2.3 Deformation cones of nestohedra

This section is a joint work with Arnau Padrol and Vincent Pilaud. It comes from the article [PPP23] (where the majority of the figures come from), enriched with some additional details and figures.

This section focuses on some specific deformed permutahedra generalizing the associahedra, namely the graph associahedra and nestohedra. Graph associahedra were defined by M. Carr and S. Devadoss [CD06] in connection to C. De Concini and C. Procesi's wonderful compactification [DCP95]. For a given graph G, the *G-associahedron* $\mathsf{Asso}(G)$ is a simple polytope whose combinatorial structure encodes the connected induced subgraphs of G and their nested structure. More precisely, the G-associahedron is a polytopal realization of the *nested complex* of G, defined as the simplicial complex of all collections of *tubes* (connected induced subgraphs) of G which are pairwise *compatible* (either nested, or disjoint and non-adjacent). As illustrated in Figure 14, the graph associahedra of certain special families of graphs coincide with well-known families of polytopes: complete graph associahedra are permutahedra, path associahedra are classical associahedra, cycle associahedra are cyclohedra, and star associahedra are stellohedra. Graph associahedra were extended to *nestohedra*, which are simple polytopes realizing the nested complex of arbitrary *building sets* [FS05, Pos09]. Graph associahedra and nestohedra have been constructed in different ways: by successive truncations of faces of the standard simplex [CD06], as Minkowski sums of faces of the standard simplex [FS05, Pos09], or from their normal fans by exhibiting explicit inequality descriptions [Zel06, Dev09]. For a given building set, the resulting polytopes all have the same normal fan, called *nested fan*, whose rays are given by the characteristic vectors of the building blocks, and whose cones are given by the nested sets. As all nested fans coarsen the braid fan, all graph associahedra and nestohedra are deformed permutahedra, and hence they can be obtained by gliding facets of the permutahedron. However, in contrast to the classical associahedron [SS93, Lod04, HL07], note that some graph associahedra and nestohedra cannot be obtained by deleting inequalities in the facet description of the permutahedron [Pil17].

In this section, we describe all realizations of the nested fans by studying the deformation cone of the G-associahedron for any graph G (Section 2.3.1) and of the $\mathcal{B}$-nestohedron of any building set $\mathcal{B}$ (Section 2.3.2). Our main contribution is an irredundant facet description of these deformation cones, characterizing which of the wall-crossing inequalities are irreplaceable (Theorems 2.27 and 2.61). Even though the graphical case is a specialization of the general case, we present it first separately, since it admits a much simpler description that serves as an introduction for the general case. This simplification relies on two pleasant properties (Proposition 2.22): first, the classical simple characterization of the pairs of exchangeable tubes, and second, the fact that the wall-crossing inequalities only depend on their exchanged tubes.

The non-graphical case is much more involved. First, we need a characterization of the pairs of exchangeable blocks (Proposition 2.48), which was surprisingly missing for arbitrary building sets (Remark 2.55). Second, the wall-crossing inequalities do not any more correspond to the pairs of exchangeable blocks. Namely, the wall-crossing inequalities do not only depend on the exchanged blocks, but also on an additional structure that we call the *frame* of the exchange. Moreover, some distinct exchange frames actually yield the same wall-crossing inequalities.

These irredundant inequality descriptions enable us to count the facets of these deformation cones and thus to determine when these deformation cones are simplicial. It turns out that the deformation cone of the G-associahedron is simplicial if and only if G is a disjoint union of paths (*i.e.* the G-associahedron is a Cartesian product of classical associahedra). In contrast, there is much more freedom for nestohedra of arbitrary building sets, and we show that the deformation cone of the nestohedron is always simplicial for an *interval building set*, that is a building set whose blocks are some intervals of $[n]$ (Proposition 2.68). As advocated in [PPPP19], the simpliciality of the deformation cone leads to an elegant description of all deformations of the polytope in the so-called kinematic space [AHBHY18]. Generalizing the kinematic associahedra of [AHBHY18], we thus define the *kinematic nestohedra* of arbitrary interval building sets (Proposition 2.72).

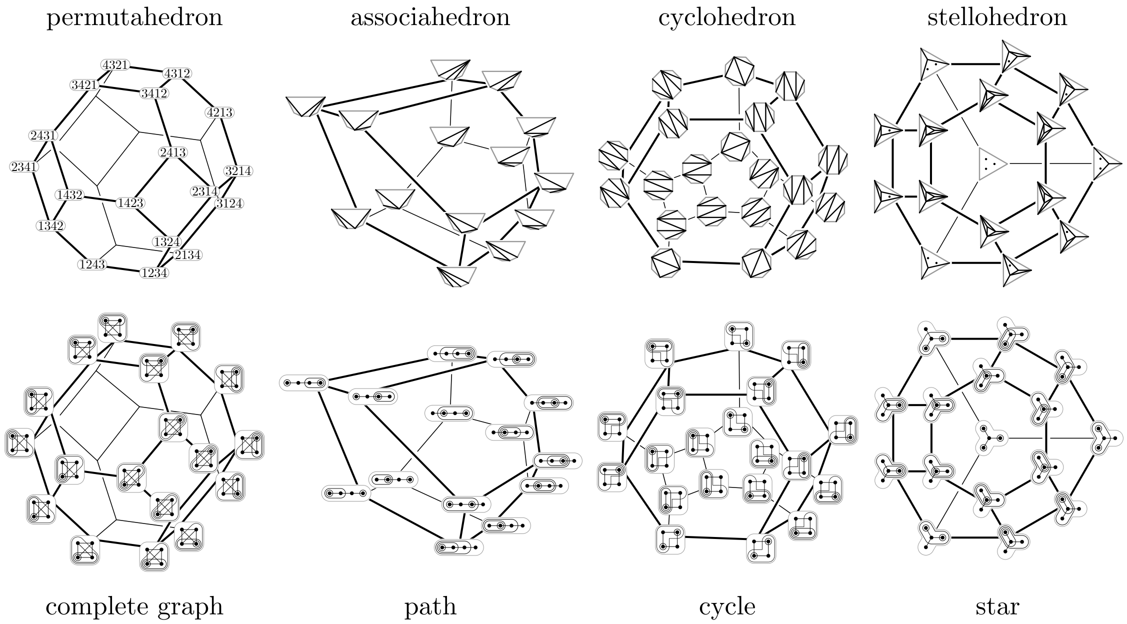


Figure 14: Some classical families of polytopes as graph associahedra. Illustration from [MP17].

2.3.1 Deformation cones of graphical nested fans

In this section, we study graphical nested fans, postponing the study of arbitrary nested fans to Section 2.3.2. While the graphical case is significantly simpler than the general case, some proof ideas presented here will be transported to Section 2.3.2. This section is thus useful both to the readers only interested in the graphical case and as a prototype for the general case.

Graphical nested complex, graphical nested fan, and graph associahedron We start with the definitions and properties of the nested complex of a graph, using material from [FS05, CD06, Zel06, Pos09, MP17].

Graphical nested complex Let G be a graph with vertex set V. A *tube* of G is a non-empty subset of vertices of G whose induced subgraph is connected. The set of tubes of G is denoted by $\mathcal{B}G$. The (inclusion) maximal tubes of G are its *connected components* $\kappa(G)$. Two tubes t, t' of G are *compatible* if they are either nested (*i.e.* $\mathsf{t} \subseteq \mathsf{t}'$ or $\mathsf{t}' \subseteq \mathsf{t}$), or disjoint and non-adjacent (*i.e.* $\mathsf{t} \cup \mathsf{t}' \notin \mathcal{B}G$). Note that any connected component of G is compatible with any other tube of G. A *tubing* on G is a set T of pairwise compatible tubes of G containing all connected components $\kappa(G)$. Examples are illustrated in Figure 15. The *nested complex* of G is the simplicial complex $\mathcal{N}(G)$ whose faces are $\mathsf{T} \smallsetminus \kappa(G)$ for all tubings T on G. If $\mathsf{T} \smallsetminus \{\mathsf{t}\} = \mathsf{T}' \smallsetminus \{\mathsf{t}'\}$ for two maximal tubings T and T' and two tubes t and t', we say that T and T' are *adjacent* and that t and t' are *exchangeable*.

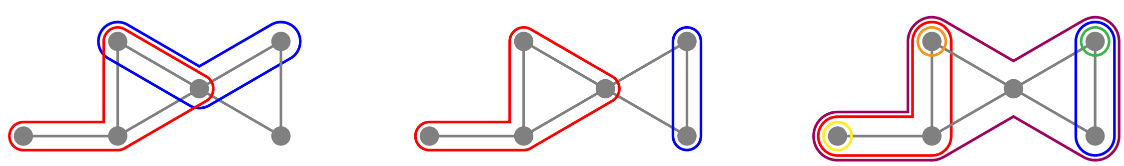


Figure 15: Some incompatible tubes (Left and Middle), and a maximal tubing (Right).

Graphical nested fan Let $(\boldsymbol{e}_v)_{v\in V}$ be the canonical basis of $\mathbb{R}^V$. We consider the subspace $\mathbb{H} := \{\boldsymbol{x} \in \mathbb{R}^V \;;\; \sum_{v\in K} x_v = 0 \text{ for all } K \in \kappa(G)\}$ and let $\pi : \mathbb{R}^V \to \mathbb{H}$ denote the orthogonal projection onto $\mathbb{H}$. The ***g*-vector** of a tube t of G is the projection $\boldsymbol{g}(\mathsf{t}) := \pi\big(\sum_{v\in\mathsf{t}} \boldsymbol{e}_v\big)$ of the characteristic vector of t. We set $\boldsymbol{g}(\mathsf{T}) := \{\boldsymbol{g}(\mathsf{t}) \;;\; \mathsf{t} \in \mathsf{T}\}$ for a tubing T on G. Note that by definition, $\boldsymbol{g}(\varnothing) = \boldsymbol{0}$ and $\boldsymbol{g}(K) = \boldsymbol{0}$ for all connected components $K \in \kappa(G)$. The vectors $\boldsymbol{g}(\mathsf{t})$ with $\mathsf{t} \in \mathcal{B}G$ support a complete simplicial fan realization of the nested complex, see Figure 16.

Theorem 2.18 ([FS05, CD06, Zel06, Pos09])**.** *For any graph G, the set of cones*

$$\mathcal{F}(G) := \{\mathbb{R}_{\ge 0}\, \boldsymbol{g}(\mathsf{T}) \;;\; \mathsf{T} \text{ tubing on } G\}$$

is a complete simplicial fan of $\mathbb{H}$, called the nested fan of G, realizing the nested complex $\mathcal{N}(G)$.

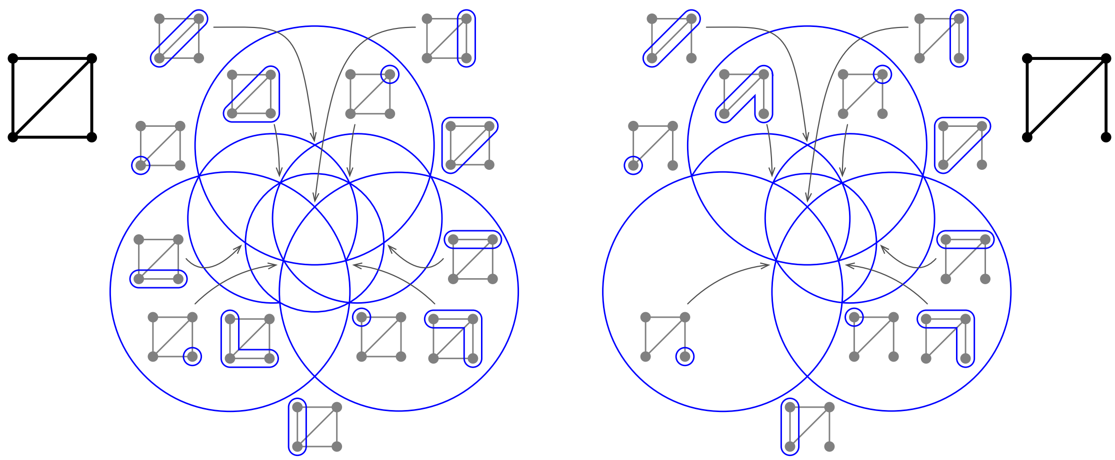


Figure 16: Two graphical nested fans. The rays are labeled by the corresponding tubes. As the fans are 3-dimensional, we intersect them with the sphere and stereographically project them from the direction $(-1,-1,-1)$.

Graph associahedron The following statement is proved in [FS05, Zel06, CD06, Dev09, Pos09]. As before, for a subset $U \subseteq V$, denote by $\Delta_U := \operatorname{conv}\{\boldsymbol{e}_u \;;\; u \in U\}$ the face of the standard simplex Δ_V corresponding to U.

Theorem 2.19 ([FS05, Zel06, CD06, Dev09, Pos09])**.** *For any graph G, the nested fan $\mathcal{F}(G)$ is the normal fan of a polytope. For instance, $\mathcal{F}(G)$ is the normal fan of*

(i) the intersection of $\mathbb{H}$ with the half-spaces $\langle \boldsymbol{g}(\mathsf{t}),\ \boldsymbol{x}\rangle \le -3^{|\mathsf{t}|}$ for all tubes $\mathsf{t} \in \mathcal{B}G$ [Dev09]

(ii) the Minkowski sum $\sum_{\mathsf{t}\in\mathcal{B}G} \Delta_{\mathsf{t}}$ of the faces of the standard simplex given by all $\mathsf{t} \in \mathcal{B}G$ [Pos09]

Definition 2.20. Any polytope whose normal fan is the nested fan $\mathcal{F}(G)$ is called *graph associahedron* and denoted by $\mathsf{Asso}(G)$.

For example, Figure 17 represents the graph associahedra realizing the graphical nested fans of Figure 16 and obtained using the construction (ii) of Theorem 2.19.

Example 2.21. For instance,

(i) for the complete graph K_n, the tubes are all non-empty subsets of $[n]$, the tubings correspond to ordered partitions of $[n]$, the maximal tubings correspond to permutations of $[n]$, the graphical nested fan $\mathcal{F}(K_n)$ is the intersection of the classical *braid fan* with $\mathbb{H} = \{\boldsymbol{x} \in \mathbb{R}^n \;;\; \sum_i x_i = 0\}$, and the graph associahedron $\mathsf{Asso}(K_n)$ is the classical *permutahedron* embedded in $\mathbb{H}$ (see *e.g.* [Zie98, Hoh12] or Section 1.2.3), this gives a slightly different point of view on Example 2.14;

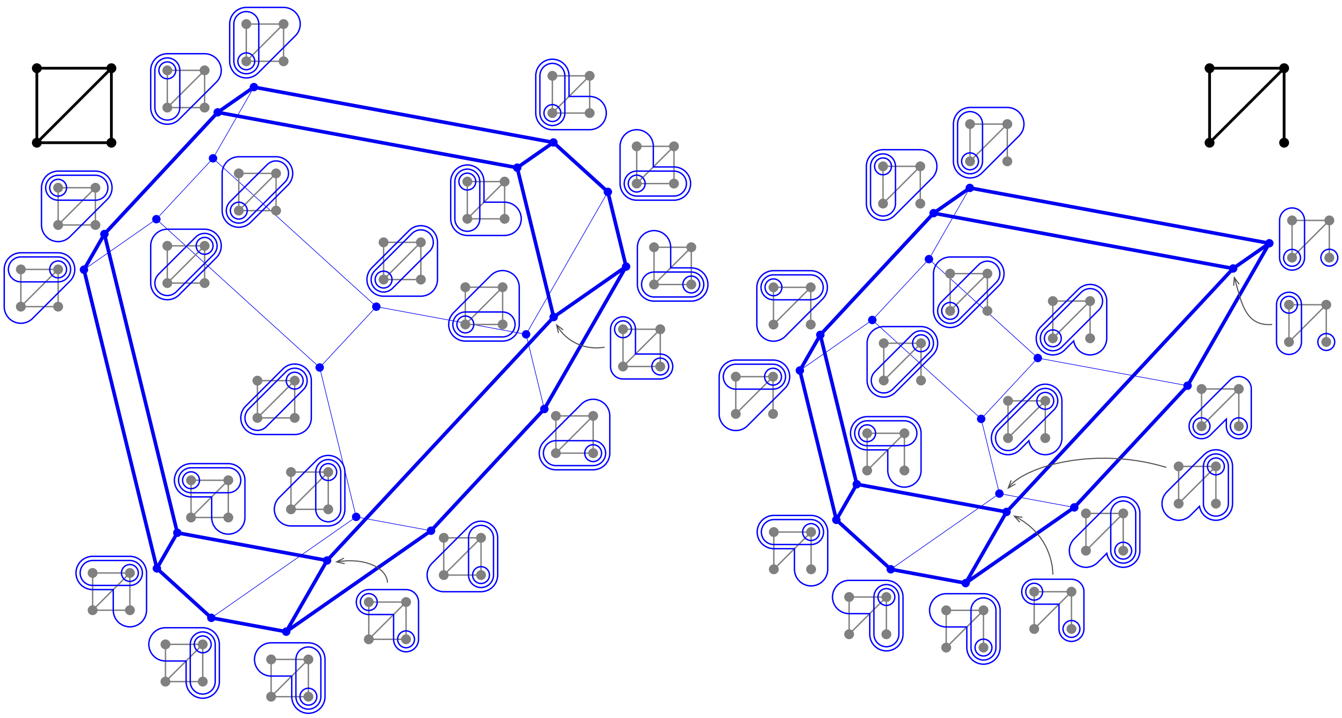


Figure 17: Two graph associahedra, realizing the graphical nested fans of Figure 16. The vertices are labeled by the corresponding maximal tubings.

(ii) for the path P_n, the tubes are all non-empty intervals of $[n]$, the tubings correspond to Schröder trees with $n+1$ leaves, the maximal tubings correspond to binary trees with $n+1$ leaves, the graphical nested fan $\mathcal{F}(P_n)$ is the classical *sylvester fan*, and the graph associahedron $\mathsf{Asso}(P_n)$ is the classical *associahedron* (see [SS93, Lod04, PSZ23] or Section 1.2.4).

Exchangeable tubes and $\boldsymbol{g}$-vector dependencies The next statement follows from [Zel06, MP17].

Proposition 2.22. *Let* t, t' *be two tubes of* G. *Then*

(i) The tubes t *and* t' *are exchangeable in* $\mathcal{F}(G)$ *if and only if* t' *has a unique neighbor* v *in* $\mathsf{t} \smallsetminus \mathsf{t}'$ *and* t *has a unique neighbor* v' *in* $\mathsf{t}' \smallsetminus \mathsf{t}$.

(ii) For any adjacent maximal tubings T, T' *on* G *with* $\mathsf{T} \smallsetminus \{\mathsf{t}\} = \mathsf{T}' \smallsetminus \{\mathsf{t}'\}$, *both* T *and* T' *contain the tube* $\mathsf{t} \cup \mathsf{t}'$ *and the connected components of* $\mathsf{t} \cap \mathsf{t}'$.

