École Doctorale Sciences Mathématiques de Paris Centre

Sorbonne Université
Insitut Mathématiques de Jussieu-Paris Rive Gauche

THÈSE DE DOCTORAT

Discipline : Mathématiques

présentée par

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Geometric combinatorics of paths and deformations of convex polytopes

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Comment remercier en si peu de mots autant de gens ?

Ma thèse, c'est avant tout 3 années de vie, à la fois trop longues et bien trop courtes, peuplées d'une myriade de rencontres, où le travail s'entremêle directement avec la vie privée. C'est pourquoi il n'y aura pas d'ordre dans ces remerciements, et pas véritablement d'organisation non plus. En outre, chacun sait à peu près ce pourquoi je tiens à le remercier : derrière chaque petite phrase se cache une longue histoire, de beaux souvenirs.

Tout d'abord, et il me semble essentiels de commencer par eux car sans eux rien n'aurait eu lieu, un gigantesque merci à mes deux directeurs de thèse, Arnau et Vincent. Merci de m'avoir fait découvrir le meilleur domaine des mathématiques qu'il soit : la combinatoire des polytopes. Merci de m'avoir encadré en thèse durant 3 ans, dans tout ce que cela implique professionnellement et personnellement. Merci pour la pédagogie dont vous avez fait preuve : j'ai grandement apprécié votre patience et votre capacité à répéter, reformuler jusqu'à ce que je comprenne. Merci pour les réussites que vous m'avez permis d'atteindre, peu de choses se seraient faites sans vous deux.

Merci à Guillaume S. pour toutes les excellentes discussions que nous avons eues, sur des centaines de sujets, et les heures de gameplay endiablées. Merci aussi à Léa H. pour son accueil toujours chaleureux.

Merci à Amaury B. et Julie A. et à leur amour invétéré des peluches. Je me souviens avec plaisir de notre voyage (enrhumé) en Alsace.

Merci à Christophe L. pour m'avoir apporté des éléments... instructifs sur les choses de la vie (dont l'administration allemande). Merci de même à Nicolas d'avoir accepté de nous accompagner à Cologne pour un sympathique séjour.

Merci à Arnaud E. de m'avoir offert son bureau et (le commencement du début de l'introduction de) ses connaissances en théorie des catégories. Merci pour tous les sous-entendus et toutes les découvertes, excellentes comme inattendues. Et merci à Grégoire et à Bojo pour l'accueil à Mareil jusque tard dans la nuit.

Merci à toute la bande : François-Marie, Nour, Sophia, Morgane, Sarah et Arthur. Nos restaurants, nos longues conversations sur nos souvenirs de prépa, nos franches tranches de rire devant les déboires des uns et des autres, nos échanges de BDs, etc. En particulier, merci à Morgane de m'avoir accueilli à Grenoble, et à François-Marie non seulement pour avoir récupéré de nombreux ouvrages, mais aussi pour tout ce qu'il m'a apporté durant ces années en prépa comme à l'X.

Merci à Thomas LM. pour m'avoir supporté en tant que colocataire pendant 2 ans. Bien peu nombreux sont ceux qui auraient été à même de le faire, surtout vu le nombre de parties de Smash et de vaisselle que cela t'a demandé !

Merci aux habitants du 93 route des Gardes, pour leur joie de vivre, leur organisation, leur aide (notamment face à une certaine porte récalcitrante), etc.

Merci à Gabriel P. de nous avoir prêté les ZomeTools indispensable (car complètement contingent) à la réalisation de magnifiques polytopes dans notre bureau.

Merci à Mathieu R. pour ses yeux plein d'étoiles.

Merci à Gabin M. pour le soutien et l'amitié qu'il m'a témoigné sans forcément s'en rendre compte.

Merci à MCV pour le savoir qu'elle m'a transmis et l'occasion qu'elle m'a offerte de me lancer dans la formidable aventure des khôlles de l'autre côté du miroir. Merci pour nos passionnants échanges de mails, des tranches de vie toujours bienvenues. Merci à Éric CV. pour son le conseil avisé qui m'a conforté dans le choix de mes directeurs.

Merci Anthony R. pour la visite du laboratoire de Chimie, et la stimulante question de géométrie plane qu'il m'a posée.

Merci à Clément R. pour sa surprenante question : non, il n'y a toujours pas de zoologie bien organisée des polyèdres.

Merci à François Ch., ce sera toujours un plaisir de jouer au frisbee avec toi ou d'admirer ta carte des courants océaniques.

Merci à Roni K. d'avoir repris l'origami et d'avoir entretenu la vitrine après mon passage.

Merci à toute ma famille. Mes parents pour m'avoir supporté (dans tous les sens du terme) durant de nombreuses années, et d'avoir accepté les changements houleux qui ont suivi mes multiples prises de consciences. Mes grands-parents pour avoir aidé durant les épreuves de ma jeunesse, pour m'avoir toujours accueilli en vacances et m'avoir prêté votre appartement alors que ma deuxième année commençait. Ma mamie de m'avoir transmis l'amour des mathématiques et de l'enseignement, et pour toujours accepter de discuter le fond des choses. Janine pour l'accueil à Bordeaux et l'ouverture d'esprit dont tu fais preuve. Mes cousins pour tous ces Noëls, ces anniversaires, ces vacances, ces jeux vidéos (en particulier V), ces repas, ces joyeux moments... et bravo à tous ! Merci à mes oncles et tantes de me permettre de contempler de multiples choix de vie, certain qui me donne envie et d'autres qui me confortent dans ma volonté de ne pas les reproduire.

Merci à ma confraternité des thésards combinatoriciens, autrement appelé "Doctorants en Géométrie Combinatoire". Doriann, Daniel, Noémie, Balthazar, Chiara, Loïc, j'ai pris grand plaisir à co-organiser le DGeCo avec vous, et si c'était à refaire, je referais ce chemin (comme disait Aragon) ! On s'est bien amusé, on a bien progressé et j'ai hâte de vivre avec vous la fin de cette grande aventure (qui n'est que le début des suivantes). J'ai gardé Eva à part, car rien ne remplace une telle co-bureau, une telle consœur ! On a partagé bien plus qu'un directeur, quelques livres, des déboires administratifs, des notions de maths et de (longs) voyages en train : tu m'as grandement motivé au sortir du Covid pour venir au bureau, et un florilège de bonnes décisions en a découlé. Merci Eva.

Merci à Viviane P. pour ses conseils, son aide et son sens de l'organisation. Merci pour ton regard pertinent sur les questions humaines : cela m'a fait beaucoup réfléchir.

Merci à toute l'équipe du LIX : outre un passionnant séminaire aux exposés toujours compréhensibles, je me souviendrai longtemps du trépidant *escape game* que nous avons résolu ensemble.

Merci Corentin de m'avoir donné dès mes début la motivation de revenir, encore et encore. Ce fut très très important pour moi.

Merci à Jimmy L. d'avoir été un chef d'équipe plein de ressources. J'ignore comment tu fais pour jongler avec autant de missions, mais tu as toujours su apporter des solutions pertinentes à mes problèmes, avec une indéfectible bonne humeur ! Et merci d'avoir accepté de diriger mon comité de suivi de thèse alors que je ne connaissais encore personne.

Merci aux gens compétents de l'administration de l'IMJ et de Sorbonne Université : Sitti M., Évariste C., Ayano T., Émilie J., vous avez été d'un recours indispensable pour l'organisation de ma thèse à de nombreux égards.

Merci aux étudiants à qui j'ai eu la chance de donner cours pour leur L1 et leur L2. Même si ma mémoire ne me permet pas de retenir vos noms, je me souviens des challenges, des épiphanies, les vôtres comme les miennes.

Merci à eux avec qui j'ai eu l'occasion de faire du frisbee. Que ce soit les sympathiques rencontres dominicales à Cité U, ou les anciens de la section, chercheurs aux SageDays, les gens de passage chez moi : j'aime toujours autant le frisbee malgré les rares opportunités que j'ai de le pratiquer.

Merci à Raphaël P. pour sa bonne humeur, son sempiternel sourire, ses discussions passionnantes et l'aide efficace qu'il distribue toujours avec gentillesse.

Merci à Jacques A. et Xenia de régulièrement avoir essayé de me sortir de ma zone de confort, avec un bilan positif.

Merci à mes co-bureaux, Thomas G., Francisco M., Pierre G., et tous ceux qui y sont passés temporairement. J'y ai toujours trouvé une ambiance à la fois chaleureuse (surtout l'été) et propice au travail. Bien des fois, j'aurais aimé que quelqu'un rentre par hasard dans notre bureau et contemple les écritures hétéroclites et ésotériques qui couvrent inlassablement le tableau.

Merci à tous ceux qui sont passés dans le couloir des doctorants à l'IMJ. Camille, Lucas, Étienne, Thomas (tous les Thomas), Mahya, Perla, Adrien, Adrien, Mattias, Christina, Mathieu, Haowen, Nastaran, Tristan, les pauses du midi n'auraient certes pas été si longues sans vous, mais mon thé aurait surtout grandement manqué de saveur ! Gracias a los hispanohablantes, Anna, Juan Felipe, Joachin, me hubiera gustado practicar más mi español con vosotros, pero ya estoy encantado de haberos conocido. Obrigado a toda comunidade brasileira, Nelson, Thiago, João, Pietro, Odylo, mesmo que você sempre chegasse barulhento, os sorrisos que os acompanhavam eram um prazer de ver. (Je suis sûr que j'ai oublié quelqu'un...)

Merci à Lionel Pournin d'avoir accepté de rapporter ma thèse : ta relecture attentive m'a beaucoup aidé à en parfaire les détails jusque-là incertains. Merci pour les questions que nous avons discutées ces dernières années, et j'espère que tu auras apprécié les améliorations survenues depuis ma soutenance de M2.

Merci à Frédéric Meunier d'avoir accepté de faire partie de mon jury de soutenance, et de mon comité de suivi de thèse. Merci aussi grandement de m'avoir conseillé la lecture très enrichissante de Matoušek, et d'avoir participé au DGeCo avec des questions toujours pertinentes (et difficiles).

Thanks to Fu Liu to have accepted to review my thesis. Thanks to Vic Reiner, Jesús De Loera and Martina Juhnke for being part of the jury of my defense. Your articles and presentations were a great source of inspiration and understanding, thus it is an honor for me to show you my modest contribution.

Thank a lot to all the mathematicians that I have met during my Ph.D. You taught me a lot about mathematics, life, people, presentations, and thinking in general. I will not list every one, as I don't want to double the length of my thesis, but I want to address special thanks to Ziegler, Sturmfels, Matoušek, Diestel, Santos for their beautiful books; to Williams, Postnikov, Hampe, Ardilla & Aguiar for their instructive articles, and all the people I have met at Kleinwalsertal, Bielefeld, PAGCAP, PolyTopics, Polytopes in Paris, and many more events.

Thank a lot to the whole Sage community : your well writen functions (and tutorials) helped me understand both maths and computer science. Especially, thank to all the team of SageDays 117 that got me involved in the project and allowed me to improve drastically me coding skills.

Vielen Dank an Raman und Aenne für das großartige Projekt, das wir gemeinsam durchgeführt haben. Dadurch konnte ich erstmals in der riesigen Welt der Optimierung von Polytopen Fuß fassen. Ich habe meinen Aufenthalt in Frankfurt, seine Atmosphäre, seine Architektur und seine Mathematiker sehr genossen.

Vielen Dank an das Osnabrücker Team, das mich als Postdoktorand willkommen heißt. Justus, ich habe mich gefreut, dich in Österreich kennenzulernen! Martina, ich hoffe, dass dich unsere zukünftige Zusammenarbeit dich genauso begeistert wie mich.

