2 Deformations of polytopes and generalized permutahedra

"Surely," said I, "surely that is something at my window lattice;
Let me see, then, what thereat is, and this mystery explore
Let my heart be still a moment and this mystery explore;"
– Edgar Allan Poe, *The Raven*

2.1 Deformations of polytopes

In Section 1.2.4, we have seen that a realization of the associahedron, Loday's associahedron, can be retrieved from the standard permutahedron by removing facets. In this construction, edge directions are preserved, and only the normal fan is coarsened. This process embodies the combinatorics of flips of Catalan families inside the graph of the weak order on permutations. This section is devoted to the presentation of the general concept of "gliding facets", and the vast family of polytopes one can obtain from the permutahedron by this construction.

Definition 2.1. A *deformation* (or *weak Minkowski summand*) of a polytope P is a polytope Q whose normal fan $\mathcal{N}_\mathsf{Q}$ coarsens the normal fan of P. The set of deformations of P is called its *deformation cone*:

$$\mathbb{DC}(\mathsf{P}) = \left\{\mathsf{Q} \subset \mathbb{R}^d \ ; \ \mathcal{N}_\mathsf{Q} \trianglelefteq \mathcal{N}_\mathsf{P}\right\}$$

The name *deformation* comes from the pictorial illustration of gliding facets along their normal vectors, see Figure 10. To justify the appellation *cone*, note the following: For Q and R two polytopes, then $\mathcal{N}_{\lambda\mathsf{Q}} = \mathcal{N}_\mathsf{Q}$ for all $\lambda > 0$, and $\mathcal{N}_{\mathsf{Q}+\mathsf{R}}$ is the common refinement of $\mathcal{N}_\mathsf{Q}$ and $\mathcal{N}_\mathsf{R}$. Consequently, $\mathcal{N}_{\lambda\mathsf{Q}}$ and $\mathcal{N}_{\mathsf{Q}+\mathsf{R}}$ coarsen $\mathcal{N}_\mathsf{P}$ when $\mathcal{N}_\mathsf{Q}$ and $\mathcal{N}_\mathsf{R}$ do, which means $\mathbb{DC}(\mathsf{P})$ is closed by dilation and Minkowski sums.

Howbeit, it is hard to understand the deformation cone as a cone of polytopes, one would prefer to parameterize it by a cone in the Euclidean space $\mathbb{R}^N$ for some N. There are several ways to do so, this thesis will focus on the so-called *height deformation cone*, and briefly present the other realizations.

Let $\mathsf{P} \subset \mathbb{R}^d$ be a polytope with normal fan $\mathcal{F}$ supported on the vector set $\boldsymbol{S}$. Let $\boldsymbol{G}$ be the $N \times d$-matrix whose rows are the vectors in $\boldsymbol{S}$. For any height vector $\boldsymbol{h} \in \mathbb{R}^N$, we define the polytope $\mathsf{P}_{\boldsymbol{h}} := \left\{\boldsymbol{x} \in \mathbb{R}^d \ ; \ \boldsymbol{G}\boldsymbol{x} \leq \boldsymbol{h}\right\}$. It is not hard to see that any weak Minkowski summand of P is of the form $\mathsf{P}_{\boldsymbol{h}}$ for some $\boldsymbol{h} \in \mathbb{R}^N$.

Moreover, for deformations $\mathsf{P}_{\boldsymbol{h}}$ and $\mathsf{P}_{\boldsymbol{h}'}$ of P, we have $\mathsf{P}_{\boldsymbol{h}} + \mathsf{P}_{\boldsymbol{h}'} = \mathsf{P}_{\boldsymbol{h}+\boldsymbol{h}'}$ and $\lambda\mathsf{P}_{\boldsymbol{h}} = \mathsf{P}_{\lambda\boldsymbol{h}}$ for any $\lambda > 0$. Hence, the deformation cone is parameterized by the *height deformation cone*:

$$\mathbb{DC}(\mathsf{P}) \simeq \mathbb{DC}_\mathrm{h}(\mathsf{P}) := \left\{\boldsymbol{h} \in \mathbb{R}^N \ ; \ \mathcal{N}_{\mathsf{P}_{\boldsymbol{h}}} \trianglelefteq \mathcal{N}_\mathsf{P}\right\}.$$

Other descriptions of the deformation cones are of theoretical importance. As we will not use them in this thesis, we restrain ourselves to a succinct glimpse of their definitions.

For a polytope P, let $E(\mathsf{P})$ be the set of its edges. To P, one can associate the vector $\boldsymbol{\ell} \in \mathbb{R}^{E(\mathsf{P})}$ of its edge-lengths, where ℓ_e is simply the length of $e \in E(\mathsf{P})$. Conversely, a vector in $\mathbb{R}_+^{E(\mathsf{P})}$ gives rise to a polytope $\mathsf{Q}_{\boldsymbol{\ell}}$: for $e \in E(\mathsf{P})$, choose a direction and denote $\boldsymbol{u}_e$ the unitary vector in this direction, then $\mathsf{Q}_{\boldsymbol{\ell}} = \operatorname{conv}\left\{\sum_{e\in\mathcal{P}} \varepsilon_e^\mathcal{P} \ell_e \boldsymbol{u}_e \ ; \ \mathcal{P} \text{ directed edge-path in } G_\mathsf{P}\right\}$ where $\varepsilon_e^\mathcal{P} = 1$ if the direction of $e \in \mathcal{P}$ is the same as in $\boldsymbol{u}_e$, and $\varepsilon_e^\mathcal{P} = -1$ else way. The deformation cone is isomorphic to the *edge deformation cone* $\left\{\boldsymbol{\ell} \in \mathbb{R}_+^{E(\mathsf{P})} \ ; \ \mathsf{Q}_{\boldsymbol{\ell}} \trianglelefteq \mathcal{N}_\mathsf{P}\right\}$, see [PRW08, Appendix 15] for instance.

On top of that, a deformation of P can also be described as a polytope whose support functional is a convex piece-wise linear continuous function supported on the face fan of P [CLS11, Section 6.1] and [DRS10, Section 9.5]. The deformation cone is isomorphic to the cone of such linear functionals.

From now on, we will slightly abuse notations by using ambiguously the word *deformation cone* to designate the cone of the deformations of a polytope or the height deformation cone. Besides, although we define the deformation cone for a polytope, it only depends on the normal equivalence class of the latter, *i.e.* of its normal fan. Consequently, we will sometimes prefer to talk about the

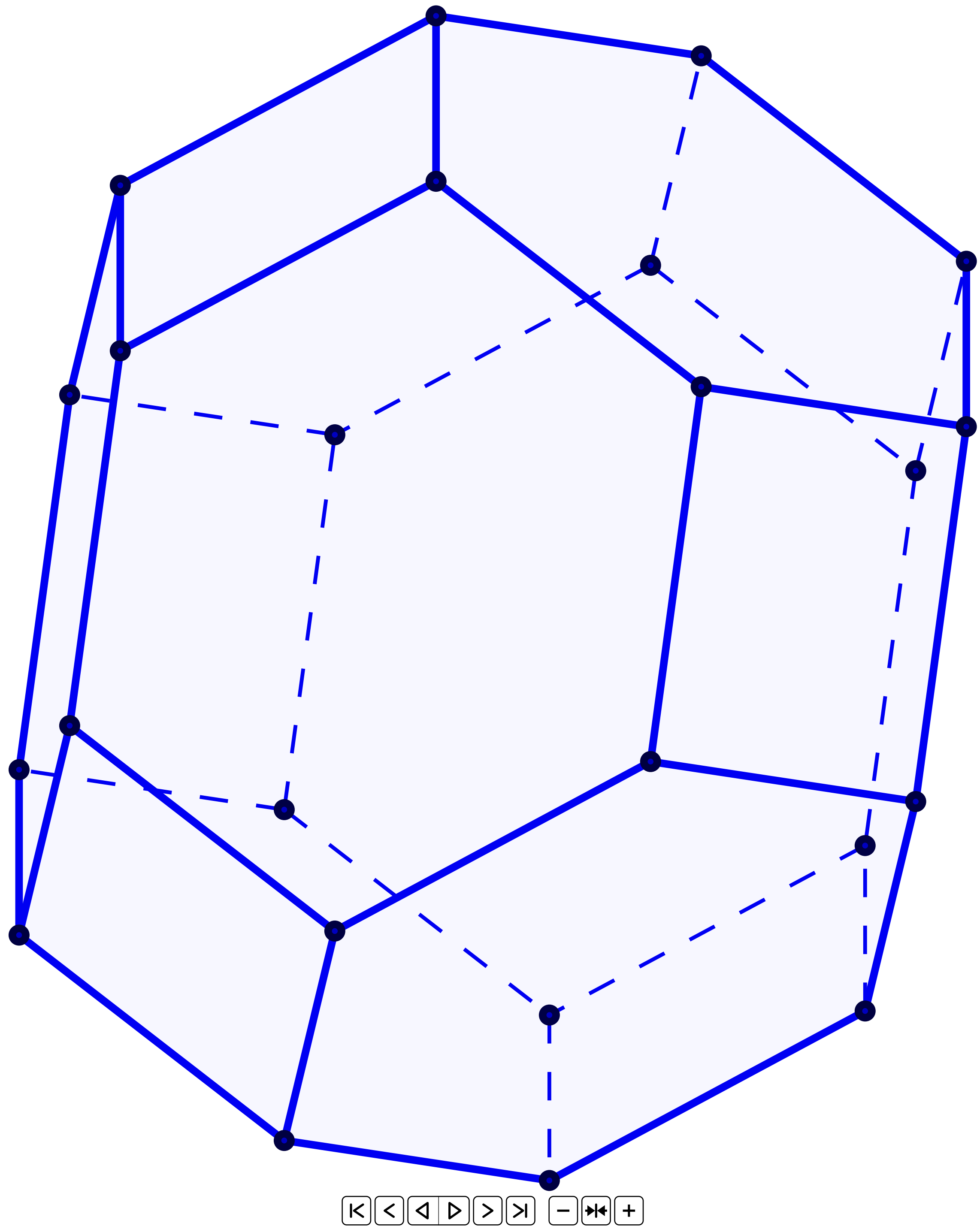

Figure 10: Animated sequence of deformations. The first polyhedron is the permutahedron Π_4 (First frame). One by one, we remove inequalities from its facet-description (by augmenting the constant b in $\langle \boldsymbol{x}, \boldsymbol{a} \rangle \leq b$) to obtain the associahedron Asso_4 (Middle pause), and then pursue the process to obtain a cube linearly isomorphic to $\square_3$ (Final frame). See also Figure 6. *(Animated figures obviously do not display on paper, and some PDF readers do not support the format: it is advised to use Adobe Acrobat Reader. If no solution is suitable, the animation can be found on my website or asked by email.)*