(iii) The linear dependence between the $\boldsymbol{g}$*-vectors of* $\mathsf{T} \cup \mathsf{T}'$ *is given by*

$$\boldsymbol{g}(\mathsf{t}) + \boldsymbol{g}(\mathsf{t}') = \boldsymbol{g}(\mathsf{t} \cup \mathsf{t}') + \sum_{\mathsf{s} \in \kappa(\mathsf{t} \cap \mathsf{t}')} \boldsymbol{g}(\mathsf{s}).$$

In particular, it only depends on the exchanged tubes t *and* t', *not on the tubings* T *and* T'.

Proof. Points (i) and (ii) were proved in [MP17]. Point (iii) follows from the fact that

$$\sum_{v \in \mathsf{t}} \boldsymbol{e}_v + \sum_{v \in \mathsf{t}'} \boldsymbol{e}_v = \sum_{v \in \mathsf{t} \cup \mathsf{t}'} \boldsymbol{e}_v + \sum_{v \in \mathsf{t} \cap \mathsf{t}'} \boldsymbol{e}_v = \sum_{v \in \mathsf{t} \cup \mathsf{t}'} \boldsymbol{e}_v + \sum_{\mathsf{s} \in \kappa(\mathsf{t} \cap \mathsf{t}')} \sum_{v \in \mathsf{s}} \boldsymbol{e}_v. \qquad \square$$

For instance, the two tubes on the left of Figure 15 are exchangeable, while the two tubes in the middle of Figure 15 are not.

Deformation cones of graphical nested fans As a direct consequence of Proposition 2.2 and Proposition 2.22, we obtain the following (possibly redundant) description of the deformation cone of the graphical nested fan $\mathcal{F}(G)$. Note that as $\mathcal{F}(G)$ is simplicial, there are (almost) no equalities to deal with, contrarily to the case of graphical zonotopes of Section 2.2. This simplifies sorely the computation of the dimension of $\mathbb{DC}(\mathcal{F}(G))$.

Corollary 2.23. *For any graph G, the deformation cone of the nested fan $\mathcal{F}(G)$ is given by*

$$\mathbb{DC}(\mathcal{F}(G)) = \left\{ \boldsymbol{h} \in \mathbb{R}^{\mathcal{B}G} \;;\; \begin{array}{l} \boldsymbol{h}_K = 0 \text{ for any connected component } K \in \kappa(G) \text{ and} \\ \boldsymbol{h}_{\mathsf{t}} + \boldsymbol{h}_{\mathsf{t}'} \ge \boldsymbol{h}_{\mathsf{t}\cup\mathsf{t}'} + \sum_{\mathsf{s}\in\kappa(\mathsf{t}\cap\mathsf{t}')} \boldsymbol{h}_{\mathsf{s}} \text{ for any exchangeable tubes } \mathsf{t}, \mathsf{t}' \end{array} \right\}.$$

We denote by $\boldsymbol{f}_{\mathsf{t}}$ for $\mathsf{t} \in \mathcal{B}G$ the canonical basis of $\mathbb{R}^{\mathcal{B}G}$ and by

$$\boldsymbol{n}(\mathsf{t}, \mathsf{t}') := \boldsymbol{f}_{\mathsf{t}} + \boldsymbol{f}_{\mathsf{t}'} - \boldsymbol{f}_{\mathsf{t}\cup\mathsf{t}'} - \sum_{\mathsf{s}\in\kappa(\mathsf{t}\cap\mathsf{t}')} \boldsymbol{f}_{\mathsf{s}}$$

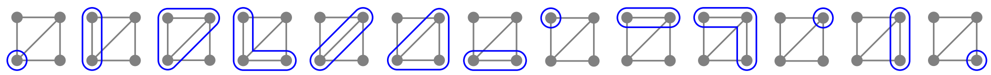
the inner normal vector of the inequality of the deformation cone $\mathbb{DC}(\mathcal{F}(G))$ corresponding to an exchangeable pair $\{\mathsf{t}, \mathsf{t}'\}$ of tubes of G. Thus $\boldsymbol{h} \in \mathbb{DC}(\mathcal{F}(G))$ if and only if $\langle \boldsymbol{n}(\mathsf{t}, \mathsf{t}'),\ \boldsymbol{h} \rangle \ge 0$ for all exchangeable tubes $\mathsf{t}, \mathsf{t}' \in \mathcal{B}G$.

Remark 2.24. For instance,

(i) for the complete graph K_n, the deformation cone $\mathbb{DC}(\mathcal{F}(K_n))$ is formed by all submodular functions, *i.e.* functions $\boldsymbol{h} : 2^{[n]} \to \mathbb{R}$ such that $\boldsymbol{h}_\varnothing = 0 = \boldsymbol{h}_{[n]}$ and $\boldsymbol{h}_A + \boldsymbol{h}_B \ge \boldsymbol{h}_{A\cap B} + \boldsymbol{h}_{A\cup B}$ for any $A, B \subseteq [n]$. The inequalities $\boldsymbol{h}_{U\smallsetminus\{v\}} + \boldsymbol{h}_{U\smallsetminus\{v'\}} \ge \boldsymbol{h}_U + \boldsymbol{h}_{U\smallsetminus\{v,v'\}}$ for $v, v' \in V$ and $\{v, v'\} \subseteq U \subseteq V$ clearly imply all submodular inequalities. This was already studied with the framework of graphical zonotopes in Example 2.14 as well as in [Pos09].

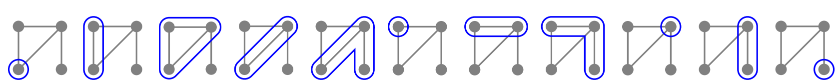
(ii) for the path P_n, the deformation cone $\mathbb{DC}(\mathcal{F}(P_n))$ is formed by the functions $\boldsymbol{h} : \{[i,j] \;;\; 1 \le i \le j \le n\} \mapsto \mathbb{R}$ such that $\boldsymbol{h}_{[1,n]} = 0 = \boldsymbol{h}_{\{i\}}$ for all $i \in [n]$ and $\boldsymbol{h}_{[i,j]} + \boldsymbol{h}_{[k,\ell]} \ge \boldsymbol{h}_{[i,\ell]} + \boldsymbol{h}_{[k,j]}$ for all $1 \le i \le j \le n$ and $1 \le k \le \ell \le n$ such that $i < k$, $j < \ell$ and $k \le j+1$ (where $\boldsymbol{h}_{[k,j]} = 0$ if $k = j+1$). This was already studied in the context of mathematical physics [AHBHY18] whose results bring back deformation cones to the fore.

Example 2.25. Consider the graphical nested fans illustrated in Figure 16. The deformation cone of the left fan lives in $\mathbb{R}^{13}$, has a lineality space of dimension 3 and 19 facet-defining inequalities (given below). In particular, it is not simplicial. Note that, as in Figure 16, we express the $\boldsymbol{g}$-vectors in the basis given by the maximal tubing containing the first three tubes below.



tubes													
$\boldsymbol{g}$-vectors	$\begin{bmatrix}1\\0\\0\end{bmatrix}$	$\begin{bmatrix}0\\1\\0\end{bmatrix}$	$\begin{bmatrix}0\\0\\1\end{bmatrix}$	$\begin{bmatrix}0\\1\\-1\end{bmatrix}$	$\begin{bmatrix}1\\-1\\1\end{bmatrix}$	$\begin{bmatrix}1\\-1\\0\end{bmatrix}$	$\begin{bmatrix}1\\0\\-1\end{bmatrix}$	$\begin{bmatrix}-1\\1\\0\end{bmatrix}$	$\begin{bmatrix}-1\\0\\1\end{bmatrix}$	$\begin{bmatrix}-1\\0\\0\end{bmatrix}$	$\begin{bmatrix}0\\-1\\1\end{bmatrix}$	$\begin{bmatrix}0\\-1\\0\end{bmatrix}$	$\begin{bmatrix}0\\0\\-1\end{bmatrix}$
facet defining inequalities	0	0	0	0	0	0	0	0	0	0	1	−1	1
	0	0	0	0	0	1	0	0	0	1	0	−1	0
	−1	1	0	−1	0	0	1	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	1	−1	−1	1	0
	0	0	0	0	0	−1	1	0	0	0	0	1	−1
	−1	1	−1	0	1	0	0	0	0	0	0	0	0
	−1	0	0	0	1	−1	1	0	0	0	0	0	0
	0	1	−1	0	0	0	0	−1	1	0	0	0	0
	0	0	−1	0	1	0	0	0	1	0	−1	0	0
	0	0	0	0	1	−1	0	0	0	0	−1	1	0
	0	0	1	0	0	0	0	0	−1	1	0	0	0
	0	0	1	0	−1	1	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	1	−1	0	1	0	0
	1	0	0	0	0	0	−1	0	0	0	0	0	1
	1	0	0	0	−1	0	0	0	0	0	1	0	0
	1	−1	0	0	0	0	0	1	0	0	0	0	0
	0	0	0	1	0	0	0	−1	0	1	0	0	−1
	0	0	0	1	0	1	−1	0	0	0	0	0	0
	0	−1	1	1	0	0	0	0	0	0	0	0	0

The deformation cone of the right fan lives in $\mathbb{R}^{11}$, has a lineality space of dimension 3 and 12 facet-defining inequalities (given below). In particular, it is not simplicial. Note that, as in Figure 16, we express the $\boldsymbol{g}$-vectors in the basis given by the maximal tubing containing the first three tubes below.



tubes											
$\boldsymbol{g}$-vectors	$\begin{bmatrix}1\\0\\0\end{bmatrix}$	$\begin{bmatrix}0\\1\\0\end{bmatrix}$	$\begin{bmatrix}0\\0\\1\end{bmatrix}$	$\begin{bmatrix}1\\-1\\1\end{bmatrix}$	$\begin{bmatrix}1\\-1\\0\end{bmatrix}$	$\begin{bmatrix}-1\\1\\0\end{bmatrix}$	$\begin{bmatrix}-1\\0\\1\end{bmatrix}$	$\begin{bmatrix}-1\\0\\0\end{bmatrix}$	$\begin{bmatrix}0\\-1\\1\end{bmatrix}$	$\begin{bmatrix}0\\-1\\0\end{bmatrix}$	$\begin{bmatrix}0\\0\\-1\end{bmatrix}$
facet defining inequalities	−1	1	−1	1	0	0	0	0	0	0	0
	1	−1	0	0	0	1	0	0	0	0	0
	0	1	−1	0	0	−1	1	0	0	0	0
	1	0	0	−1	0	0	0	0	1	0	0
	0	0	−1	1	0	0	1	0	−1	0	0
	0	0	0	1	−1	0	0	0	−1	1	0
	0	0	0	0	0	1	−1	0	1	0	0
	0	0	0	0	0	0	1	−1	−1	1	0
	0	0	0	0	0	0	0	0	1	−1	1
	0	0	0	0	1	0	0	1	0	−1	0
	0	0	1	0	0	0	−1	1	0	0	0
	0	0	1	−1	1	0	0	0	0	0	0

Example 2.26. We can exploit Corollary 2.23 to show that certain height functions belong to (the interior of) the deformation cone of $\mathcal{F}(G)$ and recover some classical constructions of the graph associahedron.

(i) Consider the height function $\boldsymbol{h} \in \mathbb{R}^{\mathcal{B}G}$ given by $\boldsymbol{h}_\mathsf{t} := -3^{|\mathsf{t}|}$. Then for any exchangeable tubes t and t', we have

$$\langle \boldsymbol{n}(\mathsf{t},\mathsf{t}'),\ \boldsymbol{h}\rangle = -3^{|\mathsf{t}|} - 3^{|\mathsf{t}'|} + 3^{|\mathsf{t}\cup\mathsf{t}'|} + \sum_{\mathsf{s}\in\kappa(\mathsf{t}\cap\mathsf{t}')} 3^{|\mathsf{s}|} \geq -2\cdot 3^{|\mathsf{t}\cup\mathsf{t}'|-1} + 3^{|\mathsf{t}\cup\mathsf{t}'|} > 0.$$

Therefore, the height function $\boldsymbol{h}$ belongs to the interior of the deformation cone $\mathbb{DC}(\mathcal{F}(G))$. The corresponding polytope $P_{\boldsymbol{h}} := \big\{\boldsymbol{x} \in \mathbb{R}^V\ ;\ \langle \boldsymbol{g}(\mathsf{t}),\ \boldsymbol{x}\rangle \leq \boldsymbol{h}_\mathsf{t} \text{ for } \mathsf{t} \in \mathcal{B}G\big\}$ is the graph associahedron constructed by S. Devadoss's in [Dev09].

(ii) Consider the height function $\boldsymbol{h} \in \mathbb{R}^{\mathcal{B}G}$ given by $\boldsymbol{h}_\mathsf{t} := -\big|\{\mathsf{s}\in\mathcal{B}G\ ;\ \mathsf{s}\subseteq\mathsf{t}\}\big|$. Then for any exchangeable tubes t and t', we have

$$\langle \boldsymbol{n}(\mathsf{t},\mathsf{t}'),\ \boldsymbol{h}\rangle = \big|\{\mathsf{s}\in\mathcal{B}G\ ;\ \mathsf{s}\not\subseteq\mathsf{t} \text{ and } \mathsf{s}\not\subseteq\mathsf{t}' \text{ but } \mathsf{s}\subseteq\mathsf{t}\cup\mathsf{t}'\}\big| > 0$$

since $\mathsf{t}\cup\mathsf{t}'$ fulfills the conditions on s. Thus, the height function $\boldsymbol{h}$ belongs to the interior of the deformation cone $\mathbb{DC}(\mathcal{F}(G))$. The polytope $P_{\boldsymbol{h}} := \big\{\boldsymbol{x} \in \mathbb{R}^V\ ;\ \langle \boldsymbol{g}(\mathsf{t}),\ \boldsymbol{x}\rangle \leq \boldsymbol{h}_\mathsf{t} \text{ for } \mathsf{t} \in \mathcal{B}G\big\}$ is the graph associahedron constructed by A. Postnikov's in [Pos09].

Note that many inequalities of Corollary 2.23 are redundant. In the remaining of this section, we describe the facet-defining inequalities of the deformation cone of the graphical nested fans. We say that an exchangeable pair $\{\mathsf{t},\mathsf{t}'\}$ of tubes of G is

- *extremal* if its corresponding inequality in Corollary 2.23 defines a facet of $\mathbb{DC}(\mathcal{F}(G))$,
- *maximal* if $\mathsf{t}\smallsetminus\{v\} = \mathsf{t}'\smallsetminus\{v'\}$ for some neighbor v of t' and some neighbor v' of t.

We can now state our main result on graphical nested complexes.

Theorem 2.27. *An exchangeable pair is extremal if and only if it is maximal.*

Proof. We treat separately the two implications:

Extremal ⇒ maximal. Consider an exchangeable pair $\{\mathsf{t},\mathsf{t}'\}$ of tubes of G. By Proposition 2.22, t' has a unique neighbor v in $\mathsf{t}\smallsetminus\mathsf{t}'$ and t has a unique neighbor v' in $\mathsf{t}'\smallsetminus\mathsf{t}$. Therefore, $\mathsf{t}\smallsetminus\mathsf{t}'$ and $\mathsf{t}'\smallsetminus\mathsf{t}$ are both connected. Assume that $\{\mathsf{t},\mathsf{t}'\}$ is not maximal, for instance that $\mathsf{t}\smallsetminus\mathsf{t}' \neq \{v\}$, and let $w \neq v$ be a non-disconnecting node of $\mathsf{t}\smallsetminus\mathsf{t}'$. By Proposition 2.22, $\tilde{\mathsf{t}} := \mathsf{t}\smallsetminus\{w\}$ and t' are exchangeable, and $\tilde{\mathsf{t}}' := (\mathsf{t}\cup\mathsf{t}')\smallsetminus\{w\}$ and t are exchangeable as well. Moreover, we have

$$\begin{aligned}\boldsymbol{n}(\tilde{\mathsf{t}},\mathsf{t}') + \boldsymbol{n}(\mathsf{t},\tilde{\mathsf{t}}') &= \Big(\boldsymbol{f}_{\tilde{\mathsf{t}}} + \boldsymbol{f}_{\mathsf{t}'} - \boldsymbol{f}_{\tilde{\mathsf{t}}\cup\mathsf{t}'} - \sum_{\mathsf{s}\in\kappa(\tilde{\mathsf{t}}\cap\mathsf{t}')} \boldsymbol{f}_\mathsf{s}\Big) + \Big(\boldsymbol{f}_{\mathsf{t}} + \boldsymbol{f}_{\tilde{\mathsf{t}}'} - \boldsymbol{f}_{\mathsf{t}\cup\tilde{\mathsf{t}}'} - \sum_{\mathsf{s}\in\kappa(\mathsf{t}\cap\tilde{\mathsf{t}}')} \boldsymbol{f}_\mathsf{s}\Big)\\ &= \boldsymbol{f}_{\mathsf{t}} + \boldsymbol{f}_{\mathsf{t}'} - \boldsymbol{f}_{\mathsf{t}\cup\mathsf{t}'} - \sum_{\mathsf{s}\in\kappa(\mathsf{t}\cap\mathsf{t}')} \boldsymbol{f}_\mathsf{s} = \boldsymbol{n}(\mathsf{t},\mathsf{t}'),\end{aligned}$$

as $\tilde{\mathsf{t}}\cup\mathsf{t}' = \tilde{\mathsf{t}}'$, $\tilde{\mathsf{t}}\cap\mathsf{t}' = \mathsf{t}\cap\mathsf{t}'$, $\mathsf{t}\cup\tilde{\mathsf{t}}' = \mathsf{t}\cup\mathsf{t}'$ and $\kappa(\mathsf{t}\cap\tilde{\mathsf{t}}') = \kappa(\tilde{\mathsf{t}}) = \tilde{\mathsf{t}}$. Therefore $\boldsymbol{n}(\mathsf{t},\mathsf{t}')$ defines a redundant inequality and $\{\mathsf{t},\mathsf{t}'\}$ is not an extremal exchangeable pair. The proof is symmetric if $\mathsf{t}'\smallsetminus\mathsf{t} \neq \{v'\}$.

Maximal ⇒ extremal. Let $\{\mathsf{t},\mathsf{t}'\}$ be a maximal exchangeable pair. To prove that $\{\mathsf{t},\mathsf{t}'\}$ is extremal, we will construct a vector $\boldsymbol{w} \in \mathbb{R}^{\mathcal{B}G}$ such that $\langle \boldsymbol{n}(\mathsf{t},\mathsf{t}'),\ \boldsymbol{w}\rangle < 0$, but $\big\langle \boldsymbol{n}(\tilde{\mathsf{t}},\tilde{\mathsf{t}}'),\ \boldsymbol{w}\big\rangle > 0$ for any other maximal exchangeable pair $\{\tilde{\mathsf{t}},\tilde{\mathsf{t}}'\}$. This will show that the inequality induced by $\{\mathsf{t},\mathsf{t}'\}$ is not redundant.