Vielen Dank an die Verwaltung der Universität Osnabrück, Frau Mallek und Frau Wolterink, die mir die Anreise erheblich erleichtert haben.

感林岳教我中文... 取得了一定的成功！恭喜您完成文，我很快就遇到其他令人的言。

また折りをやりたいと思わせてくれた早苗さんに感します。アパトをったり、友への完璧なり物を作ったりするだけでなく、折りは折るときにいつもしみの源です。

Introduction

That the powerful play goes on, and you may contribute a verse.
– Walt Whitman, "Oh Me! Oh Life!" in *Leaves of Grass*

Mathematicians study geometry for more than 2 500 years. Even though they may not be the first to have explored such concepts, Greek scholars are renown for having introduced both the art of the proof and the formalization of abstract geometry. In particular, polygons and polyhedra seem to have held a very special place in their representation of the world.

Polygons, for instance, were at the heart of a burning controversy about the essence of Nature. At first, integers were thought to be the *natural numbers*. Furthermore, as ratios of integers, the *rational numbers* were also thought as perfect: a fraction is no more than two commensurable (integer) lengths, that is two lengths that can be drawn as integer lengths in a well-chosen scaling. Greek philosophers thought for a time that perfect objects can only involve such numbers, but regular polygons seem to be perfect as well: nevertheless, the length of the diagonal of a square with unit-length side is not a rational number, and more and more irrational numbers appear when considering all regular polygons. This cataclysmic discovery led the majority of Greek mathematicians and philosophers to accept *constructible numbers* as (somewhat) perfect, and to state the famous three problems of Ancient history: trisecting the angle, doubling the cube, and squaring the circle.

Besides, 3-dimensional studies were not outdone. Pythagoras, and his school after him, discovered the tetrahedron, the hexahedron (*a.k.a.* the cube), and the dodecahedron. Later on, Theaetetus constructed the two last of the five *Platonic solids*, namely the octahedron and the icosahedron. Plato, in the *Timaeus*, associated each of these five polyhedra with an element: Fire (tetrahedron), Earth (cube), Air (octahedron), Water (icosahedron), and the element "the demiurge used for arranging the constellations on the whole heaven" (dodecahedron).

The Greek history of geometry is synthesized by Euclid. Books IV and XIII of his famous *Elements* ($\Sigma\tau o\iota\chi\varepsilon\iota\alpha$) are devoted to regular polygons and regular polyhedra, respectively. He proved numerous and numerical relations between these objects, gave explicit constructions, stated and solved problems and puzzles. In his honor, the name *Euclidean geometry* has ever since designated the usual geometry (which was the only existing one before the Euclid's fifth postulate was put into question, and Gauss defined non-Euclidean geometries).

Researches around polygons and polyhedra have never stopped since, and it is hard to find a great mathematician that has spoken no word about them. On top of that, polyhedra are among the very few abstract mathematical objects to make repeated appearances in Arts and Letters. To begin with, not only did Plato give his solids a central role in his cosmogony, but Johannes Kepler, in his *Mysterium Cosmographicum*, also thought that the distances between the planets in the solar system were explained by the possibility to circumscribe Platonic solids around each of their orbits. He soon abandoned this idea when realizing planet's orbits were not circular. Architecture obviously benefited from the study of plane and spacial geometry, while making contributions to it. Especially, it is worth mentioning the *Grande Arche de La Défense*, designed by Johan Otto von Spreckelsen as a gigantic 3-dimensional embodiment of a 4-dimensional cube. Painting as well, above all during the Renaissance period, was largely inspired by plane geometry and the symmetry regular polygons and polyhedra exhibit. Albrecht Dürer famously portrayed an eponymous polyhedron in *Melancolia I*, and Platonc solids appear in the *Portrait of Luca Pacioli* from Jacopo de' Barbari. Even music is imbued with geometry: the *Timaeus* is a dialogue on music, and the recent piece *Polytopes* from Iannis Xenakis is constructed thanks to polytopal ideas.

By dint of all these cultural occurrences, polyhedra are well known to the public. It is then quite remarkable that the research on polygons, polyhedra, and polytopes, is still greatly flourishing: after thousands of years and thousands of contributors, some questions are still open, and new exciting ones keep being thought of.

Such a keen interest for polyhedra may be explained by their duality. Indeed, polyhedra are both simple to define, and rich in the behaviors they can express; they are sitting on the fence between concrete numerical geometry and purely abstract topology; they are drawn and visualized by everyone but elementary properties can be hard to prove; they seem alighting from the realm of ideas, although they can be directly encountered in nature (from capsids of viruses to furnishing). If this has participated in draping polyhedra with a mystical allure, it has also exerted a prolific fascination on numerous mathematicians. In particular, the help of modern computers have greatly improved our capacity to construct polyhedra and play with them, leading to numerous conjectures and open problems. In return, the discrete nature of computer science's problems has paved the way for a large panel of new challenges and applications of polyhedral geometry.

The main objects of the present manuscript are polytopes: a polytope is defined as the convex hull of finitely many points in the Euclidean space $\mathbb{R}^d$. As such, polytopes are the generalization of polygons and polyhedra to higher dimensions. In this thesis, I will try to unveil some links between the geometric aspects of polytopes and their combinatorial behaviors. We give hereafter a precise description of the context each chapter contributes to, and the new results proven. Two concepts will be at the center of this polytopal journey: *generalized permutahedra* and *linear programs*.

The first notion arises from the systematic research of the combinatorial properties of polytopes, which have played a great role in the development of the field since their (re)popularization during the 20th century, see [Grü03, Zie98] for the history of the subject. Polytopes naturally come with various combinatorial properties: foremost, one can try to understand their faces (which are themselves polytopes), and how its faces are included one in another, leading to the definition of the *face lattice* of a polytope. If exploring the face lattice of a polytope is already fascinating, the reverse question turns out to be even more fecund: given a combinatorial structure, how to construct a polytope to embody it? An epitome of such quest is surely the construction of the *permutahedron*. Discovered by Schoute in 1911 [Sch11], the vertices of the permutahedron are in one-to-one correspondence with the permutations. Moreover, the faces of it can be labelled by the *ordered partitions*, while its (oriented) graph naturally describes the *Bruhat order* on permutations.

But this is only the tip of the iceberg: the permutahedron can be *deformed* (in a sense that will be made clear in Section 2) to create *generalized permutahedra*. Originally defined by Edmonds [Edm70] under the name *polymatroids*, their rediscovery by Postnikov in 2009 [Pos09] was the starting point of a myriad of researches. In particular, various combinatorial families can be encapsulated in the combinatorics of certain generalized permutahedra. A first example is the (hyper)cube whose vertices are in bijection with binary sequences, and help for instance to understand Hamming codes [Ham50]. Besides, the matroid polytopes also arises as generalized permutahedra [ABD10, Ard21], and play an important role in the search for Minkowski indecomposable generalized permutahedron. But the one and foremost example of generalized permutahedra is probably the *associahedron*: known as the "mythical polytope" [Hai84] and introduced by Tamari [Tam51] and Stasheff [Sta63], its first realizations were given by Milnor (unpublished), Haiman [Hai84], Lee [Lee89], and then Loday [Lod04, PSZ23]. Its combinatorics encapsulate the one of *Catalan families*, that is to say triangulations of a polygon, binary search trees, subpartitions of the staircase partition, non-crossing arborescences, etc. The realization of the associahedron as a generalized permutahedron allows for numerous links between the combinatorics of permutations and Catalan families, this is nowadays the subject of an abundant literature ranging from mathematical physics [AHBHY18] to cluster algebra [FZ02] and moduli space [Sta63, Kel01].

On top of that, the set of generalized permutahedra is not only a list of relevant examples, it is also endowed with its own structure: it forms a cone, called the *submodular cone*. This cone is the type cone of the permutahedron, in the sense of McMullen [McM73], and as such have been studied from a wide variety of perspectives: for instance, the name *submodular cone* comes from the notion of diminishing returns in economy [JKS22], while in the domain of toric varieties, it is known as the *numerically effective cone* [CLS11]. Furthermore, it is the natural (and universal) setting for the construction of a Hopf monoid of polytopes [AA17]; it appears in the study of the Grassmannian, especially in positive geometry [AHBL17, LP20], and the amplituahedron program (with links to mathematical physics) [AHT14, AHBC+16].

On the other side, linear programming delves into the geometrical aspects of polytopes. Optimization is known for being a supremely useful but notably difficult theory, and linear optimization encompasses the optimization problems in which both the constraints and the quantity to optimize are linear in the involved variables. This kind of problems originally appeared for logistic grounds: Dantzig [Dan63] was working for the U.S. Air Force in 1947 when he introduced the general concept, and Kantorovich already thought about some specific cases in 1939 for the timber industry of U.S.S.R. However, the field grew only slowly at first, until two crucial breakthroughs: Kantorovich and Koopmans earned the Nobel Prize of Economy in 1975 for their work on resource allocation, and computers became a growing part of the organization of our civilizations throughout the end of the century.

There are several methods to solve a linear problem, among which some are known to be of polynomial complexity (see [MG07, Chapter 7] and [RTV05, DNT08]), but the original method, which is still of prime importance, is the simplex method, whose complexity class is not fully understood for now [KM72, DS14]. The *simplex method* can be thought of as the counterpart of the Gaussian elimination, but when dealing with linear inequalities (and a linear functional to optimize). In broad, the key idea is to consider the set of solutions of your system of inequalities as a polytope (or an unbounded polyhedra), and to jump from one vertex onto one of its neighbors, increasing the value of the linear functional at each step. This method will end at a point maximizing the linear functional, thanks to the convexity of the polytope. Nevertheless, one needs to set up a rule on how to choose the neighbor to jump onto: this is the *pivot rule*. Pivot rules and how to elect the right one have been written about extensively (see [MG07, APR14, DS14, FS14] among many other), and we certainly do not intend to fully answer this question here.

Instead of taking a computer science approach, the point of view we would like to develop on the simplex method centers around its combinatorial behavior: given a polytope and a direction, what can be said about the structure of the set of (edge) paths on the polytope that are increasing for this direction? There are several ways to understand (and to tackle) this question. When focusing on the worst case scenario, it is natural to explore monotone paths [AER00, BLL20], and even monotone path polytopes [BS92, ALRS00, MSS20, BL21], while new works focus on the whole decision tree that a (memory-less) pivot rule gives rise to, and construct several pivot rule polytopes [BDLLS22, BDLLSon]. Among the pivot rule polytopes, the *max-slope pivot rule polytope* is a generalization of the monotone path polytope, moreover, the latter is a deformation of the first.

In addition, the pivot rules that max-slope pivot rule polytopes and monotone path polytopes casts about, namely the *max-slope pivot rules* (a generalisation of *shadow vertex rules*), echo more algebraic researches. More precisely, *fiber polytopes* were defined by Billera and Sturmfels [BS92] to understand both projections and subdivisions of polytopes. While pivot rules address edge paths on polytopes, fiber polytopes aim at encompassing triangulations which can be thought of as their higher-dimensional counterpart. This construction opened the door to new ways of thinking about classical polytopes: for instance, the permutahedron is the monotone path polytope of the cube [BS92], and the fiber polytope for the projection from a simplex is deeply linked to the triangulations of points configuration through the secondary polytope of Gelfand, Kapranov and Zelevinski [GKZ90, GKZ91]. Furthermore, it led to fruitful developments in a wide variety of research areas such as convex geometry [Est08, Mer22], type B Coxeter associahedron [Rei02], and even power series [McD95].

