Dimers and related combinatorial models of statistical mechanics

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Plan

1. Introduction
   - Notions from graph theory
   - The dimer model and some of its avatars
2. Dimers on rectangular grids and the Temperley bijection

What are we going to talk about?

In these lectures I want to talk about some combinatorial models of statistical mechanics.

Statistical mechanics aims at describing the properties of systems which are made of a very large number of *microscopic* entities (atoms, molecules...). It comes from the observation that the *macroscopic* state of such a system can usually be described by a small number of parameters (temperature, pressure...) and that there is no need (and usually no way) to keep track of each single entity.

As often in physics we often rely on *models* which simplify the rules governing the evolution of the microscopic entities. This works well in practice.

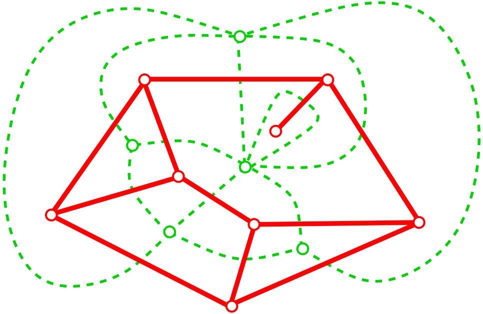
Some of these models are of *combinatorial* nature. They involve a finite or countable *state space* and studying the models boils down to questions of enumerative combinatorics. Sometimes it is possible to come up with an exact answer to these questions: the models are said *exactly solvable* or *integrable*.

In these lectures we will talk about some of these models, and notably the dimer model, to see some of the nice combinatorial tools which are used to study them.

Notions from graph theory

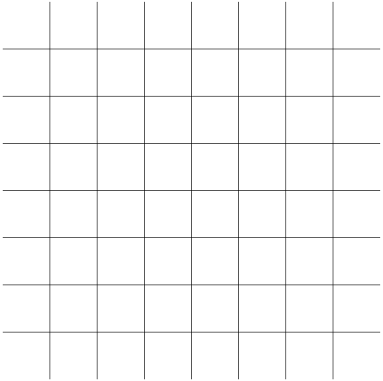
A graph $G = (V, E)$ consists of the data of a set V of vertices, and of a set E of edges. An edge consists of a pair of vertices (its *endpoints*), which may be unordered (undirected graph) or ordered (directed graph). In a *multigraph* we allow *loops* (edges whose two endpoints are the same) and *multiple edges* (several edges having the same two endpoints).

A plane graph is a graph drawn in the plane in such a way that the edges meet only at their endpoints. The drawing cuts the plane in connected regions called faces. For a connected plane graph, the regions are homeomorphic to disks (with a "point at infinity" added to the unbounded face).

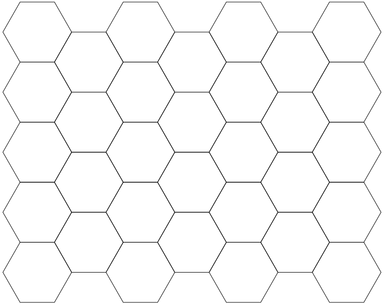


A connected plane graph G admits a dual graph G^* whose vertices, edges and faces are in bijection with respectively the faces, edges and vertices of G. Duality is "involutive".

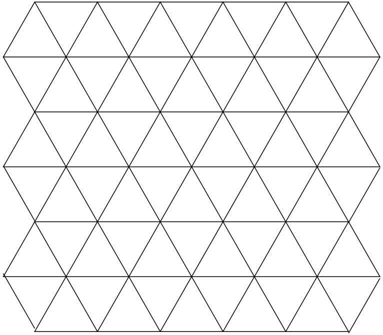
Often the plane graphs that we consider will be (portions of) regular lattices.



square lattice ($\mathbb{Z}^2$)



honeycomb/hexagonal lattice



triangular lattice

What are the duality relations between these lattices?

Consider the $m \times n$ *rectangular grid*, i.e. the portion of the square lattice fitting within the rectangle $[0, m-1] \times [0, n-1]$. What is its dual graph?

Dimer configurations

Definition

Given a graph $G = (V, E)$, a dimer configuration on G is a subset C of E such that every vertex in V is incident to exactly one edge in C.

Remark: the number of vertices must be even for a dimer configuration to exist!

