4.3 Fiber polytopes for the projection from $\mathsf{Cyc}_d(\boldsymbol{t})$ to $\mathsf{Cyc}_2(\boldsymbol{t})$

This section is a joint work with Aenne Benjes and Raman Sanyal. An article is in preparation, containing this section, together with Section 3.2.

We have seen that the study of fiber polytopes has drawn a lot of attention in recent years. They were, at the beginning, constructed in order to give a positive answer to the generalized Baues problem [BS92, Rei99] which can be thought of as the problem of structuring the set of subdivisions of a given polytope. From there, fiber polytopes made an appearance in a myriad of domains, ranging from linear optimization up to triangulations, and the search for a category of polytopes. We have already discussed their link with linear programming in Section 4.2, and briefly mentioned the longing for a category of polytope in Section 4.1, and we emphasize here their relationship with triangulations.

In particular, the secondary polytope introduced by Gelfand, Kapranov and Zelevinsky [GKZ90, GKZ91] (see also [BFS90]) encapsulates (regular) triangulations of a point configuration into the vertices of a polytope. It is worth noting that when the points of the configuration are in convex position, *i.e.* when looking at a polytope, the secondary polytope is (a dilation of) the fiber polytope for the projection of a simplex onto these points. In this context, the associahedron again appears, as the secondary polytope of a polygon [DRS10, Chapter 5]. Polygons and simplices are the two extreme cases of cyclic polytopes, so the aim of the section is to factor the projection from a simplex onto a polygon by an intermediate projection from a cyclic polytope onto a polygon: for $\boldsymbol{t} = (t_1, \dots, t_n)$, one consider the sequence of projections $\mathsf{Cyc}_n(\boldsymbol{t}) \to \mathsf{Cyc}_d(\boldsymbol{t}) \to \mathsf{Cyc}_2(\boldsymbol{t})$ where $\mathsf{Cyc}_n(\boldsymbol{t}) \simeq \Delta_n$ and $\mathsf{Cyc}_2(\boldsymbol{t})$ is a n-gon (and $2 \leq d \leq n$). The vertices of the fiber polytope for the projection $\mathsf{Cyc}_d(\boldsymbol{t}) \to \mathsf{Cyc}_2(\boldsymbol{t})$ will naturally associate to triangulations of $\mathsf{Cyc}_2(\boldsymbol{t})$, prompting a notion of degree on triangulation (and hence Catalan families). This way, we will analyze a fiber polytope that is not a monotone path polytope (projection onto a segment), nor a secondary polytope (projection from a simplex), but a more general case.

It is not an accident that the present framework resemble the one of Section 3.2: even if the motivations and context are different, the tools and techniques developed are the same, and the results similar. We will widely reuse the material of this section and the ideas of [ALRS00] who began the exploration of fiber polytopes between cyclic polytopes. The present section starts with a short preliminary on triangulations (Section 4.3.1), gathering the useful vocabulary and constructing the bijection from triangulations to non-crossing arborescences. We pursue with the main result (Section 4.3.2) that determines how to know if a triangulation appears or not as a vertex of the fiber polytope for the projection $\mathsf{Cyc}_d(\boldsymbol{t}) \to \mathsf{Cyc}_2(\boldsymbol{t})$, and we then focus on the case $d = 4$ (Section 4.3.3). Again, quite surprisingly, we obtain that the number of vertices of the fiber polytope $\Sigma_\pi(\mathsf{Cyc}_4(\boldsymbol{t}), \mathsf{Cyc}_2(\boldsymbol{t}))$ is $\binom{n}{2} - 1$, independently of $\boldsymbol{t}$, see Theorem 4.60.

4.3.1 Bijection between triangulations and non-crossing arborescences

Triangulations of a $(n+1)$-gon and non-crossing arborescences on n nodes are both Catalan families, as presented in Section 1.2.4. We exhibit an explicit bijection between these families that will allow us to link fiber polytope for the projection from $\mathsf{Cyc}_d(\boldsymbol{t})$ to $\mathsf{Cyc}_2(\boldsymbol{t})$ and cyclic associahedra $\Pi^d_{\boldsymbol{t}}$ of Section 3.2.

A vast study of triangulations, adorned by plenty of figures, can be found in [LRS10], especially Chapters 3 and 5 for what concerns us here.

Definition 4.42. Let P be a $(n+1)$-gon whose vertices are labelled clockwise from 0 to n in circular order. A *triangle* in P is a triplet of distinct indices $\delta = (i, j, k) \in [0, n]^3$ with $i < j < k$. Such a triangle splits the (cyclic) interval $[n]$ into three *pieces of circle*: $[i, j]$, $[j, k]$ and $[k, n] \cup [0, i]$. Two triangles δ_1, δ_2 in P don't intersect when all three indices of δ_2 belong to the same piece of circle of δ_1. A *triangulation* T of P is a family of $n-1$ (pairwise) non-intersecting triangles, see Figure 70 ($n-1$ being the maximum number of non-intersecting triangles that a $(n+1)$-gon can welcome).

An *edge* of a triangulation T is a couple (x, y) with $x < y$ that appears in a triangle of T: $(x, y) \subset \delta$ for some $\delta \in T$. An edge (x, y) is *exterior* when $y = x + 1$ or $(x, y) = (0, n)$ (meaning

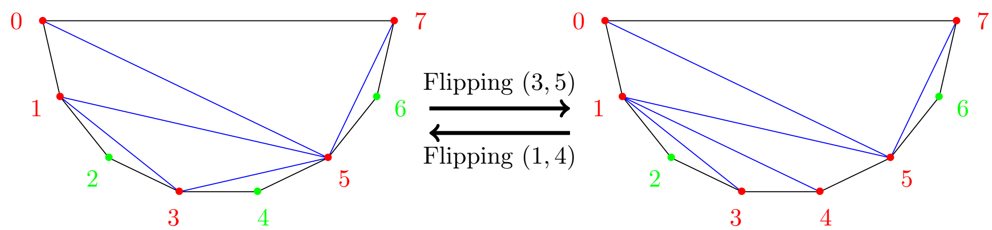

Figure 70: (Left) The triangulation $T = \big((0,5,7),(0,1,5),(1,3,5),(1,2,3),(3,4,5),(5,6,7)\big)$. Its 8 exterior edges are in black, while its 5 interior ones are in blue. It has 2 positive quadrangles and 3 negative ones: $Q^+(T) = \big((0,1,5,7),(1,2,3,5)\big)$ and $Q^-(T) = \big((0,1,3,5),(0,5,6,7),(1,3,4,5)\big)$. Its immediate vertices are in green: $\mathbb{L}(T) = \{2,4,6\}$. Flipping the edge $(3,5)$ gives the triangulation T' on the Right. The new quadrangles are $Q^+(T') = \big((0,1,5,7),(1,3,4,5),(1,2,3,4)\big)$ and $Q^-(T') = \big((0,5,6,7),(0,1,4,5)\big)$, and new immediate vertices are $\mathbb{L}(T') = \{2,6\}$. Flipping the edge $(1,4)$ in T' gives back T.

it is an edge of the polygon P), *interior* otherwise. We denote by $E(T)$ the set of edges of T and $E^\circ(T)$ the set of interior edges of T. Note that in a triangulation, interior edges appear in exactly two triangles, while exterior ones appear in exactly one triangle.

A *quadrangle* in a triangulation T is a quadruplet of indices $\kappa = \delta_1 \cup \delta_2$ corresponding to two adjacent triangles $\delta_1, \delta_2 \in T$, *i.e.* $|\kappa| = 4$. The *edge e_κ of the quadrangle κ* is the (interior) edge shared by the two adjacent triangles. Note that $\kappa \mapsto e_\kappa$ is a bijection between interior edges $E^\circ(T)$ and the set of quadrangles of T. A quadrangle (i,j,k,l) is *positive* when its edge is (i,k), and *negative* when its edge is (j,l). The family of positive quadrangles is denoted $Q^+(T)$, and the family of negative ones $Q^-(T)$.

A *flip* in a triangulation T consists in removing one interior edge and adding back the only other interior edge possible. Namely, if $(x,y) \in E^\circ(T)$, then (x,y) belongs to two triangles, forming a quadrangle κ: flipping (x,y) amounts to changing κ from positive to negative or the reverse (the Tamari orientation consists in changing from negative to positive). This changes one edge, two triangles and at most five quadrangles. It is well known that the graph of flips of the triangulations of a $(n+1)$-gon is precisely the graph of the associahedron Asso_{n-1}.

An *immediate vertex* of T is some index $\ell \in [n-1]$ such that $(\ell-1,\ell,\ell+1) \in T$. When immediate vertices, 1 and $n-1$ are called *exterior*, while other immediate vertices are called *interior* ones. We denote by $\mathbb{L}(T)$ the set of all immediate vertices of T, and by $\mathbb{L}^\circ(T)$ the set of interior ones.

The corresponding super-Catalan family is the family of all subdivisions of P.

There is a very easy way to construct a bijection from the set of triangulations of a $(n+1)$-gon to the set of non-crossing arborescences on n nodes:

Proposition 4.43. *Let T be a triangulation of a $(n+1)$-gon and $E(T)$ its set of edges. Then the map $A_T : [n-1] \to [n-1]$ defined by $A_T(i) = \max\{j \ ; \ j > i \text{ and } (i,j) \in E(T)\}$ is a non-crossing arborescence on n nodes*[16]*, see Figure 71. The application $T \mapsto A_T$ is a bijection between the set of triangulations of a $(n+1)$-gon and the set of non-crossing arborescences on n nodes.*[17]

Proof. First, notice that $T \mapsto A_T$ sends a triangulation of a $(n+1)$-gon to a non-crossing arborescence on n nodes. Indeed, edges of A_T are (some) edges of T, so if A_T were crossing, then two triangles of T would intersect. Furthermore, flipping the edge (j,l) in a negative quadrangle (i,j,k,l) of T changes the (j,l) in A_T to (j,k) because of the non-intersecting property: flips for triangulations correspond to flips for non-crossing arborescences. Thus, the application $T \mapsto A_T$

[16] As usual, it shall be complete with $A_T(n) = n$.

[17] Triangulations are defined on $[0,n]$ while arborescences are defined on $[1,n]$: 0 is not mapped by A_T.

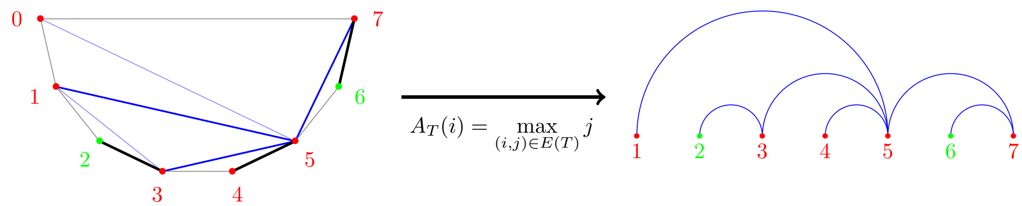

Figure 71: The bijection $T \mapsto A_T$ between triangulations of a $(n+1)$-gon and non-crossing arborescences on n nodes. (Left) In bold are drawn the edges of the triangulation kept in the non-crossing arborescence.

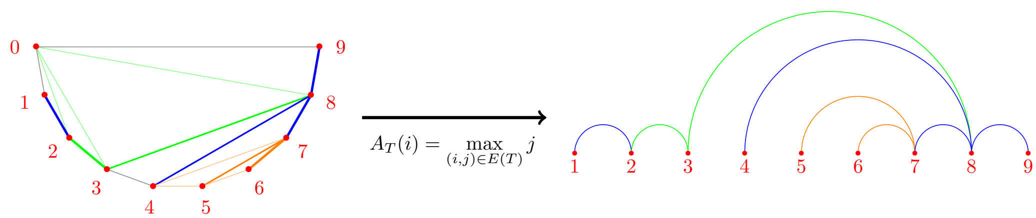

Figure 72: In the left triangulation T, the positive quadrangle $(0,2,3,8) \in Q^+(T)$ is sent to the forward-sliding node $2 \in \mathcal{I}^f_{A_T}$, while the negative quadrangle $(4,5,6,7) \in Q^-(T)$ is sent to the backward-sliding node $5 \in \mathcal{I}^b_{A_T}$.

is surjective as the graph of flips of non-crossing arborescences is connected. As the numbers of triangulations of a $(n+1)$-gon and the number of non-crossing arborescences on n nodes is the same, $T \mapsto A_T$ is a bijection. $\square$

Note that, in particular, immediate vertices of T are sent bijectively through the bijection $T \mapsto A_T$ to immediate leaves of A_T. We refer to Section 3.2.2 for the definitions of forward- and backward-sliding nodes (and consort).