Define $\alpha(\mathsf{t},\mathsf{t}') := \{\mathsf{s} \in \mathcal{B}G \;;\; \mathsf{s} \not\subseteq \mathsf{t} \text{ and } \mathsf{s} \not\subseteq \mathsf{t}' \text{ but } \mathsf{s} \subseteq \mathsf{t} \cup \mathsf{t}'\}$. Note that $\alpha(\mathsf{t},\mathsf{t}')$ is non-empty since it contains $\mathsf{t} \cup \mathsf{t}'$. Define three vectors $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \mathbb{R}^{\mathcal{B}G}$ by, for each tube $\mathsf{s} \in \mathcal{B}G$:

$$\begin{aligned}
\boldsymbol{x}_\mathsf{s} &:= -\big|\big\{\mathsf{r} \in \mathcal{B}G \smallsetminus \alpha(\mathsf{t},\mathsf{t}') \;;\; \mathsf{r} \subseteq \mathsf{s}\big\}\big|,\\
\boldsymbol{y}_\mathsf{s} &:= -\big|\big\{\mathsf{r} \in \alpha(\mathsf{t},\mathsf{t}') \;;\; \mathsf{r} \subseteq \mathsf{s}\big\}\big|,\\
\boldsymbol{z}_\mathsf{s} &:= \begin{cases} -1 & \text{if } \mathsf{t} \subseteq \mathsf{s} \text{ or } \mathsf{t}' \subseteq \mathsf{s},\\ 0 & \text{otherwise,}\end{cases}
\end{aligned}$$

We will prove below that their scalar products with $\boldsymbol{n}(\tilde{\mathsf{t}},\tilde{\mathsf{t}}')$ for any maximal exchangeable pair $\{\tilde{\mathsf{t}},\tilde{\mathsf{t}}'\}$ satisfy the following inequalities

	$\langle \boldsymbol{n}(\tilde{\mathsf{t}},\tilde{\mathsf{t}}'),\ \boldsymbol{x}\rangle$	$\langle \boldsymbol{n}(\tilde{\mathsf{t}},\tilde{\mathsf{t}}'),\ \boldsymbol{y}\rangle$	$\langle \boldsymbol{n}(\tilde{\mathsf{t}},\tilde{\mathsf{t}}'),\ \boldsymbol{z}\rangle$
if $\{\mathsf{t},\mathsf{t}'\} = \{\tilde{\mathsf{t}},\tilde{\mathsf{t}}'\}$	$= 0$	$= \lvert\alpha(\mathsf{t},\mathsf{t}')\rvert$	$= -1$
if $\alpha(\tilde{\mathsf{t}},\tilde{\mathsf{t}}') \not\subseteq \alpha(\mathsf{t},\mathsf{t}')$	≥ 1	≥ 0	≥ -1
otherwise	$= 0$	≥ 1	≥ 0

It immediately follows from this table that the vector $\boldsymbol{w} := \boldsymbol{x} + \delta\boldsymbol{y} + \varepsilon\boldsymbol{z}$ fulfills the desired properties for any δ, ε such that $0 < \delta \cdot |\alpha(\mathsf{t},\mathsf{t}')| < \varepsilon < 1$.

To prove the inequalities of the table, observe that for any maximal exchangeable pair $\{\tilde{\mathsf{t}},\tilde{\mathsf{t}}'\}$,

- $\langle \boldsymbol{n}(\tilde{\mathsf{t}},\tilde{\mathsf{t}}'),\ \boldsymbol{x}\rangle = |\alpha(\tilde{\mathsf{t}},\tilde{\mathsf{t}}') \smallsetminus \alpha(\mathsf{t},\mathsf{t}')|$,
- $\langle \boldsymbol{n}(\tilde{\mathsf{t}},\tilde{\mathsf{t}}'),\ \boldsymbol{y}\rangle = |\alpha(\tilde{\mathsf{t}},\tilde{\mathsf{t}}') \cap \alpha(\mathsf{t},\mathsf{t}')|$,
- $\langle \boldsymbol{n}(\tilde{\mathsf{t}},\tilde{\mathsf{t}}'),\ \boldsymbol{z}\rangle \ge -1$ since $\boldsymbol{z}_{\tilde{\mathsf{t}}} = -1$ or $\boldsymbol{z}_{\tilde{\mathsf{t}}'} = -1$ implies $\boldsymbol{z}_{\tilde{\mathsf{t}} \cup \tilde{\mathsf{t}}'} = -1$,
- $\langle \boldsymbol{n}(\tilde{\mathsf{t}},\tilde{\mathsf{t}}'),\ \boldsymbol{z}\rangle \ge 0$ when $\{\mathsf{t},\mathsf{t}'\} \ne \{\tilde{\mathsf{t}},\tilde{\mathsf{t}}'\}$ but $\alpha(\tilde{\mathsf{t}},\tilde{\mathsf{t}}') \subseteq \alpha(\mathsf{t},\mathsf{t}')$. Indeed $\alpha(\tilde{\mathsf{t}},\tilde{\mathsf{t}}') \subseteq \alpha(\mathsf{t},\mathsf{t}')$ implies $\tilde{\mathsf{t}} \cup \tilde{\mathsf{t}}' \subseteq \mathsf{t} \cup \mathsf{t}'$. If $\mathsf{t} \subseteq \tilde{\mathsf{t}}$, then $\mathsf{t} \subseteq \tilde{\mathsf{t}} \subsetneq \tilde{\mathsf{t}} \cup \tilde{\mathsf{t}}' \subseteq \mathsf{t} \cup \mathsf{t}'$, which implies that $\mathsf{t} = \tilde{\mathsf{t}}$ by maximality of t in $\mathsf{t} \cup \mathsf{t}'$. Similarly, $\mathsf{t}' \subseteq \tilde{\mathsf{t}}$ implies $\mathsf{t}' = \tilde{\mathsf{t}}$. Hence, if $\boldsymbol{z}_{\tilde{\mathsf{t}}} = -1$, then by definition $\mathsf{t} \subseteq \tilde{\mathsf{t}}$ or $\mathsf{t}' \subseteq \tilde{\mathsf{t}}$, which implies that $\tilde{\mathsf{t}} \in \{\mathsf{t},\mathsf{t}'\}$. Similarly, $\boldsymbol{z}_{\tilde{\mathsf{t}}'} = -1$ implies $\tilde{\mathsf{t}}' \in \{\mathsf{t},\mathsf{t}'\}$. Hence, $\boldsymbol{z}_{\tilde{\mathsf{t}}} = -1 = \boldsymbol{z}_{\tilde{\mathsf{t}}'}$ implies $\tilde{\mathsf{t}} = \tilde{\mathsf{t}}'$ (impossible since $\tilde{\mathsf{t}}$ and $\tilde{\mathsf{t}}'$ are exchangeable) or $\{\mathsf{t},\mathsf{t}'\} = \{\tilde{\mathsf{t}},\tilde{\mathsf{t}}'\}$ (contradicting our assumption). Therefore, at most one of $\boldsymbol{z}_{\tilde{\mathsf{t}}}$ and $\boldsymbol{z}_{\tilde{\mathsf{t}}'}$ equals to -1, and if exactly one does, then $\boldsymbol{z}_{\tilde{\mathsf{t}} \cup \tilde{\mathsf{t}}'} = -1$. We conclude that $\langle \boldsymbol{n}(\tilde{\mathsf{t}},\tilde{\mathsf{t}}'),\ \boldsymbol{z}\rangle \ge 0$. □

The following statement reformulates Theorem 2.27.

Corollary 2.28. *The extremal exchangeable pairs for the nested fan of G are precisely the pairs of tubes $\mathsf{s} \smallsetminus \{v'\}$ and $\mathsf{s} \smallsetminus \{v\}$ for any tube $\mathsf{s} \in \mathcal{B}G$ and distinct non-disconnecting vertices v, v' of s.*

We derive from Theorem 2.27 and Corollary 2.28 the irredundant facet description of the deformation cone $\mathbb{DC}(\mathcal{F}(G))$.

Corollary 2.29. *For any graph G, the deformation cone of the nested fan $\mathcal{F}(G)$ is given by the following irredundant facet description*

$$\mathbb{DC}(\mathcal{F}(G)) = \left\{ \boldsymbol{h} \in \mathbb{R}^{\mathcal{B}G} \;;\; \begin{array}{l} \boldsymbol{h}_K = 0 \textit{ for any connected component } K \in \kappa(G), \textit{ and} \\ \boldsymbol{h}_{\mathsf{s} \smallsetminus \{v'\}} + \boldsymbol{h}_{\mathsf{s} \smallsetminus \{v\}} \ge \boldsymbol{h}_\mathsf{s} + \boldsymbol{h}_{\mathsf{s} \smallsetminus \{v,v'\}} \textit{ for any tube } \mathsf{s} \in \mathcal{B}G \\ \textit{and distinct non-disconnecting vertices } v, v' \in \mathsf{s} \end{array} \right\}.$$

Remark 2.30. For instance,

(i) for the complete graph K_n, all the inequalities $\boldsymbol{h}_{U \smallsetminus \{v\}} + \boldsymbol{h}_{U \smallsetminus \{v'\}} \ge \boldsymbol{h}_U + \boldsymbol{h}_{U \smallsetminus \{v,v'\}}$ for $v, v' \in V$ and $\{v, v'\} \subseteq U \subseteq V$ are facet defining inequalities of $\mathbb{DC}(\mathcal{F}(K_n))$ (fortunately, this result is coherent with Example 2.14).

(ii) for the path P_n, only the inequalities $\boldsymbol{h}_{[i,j-1]} + \boldsymbol{h}_{[i+1,j]} \ge \boldsymbol{h}_{[i,j]} + \boldsymbol{h}_{[i+1,j-1]}$ for $1 \le i < j \le n$ are facet defining inequalities of $\mathbb{DC}(\mathcal{F}(P_n))$ (where $\boldsymbol{h}_\varnothing = 0$ by convention).

We derive from Corollary 2.28 the number of facets of the deformation cone $\mathbb{DC}(\mathcal{F}(G))$. For a tube t of G, we denote by $\mathrm{nd}(\mathsf{t})$ the number of non-disconnecting vertices of t. In other words, $\mathrm{nd}(\mathsf{t})$ is the number of tubes covered by t in the inclusion poset of all tubes of G.

Corollary 2.31. *The deformation cone $\mathbb{DC}(\mathcal{F}(G))$ has $\sum_{\mathsf{s}\in\mathcal{B}G}\binom{\mathrm{nd}(\mathsf{s})}{2}$ facets. This cone has dimension $|\mathcal{B}G| - |\kappa(G)|$ and lineality $|V| - |\kappa(G)|$.*

We call proper dimension of $\mathbb{DC}(\mathcal{F}(G))$ the value $|\mathcal{B}G| - |V|$ (*i.e.* the number of tubes that not singletons) which is its dimension as a cone once quotiented out its lineality. The formula of Corollary 2.31 can be made more explicit for specific families of graph associahedra discussed in the introduction and illustrated in Figure 14.

Proposition 2.32. *The number of facets of the deformation cone $\mathbb{DC}(\mathcal{F}(G))$ is:*

- $2^{n-2}\binom{n}{2}$ *for the permutahedron (complete graph associahedron) in proper dimension* 2^n-n-1*;*
- $\binom{n}{2}$ *for the associahedron (path associahedron) in proper dimension* $\binom{n}{2}$*;*
- $3\binom{n}{2} - n$ *for the cyclohedron (cycle associahedron) in proper dimension* $(n-1)^2$*;*
- $n - 1 + 2^{n-3}\binom{n-1}{2}$ *for the stellohedron (star associahedron)in proper dimension* $2^{n-1} - 1$*.*

Proof. As the proper dimension is straightforward, we detail the computation of the number of facets. For the permutahedron, choose any two vertices v, v', and complete them into a tube by selecting any subset of the $n-2$ remaining vertices. For the associahedron, choose any two vertices v, v', and complete them into a tube by taking the path between them. For the cyclohedron, choose the two vertices v, v', and complete them into a tube by taking either all the cycle, or one of the two paths between v and v' (this gives three options in general, but only two when v, v' are neighbors). For the stellohedron, choose either v as the center of the star and v' as one of the $n-1$ leaves, or v and v' as leaves of the star and complete them into a tube by taking the center and any subset of the $n-3$ remaining leaves. □

Example 2.33. We would like to draw a figure similar to Figures 12 and 13. The proper dimension of such example shall be at most 4, so that we can intersect $\mathbb{DC}(\mathcal{F}(G))$ with a hyperplane and embed it in 3 dimensions. Unfortunately, the path P_4 already contains 6 tubes that are not singletons, and the star on 4 vertices contains 7 (non-singleton) tubes, so all connected graphs G with 4 vertices or more yield a deformation cone with proper dimension at least 6 and can not be drawn.

There are two connected graphs on 3 vertices: the complete graph K_3 and the path P_3. The deformation cone $\mathbb{DC}(\mathcal{F}(K_3))$ is depicted in Figure 12, and the deformation cone $\mathbb{DC}(\mathcal{F}(P_3))$ is the above left 2-dimensional face of the latter, as the associahedron can be written $\mathsf{Asso}_3 = \Delta_{13} + \Delta_{23} + \Delta_{123}$ in the setting of graphical zonotopes.

Simplicial deformation cone To conclude on graphical nested fans, we characterize the graphs G whose nested fan $\mathcal{F}(G)$ has a simplicial deformation cone.

Proposition 2.34. *The deformation cone $\mathbb{DC}(\mathcal{F}(G))$ is simplicial if and only if G is a disjoint union of paths.*

Proof. Observe first that the graphical nested fan $\mathcal{F}(G)$ has $N = |\mathcal{B}G| - |\kappa(G)|$ rays and dimension $n = |V| - |\kappa(G)|$. Moreover, any tube t with $|\mathsf{t}| \geq 2$ has two non-disconnecting vertices when it is a path, and at least three non-disconnecting vertices otherwise (the leaves of an arbitrary spanning tree of t, or any vertex if it is a cycle). Therefore, each tube of $\mathcal{B}G$ which is not a singleton contributes to at least one extremal exchangeable pair. We conclude that the number of extremal exchangeable pairs is at least

$$|\mathcal{B}G| - |V| = (|\mathcal{B}G| - |\kappa(G)|) - (|V| - |\kappa(G)|) = N - n,$$

with equality if and only if all tubes of G are paths, *i.e.* if and only if G is a collection of paths. Hence, $\mathbb{DC}(\mathcal{F}(G))$ is simplicial if and only if G is a disjoint union of paths. $\square$

The motivation to study the simpliciality of the deformation cone $\mathbb{DC}(\mathcal{F}(G))$ stems from the *kinematic associahedra* of [AHBHY18, Sect. 3.2]. These polytopes are alternative realizations of the associahedron obtained as sections of the kinematic space (the positive orthant in $\mathbb{R}^{\binom{n}{2}}$) by a well-chosen affine subspace parametrized by positive vectors. While these polytopes are just affinely equivalent to the realizations in $\mathbb{R}^V$, they have the advantage of being more natural from the scattering amplitudes' perspective [AHBHY18]. As observed in [PPPP19], such realizations can be directly obtained from the facet description of the deformation cone, when the latter is simplicial. Hence, combining Proposition 2.34 and Corollary 2.29 produces kinematic realizations of all graph associahedra of disjoint union of paths (*i.e.* all Cartesian products of associahedra). Our next statement only recalls the construction of the kinematic associahedron as it serves as a prototype for Proposition 2.72 that will describe new families of kinematic nestohedra.

Proposition 2.35. *For any $\boldsymbol{p} \in \mathbb{R}_{>0}^{\binom{[n]}{2}}$, the polytope $R_{\boldsymbol{p}}(n)$ defined as the intersection of the positive orthant $\{\boldsymbol{z} \in \mathbb{R}^{\{[i,j] \;;\; 1 \le i \le j \le n\}} \;;\; \boldsymbol{z} \ge 0\}$ with the hyperplanes*

- $\boldsymbol{z}_{[1,n]} = 0$ *and* $\boldsymbol{z}_{[i,i]} = 0$ *for* $i \in [n]$,
- $\boldsymbol{z}_{[i,j-1]} + \boldsymbol{z}_{[i+1,j]} - \boldsymbol{z}_{[i,j]} + \boldsymbol{z}_{[i+1,j-1]} = \boldsymbol{p}_{[i,j]}$ *for all* $1 \le i < j \le n$,

is an associahedron whose normal fan is $\mathcal{F}(P_n)$. Moreover, the polytopes $R_{\boldsymbol{p}}(n)$ for $\boldsymbol{p} \in \mathbb{R}_{>0}^{\binom{[n]}{2}}$ describe all polytopal realizations of $\mathcal{F}(P_n)$ (up to translations).

2.3.2 Deformation cones of arbitrary nested fans

We now extend our results from graph associahedra to nestohedra. In the general situation, the set of tubes is replaced by a building set $\mathcal{B}$, and the tubings are replaced by $\mathcal{B}$-nested sets (this generalization can equivalently be interpreted as replacing the graph by an arbitrary hypergraph). As in the graphical case, the nested sets define a nested complex and a nested fan, which is the normal fan of the nestohedron. In this section, we describe the deformation cones of arbitrary nested fans. We follow the same scheme as in the previous Section 2.3.1, even if the general situation is significantly more intricate (Remarks 2.49 and 2.54 highlight some of the complications of the general case).

Nested complex, nested fan, and nestohedron We first recall the definitions of arbitrary building sets, nested complexes, nested fans and nestohedra, following [FS05, Zel06, Pos09, Pil17].

Building sets A *building set* $\mathcal{B}$ on a ground set V is a set of non-empty subsets of V such that

- if $B, B' \in \mathcal{B}$ and $B \cap B' \neq \varnothing$, then $B \cup B' \in \mathcal{B}$, and
- $\mathcal{B}$ contains all singletons $\{v\}$ for $v \in V$.

We denote by $\kappa(\mathcal{B})$ the set of *connected components* of $\mathcal{B}$, defined as the (inclusion) maximal elements of $\mathcal{B}$. We denote by $\varepsilon(\mathcal{B})$ the set of *elementary blocks* of $\mathcal{B}$, defined as the blocks $B \in \mathcal{B}$ such that $|B| > 1$, and $B = B' \cup B''$ implies $B' \cap B'' = \varnothing$ for any $B', B'' \in \mathcal{B} \smallsetminus \{B\}$. For instance, consider the building set $\mathcal{B}_\circ$ on $[9]$ defined by

$$\mathcal{B}_\circ := \{1, 2, 3, 4, 5, 6, 7, 8, 9, 14, 25, 123, 456, 789, 1234, 1235, 1456, 2456, 12345, 12456, 123456\}$$

(since all labels have a single digit, we abuse notation and write 123 for $\{1, 2, 3\}$). Its connected components are $\kappa(\mathcal{B}_\circ) = \{123456, 789\}$, and its elementary blocks are $\varepsilon(\mathcal{B}_\circ) = \{14, 25, 123, 456, 789\}$, which are represented in Figure 18 (left).

