Pivot rule polytope of cyclic polytopes

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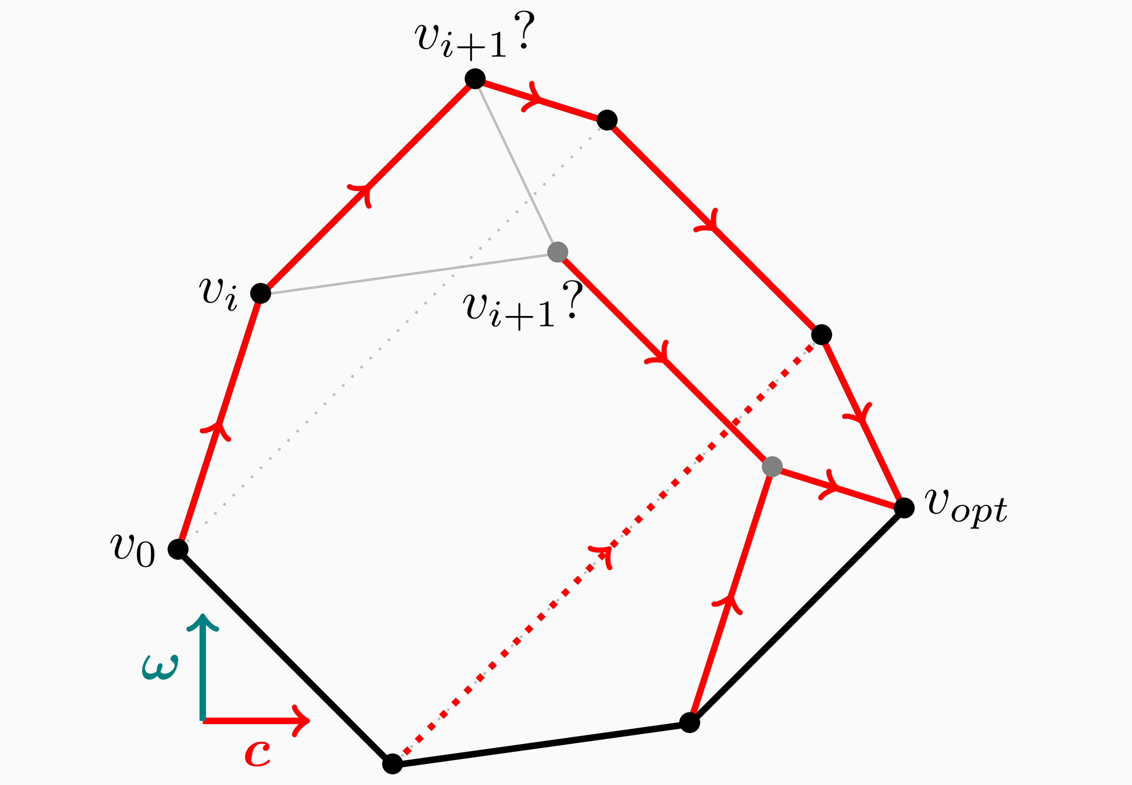
Shadow vertex rule

Linear program $(\mathsf{P}, \boldsymbol{c})$: how to choose next vertex in simplex method?



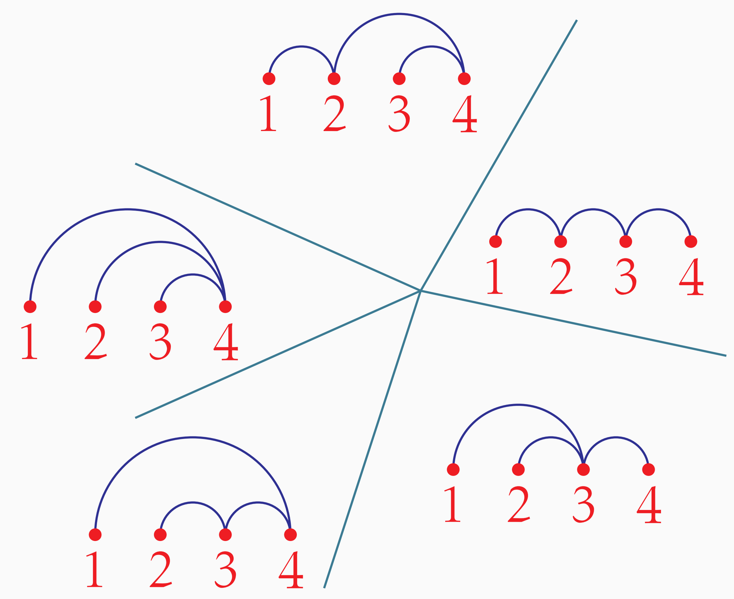
Shadow vertex: for ω, project in plane $(\boldsymbol{c}, \omega)$, take the neighbor with the best slope:

$$A^{\omega}(v) = \operatorname{argmax}\left\{ \frac{\langle \omega, u - v\rangle}{\langle \boldsymbol{c}, u - v\rangle} \; ; \; \begin{matrix} u \text{ improving} \\ \text{neighbor of } v \end{matrix} \right\}$$