deformation cone of a fan (especially in Section 2.3): when the fan $\mathcal{F}$ is polytopal, then $\mathbb{DC}(\mathcal{F})$ would be the deformation cone $\mathbb{DC}(\mathsf{P})$ of any polytope P with $\mathcal{N}_{\mathsf{P}} = \mathcal{F}$; and $\mathbb{DC}(\mathcal{F}) = \varnothing$ when $\mathcal{F}$ is not polytopal.

The height deformation cone is a polyhedral cone, and the two following propositions give an inequality description of it. The first one is devoted to simple polytopes and will be used for describing the deformation cones of nestohedra in Section 2.3, while the second one deals with general polytopes exploiting a triangulation of their normal fan, allowing a description of the deformation cones of graphical zonotopes in Section 2.2. Note that, in general, these propositions give an inequality description far from being a facet-description: namely, many inequalities are actually redundant.

Proposition 2.2 ([CFZ02, GKZ08]). *Let $\mathsf{P} \subset \mathbb{R}^d$ be a simple polytope with simplicial normal fan $\mathcal{F}$ supported on the rays $\boldsymbol{S}$. Then the deformation cone $\mathbb{DC}(\mathsf{P})$ is the set of polytopes $\mathsf{P}_{\boldsymbol{h}}$ for all $\boldsymbol{h}$ in the cone of $\mathbb{R}^{\boldsymbol{S}}$ defined by the inequalities*

$$\sum_{\boldsymbol{s} \in \boldsymbol{R} \cup \boldsymbol{R}'} \alpha_{\boldsymbol{R},\boldsymbol{R}'}(\boldsymbol{s})\, \boldsymbol{h}_{\boldsymbol{s}} \geq 0$$

for all adjacent maximal cones $\mathbb{R}_{\geq 0}\boldsymbol{R}$ and $\mathbb{R}_{\geq 0}\boldsymbol{R}'$ of $\mathcal{F}$ with $\boldsymbol{R} \smallsetminus \{\boldsymbol{r}\} = \boldsymbol{R}' \smallsetminus \{\boldsymbol{r}'\}$, where $\alpha_{\boldsymbol{R},\boldsymbol{R}'}(\boldsymbol{s})$ denote the coefficients in the unique linear dependence[5]

$$\sum_{\boldsymbol{s} \in \boldsymbol{R} \cup \boldsymbol{R}'} \alpha_{\boldsymbol{R},\boldsymbol{R}'}(\boldsymbol{s})\, \boldsymbol{s} = \boldsymbol{0}$$

among the rays of $\boldsymbol{R} \cup \boldsymbol{R}'$ such that $\alpha_{\boldsymbol{R},\boldsymbol{R}'}(\boldsymbol{r}) + \alpha_{\boldsymbol{R},\boldsymbol{R}'}(\boldsymbol{r}') = 2$.

The edge deformation cone also enjoys an inequality description. Indeed, an edge vector $\boldsymbol{\ell} \in \mathbb{R}_+^E$ corresponds to a deformation of a simple polytope P when it satisfies the *polygonal face equations*: for each 2-dimensional face F of P, $\sum_{e \in E(\mathsf{F})} \ell_e \boldsymbol{u}_e = \boldsymbol{0}$ where the sum is on the edges of F. The edge deformation cone of a simple polytope P is the intersection of $\mathbb{R}_+^{E(\mathsf{P})}$ with the kernel of polygonal face equations [PRW08, Pos09].

Similarly, the cone of convex piece-wise linear continuous functions on the face fan of a simple polytope has an inequality description.

The characterization of the height deformation cone can be extended to general (not necessarily simple) polytopes. One straightforward way to do so is via a simplicial refinement of the normal fan. If such a simplicial refinement contains additional rays, then the type cone will be embedded in a higher dimensional space, but projecting out these additional coordinates gives a linear isomorphism with the standard presentation. See [PS19, Prop. 3] and [PPPP19, Prop. 1.7].

Proposition 2.3. *Let $\mathsf{P} \subset \mathbb{R}^d$ be a polytope whose normal fan $\mathcal{F}$ is refined by the simplicial fan $\mathcal{G}$ supported on the rays $\boldsymbol{S}$. Then the deformation cone $\mathbb{DC}(\mathsf{P})$ is the set of polytopes $\mathsf{P}_{\boldsymbol{h}}$ for all $\boldsymbol{h}$ in the cone of $\mathbb{R}^{\boldsymbol{S}}$ defined by*

- *the equalities $\sum_{\boldsymbol{s} \in \boldsymbol{R} \cup \boldsymbol{R}'} \alpha_{\boldsymbol{R},\boldsymbol{R}'}(\boldsymbol{s})\, \boldsymbol{h}_{\boldsymbol{s}} = 0$ for any adjacent maximal cones $\mathbb{R}_{\geq 0}\boldsymbol{R}$ and $\mathbb{R}_{\geq 0}\boldsymbol{R}'$ of $\mathcal{G}$ belonging to* ***the same*** *maximal cone of $\mathcal{F}$,*
- *the inequalities $\sum_{\boldsymbol{s} \in \boldsymbol{R} \cup \boldsymbol{R}'} \alpha_{\boldsymbol{R},\boldsymbol{R}'}(\boldsymbol{s})\, \boldsymbol{h}_{\boldsymbol{s}} \geq 0$ for any adjacent maximal cones $\mathbb{R}_{\geq 0}\boldsymbol{R}$ and $\mathbb{R}_{\geq 0}\boldsymbol{R}'$ of $\mathcal{G}$ belonging to* ***distinct*** *maximal cones of $\mathcal{F}$,*

where $\sum_{\boldsymbol{s} \in \boldsymbol{R} \cup \boldsymbol{R}'} \alpha_{\boldsymbol{R},\boldsymbol{R}'}(\boldsymbol{s})\, \boldsymbol{s} = \boldsymbol{0}$ is the unique linear dependence with $\alpha_{\boldsymbol{R},\boldsymbol{R}'}(\boldsymbol{r}) + \alpha_{\boldsymbol{R},\boldsymbol{R}'}(\boldsymbol{r}') = 2$ among the rays of two adjacent maximal cones $\mathbb{R}_{\geq 0}\boldsymbol{R}$ and $\mathbb{R}_{\geq 0}\boldsymbol{R}'$ of $\mathcal{G}$ with $\boldsymbol{R} \smallsetminus \{\boldsymbol{r}\} = \boldsymbol{R}' \smallsetminus \{\boldsymbol{r}'\}$.

[5] The linear dependence is unique up to rescaling, and we fix this arbitrary positive rescaling for convenience in the exposition.

As a polyhedral cone, the height deformation cone possesses a face lattice. In particular, if $\boldsymbol{h} \in \mathbb{DC}_{\mathrm{h}}(\mathsf{P})$ is in the interior of the deformation cone, then $\mathsf{P}_{\boldsymbol{h}}$ has the same normal fan as P, *i.e.* is normally equivalent to P. The interior of the deformation cone is sometimes called the *type cone* in the literature, while the word *deformations* can refer to non-(normally)-equivalent deformations of P. Consequently, (the interior of) each face of $\mathbb{DC}_{\mathrm{h}}(\mathsf{P})$ is associated to a class of normally equivalent polytopes, and the face lattice of $\mathbb{DC}_{\mathrm{h}}(\mathsf{P})$ gives rise to a lattice of (classes of normally equivalent) deformations of P. The following proposition grants us access to the faces of the deformation cone.

Proposition 2.4. *If* Q *is a deformation of* P*, then* $\mathbb{DC}(\mathsf{Q})$ *is a face of* $\mathbb{DC}(\mathsf{P})$*.*

Though simple, this proposition is of great importance. As a first application, suppose we want to study the deformation cone of P and we know one of its deformations, Q, then studying the deformation cone of Q is a simpler problem (because Q is of lower dimension than P) which describes a face of $\mathbb{DC}(\mathsf{P})$. A second purpose of this proposition is to measure how deformed is Q with respect to P. For example, the associahedron is a deformation of the permutahedron, and we will see in Proposition 2.32 that the respective dimensions of $\mathbb{DC}\big(\Pi_n\big)$ and $\mathbb{DC}(\mathsf{Asso}_n)$ are $2^n - n - 1$ and $\binom{n}{2}$: in high dimension, the associahedron is very low in the lattice of deformations of Π_n.