Remark 2.36. If $B \in \mathcal{B}$ is elementary, then the maximal blocks of $\mathcal{B}$ strictly contained in B are disjoint. Conversely, if there exist two disjoint maximal blocks $M, N \in \mathcal{B}$ strictly contained in $B \in \mathcal{B}$, then B is elementary. Otherwise, there would be $B', B'' \in \mathcal{B} \smallsetminus \{B\}$ such that $B = B' \cup B''$ and $B' \cap B'' \neq \varnothing$. By maximality, M and N are not strict subsets of B' and B'', hence M and N intersect both B' and B''. Since $M \cap B' \neq \varnothing$, we have $M \cup B' \in \mathcal{B}$. As $M \subseteq M \cup B' \subseteq B$, we obtain again by maximality of M that $M = M \cup B'$ or $M \cup B' = B$. In the former case, we have $\varnothing \neq B' \cap N \subseteq M \cap N$ contradicting our assumption on M and N. In the latter case, we have $N \subseteq B' \smallsetminus M$ contradicting the maximality of N.

Remark 2.37. For a graph G with vertex set V, the set $\mathcal{B}G$ of all tubes of G is a *graphical building set*. The blocks of $\kappa(\mathcal{B}G)$ are the vertex sets of the connected components $\kappa(G)$ of G, and the blocks of $\varepsilon(\mathcal{B}G)$ are the edges of G.

Remark 2.38. Note that not all building sets are graphical building sets. It was in fact proved in [Zel06, Prop. 7.3] that a building set is graphical if and only if for any $B \in \mathcal{B}$ and $\mathcal{C} \subset \mathcal{B}$, if $B \cup \bigcup \mathcal{C} \in \mathcal{B}$, then there is $C \in \mathcal{C}$ such that $B \cup C \in \mathcal{B}$. However, arbitrary building sets can be interpreted using hypergraphs [Ber89] instead of graphs. More precisely, a hypergraph H on V defines a building set $\mathcal{B}H$ on V given by all non-empty subsets of V which induce connected subhypergraphs of H (a path in H is a sequence of vertices where any two consecutive ones belong to a common hyperedge of H). Conversely, a building set $\mathcal{B}$ on V is the building set of various hypergraphs on V, all containing the hypergraph with hyperedge set $\varepsilon(\mathcal{B})$. See [DP11] for details.

Nested complex Given a building set $\mathcal{B}$, a *$\mathcal{B}$-nested set* $\mathcal{N}$ is a subset of $\mathcal{B}$ such that

- for any $B, B' \in \mathcal{N}$, either $B \subseteq B'$ or $B' \subseteq B$ or $B \cap B' = \varnothing$,
- for any $k \geq 2$ pairwise disjoint $B_1, \dots, B_k \in \mathcal{N}$, the union $B_1 \cup \dots \cup B_k$ is not in $\mathcal{B}$, and
- $\mathcal{N}$ contains $\kappa(\mathcal{B})$.

These are the original conditions that appeared for instance in [Pos09]. In this paper, we prefer to use the following convenient reformulation, similar to that of [Zel06]: $\mathcal{N} \subseteq \mathcal{B}$ is a $\mathcal{B}$-nested set if and only if $\kappa(\mathcal{B}) \subseteq \mathcal{N}$ and the union $\bigcup \mathcal{X}$ of any subset $\mathcal{X} \subseteq \mathcal{N}$ does not belong to $\mathcal{B} \smallsetminus \mathcal{X}$. It is known that all inclusion maximal nested sets have $|V|$ blocks. The *$\mathcal{B}$-nested complex* is the simplicial complex $\mathcal{N}(\mathcal{B})$ whose faces are $\mathcal{N} \smallsetminus \kappa(\mathcal{B})$ for all $\mathcal{B}$-nested sets $\mathcal{N}$. It is a simplicial sphere of dimension $|V| - |\kappa(\mathcal{B})|$. Note that it is convenient to include $\kappa(\mathcal{B})$ in all $\mathcal{B}$-nested sets as in [Pos09] for certain combinatorial manipulations, but to remove $\kappa(\mathcal{B})$ from all $\mathcal{B}$-nested sets as in [Zel06] when defining the $\mathcal{B}$-nested complex. If $\mathcal{N} \smallsetminus \{B\} = \mathcal{N}' \smallsetminus \{B'\}$ for two maximal $\mathcal{B}$-nested sets $\mathcal{N}$ and $\mathcal{N}'$ and two building blocks B and B', we say that $\mathcal{N}$ and $\mathcal{N}'$ are *adjacent* and that B and B' are *exchangeable*.

For instance, Figure 18(Middle) represents the two adjacent maximal $\mathcal{B}_\circ$-nested sets

$$\mathcal{N}_\circ := \{3, 4, 5, 7, 8, 14, 789, 12345, 123456\} \quad \text{and} \quad \mathcal{N}'_\circ := \{3, 4, 5, 7, 8, 25, 789, 12345, 123456\}.$$

Remark 2.39. For a graph G, a set of tubes of $\mathcal{B}G$ is nested if and only if its tubes are pairwise compatible in the sense of Section 2.3.1 (either nested or non-adjacent). The nested complex $\mathcal{N}(\mathcal{B}G)$ thus coincides with the graphical nested complex $\mathcal{N}(G)$ introduced in Section 2.3.1 (which justifies our notation there). Note that, in contrast to the graphical nested complexes, not all nested complexes are flag (*i.e.* clique complexes of their graphs).

For a $\mathcal{B}$-nested set $\mathcal{N}$ and $B \in \mathcal{N}$, we call *root* of B in $\mathcal{N}$ the set $\boldsymbol{r}(B, \mathcal{N}) := B \smallsetminus \bigcup_C C$ where the union runs over $C \in \mathcal{N}$ such that $C \subsetneq B$. The $\mathcal{B}$-nested set $\mathcal{N}$ is maximal if and only if all $\boldsymbol{r}(B, \mathcal{N})$ are singletons for $B \in \mathcal{N}$. In that case, we abuse notation writing $\boldsymbol{r}(B, \mathcal{N})$ for the only element of this singleton. For instance, in the maximal $\mathcal{B}_\circ$-nested sets $\mathcal{N}_\circ$ and $\mathcal{N}'_\circ$ represented in Figure 18 (middle), we have $\boldsymbol{r}(14, \mathcal{N}_\circ) = 1 = \boldsymbol{r}(12345, \mathcal{N}'_\circ)$ and $\boldsymbol{r}(12345, \mathcal{N}_\circ) = 2 = \boldsymbol{r}(25, \mathcal{N}'_\circ)$.

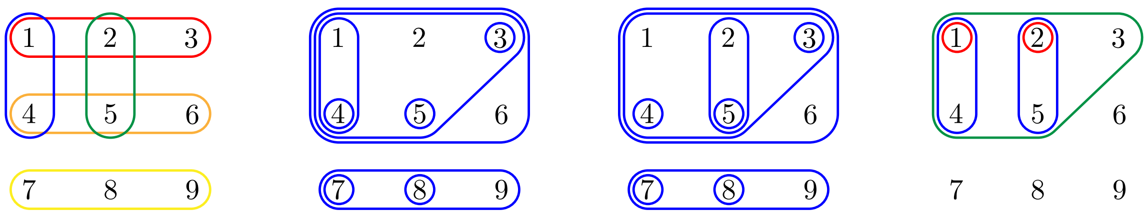


Figure 18: The elementary blocks of a building set $\mathcal{B}_\circ$ (Left), two adjacent maximal $\mathcal{B}_\circ$-nested sets (Middle), and the corresponding frame (Right).

Nested fan We still denote by $(\boldsymbol{e}_v)_{v\in V}$ the canonical basis of $\mathbb{R}^V$. We consider the subspace $\mathbb{H} := \big\{\boldsymbol{x} \in \mathbb{R}^V \;;\; \sum_{v\in B} x_v = 0 \text{ for all } B \in \kappa(\mathcal{B})\big\}$ and let $\pi : \mathbb{R}^V \to \mathbb{H}$ denote the orthogonal projection onto $\mathbb{H}$. The ***g*-vector** of a building block B of $\mathcal{B}$ is the projection $\boldsymbol{g}(B) := \pi\big(\sum_{v\in B} \boldsymbol{e}_v\big)$ of the characteristic vector of B. We set $\boldsymbol{g}(\mathcal{N}) := \{\boldsymbol{g}(B) \;;\; B \in \mathcal{N}\}$ for a $\mathcal{B}$-nested set $\mathcal{N}$. Note that by definition, $\boldsymbol{g}(K) = \boldsymbol{0}$ for all connected components $K \in \kappa(\mathcal{B})$. The vectors $\boldsymbol{g}(B)$ with $B \in \mathcal{B}$ support a complete simplicial fan realization of the nested complex. See Figure 19.

Theorem 2.40 ([FS05, Zel06, Pos09])**.** *For any building set $\mathcal{B}$, the set of cones*

$$\mathcal{F}(\mathcal{B}) := \{\mathbb{R}_{\geq 0}\, \boldsymbol{g}(\mathcal{N}) \;;\; \mathcal{N} \text{ nested set of } \mathcal{B}\}$$

is a complete simplicial fan of $\mathbb{H}$, called the nested fan of $\mathcal{B}$, which realizes the nested complex $\mathcal{N}(\mathcal{B})$.

Remark 2.41. For a graph G, the nested fan $\mathcal{F}(\mathcal{B}G)$ coincides with the graphical nested fan $\mathcal{F}(G)$ introduced in Section 2.3.1 (which justifies our notation there).

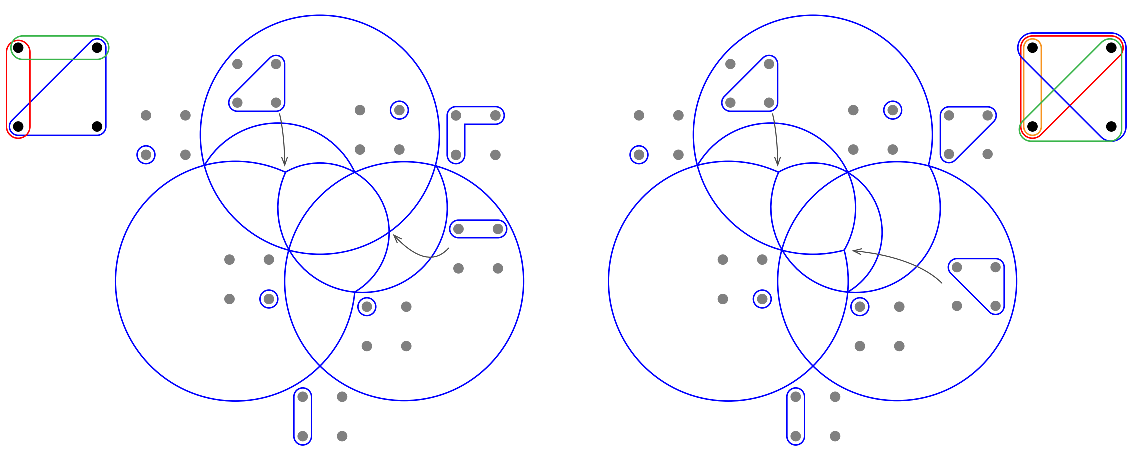


Figure 19: Two nested fans. The rays are labeled by the corresponding blocks. As the fans are 3-dimensional, we intersect them with the sphere and stereographically project them from the direction $(-1, -1, -1)$.

Nestohedron Again, the $\mathcal{B}$-nested fan is always the normal fan of a polytope, as shown in [FS05, Zel06, Pos09]. As usual, we still denote by $\Delta_U := \operatorname{conv}\{\boldsymbol{e}_u \;;\; u \in U\}$ the face of the standard simplex Δ_V corresponding to a subset U of V.

Theorem 2.42 ([FS05, Zel06, Pos09])**.** *For any building set $\mathcal{B}$, the nested fan $\mathcal{F}(\mathcal{B})$ is the normal fan of a polytope. For instance, $\mathcal{F}(\mathcal{B})$ is the normal fan of*

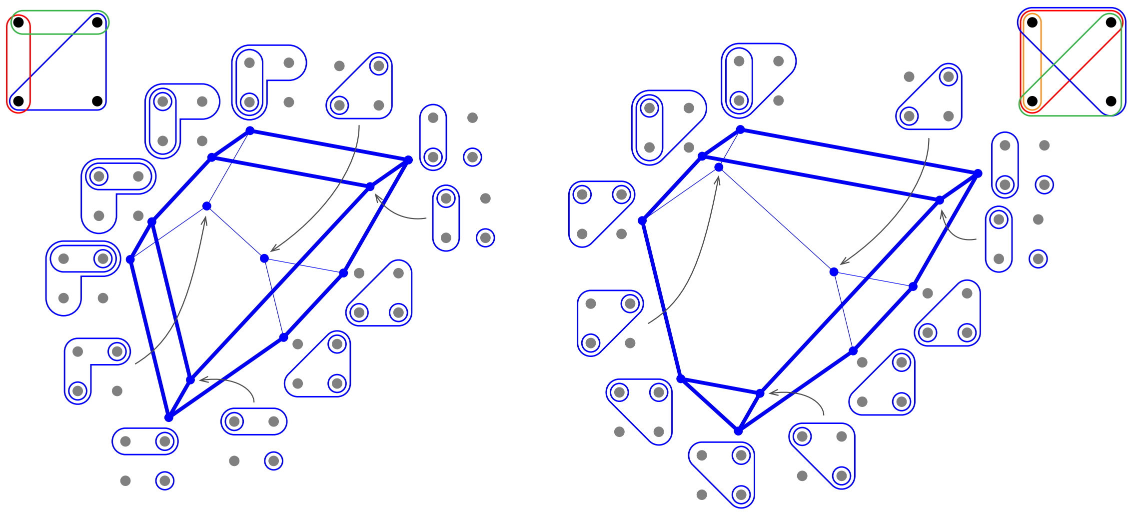


Figure 20: Two nestohedra, realizing the nested fans of Figure 19. The vertices are labeled by the corresponding maximal nested sets.

(i) the intersection of $\mathbb{H}$ with the hyperplanes $\langle \boldsymbol{g}(B),\ \boldsymbol{x}\rangle \le -3^{|B|}$ for all $B \in \mathcal{B}$ [Dev09, Pil17],

(ii) the Minkowski sum $\sum_{B\in\mathcal{B}} \Delta_B$ of faces of the standard simplex given by all $B \in \mathcal{B}$ [Pos09].

Definition 2.43. Any polytope whose normal fan is the nested fan $\mathcal{F}(\mathcal{B})$ is called a *nestohedron* of $\mathcal{B}$ and denoted by $\mathsf{Nest}_{\mathcal{B}}$.

For instance, Figure 20 represents the nestohedra realizing the nested fans of Figure 19 and obtained using the construction (ii) of Theorem 2.42.

Remark 2.44. For a graph G, the nestohedra of $\mathcal{B}G$ are the graph associahedra of G.

Restrictions and contractions Following [Zel06], we describe a structural decomposition of links in nested complexes. For any $U \subseteq V$, define

- the *restriction* of $\mathcal{B}$ to U as the building set $\mathcal{B}_{|U} := \{B \in \mathcal{B} \ ; \ B \subseteq U\}$,
- the *contraction* of U in $\mathcal{B}$ as the building set $\mathcal{B}_{/U} := \{C \subseteq V \smallsetminus U \ ; \ C \in \mathcal{B} \text{ or } C \cup U \in \mathcal{B}\}$.

Proposition 2.45 ([Zel06, Prop. 3.2])**.** *For $U \in \mathcal{B} \smallsetminus \kappa(\mathcal{B})$, the link $\{C \subseteq \mathcal{B} \smallsetminus \{U\} \ ; \ C \cup \{U\} \in \mathcal{N}(\mathcal{B})\}$ is isomorphic to the Cartesian product $\mathcal{N}(\mathcal{B}_{|U}) \times \mathcal{N}(\mathcal{B}_{/U})$.*

In particular, two building blocks B and B' in U (resp. in $V \smallsetminus U$) are exchangeable in $\mathcal{N}(\mathcal{B})$ if and only if they are exchangeable in $\mathcal{N}(\mathcal{B}_{|U})$ (resp. in $\mathcal{N}(\mathcal{B}_{/U})$).

Slightly abusing notation when $\mathcal{B}$ is clear from the context, we define the *connected components* of U as $\kappa(U) := \kappa(\mathcal{B}_{|U})$. For instance, for the building set $\mathcal{B}_\circ$ whose elementary blocks are represented in Figure 18(Left) and $U = \{1, 2, 4, 5, 7, 8\}$, we have $\mathcal{B}_{\circ|U} = \{1, 2, 4, 5, 7, 8, 14, 25\}$ so that $\kappa(U) = \{14, 25, 7, 8\}$. Note that the definition of building sets implies that

- for any $U \subseteq V$, the connected components $\kappa(U)$ define a partition of U,
- for any $U, U' \subseteq V$ such that $U \cap U' = \varnothing$ and there is no $B \in \mathcal{B}$ with $B \subseteq U \sqcup U'$ and $U \cap B \neq \varnothing \neq U' \cap B$, we have $\kappa(U \sqcup U') = \kappa(U) \sqcup \kappa(U')$.

Exchangeable building blocks and exchange frames We now provide an analogue of Proposition 2.22 characterizing the exchangeable blocks for arbitrary building sets. The situation is however much more technical, as highlighted in Remarks 2.49 and 2.54. We start with two useful lemmas.