Lemma 4.44. *Let T be triangulation and A_T the associated non-crossing arborescence. Positive quadrangles of T are sent bijectively to forward-sliding nodes A_T through $(i,j,k,l) \mapsto j$; and negative quadrangles of T are sent bijectively to backward-sliding nodes A_T through $(i,j,k,l) \mapsto j$, see Figure 72.*

Proof. Fix $(i,j,k,l) \in Q^+(T)$. Then $k = \max\{y \ ; \ (j,y) \in E(T)\}$ because otherwise (j,y) would cross the edge (i,k). Thus $A_T(j) = k$. Besides $l = \max\{y \ ; \ (k,y) \in E(T)\}$ because otherwise (k,y) would cross the edge (i,l). Thus $A_T(k) = l$. Finally, an edge (a,k) with $a < j$ would cross either the edge (i,j) or the edge (i,l). So $j = \min\{x \ ; \ A_T(x) = k\}$, which fulfills the definition for $j \in \mathcal{I}^f_{A_T}$.

The proof is similar for the negative quadrangles. As $|Q^+(T)|+|Q^-(T)| = n-1 = |\mathcal{I}^f_{A_T}|+|\mathcal{I}^b_{A_T}|$, the two bijections holds. $\square$

4.3.2 Fiber polytopes for the projection $\mathsf{Cyc}_d(\boldsymbol{t}) \xrightarrow{\pi} \mathsf{Cyc}_2(\boldsymbol{t})$

In the remaining of this section, we present some new results on a family of fiber polytopes associated to cyclic polytopes. These results extend the one of [ALRS00]. It will be the opportunity to use the tools and ideas developed in Section 3.2.

In the latter, we designated the order cone by $\mathsf{O}_n = \{\boldsymbol{t} \in \mathbb{R}^n \ ; \ t_1 \leq \cdots \leq t_n\}$. In what follows, we will slightly abuse notations: for a fixed $n \geq 1$, if $\boldsymbol{t} \in \mathsf{O}_{n+1}$, then its coordinates will be denoted $\boldsymbol{t} = (t_0, t_1, \dots, t_n)$, and triangulations will be on a $(n+1)$-gon ; while if $\boldsymbol{t} \in \mathsf{O}_n$, then its coordinates will be denoted $\boldsymbol{t} = (t_1, \ldots, t_n)$ and non-crossing arborescences will take place on n nodes.

Definition 4.45. For $d \geq 2$ and $\boldsymbol{t} \in \mathsf{O}^{\circ}_{n+1}$, the *fiber associahedron* $\Sigma^d_2(\boldsymbol{t})$ is the fiber polytope $\Sigma_\pi(\mathsf{Cyc}_d(\boldsymbol{t}), \mathsf{Cyc}_2(\boldsymbol{t}))$ where π is the projection that forgets all but the two first coordinates: $\pi(\boldsymbol{x}) = (\langle \boldsymbol{x}, \boldsymbol{e}_1 \rangle\,,\ \langle \boldsymbol{x}, \boldsymbol{e}_2 \rangle)$.

By Corollary 4.10, $\Sigma^d_2(\boldsymbol{t})$ is a projection of the secondary polytope $\Sigma(\mathsf{Cyc}_2(\boldsymbol{t}))$. As $\mathsf{Cyc}_2(\boldsymbol{t})$ is a polygon with n vertices, its secondary polytope $\Sigma(\mathsf{Cyc}_2(\boldsymbol{t}))$ is an associahedron Asso_{n-2}, see [LRS10]. For $\boldsymbol{t} \in \mathsf{O}^{\circ}_{n+1}$, as vertices of $\Sigma(\mathsf{Cyc}_2(\boldsymbol{t}))$ are naturally associated to triangulations of $\mathsf{Cyc}_2(\boldsymbol{t})$, one can associate a triangulation to each vertex of $\Sigma^d_2(\boldsymbol{t})$. We now establish a criterion for a triangulation to be associated to a vertex of $\Sigma^d_2(\boldsymbol{t})$.

Proposition 4.46. *For $(i, j, k, l) \in [0, n]^4$ with $i < j < k < l$, $\boldsymbol{t} \in \mathsf{O}^{\circ}_n$ and a polynomial P, we denote*

$$\tau(i, j, k, l) = \det \begin{pmatrix} 1 & 1 & 1 & 1 \\ t_i & t_j & t_k & t_l \\ t_i^2 & t_j^2 & t_k^2 & t_l^2 \\ P(t_i) & P(t_j) & P(t_k) & P(t_l) \end{pmatrix}$$

For $\boldsymbol{t} \in \mathsf{O}^{\circ}_{n+1}$ and $d \geq 2$, a triangulation T of $\mathsf{Cyc}_2(\boldsymbol{t})$ corresponds to a vertex of $\Sigma^d_2(\boldsymbol{t})$ if and only if there exists a polynomial P of degree at most d such that $\tau(\kappa) > 0$ for all $\kappa \in Q^+(T)$ and $\tau(\kappa) < 0$ for all $\kappa \in Q^-(T)$.

When these conditions are satisfied, we say that P captures the triangulation T on $\boldsymbol{t}$.

To prove this property, we need a classical lemma, of which we give a short proof for the sake of completeness.

Lemma 4.47. *For a 3-dimensional polytope P with vertices $\boldsymbol{v}_0 = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}, \ldots, \boldsymbol{v}_n = \begin{pmatrix} x_n \\ y_n \\ z_n \end{pmatrix}$, a triangle (i, j, k) corresponds to a lower face of P if and only if, for all $l \in [0, n] \smallsetminus \{i, j, k\}$ one has:*

$$\det \begin{pmatrix} 1 & 1 & 1 & 1 \\ x_i & x_j & x_k & x_l \\ y_i & y_j & y_k & y_l \\ z_i & z_j & z_k & z_l \end{pmatrix} > 0$$

Proof of Lemma 4.47. The positivity of this determinant is equivalent to the fact that $\boldsymbol{v}_l \in H^+_{(i,j,k)}$ where $H_{(i,j,k)}$ is the plane of $\mathbb{R}^3$ containing the points $\boldsymbol{v}_i$, $\boldsymbol{v}_j$ and $\boldsymbol{v}_k$. □

Proof of Proposition 4.46. Fix $\boldsymbol{t} \in \mathsf{O}^{\circ}_{n+1}$. By Theorem 4.6, vertices of $\Sigma^d_2(\boldsymbol{t})$ are in bijection with π-coherent triangulations of $\mathsf{Cyc}_2(\boldsymbol{t})$. Pick $\boldsymbol{w} = (w_1, ..., w_d) \in \mathbb{R}^d$ generic with respect to $\mathsf{Cyc}_d(\boldsymbol{t})$ and construct $\pi^{\boldsymbol{w}} : \boldsymbol{x} \mapsto \begin{pmatrix} \pi(\boldsymbol{x}) \\ \langle \boldsymbol{w}, \boldsymbol{x} \rangle \end{pmatrix}$ as in Definition 4.4. Then the vertices of $\pi^{\boldsymbol{w}}(\mathsf{Cyc}_d(\boldsymbol{t}))$ come from vertices $\boldsymbol{\gamma}_d(t_i)$ of $\mathsf{Cyc}_d(\boldsymbol{t})$: they are thus of the form $p_i := \begin{pmatrix} t_i \\ t_i^2 \\ \langle \boldsymbol{w}, \boldsymbol{\gamma}_d(t_i) \rangle \end{pmatrix}$. Denoting $P(t) = w_1 t + \cdots + w_d t^d$, one has $\langle \boldsymbol{w}, \boldsymbol{\gamma}_d(t_i) \rangle = P(t_i)$. The family of lower faces of $\pi^{\boldsymbol{w}}(\mathsf{Cyc}_d(\boldsymbol{t}))$ projects down to a triangulation of $\mathsf{Cyc}_2(\boldsymbol{t})$ (by forgetting the last coordinate).

Consequently, a triangulation T of $\mathsf{Cyc}_2(\boldsymbol{t})$ appears as such a projection if and only if there exists a polynomial P of degree at most d satisfying that for all triangle $\delta = (j, k, l) \in T$, the points $\boldsymbol{p}_j, \boldsymbol{p}_k, \boldsymbol{p}_l$ are the vertices of a lower face of $\operatorname{conv}\{\boldsymbol{p}_i\ ;\ i \in [0, n]\}$. By Lemma 4.47, this amount to having $\tau(j, k, l, m) > 0$ for all $m \in [0, n] \smallsetminus \{j, k, l\}$.

In the associahedron Asso_{n-2}, the vertex associated with T is adjacent to the vertices associated with the triangulations T' obtained by flipping any quadrangle in T. Hence, by convexity, it is equivalent to test the positivity of $\tau(j,k,l,m)$ for (j,k,l,m) a quadrangle of T, than to test the positivity of $\tau(j,k,l,m)$ for all $(j,k,l) \in T$ and $m \in [0,n] \smallsetminus \{j,k,l\}$.

Thus, by re-ordering the columns of the determinant $\tau(\kappa)$, the triangulation T appears if and only P satisfies that $\tau(\kappa) > 0$ for all $\kappa \in Q^+(T)$ and $\tau(\kappa) < 0$ for all $\kappa \in Q^-(T)$. $\square$

The above Proposition 4.20 gives a simple criterion for determining the triangulation captured by a polynomial on a given $\boldsymbol{t}$. Moreover, Lemma 4.47 ensures that if P captures T on $\boldsymbol{t}$, then we know the value of $\tau(i,j,k,l)$ for all quadruples $(i,j,k,l) \in [0,n]^4$, not only for the quadrangles of T.

Recall from Appendix A that the complete symmetric polynomial of degree s on 4 variables is:

$$h_s(X,Y,Z,U) = \sum_{a+b+c+d=s} X^a Y^b Z^c U^d$$

For a quadrangle $\kappa = (i,j,k,l)$ in a triangulation T and $\boldsymbol{t} \in \mathsf{O}^\circ_{n+1}$, we construct $\Omega^d_\kappa(\boldsymbol{t}) \in \mathbb{R}^d$ defined by $\Omega^d_\kappa(\boldsymbol{t})_s = h_{s-3}(t_i,t_j,t_k,t_l)$, together with $\overline{\Omega}^d_\kappa(\boldsymbol{t}) = \big(h_s(t_i,t_j,t_k,t_l)\big)_{s=1,\dots,d-3} \in \mathbb{R}^{d-2}$. As for Theorem 3.16, these points allow us to reformulate Proposition 4.46 into a more handy criterion, that hinges on intersection of polytopes.

Theorem 4.48. *For $\boldsymbol{t} \in \mathsf{O}^\circ_{n+1}$, a triangulation T of $\mathsf{Cyc}_2(\boldsymbol{t})$ can be captured on $\boldsymbol{t}$ if and only if the following polytopes **do not** intersect:*

$$\mathsf{Q}^+_d(T,\boldsymbol{t}) = \operatorname{conv}\left\{\overline{\Omega}^d_\kappa(\boldsymbol{t}) \ ; \ \kappa \in Q^+(T)\right\} \quad \text{and} \quad \mathsf{Q}^-_d(T,\boldsymbol{t}) = \operatorname{conv}\left\{\overline{\Omega}^d_\kappa(\boldsymbol{t}) \ ; \ \kappa \in Q^-(T)\right\}$$

Proof. Fix $\boldsymbol{t} \in \mathsf{O}^\circ_{n+1}$. A triangulation T can be captured on $\boldsymbol{t}$ if and only if there exists $P(t) = w_d t^d + \dots + w_1 t$ such that $\tau(\kappa) > 0$ for all $\kappa \in Q^+(T)$, and $\tau(\kappa) < 0$ for all $\kappa \in Q^-(T)$. Remark that, by linearity of the determinant and Theorem A.2:

$$\tau(\kappa) = \sum_{s=1}^{d} w_s \det\begin{pmatrix} 1 & 1 & 1 & 1 \\ t_i & t_j & t_k & t_l \\ t_i^2 & t_j^2 & t_k^2 & t_l^2 \\ t_i^s & t_j^s & t_k^s & t_l^s \end{pmatrix} = \mathrm{VdM}_4(t_i,t_j,t_k,t_l) \sum_{s=1}^{d} w_s h_{s-3}(t_i,t_j,t_k,t_l)$$

where the Vandermonde determinant $\mathrm{VdM}_4(t_i,t_j,t_k,t_l) = (t_l - t_k)(t_l - t_j)(t_l - t_i)(t_k - t_j)(t_k - t_i)(t_j - t_i) > 0$ (as $i<j<k<l$ and $\boldsymbol{t} \in \mathsf{O}^\circ_{n+1}$).