In this manuscript, we study on the one hand generalized permutahedra and the submodular cone, and on the other hand max-slope pivot rule polytopes and fiber polytopes. Although the domains undeniably interact all along the present thesis, ideas coming from one side being steadily applied to the other, the pre-eminent result creating a neat bridge between these two realms is Section 3.3: we state that the combinatorial behavior of the class of max-slope pivot rules can be handled more easily by embedding the question inside the realm of generalized permutahedra. We hope that such a new insight may open the way to a better understanding of (memory-less) pivot rules.

The rest of this introduction details the content of the present thesis, especially the contexts and results of each following section.

Section 1 is dedicated to a short introduction to the basic notions we will need on polytopes, order theory and linear programming. The three main chapters (Sections 2 to 4) begin by their own preliminaries, before presenting two new results. For each result, we have implemented the key objects thanks to the open source software Sage. We end each sub-section by discussing these implementations and putting some light on possible mathematical perspectives.

After the preliminary Section 1, each chapter can be read independently, even though they are thought to be read in order. Likewise, inside each section, the first sub-section shall be read first, but the two others can be read independently. Note however that Section 4.3 deeply relies on Section 3.2.

Section 1: Preliminaries. The reader probably already knows what is presented in this introductory chapter. The crucial definitions are highlighted so he or she can swiftly consult them and look at the figures. Yet, we may emphasize some key elements:

- Partially ordered sets: The notion of pre-orders is probably the less known among the presented notions, while ordered partitions and permutations are the most important ones.
- Polytopes: The definition of polytopes, faces and lattices of faces are of prime importance, but the dual notion of fans and normal fans is more central in this thesis. The examples, especially the permutahedron and the associahedron, will be the key objects for the next chapters.
- Linear programming: As this thesis is not about linear programming itself, its presentation will be succinct and only aims at giving a different yet enlightening point of view on the geometry of polytopes.

Section 2: Deformations of polytopes. The first main section is devoted to the study of deformations of polytopes. A *deformation* of a polytope P can be equivalently described as (i) a polytope whose normal fan coarsens the normal fan of P [McM73], (ii) a Minkowski summand of a dilate of P [Mey74, She63], (iii) a polytope obtained from P by perturbing the vertices so that the directions of all edges are preserved [Pos09, PRW08], (iv) a polytope obtained from P by gliding its facets in the direction of their normal vectors without passing a vertex [Pos09, PRW08]. A sequence of deformations is illustrated in Figure 10. The deformations of P form a polyhedral cone under dilation and Minkowski addition, called the *deformation cone* of P [Pos09]. The interior of the deformation cone of P, called the *type cone* [McM73], contains those polytopes with the same normal fan as P. When P is a rational polytope, it has an associated toric variety [CLS11], and the type cone (also known as the *numerically effective cone* or *nef cone*) encodes its embeddings into projective space [CLS11, Sect. 6.3]. Among the different ways to parametrize and describe the deformation cone of a polytope P (see *e.g.* [PRW08, App. 15]), we use the parametrization by the *heights* corresponding to the facets of P and the description given by the *wall-crossing inequalities* corresponding to the edges of P [CFZ02]. While this inequality description is immediately derived from the linear dependencies among certain normal vectors of P, it is in general more difficult to extract the irredundant facet inequality description of the deformation cone.

Fundamental examples of deformations of polytopes are the *deformed permutahedra* (*a.k.a.* generalized permutahedra or polymatroids) studied in [Edm70, Pos09, PRW08], which are classically parametrized by *submodular functions*. The set of deformed permutahedra is the set of deformations of the standard permutahedron. As such, it forms a cone, namely the *submodular cone*. In the present thesis, we give the facet-description of some faces of the submodular cone. The key idea lies in the following fact: if Q is a deformation of P, then the deformation cone of Q is a face of the deformation cone of P, see Section 2.1. As numerous deformations of the standard permutahedron are already well studied, this fact grants us access to faces of the submodular cone. We present two new results in this domain: the facet-description of the deformation cone of graphical zonotopes in Section 2.2, and the facet-description of the deformation cone of nestohedra in Section 2.3. Moreover, we characterize which graphical zonotopes and which nestohedra have a simplicial type cone.

Section 2.2 is directly adapted from our paper [PPP22b] (accepted for publication), while Section 2.3 is adapted from our published paper [PPP23].

Section 3: Max-slope pivot rule polytopes. The second main section is dedicated to max-slope pivot rule polytopes. For solving linear programs, the *simplex method* has been used since its introduction by Dantzig [Dan63]. This algorithm was not only used for numerical computing, it also brought new understanding to the combinatorial and geometrical problems, such as finding a flow in a graph or the largest circle in a polygon [MG07]. The simplex method requires a *pivot rule* to guide the consecutive choices that are to be made. Understanding pivot rules is crucial to discuss the complexity of the simplex method [KM72, APR14, DS14, FS14]. An important class of pivot rules are the *max-slope pivot rules*, introduced in [BDLLS22] to generalize the *shadow vertex rule*. A given max-slope pivot rule is encoded by the *arborescence* (directed tree) it induces on the graph of the feasibility domain. Remarkably, when the feasibility domain is a polytope P, the combinatorial behavior of the max-slope pivot rules can be captured by the face lattice of a polytope, called the *max-slope pivot rule polytope* of P.

These polytopes are not yet well understood. The case of the standard cube has been detailed in the original article [BDLLS22], and the simplex will be dealt with in the upcoming [BDLLSon]. In the present thesis, we explore in Section 3.2 the max-slope pivot rule polytope of the cyclic polytopes, that we call *cyclic associahedra*, helped by ideas developed in [ALRS00] for fiber polytopes between cyclic polytopes. They generalize the standard associahedron Asso_n defined in Section 1.2.4, and their genesis prompts a natural complexity parameter on Catalan families (parenthesizations, binary search trees, triangulations of polygons...).

Moreover, generalized permutahedra will step again into play in Section 3.3, and prove themselves a powerful framework to analyze max-slope pivot rule polytopes. The key idea of this part is to realize that comparing slopes of line segments between points amounts ultimately to comparing real numbers, and consequently to wander inside the *braid fan* (the normal fan of the permutahedron). This shed a new light on max-slope pivot rule polytopes, linking them with generalized permutahedra, and help us understand their behavior with respect to products of polytopes. We conclude this part by constructing the max-slope pivot rule polytopes of products of simplices. Thanks to the work of Chapoton and Pilaud [CP22] on *shuffle products* of generalized permutahedra, we show that the max-slope pivot rule polytope of the product of simplices is the shuffle product of the max-slope pivot polytope of each involved simplex. Furthermore, we explicit a piece-wise linear transformation from the max-slope pivot rule polytope of a simplex to the standard associahedron of Loday [Lod04], and point out that several basic examples of products of simplices gives rise to known polytopes, *e.g.* multiplihedra and constrainahedra.

Section 4: Fiber polytopes. The third main section focuses on fiber polytopes. In their seminal paper, Billera and Sturmfels [BS92] introduced and studied *fiber polytopes*, which have since proven to be a powerful tool to understand both projections and subdivisions of polytopes. It has also found applications in algebraic geometry [McD95], and theoretical physics as part of the *amplituhedron program* [GPW19, AHHST22]. In particular, numerous famous polytopes can be realized as fiber polytopes, prominently permutahedra, associahedra, and some other generalized permutahedra naturally appear [BS92, ALRS00, Rei02, LRS10], and new constructions come to enrich the subject, such as fiber bodies [Est08, Mer22], or (partial) sweep polytopes [PP22].

However, fiber polytopes are hard to compute in general. In their original paper [BS92], the authors link the fiber polytopes to *secondary polytopes* defined by Gelfand, Kapranov and Zelevinsky in [GKZ90, GKZ91], the vertices of which are in bijection with triangulations of a point configuration. On top of that, when one projects a polytope P onto a segment of direction $\boldsymbol{c}$, the associated fiber polytope is the *monotone path polytope* of P in direction $\boldsymbol{c}$: its vertices are in bijection with the monotone paths that the shadow vertex rules follow to go from the minimal vertex to the optimal one (*i.e.* from $\boldsymbol{v} \in \mathsf{P}$ that minimizes $\langle \boldsymbol{v}, \boldsymbol{c} \rangle$ to $\boldsymbol{v} \in \mathsf{P}$ maximizing it). Billera and Sturmfels constructed the monotone path polytopes of simplices and cubes, while recent articles solved the cases of cyclic polytopes [ALRS00], and cross-polytopes [BL21] and more. Section 4.2 centers on the monotone path polytopes of hypersimplices. We connect monotone paths on an hypersimplex with the realm of lattice paths, and conclude by giving some arguments in favor of log-concavity of the number of (coherent) monotone paths sorted by length, which was conjectured by Jesús de Loera (for all polytopes).

Numerous works on fiber polytopes either deal with projections onto a segment (*i.e.* monotone path polytopes), or examine the general case. In Section 4.3, thanks to the ideas elaborated in Section 3.2 and [ALRS00], we tackle the fiber polytope for the projection of the cyclic polytope of dimension d onto the cyclic polytope of dimension 2.

1 Preliminaries

All things are difficult before they are easy.
– Thomas Fuller

1.1 Partially ordered sets

This thesis is not on partially ordered sets, but some basic knowledge of order theory is necessary to understand combinatorial polytope theory, and especially the combinatorics of the permutahedron, see Section 1.2.3. For this reason, we will define in this section the notions of partially ordered sets, lattices, pre-orders and ordered partitions. These notions are presented for the sake of completeness and self-containment, and the reader, surely well aware of them, shall focus on pre-orders and ordered partitions.

We denote $[n] := \{1, \dots, n\} \subset \mathbb{N}$ for $n \geq 1$, and $[i, j] := \{i, \dots, j\} \subset \mathbb{N}$ for integers $i < j$.

Definition 1.1. A *binary relation* on a set E is a subset $\mathcal{E}$ of $E \times E$, and we denote $\mathcal{R}(x, y)$ instead of $(x, y) \in \mathcal{E}$. A binary relation is *total* when $\big(\mathcal{R}(x, y)$ or $\mathcal{R}(y, x)\big)$ for all $x, y \in E$, and we emphasize its possible non-totality by calling it *partial* in the general case. A binary relation is

1. *reflexive* when, for all $x \in E$, $\mathcal{R}(x, x)$.
2. *symmetric* when for all $x, y \in E$, if $\mathcal{R}(x, y)$ then $\mathcal{R}(y, x)$.
3. *anti-symmetric* when for all $x, y \in E$, if $\mathcal{R}(x, y)$ and $\mathcal{R}(y, x)$ then $x = y$.
4. *transitive* when for all $x, y, z \in E$, if $\mathcal{R}(x, y)$ and $\mathcal{R}(y, z)$ then $\mathcal{R}(x, z)$.

A binary relation is called:

(a) a *pre-order relation* when it is reflexive and transitive.
(b) an *equivalence relation* when it is reflexive, symmetric, and transitive.
(c) an *order relation* when it is reflexive, anti-symmetric and transitive.

A *partially ordered set* or *poset* is a couple $(E, \mathcal{R})$ where $\mathcal{R}$ is a (partial) order relation on E.

To a pre-order $\mathcal{R}$, we associate an equivalence relation $\mathcal{S}$ with $\mathcal{S}(x, y) = \big(\mathcal{R}(x, y)$ and $\mathcal{R}(y, x)\big)$. This creates a partition of E into *equivalence classes* $\mathrm{cl}(x) := \{y \in E \ ; \ \mathcal{S}(x, y)\}$. On the set of equivalence classes, $\mathcal{R}$ induces a pre-order $\overline{\mathcal{R}}$ defined by $\overline{\mathcal{R}}\big(\mathrm{cl}(x), \mathrm{cl}(y)\big) = \mathcal{R}(x, y)$. With this definition, $\overline{\mathcal{R}}$ is not only reflexive and transitive, but also *anti-symmetric*. When denoting a pre-order by the infix notation $\preccurlyeq$, we will denote its equivalence relation by $\simeq$ and introduce $x \prec y := \big(x \preccurlyeq y$ and not $x \simeq y\big)$. We will also slightly abuse notations by denoting again $\preccurlyeq$ the partial order associated on the set of its equivalence classes.