Lemma 2.46. *For any $\mathcal{B}$-nested set $\mathcal{N}$ and any block $B \in \mathcal{B} \smallsetminus \kappa(\mathcal{B})$, the set $\{C \in \mathcal{N} \;;\; B \subsetneq C\}$ admits a unique (inclusion) minimal element M. Moreover, if $B \notin \mathcal{N}$, then M is also the unique (inclusion) maximal element of $\{C \in \mathcal{N} \;;\; \boldsymbol{r}(C,\mathcal{N}) \cap B \neq \varnothing\}$.*

Proof. Let $\mathcal{X} := \{C \in \mathcal{N} \;;\; B \subsetneq C\}$ and $\mathcal{Y} := \{C \in \mathcal{N} \;;\; \boldsymbol{r}(C,\mathcal{N}) \cap B \neq \varnothing\}$. Note first that neither $\mathcal{X}$ nor $\mathcal{Y}$ are empty since $\varnothing \neq B \notin \kappa(\mathcal{B})$. Since all elements of $\mathcal{X}$ contain B and $\mathcal{N}$ is a $\mathcal{B}$-nested set, $\mathcal{X}$ forms a chain by inclusion, and thus admits a unique inclusion minimal element M. Moreover, any building block in $\mathcal{Y}$ intersects B so that $\bigcup \mathcal{Y} = B \cup \bigcup \mathcal{Y}$ is in $\mathcal{B}$. Hence, $\mathcal{Y}$ admits a unique maximal element $M' := \bigcup \mathcal{Y}$. By definition, $B \subseteq M'$. If $B \notin \mathcal{N}$, then $B \neq M'$ since $M' \in \mathcal{Y} \subseteq \mathcal{N}$. Hence, $M' \in \mathcal{X}$. Moreover, for any $C \in \mathcal{N}$ such that $C \subsetneq M'$, we have $C \cap \boldsymbol{r}(M',\mathcal{N}) = \varnothing$ so that $B \not\subseteq C$ and $C \notin \mathcal{X}$. We conclude that $M' = M$. $\square$

Lemma 2.47. *If $\mathcal{N}$ and $\mathcal{N}'$ are two adjacent maximal $\mathcal{B}$-nested sets with $\mathcal{N} \smallsetminus \{B\} = \mathcal{N} \smallsetminus \{B'\}$, then $\{C \in \mathcal{N} \;;\; B \subsetneq C\} = \{C' \in \mathcal{N}' \;;\; B' \subsetneq C'\}$.*

Proof. Assume for instance that there is $C \in \mathcal{N} \cap \mathcal{N}'$ such that $B \subsetneq C$ but $B' \not\subseteq C$. We then claim that $\mathcal{N} \cup \mathcal{N}'$ would be a $\mathcal{B}$-nested set, contradicting the maximality of $\mathcal{N}$ and $\mathcal{N}'$. Consider a subset $\mathcal{X}$ of $\mathcal{N} \cup \mathcal{N}'$ whose union $\bigcup \mathcal{X}$ is in $\mathcal{B}$. If $B \notin \mathcal{X}$, then $\mathcal{X} \subseteq \mathcal{N}'$, hence $\bigcup \mathcal{X}$ is in $\mathcal{X}$ as $\mathcal{N}'$ is a $\mathcal{B}$-nested set. Similarly, if $B' \notin \mathcal{X}$, then $\bigcup \mathcal{X}$ is in $\mathcal{X}$. Assume now that both B and B' belong to $\mathcal{X}$. Define $\mathcal{Y} := \{C\} \cup \mathcal{X} \smallsetminus \{B\}$. Note that $\mathcal{Y} \subseteq \mathcal{N}'$ since $B \notin \mathcal{Y}$. Moreover, $\bigcup \mathcal{Y} = C \cup \bigcup \mathcal{X}$ belongs to $\mathcal{B}$ since C and $\bigcup \mathcal{X}$ both belong to $\mathcal{B}$ and intersect B. Hence, $\bigcup \mathcal{Y}$ is in $\mathcal{Y}$ since $\mathcal{N}'$ is a $\mathcal{B}$-nested set. Note that $\bigcup \mathcal{Y} \neq C$ since $B' \not\subseteq C$ and $B' \in \mathcal{Y}$. Therefore $\bigcup \mathcal{Y}$ is in $\mathcal{X}$, and thus $\bigcup \mathcal{X} = \bigcup \mathcal{Y}$ is in $\mathcal{X}$. $\square$

For two adjacent maximal $\mathcal{B}$-nested sets $\mathcal{N}$ and $\mathcal{N}'$ with $\mathcal{N} \smallsetminus \{B\} = \mathcal{N} \smallsetminus \{B'\}$, we say that

- the unique minimal element P of $\{C \in \mathcal{N} \;;\; B \subsetneq C\} = \{C' \in \mathcal{N}' \;;\; B' \subsetneq C'\}$ is the *parent*,
- the vertices $v := \boldsymbol{r}(P,\mathcal{N}')$ and $v' := \boldsymbol{r}(P,\mathcal{N})$ are the *pivots*, and
- the triple (B, B', P) is the *frame*

of the exchange between $\mathcal{N}$ and $\mathcal{N}'$. Note that the parent is well-defined by Lemmas 2.46 and 2.47. We call an *exchange frame* a triple (B, B', P) which is the frame of an exchange between two adjacent maximal $\mathcal{B}$-nested sets. For instance, for the two adjacent maximal $\mathcal{B}_\circ$-nested sets $\mathcal{N}_\circ$ and $\mathcal{N}'_\circ$ represented in Figure 18(Middle), we have $B = 14$, $B' = 25$, $P = 12345$, $v = 1$ and $v' = 2$. The corresponding exchange frame is illustrated in Figure 18(Right).

We are now ready to characterize the pairs of exchangeable building blocks for arbitrary building sets. For three blocks $B, C, P \in \mathcal{B}$, we abbreviate the conditions $B \cap C \neq \varnothing$ and $C \subseteq P$ but $C \not\subseteq B$ into the short notation $B \vdash C \subseteq P$. The following statement generalizes Proposition 2.22 (i).

Proposition 2.48. *Two blocks $B, B' \in \mathcal{B}$ are exchangeable in $\mathcal{F}(\mathcal{B})$ if and only if there exist a block $P \in \mathcal{B}$, and some vertices $v \in B \smallsetminus B'$ and $v' \in B' \smallsetminus B$ such that*

- *$B \subsetneq P$ and $B' \subsetneq P$, and*
- *$v' \in C$ for any $B \vdash C \subseteq P$ while $v \in C'$ for any $B' \vdash C' \subseteq P$.*

Proof. Assume first that B and B' are exchangeable. Let $\mathcal{N}$ and $\mathcal{N}'$ be two adjacent maximal $\mathcal{B}$-nested sets such that $\mathcal{N} \smallsetminus \{B\} = \mathcal{N} \smallsetminus \{B'\}$. Let P be the parent and v, v' be the pivots of this exchange. Note that $v \in B$ (by Lemma 2.46) but $v \notin B'$ (by definition, since $B' \in \mathcal{N}'$ and $B' \subsetneq P$). Similarly, $v' \in B' \smallsetminus B$. Consider now a building block C such that $B \vdash C \subseteq P$. By definition,

$B \subsetneq B \cup C \subseteq P$ and $B \cup C \in \mathcal{B}$. If $B \cup C = P$, then $v' = \boldsymbol{r}(P, \mathcal{N})$ belongs to $B \cup C$ and thus to C. If $B \cup C \neq P$, then P is the inclusion minimal element of $\{D \in \mathcal{N} \ ; \ B \cup C \subsetneq D\}$. Since $B \cup C \notin \mathcal{N}$ by minimality of P in $\{D \in \mathcal{N} \ ; \ B \subsetneq D\}$, we obtain by Lemma 2.46 that $v' = \boldsymbol{r}(P, \mathcal{N})$ belongs to $B \cup C$ and thus to C. Similarly, $v \in C'$ for any $B' \vdash C' \subseteq P$.

Conversely, consider $B, B' \in \mathcal{B}$ so that there is $P \in \mathcal{B}$, $v \in B \smallsetminus B'$ and $v' \in B' \smallsetminus B$ satisfying the conditions of Proposition 2.48. Let $U := P \smallsetminus \{v, v'\}$, and $\mathcal{M}$ denote an arbitrary maximal $\mathcal{B}_{|U}$-nested set. Let $\mathcal{N} := \mathcal{M} \cup \{B\}$ and $\mathcal{N}' := \mathcal{M} \cup \{B'\}$. Consider a subset $\mathcal{X}$ of $\mathcal{N}$ whose union $\bigcup \mathcal{X}$ is in $\mathcal{B}$. If $B \notin \mathcal{X}$, then $\mathcal{X} \subseteq \mathcal{M}$, hence $\bigcup \mathcal{X}$ is in $\mathcal{X}$ since $\mathcal{M}$ is a $\mathcal{B}_{|U}$-nested set. If $B \in \mathcal{X}$, since $B \cap \bigcup \mathcal{X} \neq \varnothing$ and $\bigcup \mathcal{X} \subseteq P$ but $v' \notin \bigcup \mathcal{X}$, the conditions of Proposition 2.48 ensure that $\bigcup \mathcal{X} \subseteq B$, so that $\bigcup \mathcal{X} = B$ is in $\mathcal{X}$. Hence, $\mathcal{N}$ is a $\mathcal{B}_{|P}$-nested set. It is moreover maximal since $|\mathcal{N}| = |\mathcal{M} \cup \{B, P\}| = |\mathcal{M}| + 2 = |U| + 2 = |P|$. By symmetry, $\mathcal{N}'$ is a maximal $\mathcal{B}_{|P}$-nested set. Since $\mathcal{N} \smallsetminus \{B\} = \mathcal{N}' \smallsetminus \{B'\}$, we obtain that B and B' are exchangeable in $\mathcal{N}(\mathcal{B}_{|P})$, hence in $\mathcal{N}(\mathcal{B})$ by Proposition 2.45. The parent of this exchange is P and the pivots are v and v'. $\square$

Remark 2.49. For the graphical nested fans, Proposition 2.22 (i) ensures that if B and B' are exchangeable, then $B \cup B'$ is always a block and is the only possible parent (note however that B and B' are not necessarily exchangeable when $B \cup B'$ is a block). In contrast to the graphical case, for a general building set,

- the same exchangeable blocks may admit several possible parents and pivots,
- the set of parents does not necessarily admit a unique (inclusion) minimal element,
- $B \cup B'$ is not always a block when B and B' are exchangeable. In other words, B and B' can be exchangeable even if $\{B, B'\} \cup \kappa(\mathcal{B})$ is a $\mathcal{B}$-nested set.

For instance, in the building set $\mathcal{B}_\circ$ of Figure 18(Left), the blocks $B = 14$ and $B' = 25$ are simultaneously compatible and exchangeable. They are exchangeable with parent 12345 and pivots $(1, 2)$ or with parent 12456 and pivots $(4, 5)$.

Remark 2.50. Observe also that it follows from the definitions that

- it suffices to check the condition of Proposition 2.48 for C and C' elementary blocks of $\mathcal{B}$,
- if B and B' are exchangeable, then $B \not\subseteq B'$ and $B' \not\subseteq B$,
- if (B, B', P) is an exchange frame and $B \cup B' \subseteq P' \subseteq P$, then (B, B', P') is also an exchange frame (using the same pivots),
- if B and B' are exchangeable and $B \cup B'$ is a block (in particular if $B \cap B' \neq \varnothing$), then $(B, B', B \cup B')$ is an exchange frame.

We now apply Proposition 2.48 to identify some exchange frames that will play an important role in the description of the deformation cone of the $\mathcal{B}$-nested fan.

Proposition 2.51. *If $B, B', P \in \mathcal{B}$ are such that B and B' are two distinct blocks of $\mathcal{B}$ strictly contained in P and inclusion maximal inside P, then (B, B', P) is an exchange frame.*

Proof. Consider $C \in \mathcal{B}$ such that $B \vdash C \subseteq P$. Since $B \cap C \neq \varnothing$, we have $B \cup C \in \mathcal{B}$. Since $C \subseteq P$ and $C \not\subseteq B$, we have $B \subsetneq B \cup C \subseteq P$. By maximality of B in P, we obtain that $B \cup C = P$. Hence, $B \vdash C \subseteq P$ implies $B' \smallsetminus B \subseteq C$ and similarly $B' \vdash C' \subseteq P$ implies $B \smallsetminus B' \subseteq C'$. Therefore, choosing any $v \in B \smallsetminus B'$ and $v' \in B' \smallsetminus B$, we obtain that B, B', P, v, v' satisfy the conditions of Proposition 2.48, and thus (B, B', P) is an exchange frame. $\square$

We call *maximal exchange frames* the exchange frames defined by Proposition 2.51. For $P \in \mathcal{B}$, we will denote by $\mu(P)$ the maximal blocks of $\mathcal{B}$ strictly contained in P.

$\boldsymbol{g}$-vector dependencies We now describe the exchange relations in the $\mathcal{B}$-nested fan $\mathcal{F}(\mathcal{B})$. We first need to observe that certain building blocks are forced to belong to any two adjacent maximal nested sets with a given frame, generalizing Proposition 2.22 (ii).

Proposition 2.52. *For two adjacent maximal $\mathcal{B}$-nested sets $\mathcal{N}$ and $\mathcal{N}'$ with $\mathcal{N} \smallsetminus \{B\} = \mathcal{N}' \smallsetminus \{B'\}$ and parent P, all connected components of $\kappa(B \cap B')$ and of $\kappa\big(P \smallsetminus (B \cup B')\big)$ belong to $\mathcal{N} \cap \mathcal{N}'$.*

Proof. Even if we discuss separately the elements of $\kappa(B \cap B')$ from that of $\kappa(P \smallsetminus (B \cup B'))$, the reader will see a lot of similarities in the arguments below.

We first consider $K \in \kappa(B \cap B')$ and prove that $\mathcal{N} \cup \{K\}$ is a $\mathcal{B}$-nested set, which proves that $K \in \mathcal{N}$ by maximality of $\mathcal{N}$. Indeed, let us consider a subset $\mathcal{X}$ of $\mathcal{N} \cup \{K\}$ whose union $\bigcup \mathcal{X}$ is in $\mathcal{B}$, and prove that $\bigcup \mathcal{X}$ is in $\mathcal{X}$. We assume that $K \in \mathcal{X}$, since otherwise $\mathcal{X} \subseteq \mathcal{N}$ so that $\bigcup \mathcal{X}$ is in $\mathcal{X}$ as $\mathcal{N}$ is a $\mathcal{B}$-nested set. Assume now that $B \in \mathcal{X}$ and define $\mathcal{Y} := \mathcal{X} \smallsetminus \{K\}$. Since $K \subseteq B \in \mathcal{X}$, we have $\bigcup \mathcal{Y} = \bigcup \mathcal{X}$ in $\mathcal{B}$, thus in $\mathcal{Y} \subset \mathcal{X}$ since $\mathcal{Y} \subseteq \mathcal{N}$ and $\mathcal{N}$ is a $\mathcal{B}$-nested set. It remains to consider the case when $\mathcal{X} \subseteq (\mathcal{N} \cap \mathcal{N}') \cup \{K\}$. Assume now that $\bigcup \mathcal{X} \not\subseteq B$ and define $\mathcal{Y} := \{B\} \cup \mathcal{X} \smallsetminus \{K\}$. Since $K \subseteq B$, we have $\bigcup \mathcal{Y} = B \cup \bigcup \mathcal{X}$ which belongs to $\mathcal{B}$ since B and $\bigcup \mathcal{X}$ both belong to $\mathcal{B}$ and intersect K. Hence, $\bigcup \mathcal{Y}$ is in $\mathcal{Y}$ since $\mathcal{Y} \subseteq \mathcal{N}$ and $\mathcal{N}$ is a $\mathcal{B}$-nested set. Note that $\bigcup \mathcal{Y} \neq B$ by our assumption that $\bigcup \mathcal{X} \not\subseteq B$. Therefore, $\bigcup \mathcal{Y}$ is in $\mathcal{X}$, and thus $\bigcup \mathcal{X} = \bigcup \mathcal{Y}$ is in $\mathcal{X}$. By symmetry, we obtain that $\bigcup \mathcal{X}$ is in $\mathcal{X}$ if $\bigcup \mathcal{X} \not\subseteq B'$. Assume finally that $\bigcup \mathcal{X} \subseteq B \cap B'$. Then all the elements of $\mathcal{X}$ are in $B \cap B'$. Since $K \in \mathcal{X}$ is a connected component of $B \cap B'$ and $\bigcup \mathcal{X}$ is in $\mathcal{B}$, this implies that $\bigcup \mathcal{X} = K \in \mathcal{X}$.

We now consider $K \in \kappa(P \smallsetminus (B \cup B'))$ and prove that $\mathcal{N} \cup \{K\}$ is a $\mathcal{B}$-nested set, which proves that $K \in \mathcal{N}$ by maximality of $\mathcal{N}$. Indeed, let us consider a subset $\mathcal{X}$ of $\mathcal{N} \cup \{K\}$ whose union $\bigcup \mathcal{X}$ is in $\mathcal{B}$, and prove that $\bigcup \mathcal{X}$ is in $\mathcal{X}$. We assume that $K \in \mathcal{X}$, since otherwise $\mathcal{X} \subseteq \mathcal{N}$ so that $\bigcup \mathcal{X}$ is in $\mathcal{X}$ as $\mathcal{N}$ is a $\mathcal{B}$-nested set. Assume now that $\bigcup \mathcal{X} \not\subseteq P$ and define $\mathcal{Y} := \{P\} \cup \mathcal{X} \smallsetminus \{K\}$. Since $K \subseteq P$, we have $\bigcup \mathcal{Y} = P \cup \bigcup \mathcal{X}$ which belongs to $\mathcal{B}$ since P and $\bigcup \mathcal{X}$ both belong to $\mathcal{B}$ and intersect K. Hence, $\bigcup \mathcal{Y}$ is in $\mathcal{Y}$ since $\mathcal{Y} \subseteq \mathcal{N}$ and $\mathcal{N}$ is a $\mathcal{B}$-nested set. Note that $\bigcup \mathcal{Y} \neq P$ by our assumption that $\bigcup \mathcal{X} \not\subseteq P$. Therefore, $\bigcup \mathcal{Y}$ is in $\mathcal{X}$, and thus $\bigcup \mathcal{X} = \bigcup \mathcal{Y}$ is in $\mathcal{X}$. Assume now that $\bigcup \mathcal{X} \subseteq P \smallsetminus (B \cup B')$. Then all elements of $\mathcal{X}$ are in $P \smallsetminus (B \cap B')$. Since K is a connected component of $P \smallsetminus (B \cap B')$ and $\bigcup \mathcal{X}$ is in $\mathcal{B}$, this implies that $\bigcup \mathcal{X} = K \in \mathcal{X}$. Assume finally that $\bigcup \mathcal{X}$ is contained in P and intersects B or B'. If $B' \cap \bigcup \mathcal{X} \neq \varnothing$, then $B' \vdash \bigcup \mathcal{X} \subseteq P$, thus $v := \boldsymbol{r}(P, \mathcal{N}) \in \bigcup \mathcal{X}$ by Proposition 2.48. Hence, in both cases $B \cap \bigcup \mathcal{X} \neq \varnothing$, thus $B \vdash \bigcup \mathcal{X} \subseteq P$, and thus $v' := \boldsymbol{r}(P, \mathcal{N}) \in \bigcup \mathcal{X}$ by Proposition 2.48. Therefore, there is $C \in \mathcal{X} \smallsetminus \{K\} \subseteq \mathcal{N}$ containing v'. Since $v' = \boldsymbol{r}(P, \mathcal{N})$, we obtain that $P \subseteq C$, and hence $P = C$ because $C \subseteq \bigcup \mathcal{X} \subseteq P$. Thus $K \subseteq C$ and $\bigcup \mathcal{X} = \bigcup \mathcal{Y}$ where $\mathcal{Y} := \mathcal{X} \smallsetminus \{K\}$. Hence, $\bigcup \mathcal{Y}$ is in $\mathcal{Y}$ since $\mathcal{Y} \subseteq \mathcal{N}$ and $\mathcal{N}$ is a $\mathcal{B}$-nested set. We conclude that $\bigcup \mathcal{X} = \bigcup \mathcal{Y}$ is in $\mathcal{X}$.

