general result.

Theorem (Cauchy identity) let $X = (x_1, x_2, \ldots)$, $Y = (y_1, y_2, \ldots)$ be two "alphabets" (collections of formal variables).
Then we have:

$$\sum_{\lambda \text{ partition}} s_\lambda(\underbrace{x_1, x_2, \ldots}_{X}) s_\lambda(\underbrace{y_1, y_2, \ldots}_{Y}) = \prod_{i,j \geq 1} \frac{1}{1 - x_i y_j}$$

Theorem (RSK) There exists a bijection between the set $(\mathbb{N})^{(\mathbb{N}^*)^2}$ of 2D arrays of natural numbers

$$m = (m_{ij})_{i,j \geq 1}$$

and the set of triples (λ, P, Q) where λ is an integer partition and P, Q are SSYT's of shape λ, such that if $m \mapsto (\lambda, P, Q)$ then:

- (row sum condition) $\sum_{j \geq 1} m_{ij} = \#\{\text{entries equal to } i \text{ in } P\}$ for all i
- (column sum condition) $\sum_{i \geq 1} m_{ij} = \#\{\text{——— } j \text{ in } Q\}$ for all j

Gives the Cauchy identity since

$$\sum_\lambda s_\lambda(X) s_\lambda(Y) = \sum_{(\lambda,P,Q)} x^P y^Q \overset{RSK}{=} \sum_m \prod_{i\geq 1} (x_i)^{\sum_{j\geq 1} m_{ij}} \prod_{j \geq 1} (y_j)^{\sum_{i\geq 1} m_{ij}}$$

$$= \sum_m \prod_{i,j\geq 1} (x_i y_j)^{m_{ij}} = \prod_{i,j \geq 1} \frac{1}{1 - x_i y_j}.$$

Remark: RS case: m is a permutation matrix.
(one entry 1 in each row/col $1 \dots N$, all other zero)
$\leftrightarrow$ P, Q are standard Young tableaux.

Some of the many
Descriptions of RSK algorithm
- insertion tableau / recording tableau (simplest in RS, needs "biwords" for RSK).
- Fomin's growth diagram construction.
- Viennot's shadow lines (RS) extended by Fulton(?) to RSK
- Box ball system