MPRI course 2.10 Algorithmic aspects of combinatorics

Exam 1 — Friday 29 November 2024 — duration: 2,5 hours
Written by J. Bouttier and G. Chapuy

Please write your name on every sheet of paper you return to us. You can answer questions in either English or French.

Exercise 1 Peaks in m-ary trees

For $m \geq 2$, an m-ary tree is a rooted plane tree having only vertices of arity zero (*leaves*) or arity m (*nodes*). The *size* of a tree is its number of nodes. A node is a *peak* if its leftmost child is a leaf.

1. Explain why it is natural to use the word *peak*.

2. Let $T(z, u)$ be the generating function of m-ary trees, where z marks the size, and u the number of peaks. Write an equation for $T(z, u)$. Check that for $u = 1$ you recover the usual equation for m-ary trees.

3. Express the coefficient of z^n in $T(z, u)$ as a coefficient extraction using a tool seen in class.

4. Extract the coefficient of u^k in the previous formula, and finish the computation, to obtain an explicit formula for the number of m-ary trees of size n having k peaks. What formula do you recognize for $m = 2$?

5. (bonus, will be marked only if you did a substantial amount of other exercises) Give a bijective proof of this result.

Exercise 2 Generalized Temperley bijection

(Adapted from the paper *Trees and perfect matchings* by R. Kenyon, J. Propp and D. Wilson.)

Let G be a connected plane graph, which is unoriented for now. We denote by V, E, F its sets of vertices, faces and faces, respectively. Let G^* denote the dual of G.

1. Given a spanning tree T of G, recall how its dual tree T^* is defined. (By the way, why does G always admit a spanning tree?)

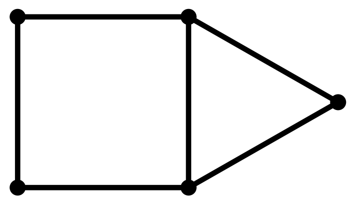
2. What is the relation between the numbers of vertices and edges of a tree?

3. By applying the result of the previous question to both T and T^*, deduce Euler's relation

$$|V| - |E| + |F| = 2.$$

Now, let G' denote the plane graph obtained by superimposing G and G^*. Since the edges of G and their duals cross each other, we create a new (tetravalent) vertex at their intersection. Observe that the vertex set of G' is in bijection with $V \cup E \cup F$.

4. Illustrate this construction by drawing the G' associated with the following G:

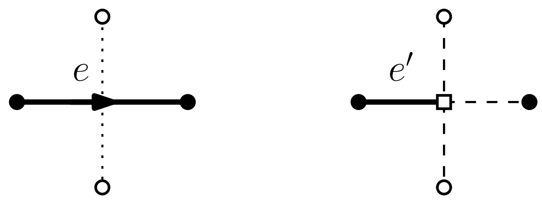


5. Returning to the general case, explain why G' does not admit any dimer configuration.

Let v (resp. f) be a vertex (resp. face) of G, such that v and f are incident to each other in G. Let v^* be the vertex of G^* corresponding to f. Let H be the plane graph obtained from G' by removing v, v^* and all their incident edges in G'.

6. Update the illustration made at question 4 by drawing H, when f is the outer face of the displayed graph, and v the vertex at the top-right corner of the square.

Let us denote by $\mathcal{T}$ the set of spanning trees of G, and by $\mathcal{D}$ the set of dimer configurations on H. Our purpose is to show that there is a bijection between $\mathcal{T}$ and $\mathcal{D}$. Let us consider a spanning tree T of G, and let T^* be its dual tree. We orient all the edges of T (resp. T^*) towards v (resp. v^*). To each edge e of T or T^*, we associate an edge e' of G' in this way:



Namely, e' is obtained by "shortening" e by a half: it remains incident to the origin of e, but its other endpoint is now the intersection of e with its dual edge. Let D denote the set of all edges e' obtained in this way.

7. Illustrate this construction by drawing your favorite spanning tree of the graph of question 4, and then by drawing the corresponding dimer configuration.

8. Show that, in general, D is a dimer configuration on H. Thus, $T \mapsto D$ is a mapping from $\mathcal{T}$ to $\mathcal{D}$. Show that it is injective.

9. To prove that the mapping is surjective, let us now consider a dimer configuration D on H. By applying the above construction backwards on each edge e' in D, we get a collection of oriented edges belonging either to G or to G^*. It is a priori not obvious that these form spanning trees: show that it is nevertheless the case. (For acyclicity, reason by contradiction, applying Euler's relation to an appropriate graph.)

10. How does does the above argument fail when the vertex v and the face f of G chosen above to construct H are not incident to each other? Find a counterexample showing that the mapping $T \mapsto D$ is no longer surjective.

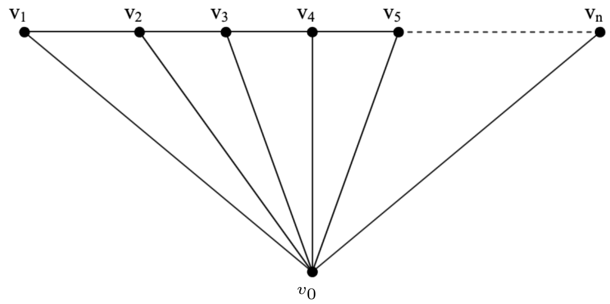
11. Let us return to the situation where v and f are incident, so that the mapping $T \mapsto D$ is a bijection between $\mathcal{T}$ and $\mathcal{D}$. To each edge e' of H, we assign a weight $w_{e'}$ with the constraint that $w_{e'} = 1$ for those edges which are incident to a vertex belonging to G^* (the weight remains arbitrary for those incident to a vertex belonging to G). Write the dimer partition function

$$Z_{H,w} = \sum_{D \in \mathcal{D}} \prod_{e' \in D} w_{e'}$$

as a determinant of a matrix which you will explicit, using the oriented version of the matrix-tree theorem.

Exercise 3 Spanning trees and rational classes

We consider the "fan graph" F_n, which is obtained from a path of n vertices $(v_1, \dots, v_n)$ by adding a vertex v_0 linked to all other vertices, as on the figure. We let a_n be the number of spanning trees of this graph.



1. Show that $a_n = a_{n-1} + \sum_{k=1}^{n} a_{n-k}$ for $n \geq 2$, with $a_0 = a_1 = 1$.

2. Convert this equation into an equation for the generating function $A(z) := \sum_{n \geq 0} a_n z^n$.

3. What is the nature of the series $A(z)$? What is the form of its coefficients? (we do not ask you to explicitly calculate all the constants).

4. The Fibonacci sequence is defined by the recurrence $F_{n+2} = F_{n+1} + F_n$ for $n \geq 0$, with $F(0) = 0$ and $F(1) = 1$. Show that $a_n = F_{2n}$ for $n \geq 1$. [Hint: introduce a combinatorial sequence that will play the role of the Fibonacci numbers of odd index.]