We obtained that all blocks of $\kappa(B \cap B')$ and of $\kappa\big(P \smallsetminus (B \cup B')\big)$ belong to $\mathcal{N}$, and thus also to $\mathcal{N}'$ by symmetry. $\square$

We are now ready to describe the exchange relations in the $\mathcal{B}$-nested fan. The main message here is that these relations only depend on the frames of the exchanges, generalizing Proposition 2.22 (iii).

Proposition 2.53. *For two adjacent maximal $\mathcal{B}$-nested sets $\mathcal{N}$ and $\mathcal{N}'$ with $\mathcal{N} \smallsetminus \{B\} = \mathcal{N}' \smallsetminus \{B'\}$ and parent P, the unique (up to rescaling) linear dependence between the $\boldsymbol{g}$-vectors of $\mathcal{N} \cup \mathcal{N}'$ is*

$$\boldsymbol{g}(B) + \boldsymbol{g}(B') + \sum_{K \in \kappa(P \smallsetminus (B \cup B'))} \boldsymbol{g}(K) = \boldsymbol{g}(P) + \sum_{K \in \kappa(B \cap B')} \boldsymbol{g}(K). \tag{9}$$

In particular, the $\boldsymbol{g}$-vector dependence only depends on the exchange frame (B, B', P).

Proof. Equation (9) is a valid linear dependence since it holds at the level of characteristic vectors, and $\boldsymbol{g}(C) := \pi\big(\sum_{v \in C} \boldsymbol{e}_v\big)$ where π is the orthogonal projection from $\mathbb{R}^V$ to $\mathbb{H}$. Since all building blocks involved in Equation (9) belong to $\mathcal{N} \cup \mathcal{N}'$ by Proposition 2.52, we conclude that Equation (9) is the unique (up to rescaling) linear dependence between the $\boldsymbol{g}$-vectors of $\mathcal{N} \cup \mathcal{N}'$. $\square$

Remark 2.54. For the graphical nested fans studied in Section 2.3.1, the parent of the exchange of B and B' is always $B \cup B'$ and we recover the $\boldsymbol{g}$-vector relation of Proposition 2.22 (iii). In contrast to the graphical case, for an arbitrary building set,

- the sum on the left of Equation (9) is empty only when $P = B \cup B'$,
- Equation (9) depends on the exchange frame (B, B', P), not only on the exchangeable building blocks B and B'.

For instance, the $\boldsymbol{g}$-vector relation of the exchange between the two adjacent maximal $\mathcal{B}_\circ$-nested sets $\mathcal{N}_\circ$ and $\mathcal{N}'_\circ$ represented in Figure 18(Middle) is $\boldsymbol{g}_{14} + \boldsymbol{g}_{25} + \boldsymbol{g}_3 = \boldsymbol{g}_{12345}$. Another $\boldsymbol{g}$-vector relation for the same exchangeable blocks $B = 14$ and $B' = 25$ is $\boldsymbol{g}_{14} + \boldsymbol{g}_{25} + \boldsymbol{g}_6 = \boldsymbol{g}_{12456}$.

Remark 2.55. The $\boldsymbol{g}$-vector dependencies were already studied in [Zel06]. Namely, our Proposition 2.52 and Equation (9) are essentially Proposition 4.5 and Equation (6.6) of [Zel06]. Our versions are however more precise since we obtained in Proposition 2.48 a complete characterization of the exchangeable building blocks of $\mathcal{B}$, which was surprisingly missing in the literature.

Note that while the $\boldsymbol{g}$-vector dependence only depends on the exchange frame, different frames may lead to the same $\boldsymbol{g}$-vector dependence. In the next two statements, we describe which of the maximal exchange frames lead to the same $\boldsymbol{g}$-vector dependence. Remember that we denote by $\mu(P)$ the maximal blocks of $\mathcal{B}$ strictly contained in a block $P \in \mathcal{B}$.

Proposition 2.56. *For an elementary block $P \in \varepsilon(\mathcal{B})$, all exchange frames (B, B', P) for $B \neq B'$ in $\mu(P)$ lead to the same $\boldsymbol{g}$-vector dependence $\sum_{B \in \mu(P)} \boldsymbol{g}(B) = \boldsymbol{g}(P)$.*

Proof. Observe first that (B, B', P) is indeed an exchange frame by Proposition 2.51. We thus apply Proposition 2.53 to describe the corresponding $\boldsymbol{g}$-vector dependence. Observe first that the sum on the right of Equation (9) is empty because $B \cap B' = \varnothing$ by Remark 2.36 since P is elementary and $B, B' \in \mu(P)$. The result thus follows from the observation that $\kappa(P \smallsetminus (B \cup B')) = \mu(P) \smallsetminus \{B, B'\}$ which we prove next.

Let us consider $K \in \kappa(P \smallsetminus (B \cup B'))$ and prove that $K \in \mu(P) \smallsetminus \{B, B'\}$. Consider $L \in \mathcal{B}$ such that $K \subseteq L \subsetneq P$. If $L \cap B \neq \varnothing$, then $L \cup B \in \mathcal{B}$ and $B \subsetneq L \cup B \subseteq P$, so that $L \cup B = P$ by maximality of B, contradicting the elementarity of P. Hence, $L \subseteq P \smallsetminus (B \cup B')$, so that $K = L$ by maximality of K in $P \smallsetminus (B \cup B')$. We conclude that $K \in \mu(P) \smallsetminus \{B, B'\}$.

Conversely, let us consider $C \in \mu(P) \smallsetminus \{B, B'\}$ and prove that $C \in \kappa(P \smallsetminus (B \cup B'))$. Since P is elementary and $B, B', C \in \mu(P)$, the block C is disjoint from B and B' by Remark 2.36. Hence, $C \subseteq P \smallsetminus (B \cup B')$ and thus $C \in \kappa(P \smallsetminus (B \cup B'))$ by maximality of C. □

Proposition 2.57. *If (B_1, B'_1, P) and (B_2, B'_2, P) are two distinct maximal exchange frames with the same $\boldsymbol{g}$-vector dependence, then P is elementary.*

Proof. Since the exchange relations given by Equation (9) for the exchange frames (B_1, B'_1, P) and (B_2, B'_2, P) coincide, B_2 and B'_2 belong to $\{B_1, B'_1\} \cup \kappa(P \smallsetminus (B_1 \cup B'_1))$. Since (B_1, B'_1, P) and (B_2, B'_2, P) are distinct exchange frames, we can assume for instance that B_2 does not belong to $\{B_1, B'_1\}$. Hence, B_2 belongs to $\kappa(P \smallsetminus (B_1 \cup B'_1))$, thus $B_1 \cap B_2 = \varnothing$, and therefore P is elementary by Remark 2.36 since it contains two disjoint maximal blocks. □

Deformation cone of nested fans As a consequence of Proposition 2.53, we obtain the following redundant description of the deformation cone of the nested fan $\mathcal{F}(\mathcal{B})$.

Corollary 2.58. *For any building set $\mathcal{B}$, the deformation cone of the nested fan $\mathcal{F}(\mathcal{B})$ is given by*

$$\mathbb{DC}(\mathcal{F}(\mathcal{B})) = \left\{ \boldsymbol{h} \in \mathbb{R}^{\mathcal{B}G} \;;\; \begin{array}{l} \boldsymbol{h}_B = 0 \textit{ for } B \in \kappa(\mathcal{B}) \textit{ and for any exchange frame } (B, B', P) \\ \boldsymbol{h}_B + \boldsymbol{h}_{B'} + \sum_{K \in \kappa(P \smallsetminus (B \cup B'))} \boldsymbol{h}_K \geq \boldsymbol{h}_P + \sum_{K \in \kappa(B \cap B')} \boldsymbol{h}_K \end{array} \right\}.$$

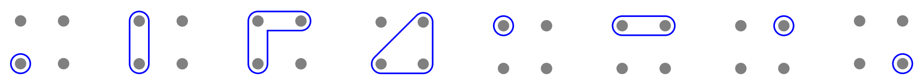
We denote by $\boldsymbol{f}_B$ for $B \in \mathcal{B}$ the canonical basis of $\mathbb{R}^{\mathcal{B}}$ and by

$$\boldsymbol{n}(B, B', P) := \Big(\boldsymbol{f}_B + \boldsymbol{f}_{B'} + \sum_{K \in \kappa(P \smallsetminus (B \cup B'))} \boldsymbol{f}_K\Big) - \Big(\boldsymbol{f}_P + \sum_{K \in \kappa(B \cap B')} \boldsymbol{f}_K\Big)$$

the inner normal vector of the inequality of the deformation cone $\mathbb{DC}(\mathcal{F}(\mathcal{B}))$ corresponding to an exchange frame (B, B', P) of $\mathcal{B}$. Thus $\boldsymbol{h} \in \mathbb{DC}(\mathcal{F}(\mathcal{B}))$ if and only if $\langle \boldsymbol{n}(B, B', P),\ \boldsymbol{h}\rangle \geq 0$ for all exchange frames (B, B', P) of $\mathcal{B}$.

Example 2.59. Consider the nested fans illustrated in Figure 19. The deformation cone of the left fan lives in $\mathbb{R}^8$, has a lineality space of dimension 3 and 5 facet-defining inequalities (given below). In particular, it is simplicial. Note that, as in Figure 16, we express the $\boldsymbol{g}$-vectors in the basis given by the maximal tubing containing the first three tubes below.

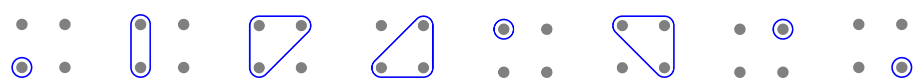
blocks



$\boldsymbol{g}$-vectors	$\begin{bmatrix}1\\0\\0\end{bmatrix}$	$\begin{bmatrix}0\\1\\0\end{bmatrix}$	$\begin{bmatrix}0\\0\\1\end{bmatrix}$	$\begin{bmatrix}1\\-1\\0\end{bmatrix}$	$\begin{bmatrix}-1\\1\\0\end{bmatrix}$	$\begin{bmatrix}-1\\0\\1\end{bmatrix}$	$\begin{bmatrix}0\\-1\\1\end{bmatrix}$	$\begin{bmatrix}0\\0\\-1\end{bmatrix}$
facet defining inequalities	1	−1	0	0	1	0	0	0
	0	0	0	0	1	−1	1	0
	1	0	0	−1	0	0	1	1
	0	1	−1	0	−1	1	0	0
	−1	0	1	1	0	0	−1	0

The deformation cone of the right fan lives in $\mathbb{R}^8$, has a lineality space of dimension 3 and 7 facet-defining inequalities (given below). In particular, it is not simplicial.

blocks



$\boldsymbol{g}$-vectors	$\begin{bmatrix}1\\0\\0\end{bmatrix}$	$\begin{bmatrix}0\\1\\0\end{bmatrix}$	$\begin{bmatrix}0\\0\\1\end{bmatrix}$	$\begin{bmatrix}1\\-1\\0\end{bmatrix}$	$\begin{bmatrix}-1\\1\\0\end{bmatrix}$	$\begin{bmatrix}-1\\0\\0\end{bmatrix}$	$\begin{bmatrix}0\\-1\\1\end{bmatrix}$	$\begin{bmatrix}0\\0\\-1\end{bmatrix}$
facet defining inequalities	1	−1	0	0	1	0	0	0
	0	1	−1	0	0	0	1	0
	1	0	0	−1	0	0	1	1
	0	0	0	0	1	−1	1	1
	−1	0	1	1	0	0	−1	0
	0	0	1	0	−1	1	−1	0
	0	0	0	1	0	1	−1	−1

Example 2.60. We can exploit Corollary 2.58 to show that certain height functions belong to the (interior of the) deformation cone of $\mathcal{F}(\mathcal{B})$ and recover some classical constructions of the nestohedron, generalizing Example 2.26.

(i) Consider the height function $\boldsymbol{h} \in \mathbb{R}^{\mathcal{B}}$ given by $\boldsymbol{h}_B := -3^{|B|}$. Then for any exchange frame (B, B', P) of $\mathcal{B}$, we have

$$\begin{aligned}\langle \boldsymbol{n}(B, B'),\ \boldsymbol{h}\rangle &= -3^{|B|} - 3^{|B'|} - \sum_{K \in \kappa(P \smallsetminus (B \cup B'))} 3^{|K|} + 3^{|P|} + \sum_{K \in \kappa(B \cap B')} 3^{|K|} \\ &\geq -2 \cdot 3^{|B \cup B'|-1} - 3^{|P \smallsetminus (B \cup B'))|} + 3^{|P|} > 0.\end{aligned}$$

Therefore, the height function $\boldsymbol{h}$ belongs to the interior of the deformation cone $\mathbb{DC}(\mathcal{F}(\mathcal{B}))$. The corresponding polytope $P_{\boldsymbol{h}} := \{\boldsymbol{x} \in \mathbb{R}^V \;;\; \langle \boldsymbol{g}(B),\ \boldsymbol{x}\rangle \le \boldsymbol{h}_B \text{ for } B \in \mathcal{B}\}$ was constructed in [Pil17], generalizing the graph associahedra of [Dev09].

(ii) Consider the height function $\boldsymbol{h} \in \mathbb{R}^{\mathcal{B}}$ given by $\boldsymbol{h}_B := -|\{C \in \mathcal{B} \;;\; C \subseteq B\}|$. Then for any exchange frame (B, B', P) of $\mathcal{B}$, we have

$$\langle \boldsymbol{n}(B, B', P),\ \boldsymbol{h}\rangle = |\,\{C \in \mathcal{B} \;;\; C \not\subseteq B,\ C \not\subseteq B' \text{ and } C \not\subseteq P \smallsetminus (B \cup B') \text{ but } C \subseteq P\}\,| > 0$$

since P fulfills the conditions on C. Thus, the height function $\boldsymbol{h}$ belongs to the interior of the deformation cone $\mathbb{DC}(\mathcal{F}(\mathcal{B}))$. The polytope $P_{\boldsymbol{h}} := \{\boldsymbol{x} \in \mathbb{R}^V \;;\; \langle \boldsymbol{g}(B),\ \boldsymbol{x}\rangle \le \boldsymbol{h}_B \text{ for } B \in \mathcal{B}\}$ is the nestohedron constructed by A. Postnikov's in [Pos09].

Note that many inequalities of Corollary 2.58 are redundant. In the remaining of this section, we describe the facet-defining inequalities of $\mathbb{DC}(\mathcal{F}(\mathcal{B}))$. We say that an exchange frame (B, B', P) is

- *extremal* if its corresponding inequality in Corollary 2.58 defines a facet of $\mathbb{DC}(\mathcal{F}(\mathcal{B}))$,
- *maximal* if B and B' are both maximal building blocks in P as in Proposition 2.51.

We can now state our main result on nested complexes, generalizing Theorem 2.27.

Theorem 2.61. *An exchange frame is extremal if and only if it is maximal.*

Proof. We treat separately the two implications:

Extremal $\Rightarrow$ *maximal.* Consider an exchange frame (B, B', P) of $\mathcal{B}$, and fix pivot vertices v, v' satisfying the conditions of Proposition 2.48. We assume that this frame is not maximal, and prove that it is not extremal by showing that the normal vector $\boldsymbol{n}(B, B', P)$ of the corresponding inequality of the deformation cone $\mathbb{DC}(\mathcal{F}(\mathcal{B}))$ is a positive linear combination of normal vectors of some other exchange frames. By symmetry, we can assume that there is $M \in \mathcal{B}$ such that $B \subsetneq M \subsetneq P$ and we can assume that M is maximal for this property. We decompose the proof into two cases, depending on whether $B' \subseteq M$ or $B' \not\subseteq M$.

Case 1: $B' \subseteq M$. Observe first that:

- (B, B', M) is an exchange frame, since (B, B', P) is an exchange frame and $B \cup B' \subseteq M \subseteq P$,
- (M, W, P) is an exchange frame for any connected component W of $P \smallsetminus (B \cup B')$ containing a vertex $w \in P \smallsetminus M$. Indeed, we just check the conditions of Proposition 2.48 for $v \in M \smallsetminus W$ and $w \in W \smallsetminus M$:
  - for any $M \vdash C \subseteq P$, we have $w \in P \smallsetminus M \subseteq C$ by maximality of M.
  - for any $W \vdash C' \subseteq P$, we have $C' \subseteq P$ and $C' \not\subseteq W$, hence $C' \cap (B \cup B') \neq \varnothing$ since W is a connected component of $P \smallsetminus (B \cup B')$. Assume for instance that $C' \cap B \neq \varnothing$ (the proof for $C' \cap B' \neq \varnothing$ is symmetric). Since $C' \cap W \neq \varnothing$, we obtain that $B \vdash C' \subseteq P$ and thus $v' \in C'$ by Proposition 2.48. We therefore obtain that $B' \vdash C' \subseteq P$ and thus $v \in C'$ by Proposition 2.48 again.

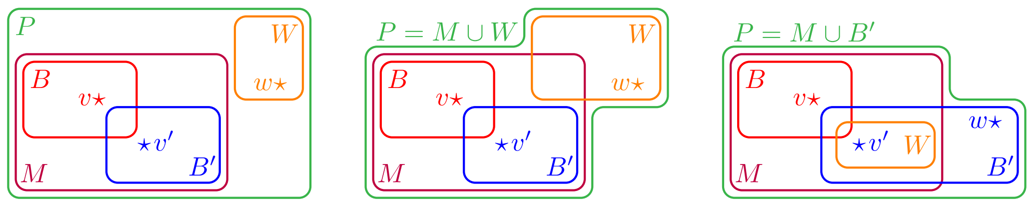
We claim that these two exchange frames enable us to write

$$\boldsymbol{n}(B, B', P) = \boldsymbol{n}(B, B', M) + \boldsymbol{n}(M, W, P).$$

Proving this identity amounts to check that

$$\kappa(P \smallsetminus (B \cup B')) \sqcup \kappa(M \cap W) = \kappa(M \smallsetminus (B \cup B')) \sqcup \kappa(P \smallsetminus (M \cup W)) \sqcup \{W\}. \tag{10}$$

For this, we distinguish two subcases, depending on whether or not M and W intersect.



Case 1.1 Case 1.2 Case 2

Figure 21: Illustrations for the case analysis of the proof of Theorem 2.61.

Subcase 1.1: $M \cap W = \varnothing$. See Figure 21 (left). First, we claim that either $C \cap M = \varnothing$ or $C \subseteq M$ for any $C \in \mathcal{B}$ with $C \subseteq P \smallsetminus (B \cup B')$. Indeed, if $C \cap M \neq \varnothing$, then $C \cap W = \varnothing$ since $M \cap W = \varnothing$ and W is a connected component of $P \smallsetminus (B \cup B')$. Hence $C \cup M \in \mathcal{B}$ and $B \subsetneq C \cup M \subsetneq P$, and thus $C \subseteq M$ by maximality of M. We therefore obtain that

$$\kappa(P \smallsetminus (B \cup B')) = \kappa(M \smallsetminus (B \cup B')) \sqcup \kappa(P \smallsetminus (M \cup W)) \sqcup \{W\}.$$

This shows Equation (10) since $M \cap W = \varnothing$.