In a poset $(E, \preccurlyeq)$, an element y *covers* an element x when $x \prec y$ and there exists no $z \in E$ with $x \prec z \prec y$. A poset is *graded* when it can be endowed with a *rank function* $r : E \to \mathbb{N}$ such that $r(y) = r(x) + 1$ when y covers x.

We say that a pre-order $\trianglelefteq$ *extends* a pre-order $\preccurlyeq$ when if $x \preccurlyeq y$, then $x \trianglelefteq y$. This extension shall be thought of as adding an order relation between some pairs (x, y) that are not already ordered by $\preccurlyeq$.

The *dual* of a poset $(E, \preccurlyeq)$ is the poset $(E, \succcurlyeq)$ where $x \succcurlyeq y$ if and only if $y \preccurlyeq x$.

Definition 1.2. In a poset $(E, \preccurlyeq)$, the *meet* of two elements $x, y \in E$ is, when it exists, the unique maximum $x \wedge y$ of the elements that are less than both x and y: $x \wedge y = \max\{z \in E \ ; \ z \preccurlyeq x, \ z \preccurlyeq y\}$.

The *join* of two elements $x, y \in E$ is, when it exists, the unique minimum $x \vee y$ of the elements that are greater than both x and y: $x \vee y = \min\{z \in E \ ; \ x \preccurlyeq z, \ y \preccurlyeq z\}$.

A poset is a *lattice* when every pairs of elements have a meet and a join.

A lattice $(\mathscr{L}, \preccurlyeq)$ always has a minimum element called 0 by convention, and a maximal element called 1 by convention. An *atom* is an element that covers 0, and the lattice is *atomic*[1] when all its elements can be written as a join of atoms. A *co-atom* is an element that is covered by 1, and the lattice is *co-atomic*[1] when all its elements can be written as a meet of co-atoms.

The dual of $(\mathscr{L}, \preccurlyeq)$ is the poset $(\mathscr{L}, \succcurlyeq)$. It is a lattice called its *dual lattice*.

[1]Some authors prefer the term *(co-)atomistic*, saving the word *(co-)atomic* for another property.

Definition 1.3. An *ordered partition* is an ordered collection of sets $I_1 \prec \cdots \prec I_k$ that partition $[n]$, *i.e.* $I_i \cap I_j = \varnothing$ for all $i \neq j$ and $\bigcup_i I_i = [n]$. Dually to ordered partitions, we define an *(integer) surjection* as a map $\sigma : [n] \to [k]$ with $k \leq n$ such that for all $a \in [k]$, $\sigma^{-1}(a) \neq \varnothing$. The collection $I_a := \sigma^{-1}(a)$ for $a \in [k]$ form an ordered partition where $I_a \preccurlyeq I_b$ when $a \leq b$. When $k = n$, the surjection is a bijection, and the ordered partition is the corresponding permutation.

A surjection σ naturally induces a total pre-order on $[n]$ by $i \preccurlyeq j$ when $\sigma(i) \leq \sigma(j)$.

1.2 Polytopes

Polytopes are beautiful objects, they are going to be at the heart of the present thesis. The aim of this section is to introduce the basic definitions and properties of polytopes that will be used all along this thesis. For a complete presentation on polytopes, the reader is invited to look at the wonderful *Lectures on Polytopes* of Ziegler [Zie98]. As what we state here are known theorems, we will mainly refer to the literature for proofs.

Polytopes come, foremost, as objects embedded in the Euclidean space $\mathbb{R}^d$. Vectors of $\mathbb{R}^d$ will usually be denoted in bold font like $\boldsymbol{x} \in \mathbb{R}^d$, and their coordinates are the real numbers $x_1, \dots, x_d$. We denote the standard basis of $\mathbb{R}^d$ by $(\boldsymbol{e}_1, \dots, \boldsymbol{e}_d)$, and for $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^d$, the standard scalar product is $\langle \boldsymbol{x}, \boldsymbol{y} \rangle := \sum_{i=1}^d x_i y_i$.

Definition 1.4. A *polytope* P is the convex hull of a finite number of points in $\mathbb{R}^d$:

$$\mathsf{P} = \mathrm{conv}\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_n\} = \left\{ \boldsymbol{x} \in \mathbb{R}^d \;;\; \begin{array}{l} \boldsymbol{x} = \sum_{i=1}^n \lambda_i \boldsymbol{v}_i \\ \forall i, \lambda_i \geq 0 \text{ and } \sum_{i=1}^n \lambda_i = 1 \end{array} \right\}$$

This endows polytopes with a first manifestation of duality, coming from the duality of $\mathbb{R}^d$ as a Euclidean space: sharing a linear dependency is equivalent to being orthogonal to a vector.

Definition 1.5. For a vector $\boldsymbol{a} \in \mathbb{R}^d \smallsetminus \{\boldsymbol{0}\}$ and a constant $b \in \mathbb{R}$, the *affine hyperplane* $H_{(\boldsymbol{a},b)}$ associated is $H := \{\boldsymbol{x} \;;\; \langle \boldsymbol{x}, \boldsymbol{a} \rangle = b\}$. An *open half-space* is the open component of $\mathbb{R}^d \smallsetminus H$ where H is an affine hyperplane of $\mathbb{R}_d$. A *closed half-space* is a closure of the latter. We denote $H^+_{(\boldsymbol{a},b)} := \{\boldsymbol{x} \;;\; \langle \boldsymbol{x}, \boldsymbol{a} \rangle \geq b\}$ and $H^-_{(\boldsymbol{a},b)} := \{\boldsymbol{x} \;;\; \langle \boldsymbol{x}, \boldsymbol{a} \rangle \leq b\}$ the two possible closed half-spaces.

Theorem 1.6. (Minkowski–Weyl, see [Zie98, Theorem 1.1]). *A polytope is a bounded intersection of closed half-spaces: There exist $\boldsymbol{v}_1, \dots, \boldsymbol{v}_n \in \mathbb{R}^d$ such that $\mathsf{P} = \mathrm{conv}\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_n\}$ if and only if there exist $\boldsymbol{a}_1, \dots, \boldsymbol{a}_m \in (\mathbb{R}^d)^m$ and $b_1, \dots, b_m \in \mathbb{R}^m$ such that P is bounded and:*

$$\mathsf{P} = \bigcap_{j=1}^m H^-_{(\boldsymbol{a}_j, b_j)} = \left\{ \boldsymbol{x} \in \mathbb{R}^d \;;\; \forall i,\; \langle \boldsymbol{x}, \boldsymbol{a}_j \rangle \leq b_j \right\}$$

Figure 1 illustrates this ambivalence. In the upcoming concept of duality, the notion of dimension and faces will play a role.

Definition 1.7. The *dimension* of P is the dimension of its affine hull, denoted $\dim(\mathsf{P})$.

Note that the dimension of a polytope is not necessary the dimension of the Euclidean space it is embedded in. Indeed, it often comes in handy to embed a polytope in an affine subspace (often a hyperplane) of $\mathbb{R}^d$, see for instance Section 1.2.3.

Definition 1.8. A *face* of a polytope P is a subset of P with, for some $\boldsymbol{c} \in \mathbb{R}^d$, the form: $\mathsf{P}^{\boldsymbol{c}} = \{\boldsymbol{x} \in \mathsf{P} \;;\; \langle \boldsymbol{x}, \boldsymbol{c} \rangle = \max_{\boldsymbol{y} \in \mathsf{P}} \langle \boldsymbol{y}, \boldsymbol{c} \rangle\}$. The polytope P itself is a face of P (with $\boldsymbol{c} = \boldsymbol{0}$), and by convention $\varnothing$ is a face of P.

For a given direction $\boldsymbol{c} \in \mathbb{R}^d$, as P is compact (it is convex and bounded), the maximum δ of $\boldsymbol{y} \mapsto \langle \boldsymbol{y}, \boldsymbol{c} \rangle$ is attained on P, so the above is well-defined and non-empty. Moreover, the face $\mathsf{P}^{\boldsymbol{c}}$ trivially satisfies $\mathsf{P}^{\boldsymbol{c}} = H_{(\boldsymbol{c},\delta)} \cap \mathsf{P} = H^+_{(\boldsymbol{c},\delta)} \cap H^-_{(\boldsymbol{c},\delta)} \cap \mathsf{P}$. The Minkoswki–Weyl theorem then ensures:

Proposition 1.9. *Every face of P is a polytope.*

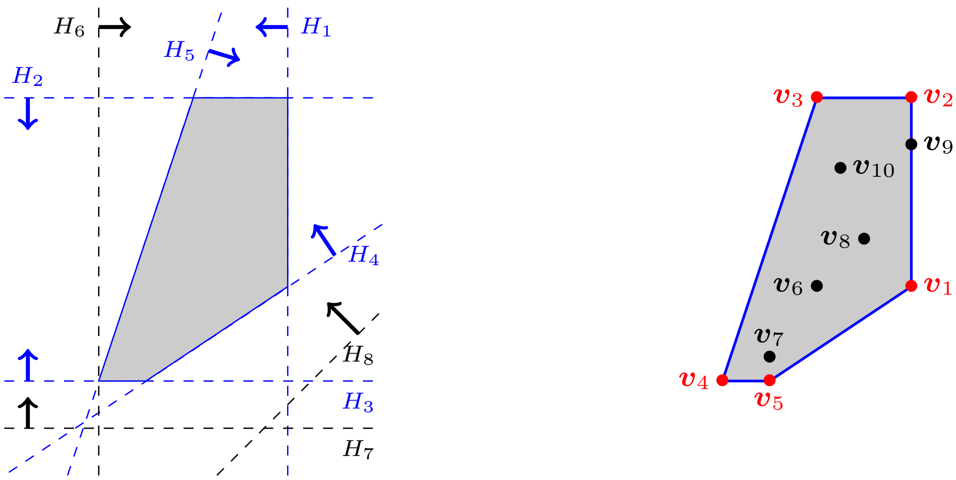

Figure 1: (Left) A polytope defined as a (bounded) intersection of half-spaces, the facet-defining ones being in blue, while black inequalities are redundant; (Right) the same polytope as the convex hull of a finite set of points, vertices are in red and edges in blue, while black points are redundant.

A hyperplane H such that $H \cap \mathsf{P}$ is a face of P will be called a *valid hyperplane*. Nevertheless, such a hyperplane does not necessarily appear in the description of P as a bounded intersection of half-spaces: a valid hyperplane sometimes shares with P a lower dimensional intersection. The dimension of a face F of P is the dimension of F as a polytope (*i.e.* the dimension of its affine hull). By convention, the empty face $\varnothing$ has dimension -1.

This naturally leads to ordering the set of faces of P by inclusion.

Theorem 1.10. ([Zie98, Theorem 2.7]). *The face lattice $\mathscr{L}(\mathsf{P})$ of P is the set of faces of P ordered by inclusion. It is a graded, atomic, co-atomic lattice with rank function $\mathsf{F} \mapsto \dim(\mathsf{F}) + 1$. The meet of two faces F and G is $\mathsf{F} \cap \mathsf{G}$ (while the join has no straightforward description).*

The faces of dimension 0, (*i.e.* the atoms of $\mathscr{L}(\mathsf{P})$) are the *vertices* of P, while the faces of co-dimension 1 (*i.e.* co-atoms of $\mathscr{L}(\mathsf{P})$) are the *facets* of P. The faces of dimension 1 are called *edges* of P, and the graph G_{P} whose vertices are the vertices of P and whose edges are the edges of P is the *graph of* P.