Deformations of the standard permutahedron Π_n are called *generalized permutahedra*. Originally introduced by Edmonds in 1970 under the name of *polymatroids* as a polyhedral generalization of matroids in the context of linear optimization [Edm70], the generalized permutahedra were rediscovered by Postnikov in 2009 [Pos09], who initiated the investigation of their rich combinatorial structure. They have since become a widely studied family of polytopes that appears naturally in several areas of mathematics, such as algebraic combinatorics [AA17, ABD10, PRW08], optimization [Fuj05], game theory [DK00], statistics [MPS+09, MUWY18], and economic theory [JKS22]. The set of deformed permutahedra can be parametrized by the cone of *submodular functions* [Edm70, Pos09].

The search for irredundant facet descriptions of deformation cones of particular families of combinatorial polytopes has received considerable attention recently [ACEP20, BMDM+18, CDG+20, CL20, PPPP19, APR21]. One of the motivations sparking this interest arises from the *amplituhedron program* to study scattering amplitudes in mathematical physics [AHT14]. As described in [PPPP19, Sec. 1.4], the deformation cone provides canonical realizations of a polytope (seen as a *positive geometry* [AHBL17]) in the positive region of the kinematic space, akin to those of the associahedron in [AHBHY18].

Contributing to this domain, Sections 2.2 and 2.3 set forth and prove the facet-descriptions of deformation cones of two families of generalized permutahedra: graphical zonotopes and nestohedra respectively.

2.2 Deformation cones of graphical zonotopes

This section is joint work with Arnau Padrol and Vincent Pilaud. It comes from our paper [PPP22b] (accepted for publication), enriched with some additional details and figures.

The *graphical zonotope* of a graph G is a convex polytope Z_G whose geometry encodes several combinatorial properties of G. For example, its vertices are in bijection with the acyclic orientations of G [Sta07, Prop. 2.5] and its volume is the number of spanning trees of G [Sta12, Ex. 4.64]. When G is the complete graph K_n, the graphical zonotope is a translation of the classical n-dimensional permutahedron, see Section 1.2.3.

The main result of this section (Theorem 2.12) presents complete irredundant descriptions of the deformation cones of graphical zonotopes. Note that, since graphical zonotopes are deformed permutahedra, their type cones appear as particular faces of the submodular cone. Faces of the submodular cone are far from being well understood. For example, determining its rays remains an open problem since the 1970s, when it was first asked by Edmonds [Edm70].

It is worth noting that most of the existing approaches to compute deformation cones only focus on simple polytopes with simplicial normal fans [CFZ02, PRW08]. Nevertheless, most graphical zonotopes are not simple. They are simple only for chordful graphs (those where every cycle induces a clique), see [PRW08, Prop. 5.2], [Kim08, Rmk. 6.2], or [Pil21, Prop. 52]. In this section, we thus use an alternative approach to describe the deformation cone of a non-simple polytope based on a simplicial refinement of its normal cone.

This section is organized as follows. We first recall in Section 2.2.1 the necessary material concerning graphical zonotopes. We then describe the deformation cone of any graphical zonotope, providing first a possibly redundant description (Section 2.2.2), then irredundant descriptions of its linear span (Section 2.2.2) and of its facet-defining inequalities (Section 2.2.3), and finally a characterization of graphical zonotopes with simplicial type cones (Section 2.2.4).

2.2.1 Graphical zonotopes

Let $G = (V, E)$ be a graph with vertex set V and edge set E. The *graphical arrangement* $\mathcal{A}_G$ is the arrangement of the hyperplanes $\{\boldsymbol{x} \in \mathbb{R}^V \;;\; \boldsymbol{x}_u = \boldsymbol{x}_v\}$ for all edges $\{u, v\} \in E$. It induces the *graphical fan* $\mathcal{F}_G$ whose cones are all possible intersections of one of the sets $\{\boldsymbol{x} \in \mathbb{R}^V \;;\; \boldsymbol{x}_u = \boldsymbol{x}_v\}$, $\{\boldsymbol{x} \in \mathbb{R}^V \;;\; \boldsymbol{x}_u \geq \boldsymbol{x}_v\}$, or $\{\boldsymbol{x} \in \mathbb{R}^V \;;\; \boldsymbol{x}_u \leq \boldsymbol{x}_v\}$ for each edge $\{u, v\} \in E$. The lineality of $\mathcal{F}_G$ is the subspace $\mathbb{K}_G$ of $\mathbb{R}^V$ spanned by the characteristic vectors of the connected components of G.

The *graphical zonotope* Z_G is the Minkowski sum of the line segments $[\boldsymbol{e}_u, \boldsymbol{e}_v]$ in $\mathbb{R}^V$ for all edges $\{u, v\} \in E$. Here, $(\boldsymbol{e}_v)_{v \in V}$ denotes the canonical basis of $\mathbb{R}^V$. Note that Z_G lies in a subspace orthogonal to $\mathbb{K}_G$. The graphical fan $\mathcal{F}_G$ is the normal fan of the graphical zonotope Z_G.

The following result is well-known. For example, it can be easily deduced from [Sta07, Proposition 2.5] or [BLS$^+$99] (for the latter, see that the graphical matroid from Section 1.1 is realized by the graphical arrangement, and use the description of the cells of the arrangement in terms of covectors from Section 1.2(c)).

An *ordered partition* (μ, ω) of G consists of a partition μ of V where each part induces a connected subgraph of G, together with an acyclic orientation ω of the quotient graph G/μ. We say that (μ, ω) refines (μ', ω') if each part of μ is contained in a part of μ' and the orientations are compatible; that is, for all $u, v \in V$ if there is a directed path in ω between the parts of μ respectively containing u and v, then there is a directed path in ω' between the parts of μ' respectively containing u and v.

Proposition 2.5. *The face lattice of $\mathcal{F}_G$ is antiisomorphic to the lattice of ordered partitions of G ordered by refinement. Explicitly, the antiisomorphism is given by the map that associates the ordered partition (μ, ω) to the cone $\mathsf{C}_{\mu,\omega}$ defined by the inequalities $\boldsymbol{x}_u \leq \boldsymbol{x}_v$ for all $u, v \in V$ such that there is a directed path in ω from the part containing u to the part containing v (in particular, $\boldsymbol{x}_u = \boldsymbol{x}_v$ if u, v are in the same part of μ).*

Some easy consequences of Proposition 2.5 are:

- The maximal cones of $\mathcal{F}_G$ are in bijection with the acyclic orientations of G. We denote by C_ω the maximal cone of $\mathcal{F}_G$ associated to the acyclic orientation ω.
- The minimal cones of $\mathcal{F}_G$, that is the rays of $\mathcal{F}_G/\mathbb{K}_G$, are in bijection with the *biconnected subsets* of G, *i.e.* non-empty subsets S of V such that there is a disjoint non-empty subset T of V such that $S \cup T$ is a connected component of G and the induced subgraphs $G[S]$ and $G[T]$ are connected.
- The rays of $\mathcal{F}_G/\mathbb{K}_G$ that belong to the maximal cone associated to an acyclic orientation are the biconnected subsets which form an upper set of the acyclic orientation (hence, they are in bijection with the minimal directed cuts of the acyclic orientation).
- Similarly, the rays of $\mathcal{F}_G/\mathbb{K}_G$ that belong to the cone associated to an ordered partition (μ, ω) are the biconnected sets that contracted by μ give rise to an upper set of ω.

Note that the natural embedding of a graphical fan $\mathcal{F}_G$ is not essential, as it has a lineality given by its connected components. This is why we cannot directly talk about the rays of the fan in the enumeration above. The usual solution to avoid this is to consider the quotient by the subspace $\mathbb{K}_G$. However, this subspace depends on the graph, and with such a quotient we would lose the capacity of uniformly treating all the graphs with a fixed vertex set. We will instead work with the natural non-essential embedding, together with a collection of vectors supporting simultaneously all graphical fans.

Example 2.6. As seen in Section 1.2.3, when G is the complete graph K_n, the graphical zonotope is the *permutahedron*. The graphical fan is the *braid fan* $\mathcal{B}_n$, induced by the *braid arrangement* consisting of the hyperplanes $\{\boldsymbol{x} \in \mathbb{R}^n \,;\, \boldsymbol{x}_i = \boldsymbol{x}_j\}$ for all $1 \le i < j \le n$. Its lineality is spanned by the all-ones vector $\mathbb{1}_n := (1, \dots, 1)$. Since all the proper subsets of $[n]$ are biconnected in K_n, the face lattice of $\mathcal{B}_n$ is isomorphic to the lattice of ordered partitions of $[n]$. The rays of $\mathcal{B}_n/\mathbb{1}_n$ correspond to proper subsets of $[n]$, and its maximal cells are in bijection with permutations of $[n]$. Each maximal cell is the positive hull of the $n-1$ rays corresponding to the proper upper sets of the order given by the permutation. In particular, $\mathcal{B}_n/\mathbb{1}_n$ is a simplicial fan.

2.2.2 Graphical deformation cones

Our main result is an irredundant facet description of the deformation cone of Z_G for every graph $G = (V, E)$. Our starting point is Proposition 2.10, which gives a (possibly redundant) description derived from Proposition 2.3. It is strongly based on the fact that the braid fan simultaneously refines all the graphical fans. Note however that the braid fan is not simplicial (due to its lineality). The classical approach to overcome this issue is to quotient the braid fan by its lineality space. However, we prefer to triangulate the braid fan, since it simplifies the presentation of the proof.