Subcase 1.2: $M \cap W \neq \varnothing$. See Figure 21 (middle). As $M \cup W \in \mathcal{B}$ and $B \subsetneq B \cup W \subseteq P$ and $W \not\subseteq M$, we have $P = M \cup W$ by maximality of M. Since $W \in \kappa(P \smallsetminus (B \cup B'))$, we have

$$\kappa(P \smallsetminus (B \cup B')) = \kappa(P \smallsetminus (B \cup B' \cup W)) \sqcup \{W\} = \kappa(M \smallsetminus (B \cup B' \cup W)) \sqcup \{W\}$$

Moreover, by maximality of W, we obtain that there is no block of $\mathcal{B}$ contained in $M \smallsetminus (B \cup B')$ and meeting both $M \cap W$ and $M \smallsetminus (B \cup B' \cup W)$. Hence

$$\kappa(M \cap W) \sqcup \kappa(M \smallsetminus (B \cup B' \cup W)) = \kappa(M \smallsetminus (B \cup B')).$$

Combining these two identities proves Equation (10) since $P = M \cup W$.

Case 2: $B' \not\subseteq M$. See Figure 21 (right). Observe that:

- (M, B', P) is an exchange frame. Indeed, we just check the conditions of Proposition 2.48 for $v \in M \smallsetminus B'$ and an arbitrary $w \in B' \smallsetminus M$:
  - for any $M \vdash C \subseteq P$, we have $w \in P \smallsetminus M \subseteq C$ by maximality of M.
  - for any $B' \vdash C' \subseteq P$, we have $v \in C'$ by Proposition 2.48.
- (B, W, M) is an exchange frame for the connected component W of $M \cap B'$ containing v'. Indeed, we just check the conditions of Proposition 2.48 for $v \in B \smallsetminus W$ and $v' \in W \smallsetminus B$:
  - for any $B \vdash C \subseteq M$, we have $B \vdash C \subseteq P$ and thus $v' \in C$ by Proposition 2.48.
  - for any $W \vdash C' \subseteq M$, we have $B' \vdash C' \subseteq P$ and thus $v \in C'$ by Proposition 2.48.

We claim that these two exchange frames enable to write

$$\boldsymbol{n}(B, B', P) = \boldsymbol{n}(M, B', P) + \boldsymbol{n}(B, W, M).$$

Proving this identity amounts to check that

$$\kappa(P \smallsetminus (B \cup B')) \sqcup \kappa(M \cap B') \sqcup \kappa(B \cap W) = \kappa(B \cap B') \sqcup \kappa(P \smallsetminus (M \cup B')) \sqcup \kappa(M \smallsetminus (B \cup W)) \sqcup \{W\}. \quad (11)$$

To prove this, we observe that:

- Since W contains v', Proposition 2.48 ensures that there is no block of $\mathcal{B}$ contained in $M \cap B'$ and meeting both B and $B' \smallsetminus (B \cup W)$. Since $W \in \kappa(M \cap B')$, we thus obtain

$$\kappa(M \cap B') = \kappa((M \cap B') \smallsetminus (B \cup W)) \sqcup \kappa(B \cap B' \smallsetminus W) \sqcup \{W\}.$$

- As $W \in \kappa(M \cap B')$, there is no block of $\mathcal{B}$ contained in $B \cap B'$ and meeting both $B \cap W$ and $B \cap B' \smallsetminus W$, hence

$$\kappa(B \cap W) \sqcup \kappa(B \cap B' \smallsetminus W) = \kappa(B \cap B').$$

- There is no block of $\mathcal{B}$ contained in $M \smallsetminus (B \cup W)$ and meeting both $M \smallsetminus (B \cup B')$ and $(M \cap B') \smallsetminus (B \cup W)$ (such a block C would satisfy $B' \vdash C \subseteq P$ and $v \notin C$, contradicting Proposition 2.48). Hence

$$\kappa(M \smallsetminus (B \cup B')) \sqcup \kappa((M \cap B') \smallsetminus (B \cup W)) = \kappa(M \smallsetminus (B \cup W)).$$

Combining these three identities proves (11) since $P = M \cup B'$ by maximality of M.

Maximal $\Rightarrow$ extremal. Let (B, B', P) be a maximal exchange frame. To prove that (B, B', P) is extremal, we will construct a vector $\boldsymbol{w} \in \mathbb{R}^{\mathcal{B}}$ such that $\langle \boldsymbol{n}(B, B', P),\ \boldsymbol{w} \rangle < 0$, but at the same time $\left\langle \boldsymbol{n}(\tilde{B}, \tilde{B}', \tilde{P}),\ \boldsymbol{w} \right\rangle > 0$ for any maximal exchange frame $(\tilde{B}, \tilde{B}', \tilde{P})$ with $\boldsymbol{n}(B, B', P) \neq \boldsymbol{n}(\tilde{B}, \tilde{B}', \tilde{P})$. This will show that the inequality induced by (B, B', P) is not redundant. Remember from Propositions 2.56 and 2.57 that, as (B, B', P) and $(\tilde{B}, \tilde{B}', \tilde{P})$ are maximal exchange frames, $\boldsymbol{n}(B, B', P) \neq \boldsymbol{n}(\tilde{B}, \tilde{B}', \tilde{P})$ if and only if $P \neq \tilde{P}$, or $P = \tilde{P}$ is not an elementary block.

Define $\alpha(B, B', P) := \{C \in \mathcal{B} \ ;\ C \not\subseteq B,\ C \not\subseteq B' \text{ and } C \not\subseteq P \smallsetminus (B \cup B') \text{ but } C \subseteq P\}$. Define three vectors $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \mathbb{R}^{\mathcal{B}}$ by

$$\begin{aligned}
\boldsymbol{x}_C &:= -\left|\left\{D \in \mathcal{B} \smallsetminus \alpha(B, B', P) \ ;\ D \subseteq C\right\}\right|, \\
\boldsymbol{y}_C &:= -\left|\left\{D \in \alpha(B, B', P) \ ;\ D \subseteq C\right\}\right|, \\
\boldsymbol{z}_C &:= \begin{cases} -1 & \text{if } B \subseteq C \text{ or } B' \subseteq C, \\ 0 & \text{otherwise,} \end{cases}
\end{aligned}$$

for each bock $C \in \mathcal{B}$.

We will prove below that their scalar products with $\boldsymbol{n}(\tilde{B}, \tilde{B}', \tilde{P})$ for any maximal exchange frame $(\tilde{B}, \tilde{B}', \tilde{P})$ satisfy the following inequalities

	$\left\langle \boldsymbol{n}(\tilde{B}, \tilde{B}', \tilde{P}),\ \boldsymbol{x} \right\rangle$	$\left\langle \boldsymbol{n}(\tilde{B}, \tilde{B}', \tilde{P}),\ \boldsymbol{y} \right\rangle$	$\left\langle \boldsymbol{n}(\tilde{B}, \tilde{B}', \tilde{P}),\ \boldsymbol{z} \right\rangle$
if $\boldsymbol{n}(B, B', P) = \boldsymbol{n}(\tilde{B}, \tilde{B}', \tilde{P})$	$= 0$	$= \lvert\alpha(B, B', P)\rvert$	$= -1$
if $\alpha(\tilde{B}, \tilde{B}', \tilde{P}) \not\subseteq \alpha(B, B', P)$	≥ 1	≥ 0	≥ -1
otherwise	≥ 0	≥ 1	≥ 0

It immediately follows from this table that the vector $\boldsymbol{w} := \boldsymbol{x} + \delta \boldsymbol{y} + \varepsilon \boldsymbol{z}$ fulfills the desired properties for any δ, ε such that $0 < \delta \cdot |\alpha(B, B', P)| < \varepsilon < 1$.

The equalities of the table are immediate. To prove the inequalities, observe that for any maximal exchange frame $(\tilde{B}, \tilde{B}', \tilde{P})$,

- $\left\langle \boldsymbol{n}(\tilde{B}, \tilde{B}', \tilde{P}),\ \boldsymbol{x} \right\rangle \geq |\alpha(\tilde{B}, \tilde{B}', \tilde{P}) \smallsetminus \alpha(B, B', P)|$,

- $\left\langle \boldsymbol{n}(\tilde{B}, \tilde{B}', \tilde{P}),\ \boldsymbol{y} \right\rangle \geq |\alpha(\tilde{B}, \tilde{B}', \tilde{P}) \cap \alpha(B, B', P)|$,

- $\left\langle \boldsymbol{n}(\tilde{B}, \tilde{B}', \tilde{P}),\ \boldsymbol{z} \right\rangle \geq -1$. Indeed, observe that $\boldsymbol{z}_{\tilde{P}} = -1$ as soon as $\boldsymbol{z}_{\tilde{K}} = -1$ for some $\tilde{K} \in \{\tilde{B}, \tilde{B}'\} \sqcup \kappa(\tilde{P} \smallsetminus (\tilde{B} \cup \tilde{B}'))$. This already implies that $\left\langle \boldsymbol{n}(\tilde{B}, \tilde{B}', \tilde{P}),\ \boldsymbol{z} \right\rangle \geq -1$ except if $\boldsymbol{z}_{\tilde{K}} = \boldsymbol{z}_{\tilde{K}'} = \boldsymbol{z}_{\tilde{K}''} = -1$ for three distinct $\tilde{K}, \tilde{K}', \tilde{K}'' \in \{\tilde{B}, \tilde{B}'\} \sqcup \kappa(\tilde{P} \smallsetminus (\tilde{B} \cup \tilde{B}'))$. But since $\tilde{B}$ and $\tilde{B}'$ are the only intersecting blocks among $\{\tilde{B}, \tilde{B}'\} \sqcup \kappa(\tilde{P} \smallsetminus (\tilde{B} \cup \tilde{B}'))$, the only option (up to permutation) is that $\tilde{K} = \tilde{B}$ and $\tilde{K}' = \tilde{B}'$ both contain B (resp. B'), K'' contains B' (resp. B), while none of the other blocks of $\{\tilde{B}, \tilde{B}'\} \sqcup \kappa(\tilde{P} \smallsetminus (\tilde{B} \cup \tilde{B}'))$ meets $B \cup B'$. This implies that $\boldsymbol{z}_{\tilde{P}} = -1 = \boldsymbol{z}_L$ for some $L \in \kappa(\tilde{B} \cap \tilde{B}')$, and thus $\left\langle \boldsymbol{n}(\tilde{B}, \tilde{B}', \tilde{P}),\ \boldsymbol{z} \right\rangle \geq -1$.

- $\left\langle \boldsymbol{n}(\tilde{B}, \tilde{B}', \tilde{P}),\ \boldsymbol{z} \right\rangle \geq 0$ when $\boldsymbol{n}(B, B', P) \neq \boldsymbol{n}(\tilde{B}, \tilde{B}', \tilde{P})$ but $\alpha(\tilde{B}, \tilde{B}', \tilde{P}) \subseteq \alpha(B, B', P)$. Indeed, $\alpha(\tilde{B}, \tilde{B}', \tilde{P}) \subseteq \alpha(B, B', P)$ implies that $\tilde{P} \subseteq P$. Let $\tilde{K} \in \{\tilde{B}, \tilde{B}'\} \sqcup \kappa(\tilde{P} \smallsetminus (\tilde{B} \cup \tilde{B}'))$. If $B \subseteq \tilde{K}$, then $B \subseteq \tilde{K} \subsetneq \tilde{P} \subseteq P$ which implies that $B = \tilde{K}$ and $P = \tilde{P}$ by maximality of B in P. Similarly, $B' \subseteq \tilde{K}$ implies $B' = \tilde{K}$ and $P = \tilde{P}$. Hence, if $\boldsymbol{z}_{\tilde{K}} = -1$, then by definition $B \subseteq \tilde{K}$ or $B' \subseteq \tilde{K}$, which implies that $\tilde{K} \in \{B, B'\}$. Hence, if $\tilde{K} \neq \tilde{K}'$ are two distinct blocks of $\{\tilde{B}, \tilde{B}'\} \sqcup \kappa(\tilde{P} \smallsetminus (\tilde{B} \cup \tilde{B}'))$ such that $\boldsymbol{z}_{\tilde{K}} = -1 = \boldsymbol{z}_{\tilde{K}'}$, then $(B, B', P) = (\tilde{K}, \tilde{K}', \tilde{P})$ and moreover either $\{B, B'\} = \{\tilde{K}, \tilde{K}'\}$, or $\tilde{K} \cap \tilde{K}' = \varnothing$, so that P is elementary by Remark 2.36 since it has two disjoint maximal blocks. In both cases, we obtain $\boldsymbol{n}(B, B', P) = \boldsymbol{n}(\tilde{B}, \tilde{B}', \tilde{P})$ by Proposition 2.57, contradicting our assumption. Therefore, at most one of $\boldsymbol{z}_{\tilde{K}}$ for $\tilde{K} \in \{\tilde{B}, \tilde{B}'\} \sqcup \kappa(\tilde{P} \smallsetminus (\tilde{B} \cup \tilde{B}'))$ equals to -1, and if exactly one does, then $\boldsymbol{z}_{\tilde{P}} = -1$. We conclude that $\left\langle \boldsymbol{n}(\tilde{B}, \tilde{B}', \tilde{P}),\ \boldsymbol{z} \right\rangle \geq 0$. □

We derive from Theorem 2.61 the facet description of the deformation cone $\mathbb{DC}(\mathcal{F}(\mathcal{B}))$. Remember that we denote by $\mu(P)$ the maximal blocks of $\mathcal{B}$ strictly contained in a block $P \in \mathcal{B}$.

Corollary 2.62. *The inequalities*

- $\sum_{B \in \mu(P)} \boldsymbol{h}_B \geq \boldsymbol{h}_P$ *for any elementary block P of $\mathcal{B}$,*
- $\boldsymbol{h}_B + \boldsymbol{h}_{B'} + \sum_{K \in \kappa(P \smallsetminus (B \cup B'))} \boldsymbol{h}_K \geq \boldsymbol{h}_P + \sum_{K \in \kappa(B \cap B')} \boldsymbol{h}_K$ *for any block P of $\mathcal{B}$ neither singleton nor elementary, and any two blocks $B \neq B'$ in $\mu(P)$,*

provide an irredundant facet description of the deformation cone $\mathbb{DC}(\mathcal{F}(\mathcal{B}))$.

Corollary 2.63. *The number of facets of the deformation cone $\mathbb{DC}(\mathcal{F}(\mathcal{B}))$ is*

$$|\varepsilon(\mathcal{B})| + \sum_P \binom{\mu(P)}{2}$$

where the sum runs over all blocks P of $\mathcal{B}$ which are neither singletons nor elementary blocks. The dimension of $\mathbb{DC}(\mathcal{F}(\mathcal{B}))$ is $|\mathcal{B}| - |\kappa(\mathcal{B})|$, and its lineality $|V| - |\kappa(\mathcal{B})|$.

Example 2.64. As for Examples 2.14 and 2.15, we can portray the deformation cone of a nestohedron. For the connected building set on 4 elements $\mathcal{B} = \{1, 2, 3, 4, 12, 123, 234, 1234\}$, the corresponding 3-dimensional nestohedron $\mathsf{Nest}_{\mathcal{B}}$ is depicted in Figure 22 as the Minkowski sum of faces of the standard simplex (faces corresponding to vertices of the standard simplex are omitted as they only account for translations in the Minkowski sum).

The deformation cone $\mathbb{DC}(\mathcal{F}(\mathcal{B}))$ has dimension $|\mathcal{B}| - |\kappa(\mathcal{B})| = 7$. Nonetheless, it has $|V| - |\kappa(\mathcal{B})| = 3$ dimensions of lineality. Thus, after intersecting it with a hyperplane, we can picture it in dimension 3 in Figure 23. By the above Corollary 2.63, it has 6 facets. Each vertex of the drawn bi-pyramid correspond to a ray of the deformation cone $\mathbb{DC}(\mathcal{F}(\mathcal{B}))$, *i.e.* to a Minkowski indecomposable polytope. Among them are the four simplices involved in the defining Minkowski sum $\mathsf{Nest}_{\mathcal{B}} = \Delta_{12} + \Delta_{134} + \Delta_{234} + \Delta_{1234}$. However, note that these polytopes do not have all the same dimension (even though they all live in $\mathbb{R}^3$ as deformations of $\mathsf{Nest}_{\mathcal{B}}$), and the last ray (rightmost polytope in Figure 23) do not correspond to a simplex. On the other side, the interior of $\mathbb{DC}(\mathcal{F}(\mathcal{B}))$ correspond to polytopes normally equivalent to $\mathsf{Nest}_{\mathcal{B}}$.

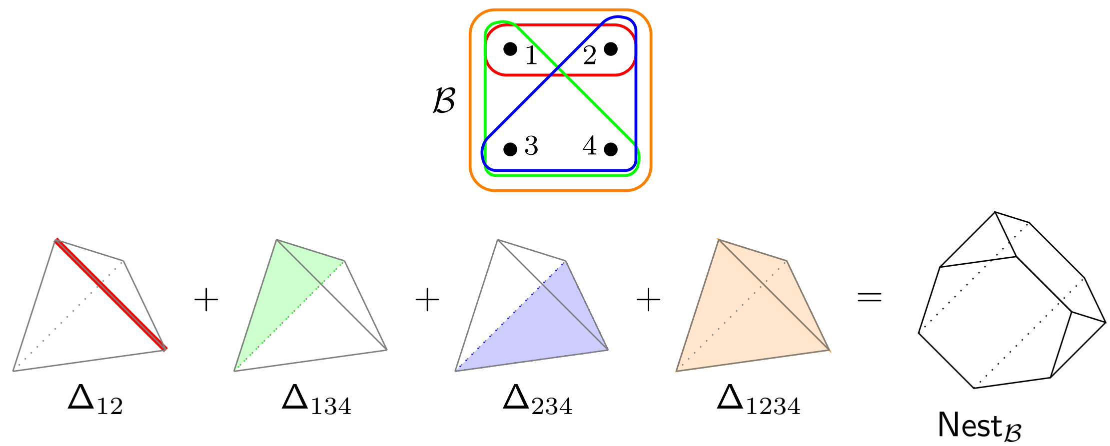


Figure 22: (Top) A building set $\mathcal{B}$ on 4 elements with 4 (no-singleton) blocks, (Bottom) the corresponding $\mathsf{Nest}_{\mathcal{B}}$ described as the Minkowski sum of faces of the standard simplex Δ_{1234}.