The set of vertices of P, denoted $V(\mathsf{P})$, is the unique set of extremal elements of P as a convex set, hence $\mathsf{P} = \mathrm{conv}\big(V(P)\big)$. The latter is called the *vertex-description* of P. In a dual fashion, the set of facets $\mathcal{H}(\mathsf{P})$ gives the unique minimal description of P as a bounded intersection of half-spaces. Such description is called the *facet-description* of P.

On that account, the description of a polytope comes across two mathematical challenges that can be solved on the algorithmic level but remain hard in high dimensions. On the one side, when a polytope is handled as a convex hull of points or as a set of linear inequalities, then extracting an extremal set from it is often arduous. On the other side, computing the vertex-description from a facet-description or the reverse can turn out to be convoluted (see the Fourier–Motzkin elimination of [Zie98, Chapter 1]).

The second notion of duality arises from a subtle embodiment of the above duality into the realm of lattices. Recall that the dual of a lattice is the lattice defined on the same set by reversing the order relation.

Theorem 1.11. ([Zie98, Section 2.3]). *The dual lattice of $\mathscr{L}(\mathsf{P})$ is (isomorphic to) the lattice of a polytope. Especially, for $\mathsf{P}^{\triangle} := \{\boldsymbol{c} \in \mathbb{R}^d \;;\; \langle \boldsymbol{c}, \boldsymbol{x} \rangle \leq 1$ for all $\boldsymbol{x} \in \mathsf{P}\}$, the lattice $\mathscr{L}(\mathsf{P}^{\triangle})$ is anti-isomorphic to $\mathscr{L}(\mathsf{P})$.*

Though prodigious, this fact will not be at the center of the current thesis, but it invites us to introduce the following.

Definition 1.12. A *polyhedral cone*[2] C is the positive span of finitely many vectors, meaning there exists $\boldsymbol{v}_1, \dots, \boldsymbol{v}_m \in \mathbb{R}^d$ such that:

$$\mathsf{C} = \left\{ \sum_{i=1}^{m} \lambda_i \boldsymbol{v}_i \ ; \ \forall i, \ \lambda_i \geq 0 \right\}$$

One can define the notion of faces for cones the same way as for polytopes.

A polyhedral cone is *simplicial* when $\boldsymbol{v}_1, \dots, \boldsymbol{v}_m$ are linearly independent. It is then a cone over a simplex (see Section 1.2.1), and all its faces are simplicial polyhedral cones.

Definition 1.13. A *fan* $\mathcal{F}$ in $\mathbb{R}^d$ is a collection of polyhedral cones of $\mathbb{R}^d$ such that if $\mathsf{C} \in \mathcal{F}$ then all faces of C belong to $\mathcal{F}$, and the intersection $\mathsf{C} \cap \mathsf{C}'$ is a face of both C and C' when $\mathsf{C}, \mathsf{C}' \in \mathcal{F}$.A fan is:
- *complete* when it covers $\mathbb{R}^d$, that is $\bigcup_{\mathsf{C} \in \mathcal{F}} \mathsf{C} = \mathbb{R}^d$.
- *essential* when none of its cones contains a line, that is $\{\mathbf{0}\} \in \mathcal{F}$.
- *simplicial* when all of its cones are simplicial.
- *polytopal* when it is the normal fan of a polytope.

The face lattice of a fan is the set of its cones, ordered by inclusion. For a polytope P, the poset of faces of its normal fan $\mathcal{N}_\mathsf{P}$ is anti-isomorphic[3] to its own face lattice $\mathscr{L}(\mathsf{P})$.

A fan $\mathcal{G}$ *coarsens* a fan $\mathcal{F}$ when the cones of $\mathcal{G}$ are unions of cones of $\mathcal{F}$. Conversely, in this setting, $\mathcal{F}$ *refines* $\mathcal{G}$.

Definition 1.14. For a face F of a polytope P, its *normal cone* is $\mathcal{N}(\mathsf{F}) = \{\boldsymbol{c} \in \mathbb{R}^d \ ; \ \mathsf{P}^{\boldsymbol{c}} = \mathsf{F}\}$. The *normal fan* of the polytope P is the collection $\mathcal{N}_\mathsf{P} = \big(\mathcal{N}(\mathsf{F}) \ ; \ \mathsf{F} \text{ face of } \mathsf{P}\big)$.

Fans and normal fans, among other appearances, will be at the heart of Sections 2 and 3.3. Given a graded, atomic and co-atomic lattice, it is NP-hard to decide whether it is the lattice of a polytope. Notwithstanding, deciding if a fan is polytopal amounts to finding a solution to a set of linear inequalities, which is far easier. The notion of normal fan is more precise than the one of face lattice: two polytopes can share the same face lattice without sharing their normal fan. This gives a first hint of what it can mean for two polytopes to be "the same":

Definition 1.15. Two polytopes P and Q are said to be:
- *combinatorially isomorphic* or *combinatorially equivalent* when $\mathscr{L}(\mathsf{P}) \simeq \mathscr{L}(\mathsf{Q})$.
- *normally equivalent* when $\mathcal{N}_\mathsf{P} = \mathcal{N}_\mathsf{Q}$.
- *affinely equivalent* when there exists an affine transformation L such that $\mathsf{Q} = L(\mathsf{P})$.

Obviously, affine equivalence implies combinatorial equivalence, and normal equivalence also implies combinatorial equivalence. But the converse does not hold, and affine equivalence does not imply normal one (rotations are not allowed in normal equivalence, for instance).

An important operation on polytopes is the Minkowski sum. Although the construction is simple to describe in the setting of convex set, the faces are more easily accessed in the framework of normal fans. This gives rise to a simple but non-trivial behavior that allows us to encapsulate various combinatorics, see Sections 1.2.3 and 2 and Figures 28 and 58 for instance.

Definition 1.16. The *Minkowski sum* of two polytopes $\mathsf{P} \subset \mathbb{R}^p$ and $\mathsf{Q} \subset \mathbb{R}^q$ is the polytope:

$$\mathsf{P} + \mathsf{Q} := \{\boldsymbol{p} + \boldsymbol{q} \ ; \ \boldsymbol{p} \in \mathsf{P}, \ \boldsymbol{q} \in \mathsf{Q}\}$$

A *zonotope* is a Minkowski sum of segments.

[2] All the cones in the present thesis will be polyhedral, except if specified else way.

[3] The face lattice of $\mathcal{N}_\mathsf{P}$ needs to be completed by a top element for this anti-isomorphism to hold.

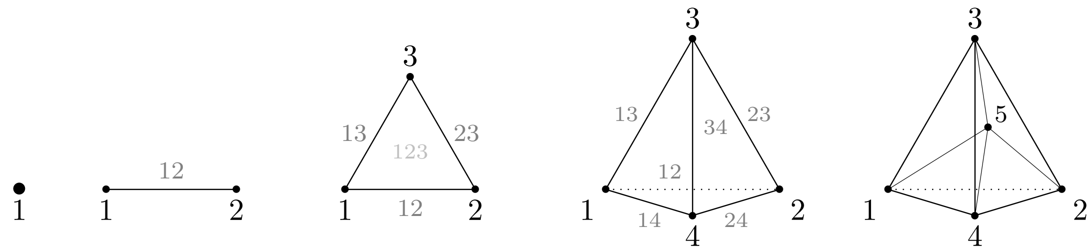

Figure 2: The standard simplices for dimension $d = 0$, 1, 2, 3 and 4 (from Left to Right). Each face is labelled by the corresponding subset of $[d]$. The 4-dimensional standard simplex (Right) is depicted thanks to its Schlegel projection.

It is immediate to see that the vertices of $\mathsf{P} + \mathsf{Q}$ are among the possible sums $\boldsymbol{v} + \boldsymbol{w}$ for $\boldsymbol{v}$ a vertex of P and $\boldsymbol{w}$ a vertex of Q (the reversed inclusion is false in general), that is:

$$V(\mathsf{P}+\mathsf{Q}) \subseteq \{\boldsymbol{v}+\boldsymbol{w} \ ; \ \boldsymbol{v} \in V(\mathsf{P}),\ \boldsymbol{w} \in V(\mathsf{Q})\}$$

The normal fan $\mathcal{N}_{\mathsf{P}+\mathsf{Q}}$ of $\mathsf{P}+\mathsf{Q}$ is the *common refinement* of $\mathcal{N}_{\mathsf{P}}$ and $\mathcal{N}_{\mathsf{Q}}$, meaning that:

$$\mathcal{N}_{\mathsf{P}+\mathsf{Q}} = \{\mathsf{C} \cap \mathsf{C}' \ ; \ \mathsf{C} \in \mathcal{N}_{\mathsf{P}},\ \mathsf{C}' \in \mathcal{N}_{\mathsf{Q}}\}$$

The rest of this section will be devoted to presenting some special yet universal polytopes.

1.2.1 Simplex

A *d-simplex* is the convex hull of $d+1$ affinely independent points. A d-simplex is d-dimensional, has $d+1$ vertices, $d+1$ facets, and $\binom{d+1}{k}$ faces of dimension k. Each face of dimension k is itself a k-simplex. The face lattice of a simplex is (isomorphic to) the lattice of subsets of $[d+1]$, called the *boolean lattice*, see Figure 2. Hence, it is a self dual lattice.

The *standard simplex* of $\mathbb{R}^{d+1}$ is $\Delta_d := \operatorname{conv}\{\boldsymbol{e}_1, \dots, \boldsymbol{e}_{d+1}\}$ where $(\boldsymbol{e}_1, \dots, \boldsymbol{e}_{d+1})$ is the canonical basis of $\mathbb{R}^{d+1}$. Note that the Δ_d has dimension d even if it lives in dimension $d+1$.

1.2.2 Cube

The *standard cube* of $\mathbb{R}^d$ is the convex hull $\square_d := \operatorname{conv}\big\{\sum_{i \in I} \boldsymbol{e}_i \ ; \ I \subseteq [d]\big\}$ where $[d] = \{1, \dots, d\}$. It is d-dimensional, has 2^d vertices and $2d$ facets. Its facet-description is given by the $2d$ inequalities $\square_d = \bigcap_{i=1}^d \{\boldsymbol{x} \in \mathbb{R}^d \ ; \ 0 \le \langle \boldsymbol{x}, \boldsymbol{e}_i \rangle \le 1\}$. Each k-dimensional face of a standard cube is (normally equivalent to) a standard cube, see Figure 3. A d-dimensional *cube* is a polytope combinatorially equivalent to the standard cube.

The standard cube is a zonotope as $\square_d = \sum_{i=1}^d [\boldsymbol{0}, \boldsymbol{e}_i]$. Moreover, any Minkowski sum of d linearly independent segments is a d-dimensional cube. Note also that all zonotopes arise as projections of the standard cube, see [Zie98, Section 7.3].

1.2.3 Permutahedron

The permutahedron Π_n is defined as the convex hull $\Pi_n = \operatorname{conv}\left\{\begin{pmatrix}\sigma(1)\\ \vdots \\ \sigma(n)\end{pmatrix} \ ; \ \sigma \in \mathcal{S}_n\right\} \subset \mathbb{R}^n$ where $\mathcal{S}_n$ is the set of all permutations of $\{1, \dots, n\}$. The permutahedron is a zonotope as it is the translation of $\sum_{i<j} \frac{1}{2}[\boldsymbol{e}_i - \boldsymbol{e}_j,\ \boldsymbol{e}_j - \boldsymbol{e}_i]$ by the vector $\frac{n+1}{2}(1, \dots, 1)$, see [Zie98, Example 7.15].