A first polyhedral description Associate to each subset $S \subseteq V$ the vector

$$\boldsymbol{\iota}_S := \sum_{v \in S} \boldsymbol{e}_v - \sum_{v \notin S} \boldsymbol{e}_v.$$

This is essentially the characteristic vector of S, but it has the advantage that $\boldsymbol{\iota}_V = \mathbb{1}_V$ and $\boldsymbol{\iota}_\varnothing = -\mathbb{1}_V$ positively span the line $\mathbb{1}_V\mathbb{R}$, which is the lineality $\mathbb{K}_{K_V}$ of the braid fan.

Lemma 2.7. *For any ordered partition (μ, ω) of a graph $G = (V, E)$, we have*

$$\mathsf{C}_{\mu,\omega} = \operatorname{cone}\{\boldsymbol{\iota}_S \,;\, S \subseteq V \text{ upper set of } \omega\}.$$

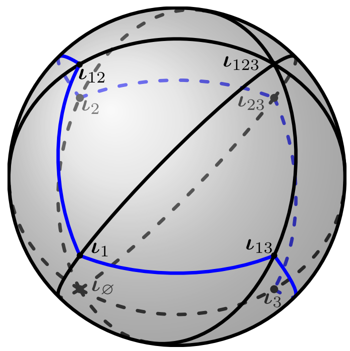

Figure 11: The fan $\widehat{\mathcal{B}}_{123}$ intersected with the unit sphere. (For brevity, here and in the labels we write 123 to denote the set $\{1,2,3\}$, and so on.) The braid fan $\mathcal{B}_{123}$ is the Cartesian product of a regular hexagonal fan with a line. To obtain $\widehat{\mathcal{B}}_{123}$, each maximal cell is divided into two simplicial cells, one containing $\boldsymbol{\iota}_\varnothing$ and one containing $\boldsymbol{\iota}_{123}$.

Here, we mean that S is an upper set of ω when contracted by μ. Note that $\varnothing$ and V are always upper sets, which is consistent with the fact that the lineality of $\mathcal{F}_G$ always contains the line spanned by $\mathbb{1}_V$.

We will work with a refined version $\widehat{\mathcal{B}}_V$ of the braid fan whose maximal cells are

$$\mathsf{C}_\sigma^\varnothing := \operatorname{cone}\{\boldsymbol{\iota}_S \;;\; S \subsetneq V \text{ upper set of } \sigma\} \quad \text{and} \quad \mathsf{C}_\sigma^V := \operatorname{cone}\{\boldsymbol{\iota}_S \;;\; \varnothing \neq S \subseteq V \text{ upper set of } \sigma\}$$

for every acyclic orientation of the complete graph K_V, which we identify with a permutation σ of V. An example is depicted in Figure 11. The following two immediate statements are left to the reader.

Lemma 2.8. *For any finite set V:*

(i) The fan $\widehat{\mathcal{B}}_V$ is an essential complete simplicial fan in $\mathbb{R}^V$ supported on the $2^{|V|}$ vectors $\boldsymbol{\iota}_S$ for $S \subseteq V$.

(ii) For any permutation σ, the maximal cones $\mathsf{C}_\sigma^\varnothing$ and C_σ^V are adjacent, and the unique linear relation supported on the rays of $\mathsf{C}_\sigma^\varnothing \cup \mathsf{C}_\sigma^V$ is $\boldsymbol{\iota}_\varnothing + \boldsymbol{\iota}_V = \mathbf{0}$.

(iii) The other pairs of adjacent maximal cells are of the form C_σ^X and $\mathsf{C}_{\sigma'}^X$, where $X \in \{\varnothing, V\}$ and $\sigma = PuvS$ and $\sigma' = PvuS$ are permutations that differ in the inversion of two consecutive elements. The two rays that are not shared by C_σ^X and $\mathsf{C}_{\sigma'}^X$ are $\boldsymbol{\iota}_{S\cup\{u\}}$ and $\boldsymbol{\iota}_{S\cup\{v\}}$, and the unique linear relation supported on the rays of $\mathsf{C}_\sigma^X \cup \mathsf{C}_{\sigma'}^X$ is given by

$$\boldsymbol{\iota}_{S\cup\{u\}} + \boldsymbol{\iota}_{S\cup\{v\}} = \boldsymbol{\iota}_S + \boldsymbol{\iota}_{S\cup\{u,v\}}.$$

Lemma 2.9. *For any graph $G = (V, E)$:*

(i) The fan $\widehat{\mathcal{B}}_V$ is a simplicial refinement of the graphical fan $\mathcal{F}_G$.

(ii) For an acyclic orientation ω of G and $S \subseteq V$, we have $\boldsymbol{\iota}_S \in \mathsf{C}_\omega$ if and only if S is an upper set of ω.

(iii) For an acyclic orientation σ of K_V and $X \in \{\varnothing, V\}$ we have $\mathsf{C}_\sigma^X \subseteq \mathsf{C}_\omega$ if and only if σ is a linear extension of ω.

We are now ready to describe the deformation cone of the graphical zonotope Z_G. For any $\boldsymbol{h} \in \mathbb{R}^{2^V}$, let $\mathsf{D}_{\boldsymbol{h}}$ be the polytope given by

$$\mathsf{D}_{\boldsymbol{h}} := \Big\{\boldsymbol{x} \in \mathbb{R}^V \;;\; \sum_{v\in S} \boldsymbol{x}_v - \sum_{v\notin S} \boldsymbol{x}_v \le \boldsymbol{h}_S \text{ for all } S \subseteq V\Big\}.$$

Proposition 2.10. *For any graph $G = (V, E)$, the deformation cone $\mathbb{DC}(\mathsf{Z}_G)$ of the graphical zonotope Z_G is the set of polytopes $\mathsf{D}_{\boldsymbol{h}}$ for all $\boldsymbol{h}$ in the cone of $\mathbb{R}^{2^V}$ defined by the following (possibly redundant) description:*

- $\boldsymbol{h}_\varnothing = -\boldsymbol{h}_V$,
- $\boldsymbol{h}_{S\cup\{u\}} + \boldsymbol{h}_{S\cup\{v\}} = \boldsymbol{h}_S + \boldsymbol{h}_{S\cup\{u,v\}}$ *for each* $\{u,v\} \in \binom{V}{2} \smallsetminus E$ *and* $S \subseteq V \smallsetminus \{u,v\}$, *and*
- $\boldsymbol{h}_{S\cup\{u\}} + \boldsymbol{h}_{S\cup\{v\}} \geq \boldsymbol{h}_S + \boldsymbol{h}_{S\cup\{u,v\}}$ *for each* $\{u,v\} \in E$ *and* $S \subseteq V \smallsetminus \{u,v\}$.

Proof. Observe first that, as stated in Lemma 2.9, $\widehat{\mathcal{B}}_V$ provides a simplicial refinement of $\mathcal{F}_G$. Following Proposition 2.3, we need to consider all pairs of adjacent maximal cones of $\widehat{\mathcal{B}}_V$, and to study which ones lie in the same cone of $\mathcal{F}_G$.

Adjacent maximal cones of $\widehat{\mathcal{B}}_V$ are described in Lemma 2.8, and the containment relations of the cones of $\widehat{\mathcal{B}}_V$ in the cones of $\mathcal{F}_G$ are described in Lemma 2.9.

For any σ, the cones $\mathsf{C}_\sigma^\varnothing$ and C_σ^V belong to the same cell of $\mathcal{F}_G$. Hence, by Proposition 2.3, the following equation holds in the deformation cone:

$$\boldsymbol{h}_\varnothing = -\boldsymbol{h}_V.$$

The remaining pairs of adjacent maximal cones of $\widehat{\mathcal{B}}_V$ correspond to pairs of acyclic orientations of K_V differing in a single edge; or equivalently, to pairs of permutations of V of the form $\sigma = PuvS$ and $\sigma' = PvuS$. The unique linear relation supported on the rays of $\mathsf{C}_\sigma^X \cup \mathsf{C}_{\sigma'}^X$ for $X \in \{\varnothing, V\}$ is then

$$\boldsymbol{\iota}_{S\cup\{u\}} + \boldsymbol{\iota}_{S\cup\{v\}} = \boldsymbol{\iota}_S + \boldsymbol{\iota}_{S\cup\{u,v\}}.$$

We consider first the case when $\{u,v\} \notin E$. Observe that both σ and σ' induce the same acyclic orientation of G, which we call ω. We have then $\mathsf{C}_\sigma^X \cup \mathsf{C}_{\sigma'}^X \subseteq \mathsf{C}_\omega$ by Lemma 2.9. Therefore, by Proposition 2.3 and Lemma 2.8, we have

$$\boldsymbol{h}_{S\cup\{u\}} + \boldsymbol{h}_{S\cup\{v\}} = \boldsymbol{h}_S + \boldsymbol{h}_{S\cup\{u,v\}}$$

for any $\boldsymbol{h}$ in $\mathbb{DC}(\mathsf{Z}_G)$. Note that, for any $\{u,v\} \notin E$ and $S \subset V \smallsetminus \{u,v\}$, we can construct such permutations σ and σ'. This gives the claimed description of the linear span of $\mathbb{DC}(\mathsf{Z}_G)$.