We will address the following questions:

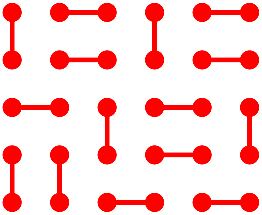
- given a finite graph G, how many dimer configurations does it admits?
- more generally, given edge weights $(w_e)_{e \in E}$, compute the dimer partition function

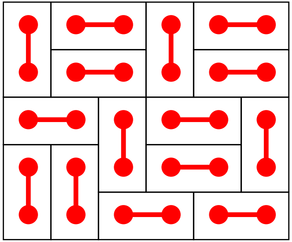
$$Z_{G,w} = \sum_{C} \prod_{e \in C} w_e.$$

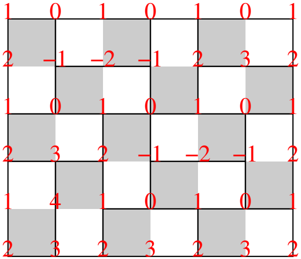
where the sum runs over all dimer configurations on G.

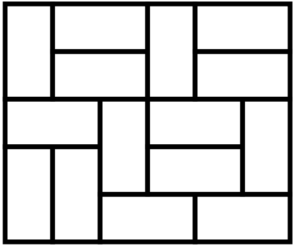
We will see that there is an "algorithmic" answer to those questions when G is plane. This is the Kasteleyn method. But before, we will see some more specific methods for particular plane graphs.

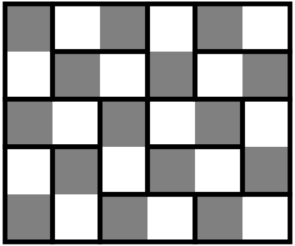
Dimers on the square lattice, domino tilings and other avatars

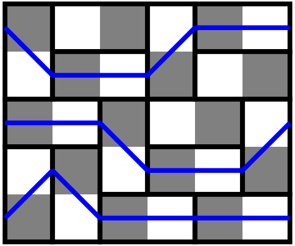




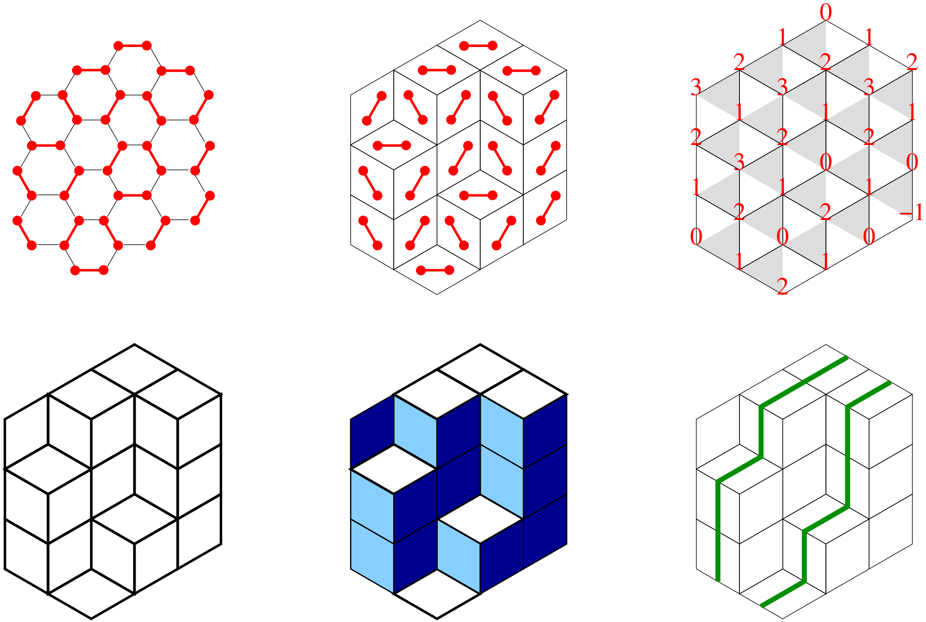








Dimers on the honeycomb lattice, lozenge tilings and other avatars



Dimers on rectangular grids

For $m, n \in \mathbb{N}$, let $G_{m,n}$ be the rectangular grid of size $m \times n$, that is the portion of the square lattice $\mathbb{Z}^2$ fitting with the rectangle $[0, m-1] \times [0, n-1]$. We let $G'_{m,n}$ be the graph obtained from $G_{m,n}$ by removing the top-right vertex $(m-1, n-1)$ when m and n are both odd.

Our purpose is to find a closed-form formula for the number of dimer configurations on $G^{(\prime)}_{m,n}$, or equivalently for the number of domino tilings of a $m \times n$ rectangle. Such a formula was found in 1961 by Temperley and Fisher, and independently by Kasteleyn, using two different methods. Let us see the first method, which relies on the matrix-tree theorem via a nice bijection.