The vertices $\boldsymbol{v}$ of the permutahedron Π_n are naturally in bijection with permutations of $\{1, \dots, n\}$ by $\sigma \mapsto \sum_i \sigma(i)\boldsymbol{e}_i$. Two vertices are adjacent when the corresponding permutations

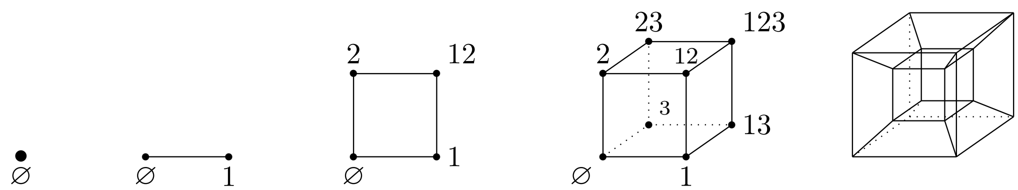

Figure 3: The standard cubes for dimension $d = 0$, 1, 2, 3 and 4 (from Left to Right). Each vertex is labelled by the corresponding subset of $[d]$. The 4-dimensional standard cube (Right) is depicted thanks to its Schlegel projection.

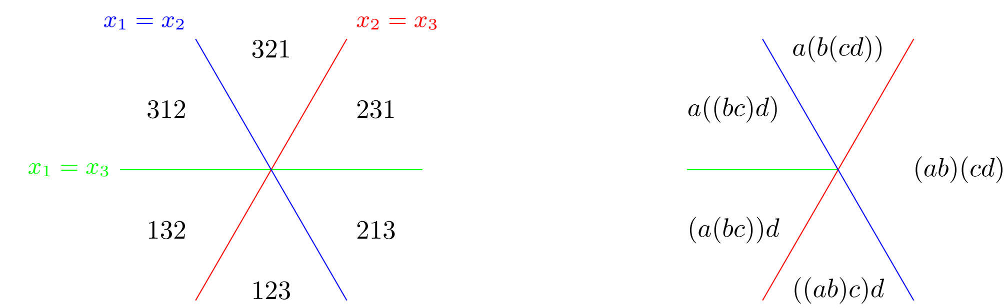

Figure 4: (Left) The braid fan of dimension 2, each maximal cone being labelled by the according permutation, (Right) the sylvester fan of dimension 2, each maximal cone being labelled by the according maximal parenthesization. These fans are the normal fans of the polytopes drawn in Figure 5(Middle Top) and Figure 7(Bottom).

σ and τ differ by an elementary transposition: $\sigma = (i \quad i+1) \circ \tau$. Consequently, the possible directions of edges of the permutahedron Π_n are $\boldsymbol{e}_j - \boldsymbol{e}_i$ for $i, j \in [n]$ with $i \neq j$, see Figure 5.

The facet-description of the permutahedron is:

$$\Pi_n = \left\{ \boldsymbol{x} \in \mathbb{R}^n \ ; \ \begin{array}{ll} \sum_{i=1}^n x_i = \binom{n+1}{2} & \\ \sum_{i \in I} x_i \geq \binom{|I|+1}{2} & \text{for all } \varnothing \subsetneq I \subsetneq [n] \end{array} \right\}$$

The face lattice of the permutahedron Π_n is (isomorphic to) the lattice of ordered partitions (or surjections) of $\{1, \dots, n\}$. Especially, it has $n!$ vertices, $2^n - 2$ facets, and dimension $n-1$. Instead of describing this face lattice more thoroughly, we focus on its normal fan. The normal fan of the permutahedron Π_n is called the *braid fan* $\mathcal{B}_n$, see Figure 4(Left). It is the fan defined by the arrangement of hyperplanes, called the *braid arrangement*, composed by all hyperplanes $\{\boldsymbol{x} \in \mathbb{R}^n \ ; \ x_i = x_j\}$ for $i, j \in [n]$ with $i \neq j$. The cones of the braid fan are in bijection with the surjections on $[n]$: to a surjection σ, one associates the cone

$$\mathsf{C}_\sigma := \left\{ \boldsymbol{x} \in \mathbb{R}^n \ ; \ \begin{array}{lll} x_i < x_j & \text{if} & \sigma(i) < \sigma(j) \\ x_i = x_j & \text{if} & \sigma(i) = \sigma(j) \end{array} \right\}$$

The braid fan, as defined here, is not essential: each cone of $\mathcal{B}_n$ contains the line in direction $(1, \dots, 1)$. This fact will be problematic for the use we want to make of it. We will solve this issue in three different ways: in Section 3.3 we will keep the non-essential braid fan as defined above; in Section 2.2 we will divide each maximal cone of the braid fan in two simplicial cones (one containing a ray in direction $(1, \dots, 1)$ and one containing a ray in direction $(-1, \dots, -1)$); in Section 2.3 we will project orthogonally the braid fan onto the hyperplane $\{\boldsymbol{x} \in \mathbb{R}^n \ ; \ \sum_i x_i = 0\}$.

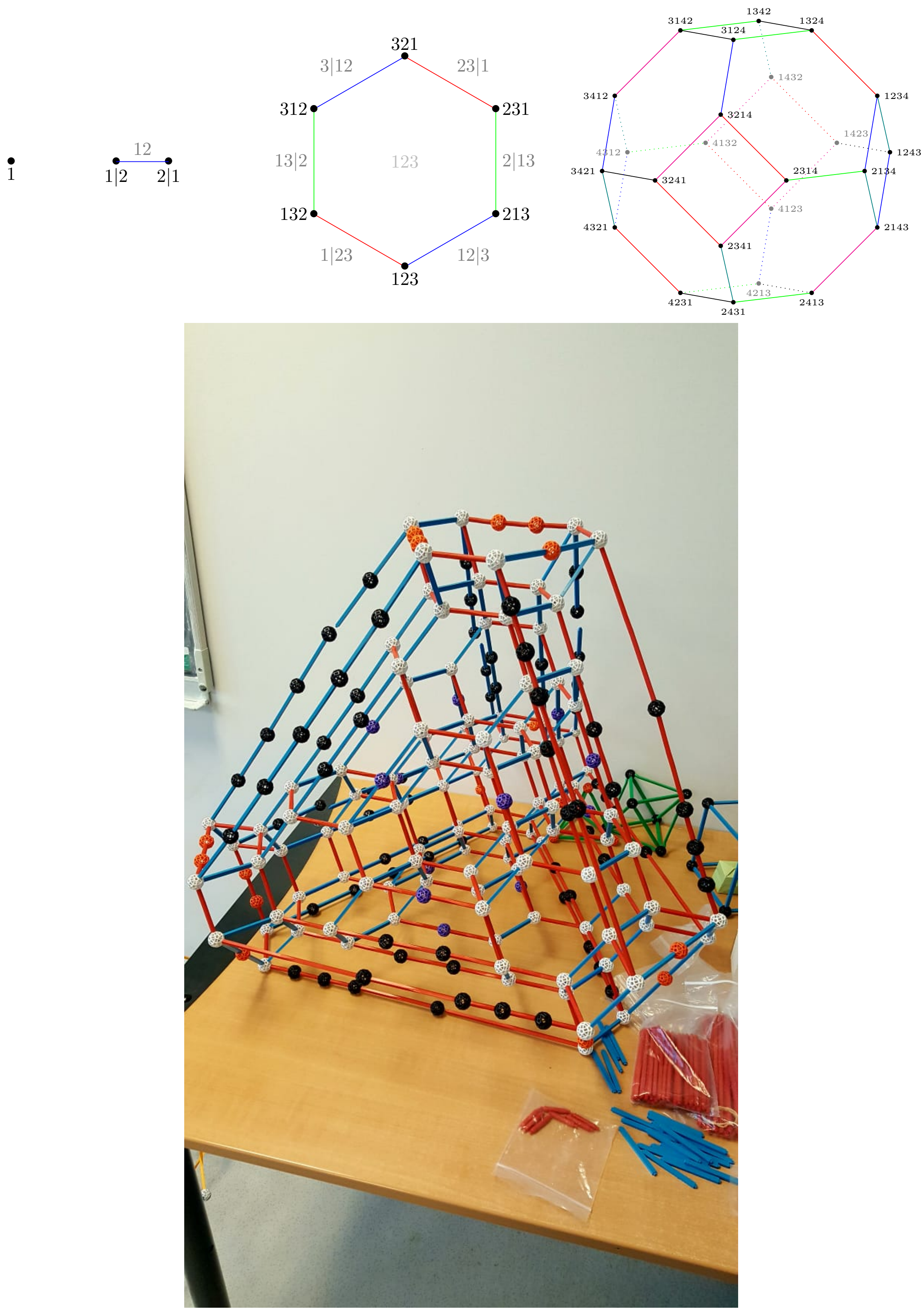

Figure 5: The permutahedra for dimensions $d = 0$, 1, 2, 3 and 4. Up to dimension 3, the vertices are labelled by the permutations, and each edge is colored according to its class of parallelism (for example, going through a blue edge amounts to exchanging the values 1 and 2). The 4-dimensional permutahedron have been made with Zometool, the vertices correspond to the white nodes.

1.2.4 Associahedron

The associahedron is the polytopal embedding of Catalan families. We will first give a quick but not-so-short overview of Catalan families and then define the associahedron as a combinatorial polytope (*i.e.* we will describe its face lattice). We will end by discussing Loday's realization of the associahedron and its link with the permutahedron.

Catalan families A Catalan family is, in first place, a family of objects counted by Catalan numbers. Neil Sloane, the founder of the (*Online*) *Encyclopedia of Integer Sequences*, claims that "the Catalan numbers are certainly the most common sequence that people don't know about" [Slo23]. A straightforward definition of the Catalan numbers C_n is their explicit formula:

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

We also mention their recursive formula:

$$C_0 = 1 \qquad \text{and} \qquad C_{n+1} = \sum_{i=0}^{n} C_i C_{n-i}$$

Note that Catalan numbers grow quite rapidly, as: $C_n \sim \frac{1}{\sqrt{\pi}} \frac{4^n}{n^{3/2}}$.

As said, a Catalan family is a family of combinatorial objects such that there are C_n objects of size n. For example, parenthesizations on $n+1$ letters, triangulations of a $(n+2)$-gon, non-crossing arborescences on $n+1$ nodes, binary search trees with n elements... The reader is invited to consult [Sta15] for a presentation of 214 Catalan families and bijections between them, we give here a shorthand on the 4 aforementioned families.

Each family is endowed with a very important notion of *flip*: they shall be thought of as a graph whose C_n vertices are the objects, and where two vertices share an edge when the corresponding objects differ by a flip. It is sometimes convenient to direct the flip, giving rise to a directed graph. Directed this way, this graph is a poset: the Tamari lattice. We will not present this lattice in details but only give a glimpse on it, see [Tam51, PSZ23]. Instead, we will be interested in the face lattice of the associahedron, which corresponds to super-Catalan families (and are counted by Schröder–Hipparchus numbers).

Before constructing the associahedron, we briefly review the four aforementioned Catalan families. The aim is not to give an extensive nor self-contained introduction, but rather a dictionary that will allow the reader to perceive the underlying general framework behind each specific lexicon (see the table of correspondences below). Therefore, we detail a bit the case of parenthesizations, as we find it easy to handle and insightful. We then give some explanations on binary search trees because it will be needed for Section 3.3.1, and we refer the reader to the indicated sections of the present thesis for non-crossing arborescences and triangulations.