In contrast, if $\{u,v\} \in E$, then σ and σ' induce different orientations of G, and hence they belong to different adjacent cones of $\mathcal{F}_G$ by Lemma 2.9. Therefore, by Proposition 2.3 and Lemma 2.8, we have

$$\boldsymbol{h}_{S\cup\{u\}} + \boldsymbol{h}_{S\cup\{v\}} \geq \boldsymbol{h}_S + \boldsymbol{h}_{S\cup\{u,v\}}$$

for any $\boldsymbol{h}$ in $\mathbb{DC}(\mathsf{Z}_G)$. As before, for any $\{u,v\} \in E$ and $S \subset V \smallsetminus \{u,v\}$, we can construct such permutations σ and σ'. This gives the claimed inequalities describing $\mathbb{DC}(\mathsf{Z}_G)$. □

The linear span of graphical deformation cones The description of the deformation cone of Proposition 2.10 is highly redundant, both in the equations describing its linear span and in the inequalities describing its facets. We will give a non-redundant description in Theorem 2.12. The first step will be to give linearly independent equations describing the linear span. As an important by-product, we will obtain the dimension and a linear basis of (the vector space generated by) the deformation cone $\mathbb{DC}(\mathsf{Z}_G)$.

It is sometimes convenient to consider the set of deformations of P embedded inside the real vector space of *virtual d-dimensional polytopes* $\mathbb{V}^d$ [PK92]. This is the set of formal differences of polytopes $\mathsf{P} - \mathsf{Q}$ under the equivalence relation $(\mathsf{P}_1 - \mathsf{Q}_1) = (\mathsf{P}_2 - \mathsf{Q}_2)$ whenever $\mathsf{P}_1 + \mathsf{P}_2 = \mathsf{Q}_1 + \mathsf{Q}_2$. Endowed with Minkowski addition, it is the Grothendieck group of the semigroup of polytopes, which are embedded into $\mathbb{V}^d$ via the map $\mathsf{P} \mapsto \mathsf{P} - \{\boldsymbol{0}\}$. It extends to a real vector space via dilation: for $\mathsf{P} - \mathsf{Q} \in \mathbb{V}^d$ and $\lambda \in \mathbb{R}$, we set $\lambda(\mathsf{P} - \mathsf{Q}) := \lambda\mathsf{P} - \lambda\mathsf{Q}$ when $\lambda \geq 0$, and $\lambda(\mathsf{P} - \mathsf{Q}) := ((-\lambda)\mathsf{Q}) - ((-\lambda)\mathsf{P})$ when $\lambda < 0$. Here, $\lambda\mathsf{P} := \{\lambda\boldsymbol{p} \,;\, \boldsymbol{p} \in \mathsf{P}\}$ denotes the dilation[6] of P by $\lambda \geq 0$.

[6]Note in particular that $-\mathsf{P}$ does not represent the reflection of P, but its group inverse.

For a polytope $\mathsf{P} \subset \mathbb{R}^d$, we define the space $\mathbb{VD}(\mathsf{P}) \subset \mathbb{V}^d$ of *virtual deformations* of P as the vector sub-space of virtual polytopes generated by the deformations of P. Equivalently, $\mathbb{VD}(\mathsf{P})$ is the linear span of the deformation cone $\mathbb{DC}(\mathsf{P})$. Every virtual polytope in $\mathbb{VD}(\mathsf{P})$ is of the form $\mathsf{P}_{\boldsymbol{h}} - \mathsf{P}_{\boldsymbol{h}'}$ for deformations $\mathsf{P}_{\boldsymbol{h}}, \mathsf{P}_{\boldsymbol{h}'} \in \mathbb{DC}(P)$. Note that the vector $\boldsymbol{h} - \boldsymbol{h}'$ uniquely describes the equivalence class of this virtual polytope, and we will use the notation $\mathsf{P}_{\boldsymbol{h}-\boldsymbol{h}'}$ to denote it.

Denote by $\Delta_U := \operatorname{conv}\{\boldsymbol{e}_u \ ; \ u \in U\} \subset \mathbb{R}^V$ the face of the standard simplex Δ_V corresponding to a subset $U \subseteq V$. These polytopes are particularly important deformed permutahedra as they form a linear basis of the deformation space of the permutahedron [DK00] (see also [ABD10, Prop. 2.4]). Namely, any (virtual) deformed permutahedron can be uniquely written as a signed Minkowski sum of dilates of Δ_I. Our first result states that this linear basis is adapted to graphical zonotopes.

Theorem 2.11. *For any graph $G = (V, E)$:*

(i) The dimension of $\mathbb{VD}(\mathsf{Z}_G)$ is the number of non-empty induced cliques in G (the vertices of G count for the dimension as they correspond to the lineality space).

(ii) The faces Δ_K of the standard simplex Δ_V corresponding to the non-empty induced cliques K of G form a linear basis of $\mathbb{VD}(\mathsf{Z}_G)$.

(iii) $\mathbb{VD}(\mathsf{Z}_G)$ is the set of virtual polytopes $\mathsf{D}_{\boldsymbol{h}}$ for all $\boldsymbol{h} \in \mathbb{R}^{2^V}$ fulfilling the following linearly independent equations:

- *$\boldsymbol{h}_\varnothing = -\boldsymbol{h}_V$ and*
- *$\boldsymbol{h}_{S \smallsetminus \{u\}} + \boldsymbol{h}_{S \smallsetminus \{v\}} = \boldsymbol{h}_S + \boldsymbol{h}_{S \smallsetminus \{u,v\}}$ for each $S \subseteq V$ with $|S| \geq 2$ not inducing a clique of G and any $\{u, v\} \in \binom{S}{2} \smallsetminus E$ (here, we only choose one missing edge for each subset S, for example, the lexicographically smallest).*

Proof. Observe first that the faces Δ_I of the standard simplex Δ_V corresponding to the induced cliques I of G are all in the deformation cone $\mathbb{DC}(\mathsf{Z}_G)$. Indeed, faces of the standard simplex Δ_I belong to the deformation cone of the complete graph K_I by [Pos09, Prop. 6.3]. The graphical zonotope $\mathsf{Z}_{G'}$ is a Minkowski summand of Z_G for any subgraph G' of G, and hence summands of $\mathsf{Z}_{G'}$ are also summands of Z_G.

Moreover, all faces Δ_I for $\varnothing \neq I \subsetneq V$ are Minkowski independent by [ABD10, Prop. 2.4]. This shows that the dimension of $\mathbb{VD}(\mathsf{Z}_G)$ is at least the number of non-empty induced cliques of G.

Let $(\boldsymbol{f}_X)_{X \subseteq V}$ be the canonical basis of $\left(\mathbb{R}^{2^V}\right)^*$. The vectors

$$\boldsymbol{o}^S := \boldsymbol{f}_S - \boldsymbol{f}_{S \smallsetminus \{u\}} - \boldsymbol{f}_{S \smallsetminus \{v\}} + \boldsymbol{f}_{S \smallsetminus \{u,v\}},$$

for all subsets $\varnothing \neq S \subseteq V$ not inducing a clique of G and one selected missing edge $\{u, v\}$ for each S, are clearly linearly independent. Indeed, if the $\boldsymbol{f}_X$ are ordered according to any linear extension of the inclusion order on the indices X, and the $\boldsymbol{o}^S$ are ordered analogously in terms of the indices S, then the equations are already in echelon form, as $\boldsymbol{f}_S$ is the greatest non-zero coordinate of $\boldsymbol{o}^S$. Finally, the vector $\boldsymbol{v} \in 2^V$ with $\boldsymbol{v}_X = |X|$ for $X \in 2^V$ is orthogonal to any $\boldsymbol{o}^S$ with $|S| \geq 2$ but not to $\boldsymbol{o}^\varnothing := \boldsymbol{f}_\varnothing + \boldsymbol{f}_V$, showing that the latter is linearly independent to the former. This proves that the dimension of $\mathbb{VD}(\mathsf{Z}_G)$ is at most the number of non-empty induced cliques of G.

We conclude that $\left\{\Delta_K \ ; \ \varnothing \neq K \subseteq V \text{ inducing a clique of } G\right\}$ is a linear basis of the deformation cone, and that $\left\{\boldsymbol{o}^S \ ; \ S = \varnothing \text{ or } S \subseteq V \text{ not inducing a clique of } G\right\}$ is a basis of its orthogonal complement (we slightly abuse notation here as $\boldsymbol{o}^S$ was defined in $\left(\mathbb{R}^{2^V}\right)^*$ instead of in $(\mathbb{V}^d)^*$, but note that each $\boldsymbol{f}_X$ can be considered as a linear functional in $(\mathbb{V}^d)^*$ if seen as a support function). $\square$

Note that the dimension of the deformation space of graphical zonotopes has been independently computed by Raman Sanyal and Josephine Yu (personal communication), who computed the space of Minkowski 1-weights of graphical zonotopes in the sense of McMullen [McM96]. Their proof also uses the basis from Theorem 2.11 (ii), but with an alternative argument to show that they are a generating family.

2.2.3 The facets of graphical deformation cones

To conclude, it remains to compute the facets of the deformation cones, *i.e.* a non-redundant inequality description.

We define the *neighborhood* of a vertex v in a graph $G = (V, E)$ as $N(v) := \{u \in V \;;\; \{u, v\} \in E\}$.