Thus, the existence of P amounts to the existence of a solution $\boldsymbol{w}$ to the system $\sum_s w_s h_{s-3}(\kappa) > 0$ if κ is positive, and negative respectively. By Gordan's lemma, this is equivalent to the existence of a $\lambda_\kappa \geq 0$, for all κ, non-identically zero, satisfying

$$\sum_{\kappa \in Q^+(T)} \lambda_\kappa \Omega^d_\kappa(\boldsymbol{t}) = \sum_{\kappa \in Q^-(T)} \lambda_\kappa \Omega^d_\kappa(\boldsymbol{t})$$

Since $\Omega^d_\kappa(\boldsymbol{t})_2 = 1$, it follows that $\Lambda = \sum_{\kappa \in Q^+(T)} \lambda_\kappa = \sum_{\kappa \in Q^-(T)} \lambda_\kappa > 0$. Dividing both sides of the previous equation by Λ yields a point in $\mathsf{Q}^+_d(T,\boldsymbol{t}) \cap \mathsf{Q}^-_d(T,\boldsymbol{t})$. $\square$

Once defined the notion of capturing a triangulation, we can define the degree of a triangulation and its realization set, mirroring the ones of non-crossing arborescences. Even if it will be slightly confusing at first glance, we adopt the same notations for triangulations and non-crossing arborescences, as the ideas concerning them are too akin to be distinguished by new notations.

Definition 4.49. Let T be a triangulation of a $(n+1)$-gon.

For $\boldsymbol{t} \in \mathsf{O}^\circ_{n+1}$, the *degree* of T on $\boldsymbol{t}$ is $\mu(T,\boldsymbol{t}) = \min\{\deg P \ ; \ T \text{ is captured by } P \text{ on } \boldsymbol{t}\}$. The *intrinsic degree* of T is $\mu(T) = \min\{\mu(T,\boldsymbol{t}) \ ; \ \boldsymbol{t} \in \mathsf{O}^\circ_{n+1}\}$.

For $d \geq 2$, the *realization set* of T of degree d is $\mathcal{T}^\circ_d(T) = \{\boldsymbol{t} \in \mathsf{O}^\circ_{n+1} \ ; \ \mu(T,\boldsymbol{t}) \leq d\}$.

A triangulation T is *universal* when $\mathcal{T}^\circ_{\mu(T)} = \mathsf{O}^\circ_{n+1}$.

These notions respect the same straightforward properties as their counterparts for non-crossing arborescences. For all $\boldsymbol{t} \in \mathsf{O}^\circ_{n+1}$, one has $\mu(T, \boldsymbol{t}) \leq n+1$, and consequently $\mu(T) \leq n+1$. Indeed, if $d \geq n+1$, then $\mathsf{Cyc}_d(\boldsymbol{t})$ is a simplex and $\Sigma^d_2(\boldsymbol{t})$ is combinatorially isomorphic to the secondary polytope $\Sigma(\mathsf{Cyc}_2(\boldsymbol{t}))$: an associahedron.

For $d \geq 2$, the closure $\mathcal{T}_d(T)$ of $\mathcal{T}^\circ_d(T)$ is a (generally non-polyhedral and even non-convex) full-dimensional subcone of the order cone O_{n+1}, because if T can be captured on $\boldsymbol{t} \in \mathsf{O}^\circ_{n+1}$, then by translation, T can be captured on $(t_0 + c, \dots, t_n + c)$ for all $c \in \mathbb{R}$, and on $\lambda \boldsymbol{t}$ for $\lambda > 0$, by a polynomial of the same degree. Furthermore, the definition ensures that:

$$\mathcal{T}_3(T) \subseteq \mathcal{T}_4(T) \subseteq \cdots \subseteq \mathcal{T}_{n+1}(T) = \mathsf{O}_{n+1}$$

In addition, Theorem 4.48 gives the following powerful reformulation.

Proposition 4.50. *Let T be a triangulation of a $(n+1)$-gon. One has, for $\boldsymbol{t} \in \mathsf{O}^\circ_{n+1}$ and $d \geq 3$: $\mu(T, \boldsymbol{t}) = \min\{d \ ; \ \mathsf{Q}^+_d(T, \boldsymbol{t}) \cap \mathsf{Q}^-_d(T, \boldsymbol{t}) = \varnothing\}$, and $\mathcal{T}^\circ_d(T) = \{\boldsymbol{t} \ ; \ \mathsf{Q}^+_d(T, \boldsymbol{t}) \cap \mathsf{Q}^-_d(T, \boldsymbol{t}) = \varnothing\}$.*

Beside these properties, $\mu(T, \boldsymbol{t})$ and $\mu(T)$ are hard to describe: Theorem 4.48 gives a nice way to check whether a triangulation can be captured in some degree, but no mean to estimate the minimal degree after which it becomes possible. In particular, note that the coordinates of $\mathsf{Q}^+_d(T, \boldsymbol{t})$ and $\mathsf{Q}^-_d(T, \boldsymbol{t})$ are polynomials of degree up to $d-3$, thus it is simple to study the case $d = 4$, but when $d \geq 5$ deciding if their intersection is empty becomes as hard as deciding if there exists a solution to a certain polynomial system (of degree at least $d-3$). In the following part of this section, we focus on the case $d = 4$.

4.3.3 Realization sets and universal triangulations for $\mathsf{Cyc}_4(\boldsymbol{t}) \xrightarrow{\pi} \mathsf{Cyc}_2(\boldsymbol{t})$

In this section, we study the fiber polytope $\Sigma^4_2(\boldsymbol{t}) = \Sigma_\pi(\mathsf{Cyc}_4(\boldsymbol{t}), \mathsf{Cyc}_2(\boldsymbol{t}))$ for $\boldsymbol{t} \in \mathsf{O}^\circ_{n+1}$ where $\pi : \mathbb{R}^4 \to \mathbb{R}^2$ is the projection forgetting all but the two first coordinates. These results extend the last example of [ALRS00]. In particular, we give a complete characterization of which triangulations of a $(n+1)$-gon can be associated to a vertex of $\Sigma^d_2(\boldsymbol{t})$ for some $\boldsymbol{t} \in \mathsf{O}^\circ_{n+1}$, then we describe their realization sets, state which of them are universal, and conclude on the number of vertices of $\Sigma^d_2(\boldsymbol{t})$. Even though the computations of the present section are different from the ones of Section 3.2, the ideas behind them clearly look alike. Note however that there seems not to be a straightforward way to deduce the following results from the theorems of Section 3.2: we will see in Example 4.61 an example indicating that cyclic associahedra and fiber associahedra are indeed dissimilar.

Even though it will not be at the center of our proofs, the bijection $T \mapsto A_T$ between triangulations of a $(n+1)$-gon will help us get a better understanding of the notions at stake. Indeed, we will prove that this bijection induces a bijection between:

(i) Triangulations T with $\mu(T) = 3$ and non-crossing arborescences A with $\mu(A) = 2$ (Corollary 4.52(i)).
(ii) Triangulations T with $\mu(T) = 4$ and non-crossing arborescences A with $\mu(A) = 3$ (Theorem 4.55).
(iii) Universal triangulations T with $\mu(T) = 4$ and universal non-crossing arborescences A with $\mu(A) = 3$ (Corollary 4.54).

Nevertheless, this bijection is not a magic wand! Some powerful properties are not shared between cyclic associahedra and fiber associahedra, in particular:

(a) We don't have a theorem that characterizes $\mu(T)$ in terms of $\mathbb{L}(T)$ and $\mathbb{L}^\circ(T)$ (a twin to Corollary 3.13). In particular, we don't know if there exists a triangulation T such that $\mu(T) = 6$ but $\mu(A_T) = 4$.
(b) The vertices of $\Sigma^4_2(\boldsymbol{u})$ for $\boldsymbol{u} \in \mathsf{O}^\circ_{n+1}$ correspond to a family of triangulations, but the associated family of non-crossing arborescences does not necessarily correspond to the vertices of $\Pi^3_{\boldsymbol{t}}$ for any $\boldsymbol{t} \in \mathsf{O}^\circ_{n+1}$, see Example 4.61.

We first would like to show that triangulations T with $|\mathbb{L}(T)| + |\mathbb{L}^\circ(T)| \leq 2$ are exactly the ones satisfying $\mu(T) \leq 4$. We will prove this in two steps. We first state one inclusion, and postpone the reciprocal for later (see Theorem 4.55).

Proposition 4.51. *For a triangulation T of a $(n+1)$-gon: if $\mu(T) \leq 4$ then $|\mathbb{L}(T)| + |\mathbb{L}^\circ(T)| \leq 2$.*

Proof. Fix a triangulation T captured by $P(t) = w_1 t + w_2 t^2 + w_3 t^3 + w_4 t^4$ on $\boldsymbol{t} \in \mathsf{O}^\circ_{n+1}$. Suppose that $|\mathbb{L}(T)| + |\mathbb{L}^\circ(T)| > 2$, then either $|\mathbb{L}^\circ(T)| \geq 2$, or $\mathbb{L}(T) \cap \{1, n-1\} \neq \varnothing$ and $\mathbb{L}^\circ(T) \neq \varnothing$.

Suppose $|\mathbb{L}^\circ(T)| \geq 2$. Let $\ell \in \mathbb{L}^\circ(T)$. For $a < \ell - 1$ and $b > \ell + 1$, Lemma 4.47 ensures that:

$$\tau(\ell-1, \ell, \ell+1, a) > 0 \quad \text{and} \quad \tau(\ell-1, \ell, \ell+1, b) > 0$$

Giving:

$$\begin{cases} \mathrm{VdM}_4(t_{\ell-1}, t_\ell, t_{\ell+1}, t_a)\,\big(w_4(t_{\ell-1} + t_\ell + t_{\ell+1} + t_a) + w_3\big) & > \; 0 \qquad \text{and} \\ \mathrm{VdM}_4(t_{\ell-1}, t_\ell, t_{\ell+1}, t_b)\,\big(w_4(t_{\ell-1} + t_\ell + t_{\ell+1} + t_b) + w_3\big) & > \; 0 \end{cases}$$

As $a < \ell - 1 < \ell < \ell + 1 < b$, the signs of the Vandermonde determinants give:

$$w_4(t_{\ell-1} + t_\ell + t_{\ell+1} + t_a) + w_3 < 0 \quad \text{and} \quad w_4(t_{\ell-1} + t_\ell + t_{\ell+1} + t_b) + w_3 > 0$$

But if $m \in \mathbb{L}^\circ(T)$ with $\ell < m$, then $w_4(t_{\ell-1} + t_\ell + t_{\ell+1} + t_m) + w_3 > 0$ as $\ell \in \mathbb{L}^\circ(T)$, and $w_4(t_{m-1} + t_m + t_{m+1} + t_\ell) + w_3 < 0$ as $m \in \mathbb{L}^\circ(T)$. But $t_{\ell-1} \leq t_{m-1}$ and $t_{\ell+1} \leq t_{m+1}$, so $w_4(t_{\ell-1} + t_\ell + t_{\ell+1} + t_m) + w_3 < w_4(t_{m-1} + t_m + t_{m+1} + t_\ell) + w_3$, which contradicts the signs of each side.

The same ideas apply when $\{1, n-1\} \cap \mathbb{L}(T) \neq \varnothing$ and $\mathbb{L}^\circ(T) \neq \varnothing$. $\square$

The above theorem can be reformulated in saying that $T \mapsto A_T$ injects the family of triangulations T with $\mu(T) \leq 4$ into the family of non-crossing arborescences A with $\mu(A) \leq 3$. This allows us to give a description of the triangulations with $\mu(T) \leq 4$.