	Max. parenthesization	Non-crossing arbo.	Triangulation	BST
Objects	parenthesis	arcs	triangles	(not defined here)
Compatibility	nested or disjoint	non-crossing	non-intersecting	(not defined here)
Flips	remove & re-add	remove & re-add	remove & re-add	rotations
Super-Catalan	parenthesizations	multi-arborescences	subdivisions	Schröder trees
See	here above	Sections 3.1 and 3.2	Section 4.3.1	Section 3.3

Parenthesizations Fix a n letter words $a_1 a_2 \dots a_n$ where a_i are just symbols. One can *parenthesize* this word by adding pairs of opening and closing parentheses $(\dots)$ where the opening one precedes the closing one: $a_1 \dots a_{i-1}(a_i \dots a_j)a_{j+1} \dots a_n$. Such a pair of parentheses can be identified by the pair (i, j) with $1 \le i < j \le n$. A pair of parentheses is valid when $(i, j) \ne (1, n)$. Two pairs of parentheses are *compatible* when the part they separate are either disjoint or included one in the other. A *maximal parenthesization* is a maximal family of pair-wise compatible pairs of parentheses.

There are C_{n-1} maximal parenthesizations for a word of n letters, each containing exactly $n-2$ pairs of parentheses.

The *flip* between two maximal parenthesizations consists in removing a pair of parentheses and adding the only other possible pair of parentheses. Such a flip is *forward* if the leftmost parenthesis moves from left to right, and *backward* else way.

In this context, the super-Catalan family is the family of all parenthesizations (not necessarily maximal ones). This family has a natural ordering: a parenthesization *refines* another one when it can be obtained by adding (compatible) pairs of parentheses to the first.

Binary search trees A *binary search tree* (BST) is a planar rooted binary tree whose nodes are labelled from 1 to n respecting that for each node i, all labels in its left sub-tree are smaller than i, and all labels in its right sub-tree are greater than i. Note that a binary search tree is fully defined by its skeleton, thus they are in bijection with binary trees and form a Catalan family.

Binary search trees are a powerful data structure for sorting the numbers from 1 to n. The key operations on this structure are insertion of a number and deletion of the root. Thanks to these operations, one can sort a list L in $O(n \log n)$ (time-)complexity by inserting all the numbers from L in order of appearance, and then extracting them. The binary search tree corresponding to L depends on which order does the numbers from 1 to n appear in L, *i.e.* its permutation. Consequently, to each permutation $\sigma \in \mathcal{S}_n$ one associates a binary search tree $T(\sigma)$ thanks to the following recursive algorithm:

- Inserting $\sigma(i)$ to an empty tree gives rise to a tree on one node $\sigma(i)$ (hence T initializes at the tree on one node labelled $\sigma(1)$);
- After having inserted $\sigma(1), \dots, \sigma(i-1)$ in T, one adds $\sigma(i)$: if $\sigma(i)$ is smaller than the root of T, then $\sigma(i)$ is recursively added to the left sub-tree of T; if $\sigma(i)$ is greater than the root of T, then $\sigma(i)$ is recursively added to the right sub-tree of T;
- Once inserted all values $\sigma(1), \dots, \sigma(n)$ (*i.e.* all numbers from 1 to n), the resulting tree is $T(\sigma)$.

A flip between binary search trees is a rotation of the tree, and the super-Catalan family is the family of Schröder trees.

We will encounter binary search trees again in Section 3.3.1.

Construction of the associahedron The associahedron was first defined by Tamari [Tam51] and later Stasheff [Sta63] as a combinatorial polytope, and then realized by Milnor (unpublished), Haiman [Hai84], Lee [Lee89] and Loday [Lod04, PSZ23]. They give a description of a super-Catalan family as defined above, and proved that this poset is the face lattice of a polytope. Hence, any polytope whose face lattice if isomorphic to the super-Catalan poset can to be called an associahedron. To this extent, several different realizations of the associahedron will appear in the present thesis, see Example 2.21 and Sections 3.2, 3.3.1 and 4.3. Nevertheless, we will emphasize a specific realization of the associahedron that, as far as we believe, would deserve to be called *standard*, namely Loday's realization of the associahedron [Lod04, PSZ23].

A straightforward way to define it and to emphasize its link with the permutahedron Π_n is thanks to its facet-description:

$$\mathsf{Asso}_n = \left\{ \boldsymbol{x} \in \mathbb{R}^n \ ; \ \begin{array}{ll} \sum_{i=1}^n x_i = \binom{n+1}{2} & \\ \sum_{i\in I} x_i \geq \binom{|I|+1}{2} & \text{for all } \varnothing \subsetneq I = [a,b] \subsetneq [n] \end{array} \right\}$$

Instead of the $2^n - 2$ facets of the permutahedron Π_n that correspond to the non-trivial subsets of $[n]$, the associahedron Asso_n is supported by the $\binom{n}{2} - 1$ facets that correspond to non-trivial sub-intervals of $[n]$. Consequently, the associahedron Asso_n is a removahedron for the permutahedron Π_n, meaning that is can be obtained by removing some inequalities in its facets-description[4], see Figure 6. A 3-dimensional model of a permutahedron inside the Loday associahedron can also be found on Viviane Pons' website [Pon18]. This will be relevant for Section 2.

[4]The standard cube $\square_n$ is linearly isomorphic to the removahedron obtained by keeping only the inequalities corresponding to I of the form $[1,i]$ or $[i,n]$ for $i \in [n]$. The latter is also a removahedron for the associahedron Asso_n.

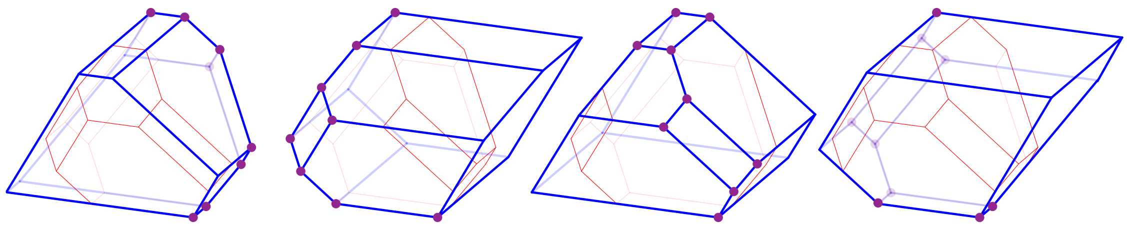

Figure 6: Four realizations of the 3-dimensional permutahedron Π_3 sitting inside an associahedron. The left-most associahedron is the Loday's associahedron Asso_3. Illustration from [PSZ23, Figure 7(Top)].

The associahedron Asso_n has C_n vertices, $\binom{n}{2} - 1$ facets, dimension $n-1$ (although being embedded in $\mathbb{R}^n$), and its face lattice is the super-Catalan poset, thus its edges correspond to the flips between Catalan objects, see Figure 7. Note that there exists an orientation of the ambient space $\mathbb{R}^n$ such that the induced orientation on the graph G_{Asso_n} is precisely the Tamari orientation of flips.

We will not provide here a (coordinate) vertex description of the associahedron Asso_n and refer to the abounding literature [Lod04, Pos09, PSZ23]. Instead, we briefly describe the normal fan of the associahedron Asso_n, called the *sylvester fan*. The maximal cones of the sylvester fan (*i.e.* the normal cones of the vertices of Asso_n) can be described easily thanks to binary search trees, see Figure 4(Right). With the definition of C_σ given above for the permutahedron, for a binary search tree T, the associated normal cone is $\mathsf{C}_T = \bigcup_{\sigma\ ;\ T(\sigma)=T} \mathsf{C}_\sigma$.

1.3 Linear programming

Linear programming has proven to be a powerful tool to tackle theoretical and applied problems. The aim is to optimize a linear functional subject to linear constraints. For example, imagine I want to breed goats and cows. My barn has 15 boxes, a goat takes 1 box and a cow 3 boxes. Milking a goat gives 4L, while a cow gives 3L, and my storage allows at most 24L. I want to maximize the number of animals I have. Denote by g the number of goats and c the number of cows, then this toy example amounts to maximize $g + c$ under the conditions:

$$g \geq 0 \qquad c \geq 0 \qquad g + 3c \leq 15 \qquad 4g + 3c \leq 24$$

Even though this optimization problem seems simple (and the numbers chosen are quite unrealistic), it encapsulates a wide variety of problems, from its prototypical purpose of optimizing a diet from a nutritional point of view [Dan63], to constructing the best line to fit data, or finding the largest disk in a polygon, see [MG07].

A linear problem can be thought of as the problem of finding the vertex (or face) of a convex polyhedron that is maximal for the scalar product with a certain direction $\boldsymbol{c}$. The polyhedron is called its *feasibility domain* while the direction $\boldsymbol{c}$ is its *objective function*. In the present thesis, the feasibility domain will be bounded (*i.e.* it will be a polytope), and the objective function will be generic (*i.e.* it will not be orthogonal to any edge of the feasibility domain, so the optimum will be attained at a vertex). Precisely, for our toy example, the four inequalities define a quadrilateral in the plane, and we want to find its furthest point in direction $\binom{1}{1}$, see Figure 8. Here, the optimal solution is 3 goats and 4 cows.

Note that we conveniently created a toy example where the optimal solution is a couple of integers. Notwithstanding, the optimal solution of a linear problem is not integral in general: the added requirement of finding the optimal integral point for a linear program is the focus of integer programming [MG07, Chapter 3], which is far more difficult than linear programming. Furthermore, a plethora of natural questions and generalizations can be asked from our toy example, that we will not address in this light introduction.