Theorem 2.12. *For any graph $G = (V, E)$, the deformation cone $\mathbb{DC}(\mathsf{Z}_G)$ of the graphical zonotope Z_G is the set of polytopes $\mathsf{D}_{\boldsymbol{h}}$ for all $\boldsymbol{h}$ in the cone of $\mathbb{R}^{2^V}$ defined by the following irredundant facet description:*

- $\boldsymbol{h}_\varnothing = -\boldsymbol{h}_V$,
- $\boldsymbol{h}_{S\smallsetminus\{u\}} + \boldsymbol{h}_{S\smallsetminus\{v\}} = \boldsymbol{h}_S + \boldsymbol{h}_{S\smallsetminus\{u,v\}}$ *for each* $\varnothing \neq S \subseteq V$ *and any* $\{u, v\} \in \binom{S}{2} \smallsetminus E$,
- $\boldsymbol{h}_{S\cup\{u\}} + \boldsymbol{h}_{S\cup\{v\}} \geq \boldsymbol{h}_S + \boldsymbol{h}_{S\cup\{u,v\}}$ *for each* $\{u, v\} \in E$ *and* $S \subseteq N(v) \cap N(v)$.

Note that this description is given as a face of the submodular cone, embedded into $\mathbb{R}^{2^V}$. One gets easily an intrinsic presentation by restricting to the space spanned by the biconnected subsets of V. However, that presentation loses its symmetry, and the explicit equations depend on the biconnected sets of G.

Proof of Theorem 2.12. We know by Proposition 2.10 that $\mathbb{DC}(\mathsf{Z}_G)$ is the intersection of the cone

$$\boldsymbol{h}_{S\cup\{u\}} + \boldsymbol{h}_{S\cup\{v\}} \geq \boldsymbol{h}_S + \boldsymbol{h}_{S\cup\{u,v\}} \tag{1}$$

for $\{u, v\} \in E$ and $S \subseteq V \smallsetminus \{u, v\}$ with the linear space given by the equations $\boldsymbol{h}_\varnothing = -\boldsymbol{h}_V$ and

$$\boldsymbol{h}_{S\cup\{u\}} + \boldsymbol{h}_{S\cup\{v\}} = \boldsymbol{h}_S + \boldsymbol{h}_{S\cup\{u,v\}} \tag{2}$$

for $\{u, v\} \in \binom{V}{2} \smallsetminus E$ and $S \subseteq V \smallsetminus \{u, v\}$.

We have already determined the equations describing the linear span in Theorem 2.11, so it only remains to provide non-redundant inequalities describing the deformation cone.

We will prove first that the inequalities from (1) indexed by $\{u, v\} \in E$ and $S \subseteq N(v) \cap N(v)$ suffice to describe $\mathbb{DC}(\mathsf{Z}_G)$. To this end, consider an inequality from (1) for which $S \not\subseteq N(v) \cap N(v)$. Without loss of generality, assume that there is some $x \in S$ such that $\{x, v\} \notin E$. We will show that this inequality is induced (in the sense that the half-spaces they define coincide on the linear span of $\mathbb{DC}(\mathsf{Z}_G)$) by the inequality

$$\boldsymbol{h}_{S'\cup\{u\}} + \boldsymbol{h}_{S'\cup\{v\}} \geq \boldsymbol{h}_{S'} + \boldsymbol{h}_{S'\cup\{u,v\}} \tag{3}$$

where $S' = S \smallsetminus \{x\}$. Our claim will then follow from this by induction on the elements of $S \smallsetminus (N(v) \cap N(v))$.

Indeed, if $\{x, v\} \notin E$, we know by (2) that the following two equations hold in the linear span of $\mathbb{DC}(\mathsf{Z}_G)$ by considering the non-edge $\{x, v\}$ with the subsets S' and $S' \cup \{u\}$, respectively:

$$\boldsymbol{h}_{S\cup\{u\}} + \boldsymbol{h}_{S'\cup\{u,v\}} = \boldsymbol{h}_{S'\cup\{u\}} + \boldsymbol{h}_{S\cup\{u,v\}}, \tag{4}$$

$$\boldsymbol{h}_S + \boldsymbol{h}_{S'\cup\{v\}} = \boldsymbol{h}_{S'} + \boldsymbol{h}_{S\cup\{v\}}, \tag{5}$$

where we used that $(S' \cup \{u\}) \cup \{x\} = S \cup \{u\}$ and $(S' \cup \{u\}) \cup \{x, v\} = S \cup \{u, v\}$ in the first equation, and that $S' \cup \{x\} = S$ and $S' \cup \{x, v\} = S \cup \{v\}$ in the second equation. To conclude, note that (1) is precisely the linear combination (3) + (4) − (5).

We know therefore that the descriptions in Proposition 2.10 and Theorem 2.12 give rise to the same cone. It remains to show that the latter is irredundant. That is, that each of the inequalities gives rise to a unique facet of $\mathbb{DC}(\mathsf{Z}_G)$.

Let $(\boldsymbol{f}_X)_{X\subseteq V}$ be the canonical basis of $\left(\mathbb{R}^{2^V}\right)^*$. For $u, v \in V$ and $S \subseteq V \smallsetminus \{u, v\}$, let

$$\boldsymbol{n}(u, v, S) := \boldsymbol{f}_{S\cup\{u\}} + \boldsymbol{f}_{S\cup\{v\}} - \boldsymbol{f}_S - \boldsymbol{f}_{S\cup\{u,v\}}.$$

Note that, if $\{u,v\} \notin E$, then $\boldsymbol{n}(u,v,S)$ is orthogonal to $\mathbb{DC}(\mathsf{Z}_G)$, whereas if $\{u,v\} \in E$, then $\boldsymbol{n}(u,v,S)$ is an inner normal vector to $\mathbb{DC}(\mathsf{Z}_G)$.

Fix $\{u,v\} \in E$ and $S \subseteq N(v) \cap N(v)$. To prove that the half-space with normal $\boldsymbol{n}(u,v,S)$ is not redundant, we will exhibit a vector $\boldsymbol{w} \in \mathbb{R}^{2^V}$ in the linear span of $\mathbb{DC}(\mathsf{Z}_G)$ that belongs to the interior of all the half-spaces describing $\mathbb{DC}(\mathsf{Z}_G)$ except for this one. That is, we will construct a vector $\boldsymbol{w} \in \mathbb{R}^{2^V}$ respecting the system:

$$\begin{cases} \langle \boldsymbol{w},\ \boldsymbol{n}(u,v,S)\rangle \le 0, & \\ \langle \boldsymbol{w},\ \boldsymbol{n}(u,v,X)\rangle > 0 & \text{for } S \neq X \subseteq N(u) \cap N(v), \\ \langle \boldsymbol{w},\ \boldsymbol{n}(a,b,X)\rangle > 0 & \text{for } \{a,b\} \in E \smallsetminus \{u,v\} \text{ and } X \subseteq N(a) \cap N(b), \text{ and} \\ \langle \boldsymbol{w},\ \boldsymbol{n}(a,b,X)\rangle = 0 & \text{for } \{a,b\} \in \binom{V}{2} \smallsetminus E \text{ and } X \subseteq V \smallsetminus \{a,b\}. \end{cases} \tag{6}$$

Denote by $T := N(u) \cap N(v) \smallsetminus S$. We will construct $\boldsymbol{w}$ as the sum $\boldsymbol{w} := \boldsymbol{t}^S - \boldsymbol{t}^T + \boldsymbol{c}$ for some vectors $\boldsymbol{t}^S$, $\boldsymbol{t}^T$, and $\boldsymbol{c} \in \mathbb{R}^{2^V}$ defined below, whose scalar products with $\boldsymbol{n}(a,b,X)$ for $\{a,b\} \in \binom{V}{2}$ and $X \subseteq V \smallsetminus \{a,b\}$ fulfill:

	$\langle \boldsymbol{t}^S,\ \boldsymbol{n}(a,b,X)\rangle$	$\langle -\boldsymbol{t}^T,\ \boldsymbol{n}(a,b,X)\rangle$	$\langle \boldsymbol{c},\ \boldsymbol{n}(a,b,X)\rangle$
if $\{a,b\} = \{u,v\}$ and $X = S$	$-\lvert S\rvert$	0	$\lvert S\rvert$
if $\{a,b\} = \{u,v\}$ and $S \neq X \subseteq N(u) \cap N(v)$	$-\lvert S \cap X\rvert$	$\lvert T \cap X\rvert$	$\lvert S\rvert$
if $\{a,b\} \in E \smallsetminus \{u,v\}$ and $X \subseteq N(a) \cap N(b)$	≥ -1	≥ 0	2
if $\{a,b\} \notin E$	0	0	0

It immediately follows from this table that the vector $\boldsymbol{w}$ will fulfill the desired properties from (6). For the second one, note that if $S \neq X \subseteq S \sqcup T$, then either $\lvert S \cap X\rvert < \lvert S\rvert$ or $\lvert T \cap X\rvert > 0$.

To define these vectors, first, for $\{x,y,z\} \in \binom{V}{3}$, let $\boldsymbol{t}^{xyz} \in \mathbb{R}^{2^V}$ be the vector such that $\boldsymbol{t}^{xyz}_X = 1$ if $\{x,y,z\} \subseteq X$ and $\boldsymbol{t}^{xyz}_X = 0$ otherwise. Note that, for any $a,b \in \binom{V}{2}$ and $X \subseteq V \smallsetminus \{a,b\}$, we have

$$\langle \boldsymbol{t}^{xyz},\ \boldsymbol{n}(a,b,X)\rangle = \begin{cases} -1 & \text{if } \{x,y,z\} = \{a,b,t\} \text{ for some } t \in X, \text{ and} \\ 0 & \text{otherwise.} \end{cases} \tag{7}$$

We define

$$\boldsymbol{t}^S := \sum_{s \in S} \boldsymbol{t}^{uvs} \qquad \text{and} \qquad \boldsymbol{t}^T := \sum_{t \in T} \boldsymbol{t}^{uvt}.$$

It is straightforward to derive the identities in the table from (7). For the inequalities, notice that if $\langle \boldsymbol{t}^{uvx},\ \boldsymbol{n}(a,b,X)\rangle = -1$ but $\{a,b\} \neq \{u,v\}$, then either $\{a,b\} = \{u,x\}$ or $\{a,b\} = \{v,x\}$, and in both cases $\langle \boldsymbol{t}^{uvy},\ \boldsymbol{n}(a,b,X)\rangle = 0$ for any $y \neq x$.