Corollary 4.52. *If T is a triangulation with $\mu(T) \leq 4$, then T falls in one of the following cases:*

- *(i) The triangulations T_m with interior edges $E^\circ(T_m) = \{(0, i) \; ; \; i \in [n-1]\}$, and T_M with interior edges $E^\circ(T_M) = \{(i, n) \; ; \; i \in [n-1]\}$. These are the only 2 triangulations with $\mu(T) = 3$. Note that $\mathbb{L}(T_m) = \{1\}$ and $\mathbb{L}(T_M) = \{n-1\}$.*
- *(ii) For $1 < k < n-1$, triangulations with triangles $(0, i, i+1)$ for $i < k$, $(0, k, n)$, and $(i, i+1, n)$ for $i \geq k$. These are $n-1$ triangulations with $\mathbb{L}(T) = \{1, n-1\}$.*
- *(iii) For $1 < \ell < n-2$, triangulations with $(\ell-1, \ell+1) \in E^\circ(T)$ and all $(x, y) \in E^\circ(T)$ satisfy $x < \ell < y$. These are $2^n - 2$ triangulations with $\mathbb{L}(T) = \{\ell\}$.*

Proof. For (i), note that $Q^-(T_m) = \varnothing$ and $Q^+(T) = \big((0, i, i+1, i+2) \; ; \; i \in [n-2]\big)$, so Theorem 4.48 ensures that T_m can be captured on any $\boldsymbol{t}$ by a degree 3 polynomial, as $\mathsf{Q}^-_4(T_m, \boldsymbol{t}) = \varnothing$ (so the intersection is empty). The case of T_M is identical.

For (ii) and (iii), note that all triangulations (on a polygon of any number of vertices) have an immediate leave, by induction. If $\ell \in \mathbb{L}^\circ(T)$, and $(x, y) \in E^\circ(T)$ with $\ell \notin [x, y]$, then the sub-triangulation $T\big|_{[x,y]}$ is the triangulation of some polygon: there is an immediate leaf $m \in \mathbb{L}^\circ(T)$ with $x \leq m \leq y$, so $m \neq \ell$. Consequently, if $\mu(T) \leq 4$, then Proposition 4.51 implies that T is of the form (ii) or (iii). $\square$

In the rest of this section, we give a description of the realization sets for triangulations T with $\mu(T) \leq 4$, and the characterization of universal triangulations.

Lemma 4.53. *Let T be a triangulation of a $(n+1)$-gon with $\mu(T) = 4$ and $\mathbb{L}(T) = \{\ell\}$, $1 < \ell < n-1$. Then $\mu(T, \boldsymbol{t}) = 4$ for all $\boldsymbol{t} \in \mathsf{O}^\circ_{n+1}$ satisfying:*

$$\max\{t_i + t_j + t_k + t_l \; ; \; (i, j, k, l) \in Q^-(T)\} \; < \; \min\{t_i + t_j + t_k + t_l \; ; \; (i, j, k, l) \in Q^+(T)\}$$

Proof. By Theorem 4.48, we know that T can be captured on $\boldsymbol{t} \in \mathsf{O}^\circ_{n+1}$ by a polynomial of degree 4 if and only if $\mathsf{Q}^+_4(T, \boldsymbol{t}) \cap \mathsf{Q}^-_4(T, \boldsymbol{t}) = \varnothing$. As they are 1-dimensional, we denote $\mathsf{Q}^+_4(T, \boldsymbol{t}) = [x_+, y_+]$ and $\mathsf{Q}^-_4(T, \boldsymbol{t}) = [x_-, y_-]$. Suppose proven that $x_- < y_+$, then $\mathsf{Q}^+_4(T, \boldsymbol{t}) \cap \mathsf{Q}^-_4(T, \boldsymbol{t}) = \varnothing$ if and only if $y_- < x_+$, which is what the lemma states.

For T non-universal with $\mu(T)=4$, let ℓ be its immediate leaf. Corollary 4.52 ensures that negative quadrangles (i,j,k,l) satisfy either $i<j<\ell<k<l$ or $i<j<k<\ell<l$. There always exists a negative quadrangle of the first kind: if (i,j,k,l) is of the second kind, then the quadrangle which edge is $(\max\{x\ ;\ (x,l)\in E(T)\},\ l)$ is of the first kind.

For (i,j,k,l) of the first kind, if (j,k) is an interior edge, then there exists $a\in]j,k[$ such that $(j,a,k)\in T$: the quadrangle (j,a,k,l) is positive with $t_i+t_j+t_k+t_l<t_j+t_a+t_k+t_l$ (as $\boldsymbol{t}\in \mathsf{O}^\circ_{n+1}$). Else, $(j,k)=(\ell-1,\ell)$, and taking $i'=\min\{x\ ;\ (x,\ell+1)\in E(T)\}$ gives a negative quadrangle $(i',i'+1,i'+2,\ell)$ and a positive quadrangle $(i',i'+1,\ell,\ell+1)$ satisfying $t_{i'}+t_{i'+1}+t_{i'+2}+t_\ell<t_{i'}+t_{i'+1}+t_\ell+t_{\ell+1}$. In all cases, we have proven that $x_-<y_+$, yielding the lemma. □

Lemma 4.53 allows us to show that universal triangulations T for $\mu(T)\leq 4$ are in bijection with universal non-crossing arborescences A with $\mu(A)\leq 3$:

Corollary 4.54. *A triangulation T of a $(n+1)$-gon is universal if and only if $\mu(T)=3$, or if $\mu(T)=4$ and one of the following holds:*

(i) $\mathbb{L}(T)=\{1,n-1\}$ *;*

(ii) $\mathbb{L}(T)=\{n-2\}$ *and interior edges of T are either $(1,3)$ and $(0,i)$ for $i\in[3,n-1]$, or $(1,3)$, $(1,4)$ and $(0,i)$ for $i\in[4,n-1]$;*

(iii) $\mathbb{L}(T)=\{2\}$ *and interior edges of T are either $(n-3,n-1)$ and (i,n) for $i\in[1,n-3]$, or $(n-3,n-1)$, $(n-4,n-1)$ and (i,n) for $i\in[1,n-3]$.*

Note that they are in bijection with universal non-crossing arborescences A with $\mu(A)=3$, through the usual bijection $T\mapsto A_T$.

Proof. If $\mu(T)=3$, then the universality of T follows directly from the proof of Corollary 4.52, as $\mathsf{Q}_4^-(T_m,\boldsymbol{t})=\varnothing$, and $\mathsf{Q}_4^-(T_M,\boldsymbol{t})=\varnothing$.

(i) In this case, by Corollary 4.52, there exists k such that $Q^+(T)=\big((0,i,i+1,i+2)\ ;\ i\in[k-2]\big)\cup\big((0,k-1,k,n)\big)$, and $Q^-(T)=\big((i,i+1,i+2,n)\ ;\ i\in[k,n-3]\big)\cup\big((0,k,k+1,,n)\big)$. Then $\mathsf{Q}_4^-(T,\boldsymbol{t})\cap\mathsf{Q}_4^+(T,\boldsymbol{t})=\varnothing$ for all $\boldsymbol{t}\in\mathsf{O}^\circ_{n+1}$, as $\sum_{i\in\kappa}t_i<\sum_{j\in\kappa'}t_j$ for all $\kappa\in Q^-(T)$ and $\kappa'\in Q^-(T)$. Thus by Theorem 4.48, T is universal.

(ii) Suppose $(1,3)\in E(T)$ but $(1,4)\notin E(T)$. In this case, $Q^-(T)=\big((0,1,2,3)\big)$ and $Q^+(T)=\big((0,i,i+1,i+2)\ ;\ i\in[3,n-2]\big)$. Thus $\mathsf{Q}_4^+(T,\boldsymbol{t})\cap\mathsf{Q}_4^-(T,\boldsymbol{t})=\varnothing$ for all $\boldsymbol{t}\in\mathsf{O}^\circ_{n+1}$.

(ii) Suppose $(1,3)\in E(T)$ and $(1,4)\in E(T)$. In this case, $Q^-(T)=\big((0,1,3,4)\big)$ and $Q^+(T)=\big((0,i,i+1,i+2)\ ;\ i\in[4,n-2]\big)\cup\big((1,2,3,4)\big)$. Thus $\mathsf{Q}_4^+(T,\boldsymbol{t})\cap\mathsf{Q}_4^-(T,\boldsymbol{t})=\varnothing$ for all $\boldsymbol{t}\in\mathsf{O}^\circ_{n+1}$.

(iii) This case is symmetric to (ii).

We finish by proving that if T does not belong to the above cases, then T is not universal, meaning there exists $\boldsymbol{t}\in\mathsf{O}^\circ_{n+1}$ with $\mathsf{Q}_4^+(T,\boldsymbol{t})\cap\mathsf{Q}_4^-(T,\boldsymbol{t})\neq\varnothing$. Fix T of the form of Corollary 4.52(iii), and for $i<\ell$, denote j_i the index $j_i>\ell$ such that $(i-1,i,j_i)\in T$. If $j_{\ell-1}\geq\ell+2$, then $(\ell-1,\ell,\ell+1,\ell+2)\in Q^+(T)$ and $(\ell-2,\ell-1,j_i,j_i-1)\in Q^-(T)$. Taking an arbitrarily high value for t_{j_i} violates the inequality of Lemma 4.53. The case $j_i=\ell+2$ is a mirror of the latter. □

With Corollary 4.52 and Lemma 4.53, we can also prove the reciprocal of Proposition 4.51:

Theorem 4.55. *For a triangulation T of a $(n+1)$-gon, $\mu(T)\leq 4$ if and only if $|\mathbb{L}(T)|+|\mathbb{L}^\circ(T)|\leq 2$.*

Proof. Proposition 4.51 states that if $\mu(T)\leq 4$ then $|\mathbb{L}(T)|+|\mathbb{L}^\circ(T)|\leq 2$. We prove the reciprocal by induction on n. The latter is clear for (the only) triangulation on $n+1=3$ vertices.

Fix a triangulation T with $|\mathbb{L}(T)|+|\mathbb{L}^\circ(T)|\leq 2$. Corollary 4.54 ensures that $\mu(T)\leq 4$ if T is of the form of Corollary 4.52(i) or Corollary 4.52(ii). Suppose T is of the form Corollary 4.52(iii), then either $(0,1,n)\in T$ or $(0,n-1,n)\in T$. In the first case, set $T'=T\big|_{[1,n]}$ and construct, by induction, $\boldsymbol{t}'\in\mathsf{O}^\circ_n$ such that $\mu(T',\boldsymbol{t}')=4$. Then, define $\boldsymbol{t}=(t_0,t'_1,\dots,t'_n)$ by choosing t_0 arbitrarily small. Then, $Q^+(T)=Q^+(T')$, and $Q^-(T)=Q^-(T')\cup\big((0,1,a,n)\big)$ with $a\in\{2,n-1\}$. As t_0 is small enough, we have $t_0+t_1+t_a+t_n<\min\{t_i+t_j+t_k+t_l\ ;\ (i,j,k,l)\in Q^+(T)\}$, so Lemma 4.53 ensures that $\mu(T,\boldsymbol{t})=4$.

The case of $(0,n-1,n)\in T$ is solved similarly by setting t_n large enough. □

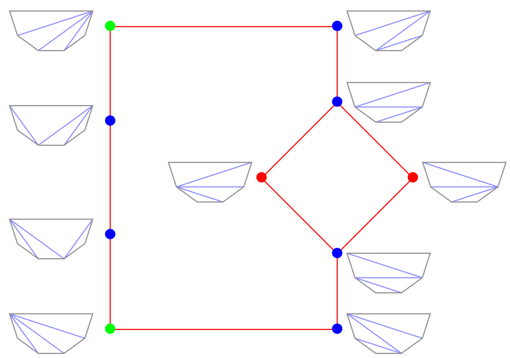

Figure 73: All triangulations T of a hexagon with $\mu(T) \leq 4$. Green and blue dots represent universal triangulations, red dots non-universal ones.

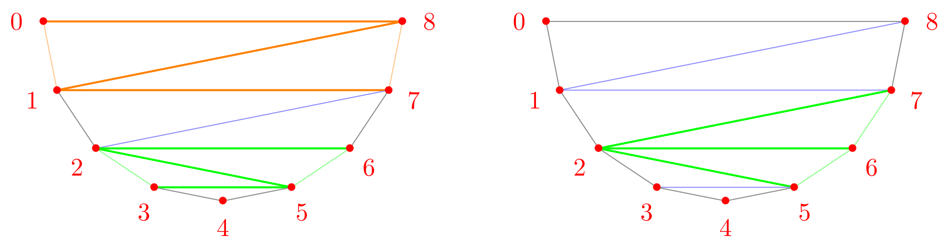

Figure 74: A non-universal triangulation T with $\mu(T) = 4$ and $\mathbb{L}(T) = \{4\}$. It has 2 minimal positive quadrangles among which $(2, 3, 5, 6)$, and 3 maximal negative quadrangles among which $(0, 1, 7, 8)$. It has 1 non-minimal positive quadrangle $(2, 5, 6, 7)$, and no non-maximal negative quadrangle.