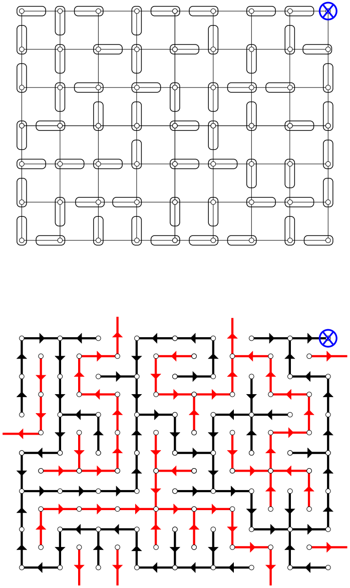
We start with the case where $m = 2M+1$ and $n = 2N+1$ are both odd:

Theorem (Temperley's bijection, odd-odd case)

There is a bijection between the set of dimer configurations on $G'_{2M+1,2N+1}$ and the set of spanning trees of $G_{M+1,N+1}$.

Here we recall that a spanning tree of a graph $G = (V, E)$ is a spanning subgraph $T = (V, E')$, with $E' \subset E$, that is both connected and acyclic.

The Temperley bijection (odd-odd case)



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The bijection is obtained as follows: consider the *even subgrid* consisting of the vertices of $G_{2M+1,2N+1}$ whose two coordinates are even. Connecting nearest neighbors by edges, we obtain a graph which is the same as $G_{M+1,N+1}$ up to scaling.

Now, fix a dimer configuration of $G'_{2M+1,2N+1}$. For each vertex of the even subgrid, consider the "direction" of the dimer that covers it. Following that direction, draw an oriented edge from that vertex to its neighbor in the even subgrid: we obtain a spanning subgraph T of $G_{M+1,N+1}$. It turns out that T is a tree:

- it cannot contain cycles (cycles on the even subgrid have an odd number of vertices in their interior, which should be covered by dimers: this is impossible),
- it is connected since, starting from any vertex and following the oriented edges, we eventually reach the top-right vertex.

To handle the general case it is useful to observe that we could have made our construction on the *odd subgrid* consisting of the vertices of $G_{2M+1,2N+1}$ whose two coordinates are odd. It is the same as the graph $G_{M,N}$ up to scaling.

When doing so, we see that some vertices have their outgoing edges pointing "outside" the grid. We consider that they connect to an external vertex "at infinity".

The odd subgrid augmented with this external vertex is precisely the dual graph $G^*_{M+1,N+1}$.

Theorem (Temperley's bijection, odd-odd case, dual version)

There is a bijection between the set of dimer configurations on $G'_{2M+1,2N+1}$ and the set of spanning trees of $G^*_{M+1,N+1}$.

Note that this is consistent with the previous version of the theorem since, for any plane graph, there is a bijection between the set of spanning trees of G and that of its dual graph G^*!

The Temperley bijection (general case)

For $(e, e') \in \{0,1\}^2$, let $G_{M,N}^{(e,e')}$ be the multigraph obtained from the rectangular grid $G_{M,N}$ by:

- adding an extra vertex ∞,
- connect it by an edge to each vertex on the left boundary of the grid,
- connect it by an edge to each vertex on the bottom boundary of the grid,
- connect it by an edge to each vertex on the right boundary of the grid if $e = 1$,
- connect it by an edge to each vertex on the top boundary of the grid if $e' = 1$.

Note that there are multiple edges, e.g. at the bottom-left vertex, and that $G_{M,N}^{(1,1)} = G^*_{M+1,N+1}$.

Theorem (Temperley's bijection, general version)

For any $M, N \in \mathbb{N}$ and $(e, e') \in \{0,1\}^2$, there is a bijection between the set of dimer configurations on $G^{(\prime)}_{2M+e,2N+e}$ and the set of spanning trees of $G_{M,N}^{(e,e')}$.

The FKT formula

Using the matrix-tree theorem and diagonalizing the graph laplacian via discrete Fourier transform, we obtain the following:

Theorem (Fisher, Kasteleyn, Temperley, 1961)

For any two integers m, n, the number of number of dimer configurations on $G^{(\prime)}_{m,n}$ is equal to

$$Z_{m,n} = \prod_{j=1}^{\lfloor \frac{m}{2} \rfloor} \prod_{k=1}^{\lfloor \frac{n}{2} \rfloor} \left(4\cos^2 \frac{\pi j}{m+1} + 4\cos^2 \frac{\pi k}{n+1} \right).$$

It is not immediately obvious that the above product evaluates to an integer but it is the case! See `https://oeis.org/A189006` for the first values.
Asymptotically we have

$$\lim_{m,n\to\infty} \frac{\ln Z_{m,n}}{mn} = \frac{1}{\pi^2} \int_0^{\pi/2} \int_0^{\pi/2} \ln(4\cos^2 x + 4\cos^2 y) dxdy = \frac{G}{\pi} \simeq 0,29156$$

where $G = 1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \cdots$ is the so-called Catalan constant.