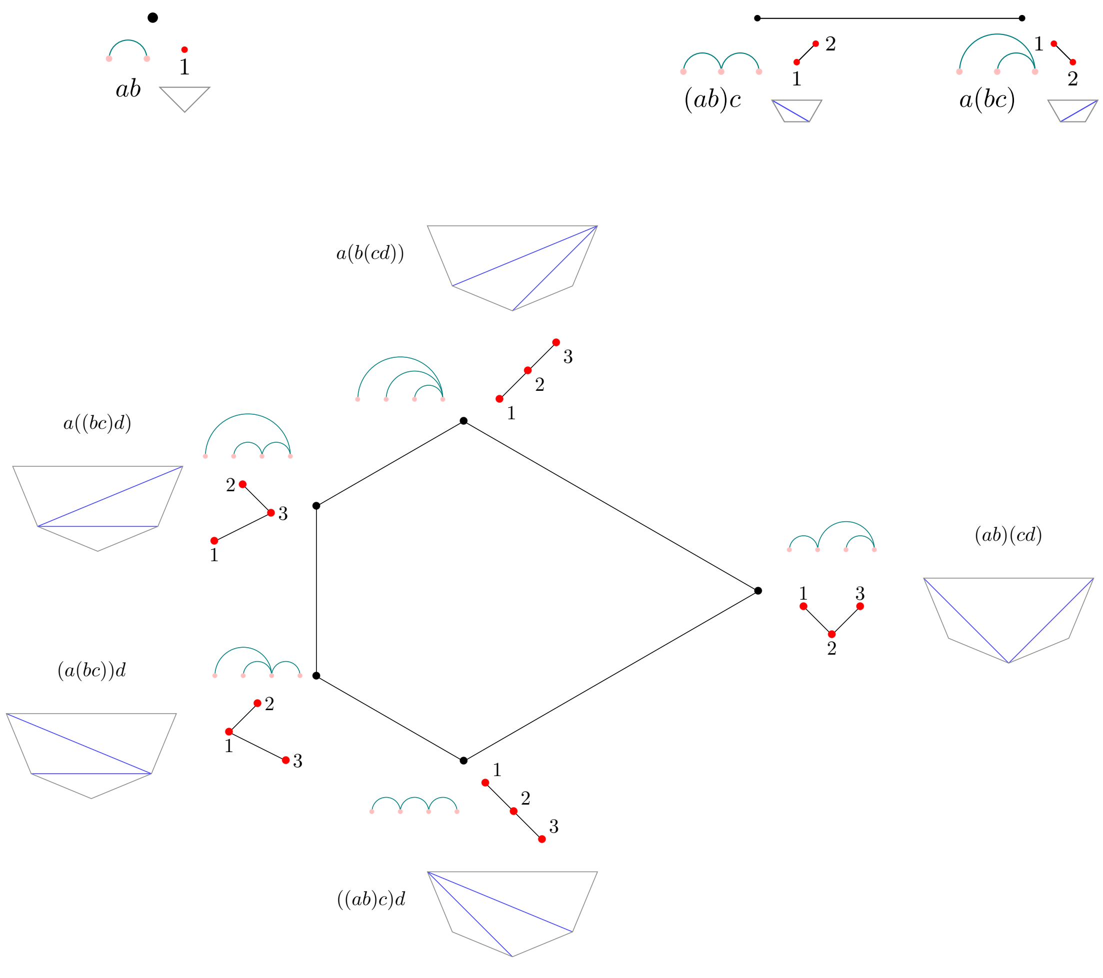

Figure 7: (Bottom) The 2-dimensional Loday's associahedron with each vertex labelled by the corresponding maximal parenthesization (on 4 letters), triangulation (of a pentagon), binary search tree (on [3], rooted at their bottoms) and non-crossing arborescence (on 4 nodes). Note that the permutations 213 and 231 both yield the same binary search tree, namely the rightmost one, thus the normal cone of the rightmost vertex is the union of the normal cone of the two right vertices of the 2-dimensional permutahedron in Figures 4 and 5. (Top) The 0 and 1-dimensional counterparts. The interested reader shall refer to page 12 of [Lod04] for the 3-dimensional picture.

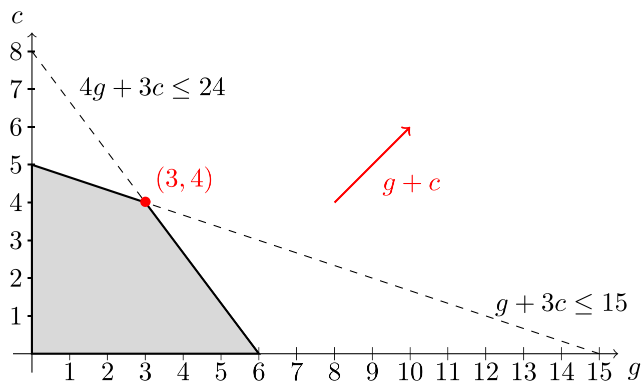

Figure 8: Feasibility domain, objective function and optimal vertex of the toy example.

A widely used method for solving linear problems is the simplex algorithm [Dan63]. Suppose known a vertex $\boldsymbol{v}$ of the feasibility domain, then it is algorithmically not difficult to find the neighbors of $\boldsymbol{v}$ that increase the scalar product $\langle \boldsymbol{v}, \boldsymbol{c} \rangle$. Choose one of these improving neighbors and pursue the algorithm from there. By convexity, this algorithm ends at a vertex $\boldsymbol{v}_{\mathrm{opt}}$ maximizing $\langle \boldsymbol{x}, \boldsymbol{c} \rangle$ for $\boldsymbol{x}$ in the feasibility domain. The path followed by the simplex method depends on the *pivot rule* used to choose the improving neighbor.

It has been proven that, given a linear problem, it is possible to solve it in polynomial time (with respect to the size of its entries) using interior-point methods [RTV05, DNT08]. Nevertheless, the polynomiality of the simplex method is still open [KM72, DS14]. The core of the simplex method is thus the choice of the pivot rule, see [MG07, Section 5.7] for a short review of the main ones, and [BDLLS22, Section 2] for a polytopal discussion on the subject. Here, we will only recall the useful definitions and detail the ideas behind the pivot rules called the shadow vertex rules and the max-slope pivot rules.

In this thesis, a *linear program* is a couple $(\mathsf{P}, \boldsymbol{c})$ where $\mathsf{P} \subset \mathbb{R}^d$ is a bounded feasibility domain (*i.e.* a polytope), and $\boldsymbol{c} \in \mathbb{R}^d$ is a generic objective function (*i.e.* a vector such that $\langle \boldsymbol{u}, \boldsymbol{c} \rangle \neq \langle \boldsymbol{v}, \boldsymbol{c} \rangle$ for $\boldsymbol{u}, \boldsymbol{v} \in V(\mathsf{P})$ with $\boldsymbol{u} \neq \boldsymbol{v}$). The vertex of P that maximizes the scalar product with respect to $\boldsymbol{c}$ is called the *optimal vertex* of the linear program. The generic objective function orients the graph G_{P} of P by orienting each edge from $\boldsymbol{u}$ to $\boldsymbol{v}$ such that $\langle \boldsymbol{u}, \boldsymbol{c} \rangle < \langle \boldsymbol{v}, \boldsymbol{c} \rangle$. This orientation is acyclic. The out-neighbors of a vertex $\boldsymbol{v} \in V(\mathsf{P})$ are called its *improving neighbors*. A directed path in this oriented graph is called a *$\boldsymbol{c}$-monotone path on* P, or simply a *monotone path* when $(\mathsf{P}, \boldsymbol{c})$ is clear from the context.

The reader shall think of a pivot rule as an oracle that, if you give it a linear program $(\mathsf{P}, \boldsymbol{c})$ and a starting vertex $\boldsymbol{v}_{\mathrm{init}}$ (not necessarily the vertex that minimizes the scalar product against $\boldsymbol{c}$), will return a monotone path that starts at $\boldsymbol{v}_{\mathrm{init}}$ and ends at $\boldsymbol{v}_{\mathrm{opt}}$. Pivot rules are hard to describe in general, see [APR14, DS14, FS14] for a study of their complexity. However, we can restrict ourselves to an easy-to-describe subclass of pivot rules, the ones that are defined by their local behavior.

Definition 1.17. A *memoryless pivot rule* R is a pivot rule that associates each non-optimal vertex $\boldsymbol{v}$ of each linear program $(\mathsf{P}, \boldsymbol{c})$ to one of its improving neighbors $R_{(\mathsf{P},\boldsymbol{c})}(\boldsymbol{v})$.

For a linear program $(\mathsf{P}, \boldsymbol{c})$ and a starting vertex $\boldsymbol{v}_{\mathrm{init}}$, its monotone path is formed by the successive images of $\boldsymbol{v}_{\mathrm{init}}$ under $R_{(\mathsf{P},\boldsymbol{c})}$, namely $\big(\boldsymbol{v}_{\mathrm{init}},\ R_{(\mathsf{P},\boldsymbol{c})}(\boldsymbol{v}_{\mathrm{init}}),\ R_{(\mathsf{P},\boldsymbol{c})}(R_{(\mathsf{P},\boldsymbol{c})}(\boldsymbol{v}_{\mathrm{init}})), \dots, \boldsymbol{v}_{\mathrm{opt}}\big)$.

For a fixed linear program $(\mathsf{P}, \boldsymbol{c})$, a memoryless pivot rule induces an arborescence A on the graph G_{P}, formed by the edges $\boldsymbol{u}\boldsymbol{v}$ such that $\boldsymbol{v} = R_{(\mathsf{P},\boldsymbol{c})}(\boldsymbol{u})$. Memoryless pivot rules are fully defined by the arborescences they induce on each linear program, as all monotone paths can be retrieved from the knowledge of this arborescence. The study of possible arborescences is at the heart of pivot rule polytopes, see [BDLLS22] and Section 3.1.

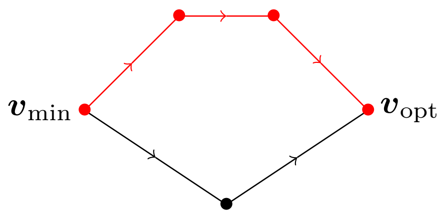

Figure 9: The two possible monotone paths starting at the minimal vertex for a 2-dimensional polytope (the objective function is from left to right).

When P is 2-dimensional, pivot rules are very simple, see Figure 9. The only vertex where a choice shall be made is the vertex $\boldsymbol{v}_{\min}$ that minimizes the scalar product against $\boldsymbol{c}$. This vertex has two possible monotone paths to choose from.

When P is higher dimensional, the situation becomes far more convoluted. An efficient idea is to simplify the problem by making it 2-dimensional again. By choosing a vector $\boldsymbol{\omega} \in \mathbb{R}^d$ linearly independent from $\boldsymbol{c} \in \mathbb{R}^d$, one can project the polytope P onto the 2-dimensional plane of basis $(\boldsymbol{c}, \boldsymbol{\omega})$, and perform the simplex algorithm on this projection. This method will find the optimal vertex $\tilde{\boldsymbol{v}}_{\text{opt}}$ in the plane $(\boldsymbol{c}, \boldsymbol{\omega})$. As $\boldsymbol{c}$ is generic, $\boldsymbol{v}_{\text{opt}}$ is the only vertex of P that projects onto $\tilde{\boldsymbol{v}}_{\text{opt}}$, solving the higher dimensional linear problem, see Figure 25(Left). Consequently, one can define a family of memoryless pivot rules as follows:

Definition 1.18. A memoryless pivot rule R is a *max-slope pivot rule* when for every linear program $(\mathsf{P}, \boldsymbol{c})$, there exists $\boldsymbol{\omega}$ linearly independent from $\boldsymbol{c}$ such that the arborescence $A^{\boldsymbol{\omega}}$ induced by R on $(\mathsf{P}, \boldsymbol{c})$ is defined by its edges between $\boldsymbol{u} \in V(\mathsf{P}) \smallsetminus \{\boldsymbol{v}_{\text{opt}}\}$ and

$$A^{\boldsymbol{\omega}}(\boldsymbol{u}) := \operatorname{argmax} \left\{ \frac{\langle \boldsymbol{\omega}, \boldsymbol{v} - \boldsymbol{u} \rangle}{\langle \boldsymbol{c}, \boldsymbol{v} - \boldsymbol{u} \rangle} \ ; \ \boldsymbol{v} \text{ improving neighbor of } \boldsymbol{u} \right\}$$

The study of max-slope pivot rules is of prime importance, as it links the world of linear optimization with the one of fiber polytopes. Presenting some aspects of this nexus is the focal point of Sections 3 and 4: especially, we will fix a linear program $(\mathsf{P}, \boldsymbol{c})$ and study all the possible arborescences that arise when varying the parameter $\boldsymbol{\omega}$ of the max-slope pivot rules.

It is worth noting that max-slope pivot rules are the memoryless version of *shadow vertex rules*, which may be more familiar to the reader. A pivot rule is a shadow vertex rule when, for every linear program $(\mathsf{P}, \boldsymbol{c})$ and every vertex $\boldsymbol{v}_{\text{int}} \in V(\mathsf{P})$, there exists $\boldsymbol{\omega}$ linearly independent from $\boldsymbol{c}$ such that the $\boldsymbol{c}$-monotone path from $\boldsymbol{v}_{\text{init}}$ to $\boldsymbol{v}_{\text{opt}}$ is the upper part of the 2-dimensional projection of P onto the plane of basis $(\boldsymbol{c}, \boldsymbol{\omega})$. Although very similar, these definitions are different: in a max-slope pivot rule, one chooses a fix $\boldsymbol{\omega}$ for all the vertices of P; whereas the choice of $\boldsymbol{\omega}$ is subordinated to the vertex $\boldsymbol{v}_{\text{init}}$ at stake for a shadow vertex rule, see [BDLLS22, Section 6.1].