As announced, we have proven that $T \mapsto A_T$ induces a bijection between:

(i) Triangulations T with $\mu(T) = 3$ and non-crossing arborescences A with $\mu(A) = 2$.

(ii) Triangulations T with $\mu(T) = 4$ and non-crossing arborescences A with $\mu(A) = 3$.

(iii) Universal triangulations T with $\mu(T) = 4$ and universal non-crossing arborescences A with $\mu(A) = 3$.

This allows us to construct in Figure 73 the graph of all triangulations T with $\mu(T) \leq 4$, similarly as in Figure 38 but with triangulations. This figure is closely related to Figure 1 of [ALRS00], and all its properties are the pendant as the one discussed in Example 3.23 about non-crossing arborescences A with $\mu(A) \leq A$.

It remains to study, for a fixed $\boldsymbol{t} \in \mathsf{O}^{\circ}_{n+1}$, the number of vertices of $\boldsymbol{\Sigma}^4_2(\boldsymbol{t})$, that is the number of triangulations T with $\mu(T, \boldsymbol{t}) \leq 4$.

Definition 4.56. In a non-universal triangulation T with $\mu(T) = 4$, a positive quadrangle $\kappa = (i, j, k, l)$ is *minimal* when $k = \min\{k' \ ; \ (i, k') \in E^{\circ}(T)\}$; a negative quadrangle $\kappa = (i, j, k, l)$ is *maximal* when $l = \max\{l' \ ; \ (j, l') \in E^{\circ}(T)\}$, see Figure 74.

Remark 4.57. Note that maximal negative quadrangles (i, j, k, l) are quadrangles that form a Z-shape, while minimal positive ones form a Ƨ-shape, see Figure 74. This illustrates the fact that flipping the edge of a minimal positive quadrangle turns it into a maximal negative quadrangle. Moreover, such flips send a triangulation T with $\mu(T) = 4$ either to another triangulation T' with $\mu(T') = 4$, or to one of the triangulations T_m or T_M of Corollary 4.52(i).

Theorem 4.58. *Let T be a non-universal triangulation with $\mu(T) = 4$, and $\boldsymbol{t} \in \mathsf{O}^\circ_{n+1}$, then $\boldsymbol{t} \in \mathcal{T}_4(T)$ if and only if $t_i + t_j + t_k + t_l < t_{i'} + t_{j'} + t_{k'} + t_{l'}$ for all $(i, j, k, l) \in Q^-(T)$ maximal and $(i', j', k', l') \in Q^+(T)$ minimal.*

Proof. Fix a non-universal triangulation T with $\mu(T) = 4$, and $\boldsymbol{t} \in \mathsf{O}^\circ_{n+1}$. By Lemma 4.53, this theorem amounts to proving that the minimum of $\{t_i + t_j + t_k + t_l \ ; \ (i, j, k, l) \in Q^+(T)\}$ is achieved when (i, j, k, l) is a minimal positive quadrangle, and conversely for negative quadrangles. Suppose that $(i, j, k, l) \in Q^+(T)$ is not minimal, and let $a = \max\{x \in]i, j[\ ; \ (i, x) \in E(T)\}$. Then $(i, a, j, k) \in Q^+(T)$ and $t_i + t_a + t_j + t_k < t_i + t_j + t_k + t_l$. A similar reasoning gives that a negative quadrangle achieves the maximum of $\{t_{i'} + t_{j'} + t_{k'} + t_{l'} \ ; \ (i', j', k', l') \in Q^-(T)\}$ only when it is maximal. $\square$

For a triangulation T, if κ is a minimal positive quadrangle and ζ a maximal negative one, such that they do not share a triangle, then one can flip the edge e_κ and the edge e_ζ independently. We say that T and T' differ by a *diagonal switch with respect to these two quadrangles* when T' can be obtained by flipping two such edges.

The *switching arrangement $\mathcal{G}_n$* is the collection of hyperplanes

$$G_{(\kappa,\zeta)} = \{\boldsymbol{t} \in \mathbb{R}^{n+1} \ ; \ t_i + t_j + t_k + t_l = t_{i'} + t_{j'} + t_{k'} + t_{l'}\}$$

for all couples of quadruples $\kappa = (i, j, k, l)$ and $\zeta = (i', j', k', l')$ such that $\kappa \in Q^+(T)$ is minimal and $\zeta \in Q^-(T)$ is maximal for some non-universal triangulation T with $\mu(T) = 4$.

Theorem 4.59. *For $\boldsymbol{t}_{lex} = (1, 2, \ldots, 2^n) \in \mathsf{O}^\circ_{n+1}$ and $\boldsymbol{u}_{lex} = (2, \ldots, 2^n) \in \mathsf{O}^\circ_n$, the triangulations T with $\mu(T, \boldsymbol{t}_{lex}) \leq 4$ are sent bijectively through $T \mapsto A_T$ to the non-crossing arborescences A with $\mu(A, \boldsymbol{u}_{lex}) \leq 3$. Informally, this amounts to say that $\Sigma^4_2(\boldsymbol{t}_{lex}) \simeq \Pi^3_{\boldsymbol{u}_{lex}}$.*

Proof. As universal triangulations and universal non-crossing arborescences are in bijection, we only focus on non-universal ones.

By Lemma 4.53, to know whether a non-universal triangulation T can be captured or not on $\boldsymbol{t}$, we need to compare the $t_i + t_j + t_k + t_l$ for different quadrangles (i, j, k, l). But comparing these values for $\boldsymbol{t}_{lex}$ amounts to comparing reverse-lexicographically the associated quadruplets (this is the principle of the binary numeral system). The lexicographic order is denoted $\leq_{lex}$.

Let T be a triangulation with $\mu(T) \leq 4$. On one side, $\boldsymbol{t}_{lex} \in \mathcal{T}^\circ_4(T)$ if and only if $(l, k, j, i) \leq_{lex} (l', k', j', i')$ for all $(i, j, k, l) \in Q^+(T)$ and $(i', j', k', l') \in Q^-(T)$. On the other side, Lemma 4.44 ensures that $(i, j, k, l) \in Q^+(T)$ if and only if $j \in \mathcal{I}^f_{A_T}$, and $(i', j', k', l') \in Q^-(T)$ if and only if $j' \in \mathcal{I}^b_{A_T}$; thus $\boldsymbol{u}_{lex} \in \mathcal{T}^\circ_3(A_T)$ if and only if $(l, k, j) \leq_{lex} (l', k', j')$ for all $(i, j, k, l) \in Q^+(T)$ and $(i', j', k', l') \in Q^-(T)$.

Moreover, if $(i, j, k, l) \in Q^+(T)$, then $(j, k, l) \notin T$, so there is no negative quadrangle in T of the form (a, j, k, l) for $a < j$. Therefore, i and i' are irrelevant in the comparison $(l, k, j, i) \leq_{lex} (l', k', j', i')$, meaning that: $(l, k, j, i) \leq_{lex} (l', k', j', i')$ if and only if $(l, k, j) \leq_{lex} (l', k', j')$.

Consequently, T can be captured on $\boldsymbol{t}_{lex}$ if and only if A_T can be captured on $\boldsymbol{u}_{lex}$. $\square$

Theorem 4.60. *For all $\boldsymbol{t} \in \mathsf{O}^\circ_{n+1} \setminus \bigcap_{G \in \mathcal{G}_n} G$, the number of vertices of $\Sigma^d_2(\boldsymbol{t})$ is $\binom{n}{2} - 1$.*

Proof. By Theorem 4.58, if $\boldsymbol{t}$ and $\boldsymbol{t}'$ belong to the same maximal cone of $\mathsf{O}^\circ_{n+1} \setminus \bigcup_{G \in \mathcal{G}_n} G$, then the triangulations captured on $\boldsymbol{t}$ and $\boldsymbol{t}'$ are the same. Thus the number of vertices of $\Sigma^d_2(\boldsymbol{t})$ and $\Sigma^d_2(\boldsymbol{t}')$ are the same.

For a maximal cone C of the arrangement $\mathcal{G}_n$, we denote by $\mathcal{V}(\mathsf{C})$ the set of triangulations T such that $\mathsf{C} \subseteq \mathcal{T}^\circ_4(T)$. Take two adjacent maximal cones C and C'. Suppose that $T \in \mathcal{V}(\mathsf{C})$ but $T \notin \mathcal{V}(\mathsf{C}')$. Then the hyperplane separating C from C' is of the form $G = \{\boldsymbol{t} \ ; \ t_i + t_j + t_k + t_l = t_{i'} + t_{j'} + t_{k'} + t_{l'}\}$ for some $\kappa = (i, j, k, l) \in Q^+(T)$ minimal in T and $\zeta = (i', j', k', l') \in Q^-(T)$ maximal in T. As the two sums are equal for $\boldsymbol{t} \in G$, κ and ζ can not share a triangle. Let T' be obtained from T by first flipping (i, k) and then (j', l'). We know that $T' \notin \mathcal{V}(\mathsf{C})$ (because C is on the wrong side of G for T' to be captured), and we want to prove that $T' \in \mathcal{V}(\mathsf{C}')$, *i.e.* $\mathsf{C}' \subseteq \mathcal{T}^\circ_4(T')$.

Fix $\boldsymbol{t} \in G$.

Then, we know that $t_{i'} + t_{j'} + t_{k'} + t_{l'} < t_\alpha + t_\beta + t_\gamma + t_\eta$ for all $(\alpha, \beta, \gamma, \eta) \in Q^+(T)$ as these inequalities are respected in C because T can be captured there. Furthermore, $t_i + t_j + t_k + t_l = t_{i'} + t_{j'} + t_{k'} + t_{l'}$ as $\boldsymbol{t} \in G$. This proves that $t_i + t_j + t_k + t_l = \min\{t_\alpha + t_\beta + t_\gamma + t_\eta \ ; \ (\alpha, \beta, \gamma, \eta) \in Q^+(T) \text{ minimal}\}$.

Now take $(e, f, g, h) \in Q^+(T')$ minimal. If $(e, f, g, h) \in Q^+(T)$, then $t_e + t_f + t_g + t_h > t_i + t_j + t_k + t_l$. Otherwise, (e, f, g, h) comes from the diagonal switch. We detail this switch, see Figure 75. If it comes from the flip of (i, k), then its edge is either $(e, g) = (i, l)$ or $(e, g) = (j, k)$. But in the first case, $(e, f, g, h) = (i, j, l, h)$ with $h > l$ because $(e, f, g, h) \in Q^+(T')$. In the second case, $(e, f, g, h) = (j, f, k, l)$ because (e, f, g, h) is minimal in T'. In both cases $t_e + t_f + t_g + t_h > t_i + t_j + t_k + t_l$. No minimal positive quadrangle can appear when flipping (j', l') (apart (i', j', k', l') itself) because if $(e, g) = (i', l')$ then (e, f, g, h) is not minimal, and if $(e, g) = (j', k')$, then (e, f, g, h) can not be positive. This establishes that $t_i + t_j + t_k + t_l = \min\{t_\alpha + t_\beta + t_\gamma + t_\eta \ ; \ (\alpha, \beta, \gamma, \eta) \in Q^+(T') \text{ minimal}\}$.

The same holds for negative quadrangles: $t_{i'} + t_{j'} + t_{k'} + t_{l'} = \max\{t_\alpha + t_\beta + t_\gamma + t_\eta \ ; \ (\alpha, \beta, \gamma, \eta) \in Q^-(T') \text{ maximal}\}$.

Thus, $\boldsymbol{t}' \in \mathsf{C}'$ taken arbitrarily close to $\boldsymbol{t} \in G$ respects all inequalities of $\mathcal{T}_3^\circ(T')$. This ensures that $T' \in \mathcal{V}(\mathsf{C}')$ but $T' \notin \mathcal{V}(\mathsf{C})$, and consequently $|\mathcal{V}(\mathsf{C}')| \geq |\mathcal{V}(\mathsf{C})|$. As a result, the cardinal $|\mathcal{V}(\mathsf{C})|$ is the same for all maximal cones C of the hyperplane arrangement $\mathcal{G}_n$ (as the graph of adjacency of its maximal cones is connected).