Pivot rule polytope

Coherent arborescence: monotone arborescence arising from shadow vertex rule

Pivot rule fan: $\omega \sim \omega'$ iff same arborescence



Pivot rule polytope $\Pi_c(\mathsf{P})$: dual to pivot rule fan

$\mathrm{Vert}\big(\Pi_c(\mathsf{P})\big) \longleftrightarrow c$-coherent arborescences

Resemble Billera–Sturmfels' fiber polytopes

Pivot rule polytope of Δ_n

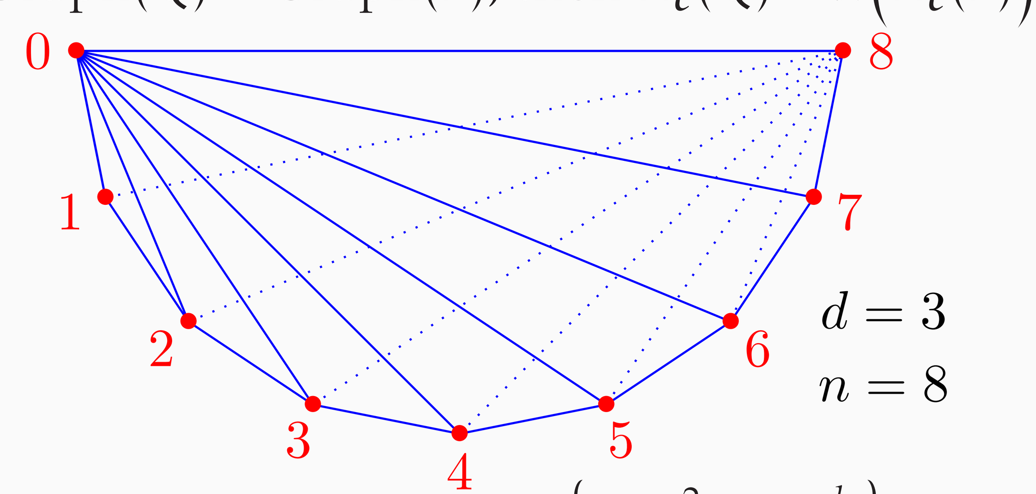
THM. [BLLS23+] for all $\boldsymbol{c}$, $\Pi_{\boldsymbol{c}}(\Delta_n) \simeq \mathsf{Asso}_{n-2}$

$\mathrm{Vert}\big(\Pi_{\boldsymbol{c}}(\Delta_n)\big)$ = non-crossing arbor. (Catalan)

Projections of associahedra

Projection of polytope $\mathsf{P} \xrightarrow{\pi} \mathsf{Q}$

If $\mathrm{Graph}(\mathsf{Q}) = \mathrm{Graph}(\mathsf{P})$, then $\Pi_{\boldsymbol{c}}(\mathsf{Q}) = \pi\big(\Pi_{\boldsymbol{c}}(\mathsf{P})\big)$