Now, for $\{x,y\} \in \binom{V}{2}$, let $\boldsymbol{c}^{xy} \in \mathbb{R}^{2^V}$ be the vector such that $\boldsymbol{c}^{xy}_X = 1$ if $\lvert\{x,y\} \cap X\rvert = 1$ (that is, if $\{x,y\}$ belongs to the cut defined by X), and $\boldsymbol{c}^{xy}_X = 0$ otherwise. Note that, for any $a,b \in \binom{V}{2}$ and $X \subseteq V \smallsetminus \{a,b\}$, we have

$$\langle \boldsymbol{c}^{xy},\ \boldsymbol{n}(a,b,X)\rangle = \begin{cases} 2 & \text{if } \{a,b\} = \{x,y\}, \text{ and} \\ 0 & \text{otherwise.} \end{cases} \tag{8}$$

We set

$$\boldsymbol{c} := \frac{\lvert S\rvert}{2} \boldsymbol{c}^{uv} + \sum_{\{a,b\} \in E \smallsetminus \{u,v\}} \boldsymbol{c}^{ab}.$$

The identities in the table are straightforward to derive from (8). □

Corollary 2.13. *For any graph $G = (V,E)$, the dimension of $\mathbb{DC}(\mathsf{Z}_G)$ is the number of induced cliques in G, the dimension of the lineality space of $\mathbb{DC}(\mathsf{Z}_G)$ is $\lvert V\rvert$, and the number of facets of $\mathbb{DC}(\mathsf{Z}_G)$ is the number of triplets (u,v,S) with $\{u,v\} \in E$ and $S \subseteq N(u) \cap N(v)$.*

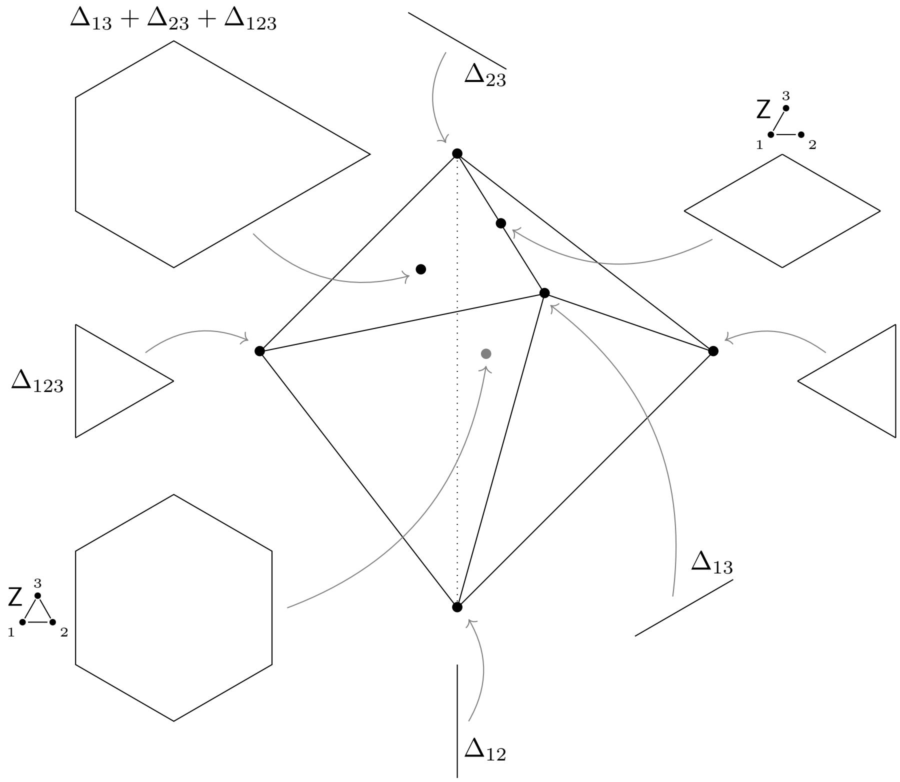

Figure 12: A 3-dimensional affine section of the deformation cone $\mathbb{DC}(\mathsf{Z}_{K_3})$ for the triangle K_3. The deformations of Z_{K_3} corresponding to some of the points of $\mathbb{DC}(\mathsf{Z}_{K_3})$ are depicted. Especially, all points in the interior correspond to polytopes normally equivalent to Π_3, while the above left polytope is the Loday associahedron Asso_3.

Example 2.14. For the complete graph K_V, the graphical zonotope Z_{K_V} is a permutahedron and the deformation cone $\mathbb{DC}(\mathsf{Z}_{K_V})$ is the submodular cone given by the irredundant inequalities $\boldsymbol{h}_{S\cup\{u\}} + \boldsymbol{h}_{S\cup\{v\}} \geq \boldsymbol{h}_S + \boldsymbol{h}_{S\cup\{u,v\}}$ for each $\{u,v\} \subseteq V$ and $S \subseteq V \smallsetminus \{u,v\}$. (The usual presentation imposes $\boldsymbol{h}_\varnothing = 0$, but both presentations are clearly equivalent up to translation). It has dimension $2^{|V|} - 1$ and $\binom{|V|}{2}2^{|V|-2}$ facets. The lineality space is $|V|$-dimensional, given by the space of translations in $\mathbb{R}^{|V|}$.

For instance, for the triangle K_3, the graphical zonotope $\mathsf{Z}_{K_3} = \Pi_3$ is the regular hexagon depicted in the bottom left of Figure 12, which arises as the Minkowski sum of 3 coplanar vectors in $\mathbb{R}^3$. Its deformation cone $\mathbb{DC}(\mathsf{Z}_{K_3})$ lives in the 8-dimensional space $\mathbb{R}^{2^{[3]}}$, has dimension 7, a lineality space of dimension 3, and 6 facets. It admits as irredundant description the equation $\boldsymbol{h}_\varnothing = -\boldsymbol{h}_{123}$ and the following 6 inequalities:

$$
\begin{array}{ccc}
\boldsymbol{h}_1 + \boldsymbol{h}_2 \geq \boldsymbol{h}_\varnothing + \boldsymbol{h}_{12} & \boldsymbol{h}_1 + \boldsymbol{h}_3 \geq \boldsymbol{h}_\varnothing + \boldsymbol{h}_{13} & \boldsymbol{h}_2 + \boldsymbol{h}_3 \geq \boldsymbol{h}_\varnothing + \boldsymbol{h}_{23} \\
\boldsymbol{h}_{12} + \boldsymbol{h}_{13} \geq \boldsymbol{h}_1 + \boldsymbol{h}_{123} & \boldsymbol{h}_{12} + \boldsymbol{h}_{23} \geq \boldsymbol{h}_2 + \boldsymbol{h}_{123} & \boldsymbol{h}_{13} + \boldsymbol{h}_{23} \geq \boldsymbol{h}_3 + \boldsymbol{h}_{123}.
\end{array}
$$

After quotienting the lineality and intersecting with an affine hyperplane, we get the bipyramid over a triangle (living in dimension $7 - 3 - 1 = 3$) illustrated in Figure 12. Note that the four rays of $\mathbb{DC}(\mathsf{Z}_{K_3})$ (*i.e.* vertices of the bipyramid) of the form Δ_K for an induced clique K of K_3 provide a linear basis of $\mathbb{DC}(\mathsf{Z}_{K_3})$ (*i.e.* an affine basis of the bipyramid). Nevertheless, the last ray can not be written as a positive Minkowski sum of Δ_K and remain thus unlabeled.