Finally, Theorem 4.59 states that this cardinal is also the number of vertices of $\Pi^3_{\boldsymbol{t}}$: $\binom{n}{2} - 1$. □

Example 4.61. One can consider the graph on triangulations T of a $(n+1)$-gon with $\mu(T) \leq 4$ with an edge between T and T' if there exists $\boldsymbol{t} \in \mathsf{O}^\circ_{n+1} \setminus \bigcup_{G \in \mathcal{G}_n} G$ such that the vertices corresponding to T and T' are neighbors in $\Sigma^4_2(\boldsymbol{t})$. This is precisely the induced sub-graph of flips of triangulations, restricted to $\{T \ ; \ \mu(T) \leq 4\}$. By Theorem 4.55, this graph is isomorphic to the graph discussed in Example 3.23, through the bijection $T \mapsto A_T$. As previously, the polygons $\Sigma^4_2(\boldsymbol{t})$ correspond to great cycles in this graph, but not all great cycles do correspond to $\Sigma^4_2(\boldsymbol{t})$. Nevertheless, a given great cycle does not give rise to the same system of inequalities for triangulations as for non-crossing arborescences. In particular, one can compute the number of combinatorially different $\Sigma^4_2(\boldsymbol{t})$, *i.e.* the number of great cycle whose associated system of inequalities has a (full-dimension set of) solution:

- For $n = 5$, there are 2 possible $\Sigma^4_2(\boldsymbol{t})$, see Figure 1 of [ALRS00] and Figure 76.
- For $n = 6$, there are 12 possible $\Sigma^4_2(\boldsymbol{t})$.
- For $n = 7$, there are 216 possible $\Sigma^4_2(\boldsymbol{t})$.
- For $n = 8$, there are 8368 possible $\Sigma^4_2(\boldsymbol{t})$

For $n = 5$ and $n = 6$, the bijection $T \mapsto A_T$ extends to a bijection between possible $\Sigma^4_2(\boldsymbol{t})$ and possible $\Pi^3_{\boldsymbol{t}}$. But for $n = 7$, as there are 216 possible $\Sigma^4_2(\boldsymbol{t})$ and only 187 possible $\Pi^3_{\boldsymbol{t}}$, this is no longer plausible (and for $n = 8$, there are only 6179 possible $\Pi^3_{\boldsymbol{t}}$). Moreover, when applying $T \mapsto A_T$, one will conclude that 181 possible $\Sigma^4_2(\boldsymbol{t})$ map to $\Pi^3_{\boldsymbol{t}}$ while 35 do not, and 6 $\Pi^3_{\boldsymbol{t}}$ are not (images of) $\Sigma^4_2(\boldsymbol{t})$.

4.3.4 Perspectives and open questions

Computational remarks As usual, the objects of this section have been implemented with Sage. Especially, to compute the fiber polytopes at stake, we chose to first compute its secondary polytope and then project it. The secondary polytope can be computed by running through all triangulations of $\mathsf{Cyc}_2(\boldsymbol{t})$, and associating to each a vertex (whose coordinates have an explicit formula involving the area of its triangles). The projection is precisely the projection $\mathsf{Cyc}_4(\boldsymbol{t}) \to \mathsf{Cyc}_2(\boldsymbol{t})$. We could also have computed the fiber polytopes as a finite Minkowski sum, see Theorem 4.7, but

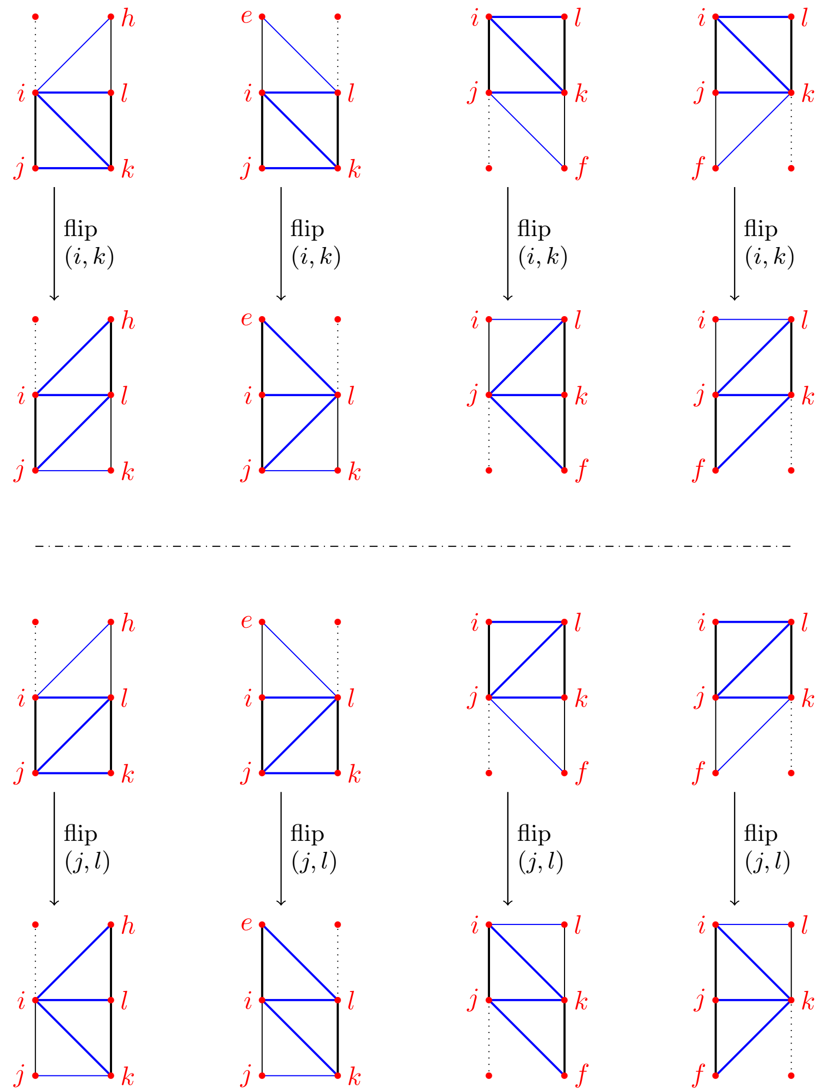

Figure 75: All possible quadrangles that can be created during a diagonal switch.

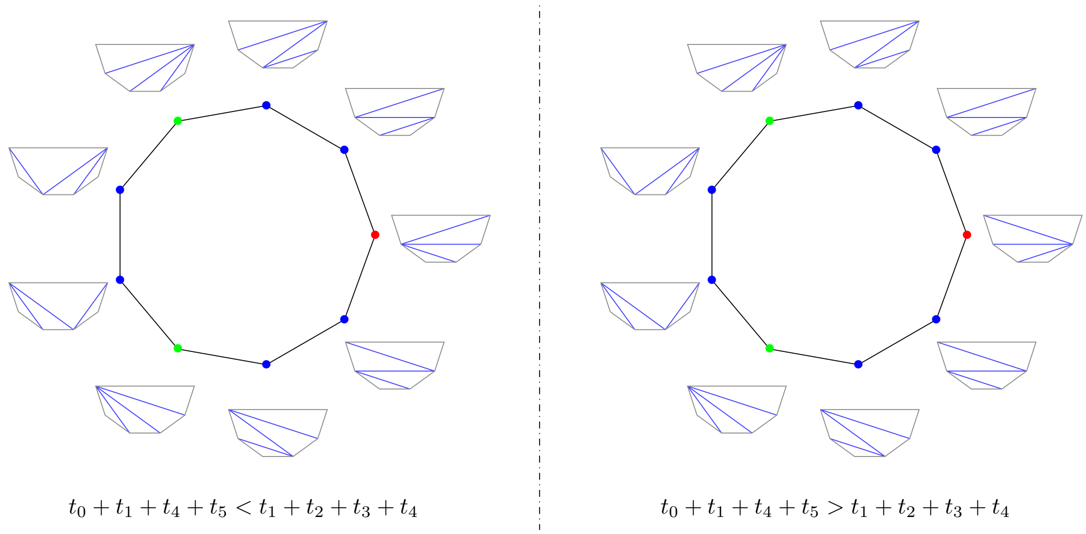

Figure 76: The 2 possible $\Sigma_2^4(\boldsymbol{t})$ for $n = 5$. Green vertices correspond to triangulations T with $\mu(T) = 3$, blue ones to universal T with $\mu(T) = 4$, and the red one to the non-universal triangulation (the only one that differs between Left and Right). Each $\Sigma_2^4(\boldsymbol{t})$ correspond to one of the two cones inside O_6 separated by the hyperplane $\{\boldsymbol{t} \ ; \ t_0 + t_1 + t_4 + t_5 = t_1 + t_2 + t_3 + t_4\}$. Contrarily to Figure 40, it is not possible to picture this subdivision of O_6 as, even when intersected with the hyperplanes $\{\boldsymbol{t} \ ; \ t_0 = 0\}$ and $\{\boldsymbol{t} \ ; \ t_5 = 1\}$, it is 4-dimensional.

the first method has the advantage to directly associate the vertices of $\Sigma_2^d(\boldsymbol{t})$ with triangulations of $\mathsf{Cyc}_2(\boldsymbol{t})$.

Besides, to calculate the values claimed in Example 4.61, one needs to construct the subdivision $\mathsf{O}_{n+1} \setminus \bigcup_{G \in \mathcal{G}_n} G$. The same issues as discussed in Section 3.2.4 occur, but note that most of the material developed for Section 3.2 can not be directly reused here, and need to be adapted.

Assets and limits of the current approach, open questions Lemma 4.44 is essential for proving Theorem 4.59, but to this end, we only use Lemma 4.44 on the triangulations with 1 interior immediate leaf or 2 exterior ones. As the lemma applies for all triangulations, we can hope for a generalization of Theorem 4.59, which would grant access to a theory of intrinsic degree for all triangulations. However, the way to do so remains unclear.

The fiber polytope we have studied in this section is very similar to the max-slope pivot rule polytope studied in Section 3.2, although they are not exactly the same. The mystery around the link between both is not totally unveiled. It would be interesting to determine whether this link is due to the combinatorics and geometry of the cyclic polytopes, or if this is but the tail of a more general phenomenon.

A A Vandermonde-like determinant

I was working on the proof of one of my poems all the morning, and took out a comma.
In the afternoon I put it back again.
– Oscar Wilde

This appendix is devoted to the proof of a formula that determines a generalization of the Vandermonde determinant. This formula is used for the proof of Theorem 4.48 and can be extracted (with some difficulties) from the literature, for example [Hei26, SV00, Jan22]. For fixed $\boldsymbol{\lambda} := (\lambda_1, \dots, \lambda_n)$, we want to compute the following determinant for $k \in \mathbb{N}$:

$$\mathrm{VdM}_{n,k}(\boldsymbol{\lambda}) := \det \begin{pmatrix} 1 & 1 & \dots & 1 \\ \lambda_1 & \lambda_2 & \dots & \lambda_n \\ \lambda_1^2 & \lambda_2^2 & \dots & \lambda_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^{n-2} & \lambda_2^{n-2} & \dots & \lambda_n^{n-2} \\ \lambda_1^k & \lambda_2^k & \dots & \lambda_n^k \end{pmatrix}$$

Remember the classic Vandermonde determinant: $\mathrm{VdM}_n(\boldsymbol{\lambda}) = \mathrm{VdM}_{n,n-1}(\boldsymbol{\lambda}) = \prod_{i<j}(\lambda_j - \lambda_i)$.