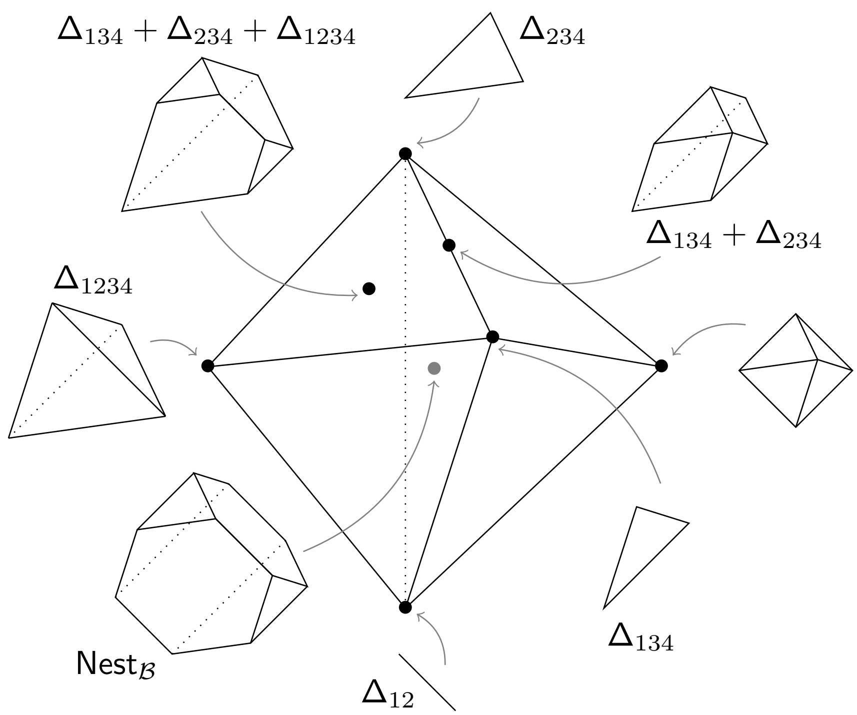


Figure 23: A 3-dimensional affine section of the deformation cone $\mathbb{DC}(\mathcal{F}(\mathcal{B}))$ for the building set of Example 2.64 and Figure 22. The deformations of $\mathsf{Nest}_{\mathcal{B}}$ corresponding to some of the points of $\mathbb{DC}(\mathcal{F}(\mathcal{B}))$ are depicted. Especially, all points in the interior correspond to polytopes normally equivalent to $\mathsf{Nest}_{\mathcal{B}}$, while the rightmost polytope is not a simplex. See also Figure 12 for the case of the complete graph on 3 vertices.

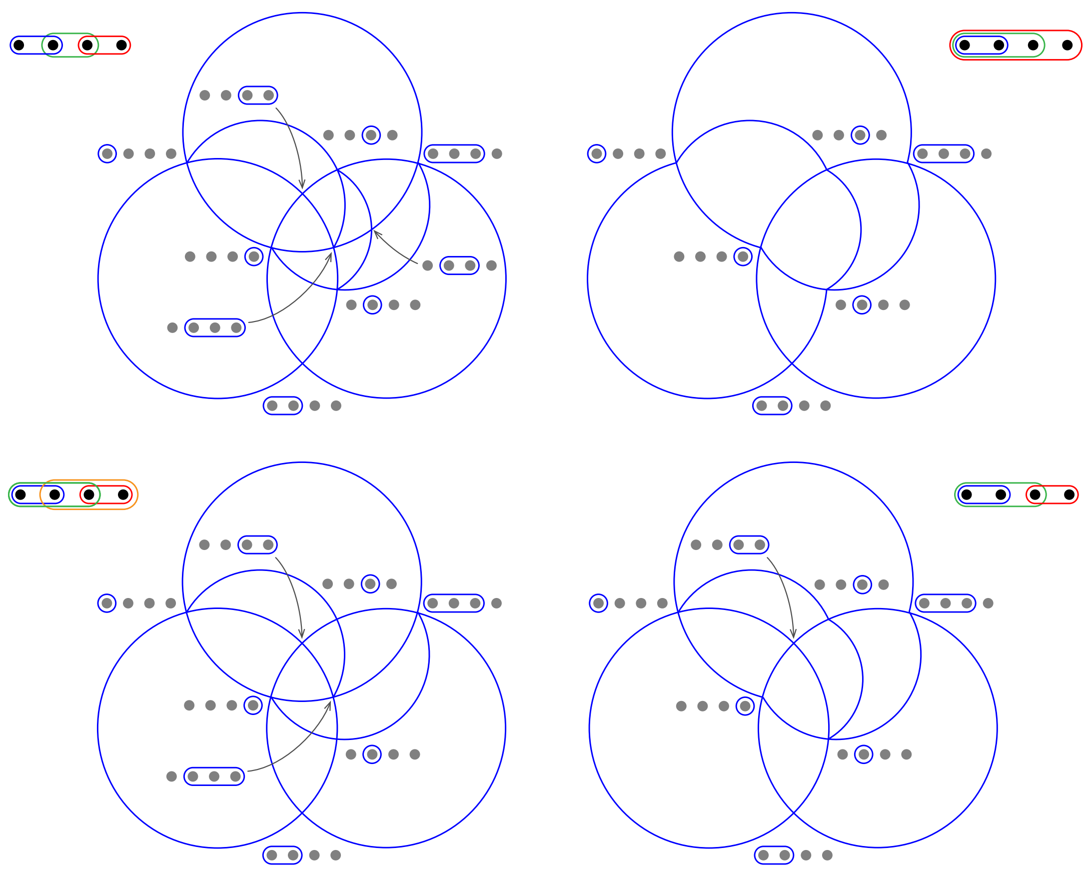


Figure 24: Four interval nested fans. The top left one is the sylvester fan, the top right one is the Pitman-Stanley fan, the bottom left one is the freehedron fan, and only the top right one is a fertilotope fan. The rays are labeled by the corresponding blocks. As the fans are 3-dimensional, we intersect them with the sphere and stereographically project them from the direction $(-1,-1,-1)$.

2.3.3 Simplicial deformation cones and interval building sets

To conclude this section, we characterize the building sets $\mathcal{B}$ whose nested fan $\mathcal{F}(\mathcal{B})$ has a simplicial deformation cone and study in more details a specific family of such building sets.

Proposition 2.65. *The deformation cone $\mathbb{DC}(\mathcal{F}(\mathcal{B}))$ is simplicial if and only if all blocks of $\mathcal{B}$ with at least three distinct maximal strict sub-blocks are elementary.*

Proof. Recall that the nested fan $\mathcal{F}(\mathcal{B})$ has dimension $|V|-|\kappa(\mathcal{B})|$ and has $|\mathcal{B}|-|\kappa(\mathcal{B})|$ rays. Hence, the deformation cone $\mathbb{DC}(\mathcal{F}(\mathcal{B}))$ is simplicial if and only if it has $|\mathcal{B}|-|V|$ facets. The statement thus immediately follows from Corollary 2.63. $\square$

We conclude this section by focussing on the following special family of building sets which fulfills Proposition 2.65 and is illustrated in Figure 24.

Definition 2.66. An *interval building set* is a building set on $[n] := \{1,\dots,n\}$ whose blocks are some intervals. We call *interval nested fan* and *interval nestohedron* the nested fan and nestohedron of an interval building set.

Example 2.67. Particularly relevant examples of interval nestohedra include:

- the classical associahedron of [SS93, Lod04, PSZ23] for the building set with all intervals of $[n]$,

- the Pitman-Stanley polytope of [SP02] for the building set with all singletons $\{i\}$ and all initial intervals $[i]$ for $i \in [n]$,
- the freehedron of [San09] for the building set with all singletons $\{i\}$, all initial intervals $[i]$ for $i \in [n]$, and all final intervals $[n] \smallsetminus [i]$ for $i \in [n-1]$,
- the fertilotopes of [Def21] for the binary building sets defined as the interval building sets where any two intervals are either nested or disjoint.

Note that, by definition, any interval nested fan coarsens the associahedron nested fan, so all the above examples are deformations of the associahedron.

Proposition 2.68. *For any interval building set $\mathcal{B}$, the deformation cone $\mathbb{DC}(\mathcal{F}(\mathcal{B}))$ is simplicial.*

Proof. We give two proofs of this fact.

(Proof with the tools of this section). Assume that $\mathcal{B}$ has a non-elementary block $[i, j]$, with at least three distinct maximal strict sub-blocks $[a, b]$, $[c, d]$ and $[e, f]$. Since $[a, b]$, $[c, d]$ and $[e, f]$ are pairwise non nested, we can assume up to permutation that $a < c < e$ and $b < d < f$. Since $[i, j]$ is not elementary, $[a, b] \cap [c, d] \neq \varnothing$ and thus $[a, b] \cup [c, d] = [a, d]$ is a block of $\mathcal{B}$. This contradicts the maximality of $[a, b]$ since $[a, b] \subsetneq [a, d] \subsetneq [i, j]$ as $b < d < f \le j$.

(Proof with using deformations). As interval nestohedra are deformation of the associahedron, by Proposition 2.4 their deformation cone is a face of the deformation cone of the associahedron which is simplicial (see Proposition 2.34). □

Remark 2.69. Note that there are building sets $\mathcal{B}$ for which the deformation cone $\mathbb{DC}(\mathcal{F}(\mathcal{B}))$ is simplicial, but which are not (isomorphic to) interval building sets. See *e.g.* Figure 19(Left).

We now translate the facet description of Corollary 2.62 to the specific case of interval building sets. We need a few additional notations. Consider an interval building set $\mathcal{B}$ on $[n]$. For $1 \le i < j \le n$, define

$$\ell(i,j) := \min\{k \in [i+1, j] \ ; \ [k, j] \in \mathcal{B}\} \qquad \text{and} \qquad r(i,j) := \max\{k \in [i, j-1] \ ; \ [i, k] \in \mathcal{B}\}.$$

Note that $\ell(i, j)$ and $r(i, j)$ are well-defined since $\mathcal{B}$ contain all singletons. Observe that $[i, r(i, j)]$ and $[\ell(i, j), j]$ are maximal strict subblocs of $[i, j]$. Therefore,

- if $[i, j] \in \mathcal{B}$ is elementary, then we have $r(i, j) < \ell(i, j)$ and the maximal strict sub-blocks of $[i, j]$ are the intervals $[s_{k-1}(i, j), s_k(i, j) - 1]$ for $k \in [p]$ where the sequence $s_0(i, j) < s_1(i, j) < \cdots < s_p(i, j)$ is defined by the boundary conditions $s_0(i, j) := i$ and $s_1(i, j) = r(i, j) + 1$ and $s_p(i, j) := j + 1$, and the induction $s_k(i, j) := r(s_{k-1}(i, j), j+1)+1$.
- if $[i, j] \in \mathcal{B}$ is not elementary, we have $\ell(i, j) \le r(i, j)$ so that

$$[i, r(i, j)] \cup [\ell(i, j), j] = [i, j] \qquad \text{and} \qquad [i, r(i, j)] \cap [\ell(i, j), j] = [\ell(i, j), r(i, j)].$$

  Thus $[i, r(i, j)]$ and $[\ell(i, j), j]$ are the only maximal strict sub-blocks of $[i, j]$. Moreover, the connected components of $[i, r(i, j)] \cap [\ell(i, j), j] = [\ell(i, j), r(i, j)]$ are the intervals $[t_{k-1}(i, j), t_k(i, j) - 1]$ for $k \in [q]$ where the sequence $t_0(i, j) < t_1(i, j) < \cdots < t_q(i, j)$ is defined by the boundary conditions $t_0(i, j) := \ell(i, j)$ and $t_q(i, j) := r(i, j) + 1$, and the induction $t_k(i, j) := r(t_{k-1}(i, j), r(i, j) + 1) + 1$.

Using these notations, the following statement is just a translation of Corollary 2.62.

Proposition 2.70. *Consider an interval building set $\mathcal{B}$ on $[n]$ and let $\mathcal{B}^\star := \mathcal{B} \smallsetminus \{\{i\} \ ; \ i \in [n]\}$ denote the blocks which are not singletons. Then the inequalities*

- $\sum_{k\in[p]} \boldsymbol{h}_{[s_{k-1}(i,j), s_k(i,j)-1]} \ge \boldsymbol{h}_{[i,j]}$ *for all* $[i, j] \in \mathcal{B}^\star$ *with* $r(i, j) < \ell(i, j)$,
- $\boldsymbol{h}_{[i,r(i,j)]} + \boldsymbol{h}_{[\ell(i,j),j]} \ge \boldsymbol{h}_{[i,j]} + \sum_{k\in[q]} \boldsymbol{h}_{[t_{k-1}(i,j), t_k(i,j)-1]}$ *for all* $[i, j] \in \mathcal{B}^\star$ *with* $\ell(i, j) \le r(i, j)$,

provide an irredundant facet description of the deformation cone $\mathbb{DC}(\mathcal{F}(\mathcal{B}))$.

Example 2.71. For instance

- for the building set containing all intervals of $[n]$, we have $\ell(i,j) = i+1$ and $r(i,j) = j-1$, so that the facet defining inequalities of the deformation cone are $\boldsymbol{h}_{[i,j-1]} + \boldsymbol{h}_{[i+1,j]} \ge \boldsymbol{h}_{[i,j]} + \boldsymbol{h}_{[i+1,j-1]}$ for all $1 \le i < j \le n$ (with the convention that $\boldsymbol{h}_{[i+1,j-1]} = 0$ for $i+1 = j$), this is the facet description of the deformation cone of the associahedron;
- for the building set containing all singletons $\{i\}$ and all intervals $[i]$ for $i \in [n]$, we have $r(1,j) = j-1 < j = \ell(1,j)$, so that the facet defining inequalities of the deformation cone are $\boldsymbol{h}_{[j-1]} + \boldsymbol{h}_{\{j\}} \ge \boldsymbol{h}_{[j]}$ for all $1 < j \le n$.

Generalizing Proposition 2.35, we finally combine Propositions 2.65 and 2.70 to define *kinematic nestohedra* for interval building sets, similar to the constructions of [AHBHY18, BMDM$^+$18, PPPP19] for associahedra, cluster associahedra and gentle associahedra. Again, these polytopes are just affinely equivalent to the realizations in $\mathbb{R}^n$, but they should be more natural from a mathematical physics' perspective.

Proposition 2.72. *Consider an interval building set* $\mathcal{B}$ *on* $[n]$ *and let* $\mathcal{B}^\star := \mathcal{B} \smallsetminus \{\{i\} \; ; \; i \in [n]\}$ *denote the blocks which are not singletons. Then for any* $\boldsymbol{p} \in \mathbb{R}^{\mathcal{B}^\star}_{>0}$*, the polytope* $R_{\boldsymbol{p}}(\mathcal{B}) \subseteq \mathbb{R}^{\mathcal{B}}$ *defined as the intersection of the positive orthant* $\left\{\boldsymbol{z} \in \mathbb{R}^{\mathcal{B}} \; ; \; \boldsymbol{z} \ge 0\right\}$ *with the hyperplanes*

- $\boldsymbol{z}_K = 0$ *for* $K \in \kappa(\mathcal{B})$*,*
- $\sum_{k \in [p]} \boldsymbol{z}_{[s_{k-1}(i,j), s_k(i,j)-1]} - \boldsymbol{z}_{[i,j]} = \boldsymbol{p}_{[i,j]}$ *for* $[i,j] \in \mathcal{B}^\star$ *with* $r(i,j) < \ell(i,j)$*,*
- $\boldsymbol{z}_{[i,r(i,j)]} + \boldsymbol{z}_{[\ell(i,j),j]} - \boldsymbol{z}_{[i,j]} - \sum_{k \in [q]} \boldsymbol{z}_{[t_{k-1}(i,j), t_k(i,j)-1]} = \boldsymbol{p}_{[i,j]}$ *for* $[i,j] \in \mathcal{B}^\star$ *with* $\ell(i,j) \le r(i,j)$*,*

is a nestohedron whose normal fan is the nested fan $\mathcal{F}(\mathcal{B})$*. Moreover, the polytopes* $R_{\boldsymbol{p}}(\mathcal{B})$ *for* $\boldsymbol{p} \in \mathbb{R}^{\mathcal{B}^\star}_{>0}$ *describe all polytopal realizations of* $\mathcal{F}(\mathcal{B})$ *(up to translations).*

2.3.4 Perspectives and open questions

Computational remarks The computation of deformation cones of nestohedra have been implemented with Sage, allowing us to construct numerous examples that helped us to build the main proofs of this section. Thanks to this code, one can input a building set $\mathcal{B}$ and compute the deformation cone of is nestohedron as the cone of heights in $\mathbb{R}^{\mathcal{B}}$, illustrating Corollary 2.62. As the fan $\mathcal{F}(\mathcal{B})$ is simplicial, the direct implementation of Proposition 2.2 can be used for this computation.

Assets and limits of the current approach, open questions We have computed in this section the deformation cones of two families of generalized permutahedra: graphical zonotopes on the one side, and nestohedra on the other. The final goal would be to understand the full submodular cone, *i.e.* the deformation cone of the permutahedron, answering a longstanding question opened since the 70s [Edm70]. However, two intermediate questions seem to be particularly interesting.

The first one is the computation of the deformation cone of hypergraphic polytopes. Hypergraphic polytopes are the pendant of graphical zonotopes for general hypergraphs: fix a collection H of subsets of $[n]$ and define the associated hypergaphic polytope as the Minkowski sum of the corresponding face of the standard simplex: $\mathsf{P}_H := \sum_{C \in H} \Delta_C$. This family encapsulates both graphical zonotopes and nestohedra. The deformation of a hypergraphic polytope is again a hypergraphic polytope, and this family can be thought of as the sub-cone of the submodular cone generated by the faces of the standard simplex. However, deformations of hypergraphic polytopes are more difficult to handle than the families studied in this section, as their normal fan is not in general simplicial, and the combinatorial resources we can use are not as rich as in the case of graphs. As stated in [PPP22a], we are able to give an explicit basis of the linear span of the deformation cone of the hypergraphic polytopes: as in the case of graphical zonotopes (see Theorem 2.11), it is formed by the *induced cliques* of the hypergraph. We also applied Proposition 2.3

to hypergraphic polytopes, but only obtained a highly redundant description of the deformation cone of hypergraphic polytopes.

Generalized permutahedra go beyond hypergraphic polytopes, and some other families could be both not-too-hard to tackle, and rich enough to improve our knowledge of the submodular cone. In particular, quotientopes [PS19] seem to fall in this category. We have already determined which quotientopes have a simplicial deformation cone, but giving a facet description of their deformation cone in general seems more involved.

Last but not least, even if we focused here on giving a facet-description of interesting faces of the submodular cone, the question of computing its rays remain open since Edmonds [Edm70], even in the cases we studied. It is worth noting that the facet-description we provide allow for computer experiments in higher dimensions than before (for the specific cases of graphical zonotopes and nestohedra), and may help to find new examples of rays for the submodular cone. These rays can be thought of as explicit height vectors, or looked upon as their polytopal counterpart.