Example 2.15. For a triangle-free graph $G = (V, E)$, the deformation cone $\mathbb{DC}(\mathsf{Z}_G)$ has dimension $|V| + |E|$ and $|E|$ facets. As before, the lineality is $|V|$-dimensional, given by the space of

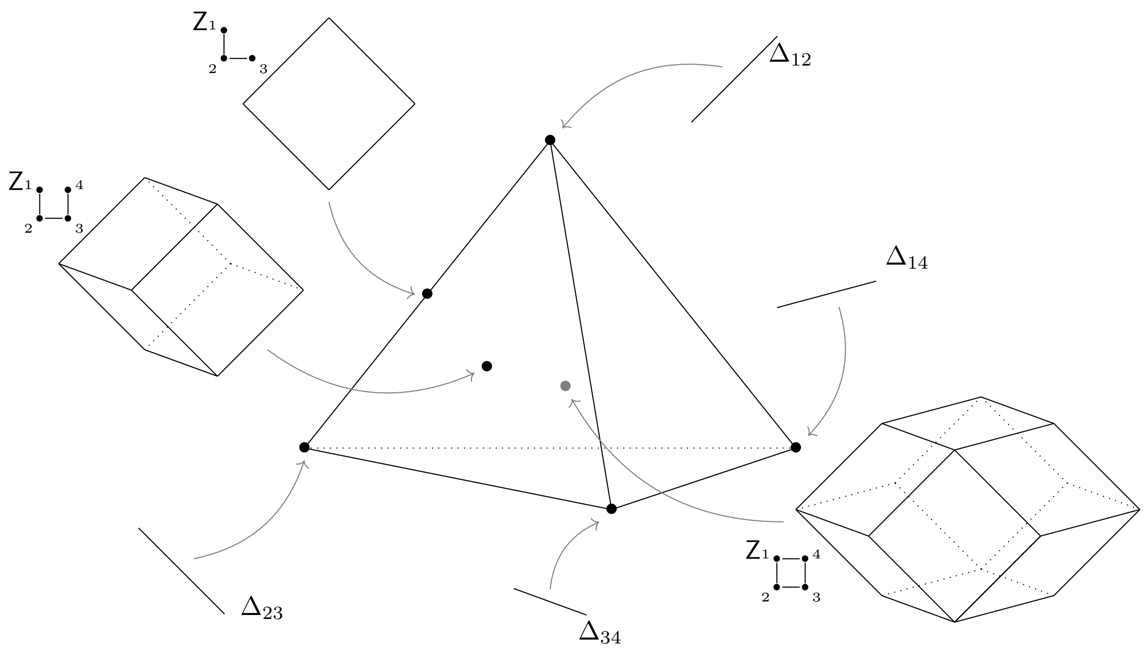

Figure 13: A 3-dimensional affine section of the deformation cone $\mathbb{DC}(\mathsf{Z}_{C_4})$ for the 4-cycle C_4. The deformations of Z_{C_4} corresponding to some of the points of $\mathbb{DC}(\mathsf{Z}_{C_4})$ are depicted. Especially, interior all points correspond to polytopes normally equivalent to Z_{C_4}. Note that as the rays of $\mathbb{DC}(\mathsf{Z}_{C_4})$ correspond to segments, all deformations of Z_{C_4} are zonotopes (which is not the case for the deformations of Π_n).

translations in $\mathbb{R}^{|V|}$. Thus $\mathbb{DC}(\mathsf{Z}_G)$ is simplicial.

For instance, for the 4-cycle C_4, the graphical zonotope Z_{C_4} is the 3-dimensional zonotope depicted in the bottom right of Figure 13 (a rhombic dodecahedron), which arises as the Minkowski sum of 4 vectors in a hyperplane of $\mathbb{R}^4$. Its deformation cone $\mathbb{DC}(\mathsf{Z}_{C_4})$ lives in the 16-dimensional space $\mathbb{R}^{2^{[4]}}$, has dimension 8, a lineality space of dimension 4, and 4 facets. It admits as irredundant description the following 8 equations and 4 inequalities:

$$
\begin{aligned}
\boldsymbol{h}_\varnothing &= -\boldsymbol{h}_{1234} & \boldsymbol{h}_{12} + \boldsymbol{h}_{14} &= \boldsymbol{h}_{124} + \boldsymbol{h}_1 & \boldsymbol{h}_1 + \boldsymbol{h}_2 &\geq \boldsymbol{h}_{12} + \boldsymbol{h}_\varnothing \\
\boldsymbol{h}_1 + \boldsymbol{h}_3 &= \boldsymbol{h}_{13} + \boldsymbol{h}_\varnothing & \boldsymbol{h}_{12} + \boldsymbol{h}_{23} &= \boldsymbol{h}_{123} + \boldsymbol{h}_2 & \boldsymbol{h}_2 + \boldsymbol{h}_3 &\geq \boldsymbol{h}_{23} + \boldsymbol{h}_\varnothing \\
\boldsymbol{h}_2 + \boldsymbol{h}_4 &= \boldsymbol{h}_{24} + \boldsymbol{h}_\varnothing & \boldsymbol{h}_{23} + \boldsymbol{h}_{34} &= \boldsymbol{h}_{234} + \boldsymbol{h}_3 & \boldsymbol{h}_3 + \boldsymbol{h}_4 &\geq \boldsymbol{h}_{34} + \boldsymbol{h}_\varnothing \\
\boldsymbol{h}_{123} + \boldsymbol{h}_{134} &= \boldsymbol{h}_{1234} + \boldsymbol{h}_{13} & \boldsymbol{h}_{14} + \boldsymbol{h}_{34} &= \boldsymbol{h}_{134} + \boldsymbol{h}_4 & \boldsymbol{h}_1 + \boldsymbol{h}_4 &\geq \boldsymbol{h}_{14} + \boldsymbol{h}_\varnothing.
\end{aligned}
$$

After quotienting the lineality and intersecting with an affine hyperplane, we get the 3-simplex (*i.e.* tetrahedron) illustrated in Figure 13.

2.2.4 Simplicial graphical deformation cones

As an immediate corollary, we obtain a characterization of those graphical zonotopes whose deformation cone is simplicial.

Corollary 2.16. *The deformation cone $\mathbb{DC}(\mathsf{Z}_G)$ is simplicial (modulo its lineality) if and only if G is triangle-free.*

Proof. If G is triangle-free, the deformation cone $\mathbb{DC}(\mathsf{Z}_G)$ has dimension $|V| + |E|$, lineality space of dimension $|V|$, and $|E|$ facets, and hence it is simplicial. If G is not triangle-free, then we claim that the number of induced cliques K of G with $|K| \geq 2$ is strictly less than the number of triples (u, v, S) with $\{u, v\} \in E$ and $S \subseteq N(u) \cap N(v)$. Indeed, each induced clique K of G with $|K| \geq 2$ already produces $\binom{|K|}{2}$ triples of the form $(u, v, K \smallsetminus \{u, v\})$ which satisfy $\{u, v\} \in E$ and $K \smallsetminus \{u, v\} \subseteq N(u) \cap N(v)$ and are all distinct. Since $\binom{|K|}{2} > |K|$ as soon as $|K| \geq 3$, by Corollary 2.13, this shows that the deformation cone $\mathbb{DC}(\mathsf{Z}_G)$ is not simplicial. $\square$

Corollary 2.17. *If G is triangle-free, then every deformation of Z_G is a zonotope, which is the graphical zonotope of a subgraph of G up to rescaling of the generators.*

Proof. For any induced clique K of G of size at least 2, Δ_K is a Minkowski indecomposable $(|K| - 1)$-dimensional polytope in the deformation cone $\mathbb{DC}(\mathsf{Z}_G)$ (see for example [Grü03, 15.1.3] for a certificate of indecomposability). It spans therefore a ray of $\mathbb{DC}(\mathsf{Z}_G)$. When G is triangle-free, the deformation cone modulo its lineality is of dimension $|E|$, and the polytopes Δ_e for $e \in E$ account for the $|E|$ rays of the simplicial deformation cone $\mathbb{DC}(\mathsf{Z}_G)$.

Therefore, each polytope $P \in \mathbb{DC}(\mathsf{Z}_G)$ can be uniquely[7] expressed as a Minkowski sum

$$P = \sum_{e \in E} \lambda_e \Delta_e$$

with non-negative coefficients λ_e. Since each Δ_e is a segment, P is a zonotope, normally equivalent to the graphical zonotope of the subgraph $G' = (V, E')$ with $E' = \{e \in E \; ; \; \lambda_e \neq 0\}$. $\square$

2.2.5 Perspectives and open questions

Computational remarks The computation of deformation cones of graphical zonotopes has been implemented with Sage, allowing us to conjecture Corollary 2.13 before proving it. Thanks to this code, one can input a graph G and compute the deformation cone of its graphical zonotope as the cone of heights in $\mathbb{R}^{2^n}$, illustrating Theorem 2.12. Although very symmetric and well suited for mathematical purposes, this first implementation has the inconvenient to live in a highly dimensional space. For this reason, I have also implemented a second version that computes the deformation cone in $\mathbb{R}^{BS(G)}$ where $BS(G)$ is the collection of biconnected subsets of G (which are in bijection with the rays of $\mathcal{F}(G)$). Some technical choices have to be made to speed up this computation, in particular by efficiently using the dual graph of $\mathcal{F}(G)$ in order to get rid of some redundant equalities.

Assets and limits of the current approach, open questions The question of the dimension space of the deformation is of prime importance for a larger subject. In [McM93, McM96], McMullen constructed several algebras associated to a polytope P: in particular, its polytope algebra and its weight algebra. This construction was used to provide an alternative proof of the famous g-theorem of Billera–Lee and Stanley [Sta80]. Both algebras are graded. When P is simple, these two algebras are isomorphic, but in general there is only an embedding of the polytope algebra in the weight algebra. For example, the permutahedron Π_n is simple and the dimension of the k-th graded piece of its polytope algebra is the Eulerian number $A(n, k)$, see [Ham17].

The first graded piece of the polytope algebra of P is the linear span of $\mathbb{DC}(\mathsf{P})$ (*i.e.* the space of virtual deformations discussed above Theorem 2.11). This means that in the present section, we have computed the dimension of the first graded piece of the polytope algebra of graphical zonotopes (and of nestohedra in the next section). On top of that, our result gives a basis of this first graded piece, and the polytope algebra is generated in degree 1.

With Arnau Padrol, we considered the second graded piece of the polytope algebra of graphical zonotopes and managed to find an explicit basis of it. We are currently attempting to extend these results to higher graded pieces. Furthermore, graphical zonotopes are (in general) non-simple polytopes: therefore, we hope to describe the gap between both algebras for graphical zonotopes.

[7] Uniqueness comes from the simpliciality of $\mathbb{DC}(\mathsf{Z}_G)$.