Definition A.1. The *elementary symmetric function* of degree s is

$$\sigma_s(X_1, \dots, X_n) := \sum_{\substack{\varepsilon_i \in \{0,1\} \\ \varepsilon_1 + \dots + \varepsilon_n = s}} X_1^{\varepsilon_1} \dots X_n^{\varepsilon_n}$$

The *complete symmetric polynomial* of degree s is (by convention if $s < 0$, $h_s(X_1, \dots, X_n) = 0$)

$$h_s(X_1, \dots, X_n) := \sum_{i_1 + \dots + i_n = s} X_1^{i_1} \dots X_n^{i_n}$$

Theorem A.2. *For any $k, n \in \mathbb{N}$, and $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n)$, one has: $VdM_{n,k}(\boldsymbol{\lambda}) = VdM_n(\boldsymbol{\lambda}) h_{k-n+1}(\boldsymbol{\lambda})$.*

Before proving this theorem, we need a useful lemma:

Lemma A.3. *For any m, n, denoting $\boldsymbol{X} = (X_1, \dots, X_n)$, the following polynomial equality holds:*

$$\sum_{q=0}^{n} (-1)^q \sigma_q(\boldsymbol{X}) h_{m-q}(\boldsymbol{X}) = 0$$

Proof. One has:

$$\sum_{q=0}^{n} (-1)^q \sigma_q(\boldsymbol{X}) h_{m-q}(\boldsymbol{X}) = \sum_{q=0}^{n} \sum_{\substack{\varepsilon_i \in \{0,1\} \\ \varepsilon_1 + \dots + \varepsilon_n = q \\ i_1 + \dots + i_n = m-q}} (-1)^q X_1^{i_1+\varepsilon_1} \dots X_n^{i_n+\varepsilon_n}$$

Fix $(j_1, \dots, j_n)$ and $I = \{i \ : \ j_i \neq 0\} = (i_1, \dots, i_r)$. We look at the coefficient of $X_1^{j_1} \dots X_n^{j_n}$ in the expression above. Such a monomial appears exactly once per choice of $(\varepsilon_{i_1}, \dots, \varepsilon_{i_r}) \in \{0,1\}^I$, with the coefficient $(-1)^q$. Thus, the coefficient on any $X_1^{j_1} \dots X_n^{j_n}$ is:

$$\sum_{\substack{q \in [0,r], \ \varepsilon_i \in \{0,1\} \\ \varepsilon_{i_1} + \dots + \varepsilon_{i_r} = q}} (-1)^q = \sum_{q \in [0,r]} \binom{r}{q} (-1)^q = 0$$

∎

Proof of Theorem A.2. We will prove the result by induction on $k-(n-1)$.

When $k \in [0, n-2]$, $\mathrm{VdM}_{n,k}(\boldsymbol{\lambda}) = 0$, as two rows are identical in the matrix.

When $k = n-1$, by definition $\mathrm{VdM}_{n,k}(\boldsymbol{\lambda}) = \mathrm{VdM}_n(\boldsymbol{\lambda})$, and the equality holds.

Suppose $\mathrm{VdM}_{n,k}(\boldsymbol{\lambda}) = \mathrm{VdM}_n(\boldsymbol{\lambda}) \times h_{k-n+1}(\boldsymbol{\lambda})$ holds for a fixed $k-(n-1)$. Denoting $\boldsymbol{\lambda}' = (\lambda_1, \dots, \lambda_{n-1})$, then the expansion of $\mathrm{VdM}_{n,k}(\boldsymbol{\lambda})$ along its last column ensures that $-\mathrm{VdM}_{n-1,k}(\boldsymbol{\lambda}')$ is the coefficient of λ_n^{n-2} in $\mathrm{VdM}_{n,k}(\boldsymbol{\lambda})$ (seen as a polynomial in λ_n). One has:

$$\begin{aligned} & \mathrm{VdM}_{n,k}(\boldsymbol{\lambda}) \\ = \; & \mathrm{VdM}_n(\boldsymbol{\lambda}) \times h_{k-n+1}(\boldsymbol{\lambda}) \\ = \; & \mathrm{VdM}_{n-1}(\boldsymbol{\lambda}') \textstyle\prod_{i=1}^{n-1}(\lambda_n - \lambda_i) \times \sum_{s=0}^{k-n+1} h_{k-n+1-s}(\boldsymbol{\lambda}')\lambda_n^s \end{aligned}$$

In the above last line, one identifies the coefficient on λ_n^{n-2} to be:

$$\begin{aligned} & \mathrm{VdM}_{n-1}(\boldsymbol{\lambda}') \times \textstyle\sum_{p+q=n-2}(-1)^{n-1-p}\sigma_{n-1-p}(\boldsymbol{\lambda}')h_{k-n+1-q}(\boldsymbol{\lambda}') \\ = \; & \mathrm{VdM}_{n-1}(\boldsymbol{\lambda}') \times \textstyle\sum_{q=1}^{n-1}(-1)^q\sigma_q(\boldsymbol{\lambda}')h_{k+1-n-(q-1)}(\boldsymbol{\lambda}') \end{aligned}$$

By Lemma A.3, we get $\mathrm{VdM}_{n-1,k}(\boldsymbol{\lambda}') = \mathrm{VdM}_{n-1}(\boldsymbol{\lambda}') \times h_{k-(n-1)+1}(\boldsymbol{\lambda}')$. This concludes the induction, as the theorem holds for $k \leq n-1$, and induces through $(n,k) \mapsto (n-1,k)$. $\square$

References

[AA17] Marcelo Aguiar and Federico Ardila. Hopf monoids and generalized permutahedra, 2017. To appear in *Mem. Amer. Math. Soc.*

[ABD10] Federico Ardila, Carolina Benedetti, and Jeffrey Doker. Matroid polytopes and their volumes. *Discrete Comput. Geom.*, 43(4):841–854, 2010.

[ACEP20] Federico Ardila, Federico Castillo, Christopher Eur, and Alexander Postnikov. Coxeter submodular functions and deformations of Coxeter permutahedra. *Adv. Math.*, 365:107039, 36, 2020.

[AER00] Christos A Athanasiadis, Paul H Edelman, and Victor Reiner. Monotone paths on polytopes. *Mathematische Zeitschrift*, 235(2):315–334, 2000.

[AHBC+16] Nima Arkani-Hamed, Jacob Bourjaily, Freddy Cachazo, Alexander Goncharov, Alexander Postnikov, and Jaroslav Trnka. *Grassmannian geometry of scattering amplitudes*. Cambridge University Press, Cambridge, 2016.

[AHBHY18] Nima Arkani-Hamed, Yuntao Bai, Song He, and Gongwang Yan. Scattering forms and the positive geometry of kinematics, color and the worldsheet. *J. High Energy Phys.*, (5):096, front matter+75, 2018.

[AHBL17] Nima Arkani-Hamed, Yuntao Bai, and Thomas Lam. Positive geometries and canonical forms. *J. High Energy Phys.*, (11):039, front matter+121, 2017.

[AHHST22] Nima Arkani-Hamed, Song He, Giulio Salvatori, and Hugh Thomas. Causal diamonds, cluster polytopes and scattering amplitudes, 2022.

[AHT14] Nima Arkani-Hamed and Jaroslav Trnka. The amplituhedron. *J. High Energy Phys.*, (10):30, 2014.

[ALRS00] Christos A. Athanasiadis, Jesús A. De Loera, Victor Reiner, and Francisco Santos. Fiber polytopes for the projections between cyclic polytopes. *European Journal of Combinatorics*, 21(1):19–47, jan 2000.

[APR14] Ilan Adler, Christos Papadimitriou, and Aviad Rubinstein. On simplex pivoting rules and complexity theory, 2014.

[APR21] Doriann Albertin, Vincent Pilaud, and Julian Ritter. Removahedral congruences versus permutree congruences. *Electron. J. Combin.*, 28(4):Paper No. 4.8, 38, 2021.

[Ard21] Federico Ardila. The geometry of geometries: matroid theory, old and new, 2021.

[Ath99] Christos A Athanasiadis. Piles of cubes, monotone path polytopes, and hyperplane arrangements. *Discrete & Computational Geometry*, 21:117–130, 1999.

[BDLLS22] Alexander E. Black, Jesús A. De Loera, Niklas Lütjeharms, and Raman Sanyal. The polyhedral geometry of pivot rules and monotone paths, 2022.

[BDLLSon] Alexander E. Black, Jesús A. De Loera, Niklas Lütjeharms, and Raman Sanyal. On the geometric combinatorics of pivot rules. in preparation.

[Ber89] Claude Berge. *Hypergraphs*, volume 45 of *North-Holland Mathematical Library*. North-Holland Publishing Co., Amsterdam, 1989. Combinatorics of finite sets, Translated from the French.

[BFS90] Louis J. Billera, Paul Filliman, and Bernd Sturmfels. Constructions and complexity of secondary polytopes. *Adv. Math.*, 83(2):155–179, 1990.

[BKS94] Louis J. Billera, Mikhail M. Kapranov, and Bernd Sturmfels. Cellular strings on polytopes. *Proc. Amer. Math. Soc.*, 122(2):549–555, 1994.

[BL21] Alexander Black and Jesús De Loera. Monotone paths on cross-polytopes, 2021.

[BLL20] Moïse Blanchard, Jesùs A. De Loera, and Quentin Louveaux. On the length of monotone paths in polyhedra, 2020.

[BLS+99] Anders Björner, Michel Las Vergnas, Bernd Sturmfels, Neil White, and Günter M. Ziegler. *Oriented matroids*, volume 46 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, second edition, 1999.

[BMDM+18] Véronique Bazier-Matte, Guillaume Douville, Kaveh Mousavand, Hugh Thomas, and Emine Yıldırım. ABHY Associahedra and Newton polytopes of F-polynomials for finite type cluster algebras.
arXiv:1808.09986, 2018.

[BML87] Roswitha Blind and Peter Mani-Levitska. Puzzles and polytope isomorphisms. *Aequationes Math.*, 34(2-3):287–297, 1987.

[Bot19] Nathaniel Bottman. 2–associahedra. *Algebraic & Geometric Topology*, 19(2):743–806, 2019.

[BPR06] Saugata Basu, Richard Pollack, and Marie-Françoise Roy. *Algorithms in Real Algebraic Geometry*, volume 2 of *Algorithms and Computation in Mathematics*. Springer Berlin, Heidelberg, 2006.

[BS92] Louis J. Billera and Bernd Sturmfels. Fiber polytopes. *Anals of Mathematics*, (135):527–549, 1992.

[CD06] Michael P. Carr and Satyan L. Devadoss. Coxeter complexes and graph-associahedra. *Topology Appl.*, 153(12):2155–2168, 2006.

[CDG+20] Federico Castillo, Joseph Doolittle, Bennet Goeckner, Michael S. Ross, and Li Ying. Minkowski summands of cubes. 2020.

[CFZ02] Frédéric Chapoton, Sergey Fomin, and Andrei Zelevinsky. Polytopal realizations of generalized associahedra. *Canad. Math. Bull.*, 45(4):537–566, 2002.

[CL20] Federico Castillo and Fu Liu. Deformation cones of nested braid fans. *Int. Math. Res. Not. IMRN*, 2020.

[CLS11] David A. Cox, John B. Little, and Henry K. Schenck. *Toric varieties*, volume 124 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2011.

[CP22] Frédéric Chapoton and Vincent Pilaud. Shuffles of deformed permutahedra, multiplihedra, constrainahedra, and biassociahedra, 2022.

[Dan63] George B. Dantzig. *Linear programming and Extensions*. Princeton Landmarks in Mathematics and Physics. Princeton University Press, 1963.

[DCP95] Conrado De Concini and Claudio Procesi. Wonderful models of subspace arrangements. *Selecta Math. (N.S.)*, 1(3):459–494, 1995.

[Def21] Colin Defant. Fertilitopes. Preprint, arXiv:2102.11836, 2021.

[Dev09] Satyan L. Devadoss. A realization of graph associahedra. *Discrete Math.*, 309(1):271–276, 2009.

[DK00] Vladimir I. Danilov and Gleb A. Koshevoy. Cores of cooperative games, superdifferentials of functions, and the Minkowski difference of sets. *J. Math. Anal. Appl.*, 247(1):pp. 1–14, 2000.

[DLST95] Jesús A De Loera, Bernd Sturmfels, and Rekha R Thomas. Gröbner bases and triangulations of the second hypersimplex. *Combinatorica*, 15:409–424, 1995.

[DNT08] Antoine Deza, Eissa Nematollahi, and Tamás Terlaky. How good are interior point methods? klee–minty cubes tighten iteration-complexity bounds. *Mathematical Programming*, 113, 2008.

[DP11] Kosta Došen and Zoran Petrić. Hypergraph polytopes. *Topology Appl.*, 158(12):1405–1444, 2011.

[DRS10] Jesus A. De Loera, Jörg Rambau, and Francisco Santos. *Triangulations: Structures for Algorithms and Applications*, volume 25 of *Algorithms and Computation in Math.* Springer, 2010.

[DS14] Yann Disser and Martin Skutella. The simplex algorithm is np-mighty, 2014.

[Edm70] Jack Edmonds. Submodular functions, matroids, and certain polyhedra. In *Combinatorial Structures and their Applications (Proc. Calgary Internat. Conf., Calgary, Alta., 1969)*, pages 69–87. Gordon and Breach, New York, 1970.

[Est08] Alexander Esterov. On the existence of mixed fiber bodies. *Mosc. Math. J.*, 8(3):433–442, 615, 2008.

[For08] Stefan Forcey. Convex hull realizations of the multiplihedra. *Topology and its Applications*, 156(2):326–347, 2008.

[FS05] Eva Maria Feichtner and Bernd Sturmfels. Matroid polytopes, nested sets and Bergman fans. *Port. Math. (N.S.)*, 62(4):437–468, 2005.

[FS14] John Fearnley and Rahul Savani. The complexity of the simplex method, 2014.

[Fuj05] Satoru Fujishige. *Submodular functions and optimization*, volume 58 of *Annals of Discrete Mathematics*. Elsevier B. V., Amsterdam, second edition, 2005.

[FZ02] Sergey Fomin and Andrei Zelevinsky. Cluster algebras. I. Foundations. *J. Amer. Math. Soc.*, 15(2):497–529, 2002.

[GKZ90] Izrail Gelfand, Mikhail Kapranov, and Andrei Zelevinski. Newton polytopes of principal A-determinants. *Soviet Math Doklady*, 1990.

[GKZ91] Izrail Gelfand, Mikhail Kapranov, and Andrei Zelevinski. Discriminants of polynomials in several variables and triangulations of Newton polyhedra. *Leningrad Math. Journal*, 1991.

[GKZ08] Israel Gelfand, Mikhail Kapranov, and Andrei Zelevinsky. *Discriminants, resultants and multidimensional determinants*. Modern Birkhäuser Classics. Birkhäuser Boston Inc., Boston, MA, 2008. Reprint of the 1994 edition.

[GPW19] Pavel Galashin, Alexander Postnikov, and Lauren Williams. Higher secondary polytopes and regular plabic graphs, 2019.

[Grü03] Branko Grünbaum. *Convex polytopes*, volume 221 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 2003. Prepared and with a preface by Volker Kaibel, Victor Klee and Günter M. Ziegler.

[Hai84] Mark Haiman. Constructing the associahedron. Unpublished manuscript, 11 pages, available at http://www.math.berkeley.edu/~mhaiman/ftp/assoc/manuscript.pdf, 1984.

[Ham50] Richard W Hamming. Error detecting and error correcting codes. *The Bell system technical journal*, 29(2):147–160, 1950.

[Ham17] Simon Hampe. The intersection ring of matroids. *Journal of Combinatorial Theory, Series B*, 122:578–614, jan 2017.

[Hei26] Ellis Richard Heineman. *Generalized Vandermonde determinants*. University of Wisconsin–Madison, 1926.

[HL07] Christophe Hohlweg and Carsten Lange. Realizations of the associahedron and cyclohedron. *Discrete Comput. Geom.*, 37(4):517–543, 2007.

[Hoh12] Christophe Hohlweg. Permutahedra and associahedra: generalized associahedra from the geometry of finite reflection groups. In *Associahedra, Tamari lattices and related structures*, volume 299 of *Prog. Math. Phys.*, pages 129–159. Birkhäuser/Springer, 2012.

[Jan22] Milan Janjic. On Vandermonde determinants via n-determinants, 2022.

[JKS22] Michael Joswig, Max Klimm, and Sylvain Spitz. Generalized permutahedra and optimal auctions. *SIAM Journal on Applied Algebra and Geometry*, 6(4):711–739, 2022.

[Kel01] Bernhard Keller. Introduction to a-infinity algebras and modules. *Homology Homotopy Appl.*, 3(1):1–35., 2001.

[Kim08] Sangwook Kim. Shellable complexes and topology of diagonal arrangements. *Discrete Comput. Geom.*, 40(2):190–213, 2008.

[KM72] Victor Klee and George J. Minty. How good is the simplex algorithm?, 1972.

[Lee89] Carl Lee. The associahedron and triangulations of the n-gon. *European J. Combin.*, 10(6):551–560, 1989.

[Lod04] Jean-Louis Loday. Realization of the Stasheff polytope. *Arch. Math. (Basel)*, 83(3):267–278, 2004.

[LP07] Thomas Lam and Alexander Postnikov. Alcoved polytopes, i. *Discrete & Computational Geometry*, 38:453–478, 2007.

[LP20] Thomas Lam and Alexander Postnikov. Polypositroids, 2020.

[LRS10] Jesús A. De Loera, Jörg Rambau, and Francisco Santos. *Triangulations*. Algorithms and Computation in Mathematics. Springer Berlin, Heidelberg, 2010.

[McD95] John McDonald. Fiber polytopes and fractional power series. *Journal of Pure and Applied Algebra*, 104(2):213–233, 1995.

[McM70] Peter McMullen. The maximum numbers of faces of a convex polytope. *Mathematika*, 17:179–184, 1970.

[McM73] Peter McMullen. Representations of polytopes and polyhedral sets. *Geometriae Dedicata*, 2:83–99, 1973.

[McM93] Peter McMullen. On simple polytopes. *Inventiones mathematicae*, 113:419–444, 1993.

[McM96] Peter McMullen. Weights on polytopes. *Discrete Comput. Geom.*, 15(4):363–388, 1996.

[Mer22] Chiara Meroni. *Semialgebraic Convex Bodies*. PhD thesis, Fakultät für Mathematik und Informatik der Universität Leipzig, 2022.

[Mey74] Walter Meyer. Indecomposable polytopes. *Trans. Amer. Math. Soc.*, 190:77–86, 1974.

[MG07] Jiří Matoušek and Bernd Gärtner. *Understanding and Using Linear Programming*. Universitext. Springer Berlin, Heidelberg, 2007.

[MP17] Thibault Manneville and Vincent Pilaud. Compatibility fans for graphical nested complexes. *J. Combin. Theory Ser. A*, 150:36–107, 2017.

[MPS+09] Jason Morton, Lior Pachter, Anne Shiu, Bernd Sturmfels, and Oliver Wienand. Convex rank tests and semigraphoids. *SIAM J. Discrete Math.*, 23(3):1117–1134, 2009.

[MSS20] Sebastian Manecke, Raman Sanyal, and Jeonghoon So. S-hypersimplices, pulling triangulations, and monotone paths, 2020.

[MUWY18] Fatemeh Mohammadi, Caroline Uhler, Charles Wang, and Josephine Yu. Generalized permutohedra from probabilistic graphical models. *SIAM J. Discrete Math.*, 32(1):64–93, 2018.

[Pil17] Vincent Pilaud. Which nestohedra are removahedra? *Rev. Colombiana Mat.*, 51(1):21–42, 2017.

[Pil21] Vincent Pilaud. Acyclic reorientation lattices and their lattice quotients, 2021.

[PK92] Aleksandr Pukhlikov and Askold Khovanskiĭ. Finitely additive measures of virtual polyhedra. *Algebra i Analiz*, 4(2):161–185, 1992.

[Pol21] Daria Poliakova. *Homotopical algebra and combinatorics of polytopes*. PhD thesis, School of The Faculty of Science, University of Copenhagen, 2021.

[Pon18] Viviane Pons, 2018. https://www.lri.fr/~pons/static/spermutahedron/.

[Pos09] Alexander Postnikov. Permutohedra, associahedra, and beyond. *Int. Math. Res. Not. IMRN*, (6):1026–1106, 2009.

[PP22] Arnau Padrol and Eva Philippe. Sweeps, polytopes, oriented matroids, and allowable graphs of permutations, 2022.

[PPP22a] Arnau Padrol, Vincent Pilaud, and Germain Poullot. Deformation cones of hypergraphic polytopes. In *FPSAC 2022-34th International Conference on Formal Power Series and Algebraic Combinatorics*, volume 86, page 71, 2022.

[PPP22b] Arnau Padrol, Vincent Pilaud, and Germain Poullot. Deformed graphical zonotopes. *accepted for publication at Discrete and Computational Geometry*, 2022.

[PPP23] Arnau Padrol, Vincent Pilaud, and Germain Poullot. Deformation cones of graph associahedra and nestohedra. *European Journal of Combinatorics*, 107:103594, 2023.

[PPPP19] Arnau Padrol, Yann Palu, Vincent Pilaud, and Pierre-Guy Plamondon. Associahedra for finite type cluster algebras and minimal relations between $\mathbf{g}$-vectors. arXiv:1906.06861, 2019.

[PRW08] Alexander Postnikov, Victor Reiner, and Lauren K. Williams. Faces of generalized permutohedra. *Doc. Math.*, 13:207–273, 2008.

[PS19] Vincent Pilaud and Francisco Santos. Quotientopes. *Bull. Lond. Math. Soc.*, 51(3):406–420, 2019.

[PSZ23] Vincent Pilaud, Francisco Santos, and Günter M. Ziegler. Celebrating loday's associahedron, 2023.

[Rei99] Victor Reiner. The generalized baues problem. *New perspectives in algebraic combinatorics*, 38:293–336, 1999.

[Rei02] Victor Reiner. Equivariant fiber polytopes. *Documenta Mathematica*, 7:113–132, 2002.

[RSS03] Günter Rote, Francisco Santos, and Ileana Streinu. Expansive motions and the polytope of pointed pseudo-triangulations. In *Discrete and computational geometry*, volume 25 of *Algorithms Combin.*, pages 699–736. Springer, Berlin, 2003.

[RTV05] Cornelis Roos, Tamás Terlaky, and Jean-Philipe Vial. *Interior Point Methods for Linear Optimization (2nd edition)*. Springer, 2005.

[San09] Samson Saneblidze. The bitwisted Cartesian model for the free loop fibration. *Topology Appl.*, 156(5):897–910, 2009.

[Sch11] Pieter Hendrik Schoute. *Analytical treatment of the polytopes regularly derived from the regular polytopes. Section I: The simplex.*, volume 11. 1911.

[Sch98] Alexander Schrijver. *Theory of linear and integer programming*. Wiley-Interscience, 1998.

[She63] Geoffrey C. Shephard. Decomposable convex polyhedra. *Mathematika*, 10:89–95, 1963.

[Slo23] Neil Sloane. "a handbook of integer sequences" fifty years later, 2023.

[SP02] Richard P. Stanley and Jim Pitman. A polytope related to empirical distributions, plane trees, parking functions, and the associahedron. *Discrete Comput. Geom.*, 27(4):pp. 603–634, 2002.

[SS93] Steve Shnider and Shlomo Sternberg. *Quantum groups: From coalgebras to Drinfeld algebras*. Series in Mathematical Physics. International Press, Cambridge, MA, 1993.

[Sta63] Jim Stasheff. Homotopy associativity of H-spaces I & II. *Trans. Amer. Math. Soc.*, 108(2):275–312, 1963.

[Sta70] James Stasheff. *H-Spaces from a Homotopy Point of View*, volume 161 of *Lecture Notes in Mathematics*. Springer Berlin, Heidelberg, 1970.

[Sta80] Richard P Stanley. The number of faces of a simplicial convex polytope. *Advances in Mathematics*, 35(3):236–238, 1980.

[Sta07] Richard P. Stanley. An introduction to hyperplane arrangements. In *Geometric combinatorics*, volume 13 of *IAS/Park City Math. Ser.*, pages 389–496. Amer. Math. Soc., Providence, RI, 2007.

[Sta12] Richard P. Stanley. *Enumerative combinatorics. Volume 1*, volume 49 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, second edition, 2012.

[Sta15] Richard P. Stanley. *Catalan Numbers*. Cambridge University Press, 2015.

[SV00] Hans Schlickewei and Carlo Viola. Generalized Vandermonde determinants. *Acta Arithmetica*, 95(2):123–137, 2000.

[Tam51] Dov Tamari. *Monoides préordonnés et chaînes de Malcev.* PhD thesis, Université Paris Sorbonne, 1951.

[Zel06] Andrei Zelevinsky. Nested complexes and their polyhedral realizations. *Pure Appl. Math. Q.*, 2(3):655–671, 2006.

[Zie98] Günter M. Ziegler. *Lectures on Polytopes*, volume 152 of *Graduate texts in Mathematics.* Springer-Verlag, New York, 1998.