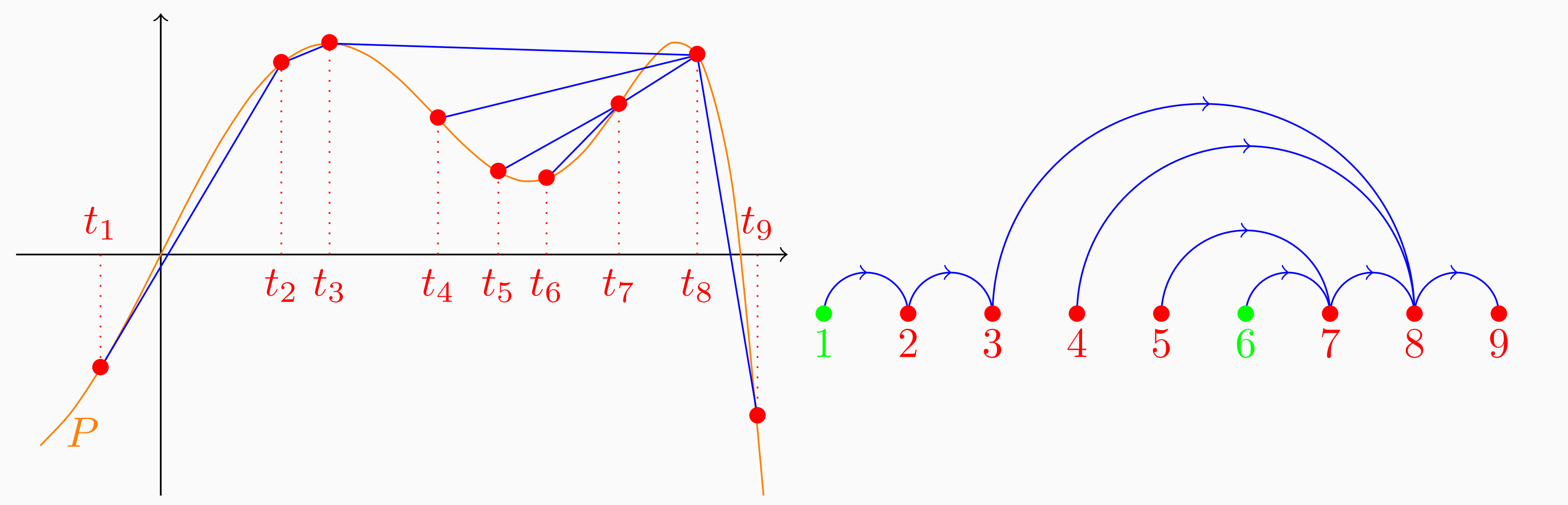


Cyclic poly. $\mathsf{Cyc}_d(\boldsymbol{t}) = \mathrm{conv}\big\{(t_i, t_i^2, \dots, t_i^d)\big\}_{1<i<n}$

CORO. for $d \geq 4$, as $\mathrm{Graph}(\mathsf{Cyc}_d(\boldsymbol{t}))$ complete, then $\Pi_{\boldsymbol{c}}(\mathsf{Cyc}_d(\boldsymbol{t}))$ = projection of Asso_{n-1}

Degrees of non-crossing arborescences

A is captured by P on $\boldsymbol{t}$: best slopes between $(t_i, P(t_i))$ are edges of A
$\Leftrightarrow$ vertex of $\Pi_{e_1}(\mathsf{Cyc}_d(\boldsymbol{t}))$



Degree $\mu(A, \boldsymbol{t}) = \min\{d \;:\; A \text{ captured by } P \in \mathbb{R}_d[X] \text{ on } \boldsymbol{t}\}$

Intrinsic degree $\mu(A) = \min_{\boldsymbol{t}} \mu(A, \boldsymbol{t})$

Immediate leaves $\mathbb{L}(A)$: i leaf with $A(i) = i + 1$ (above), *interior* if $\neq 1, n-1$

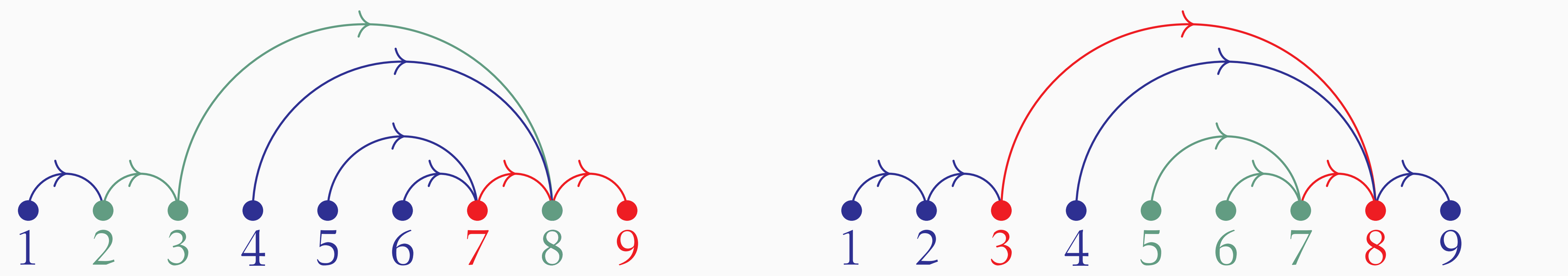
THM. $\mu(A) = |\mathbb{L}(A)| + |\mathbb{L}^{\text{interior}}(A)| + 1$

2 arborescences with $\mu(A) = 2$; $2^{n-2} + n - 5$ arborescences with $\mu(A) = 3$

Realization sets and universal arborescences

Realization set $\mathcal{T}_d^{\circ}(A) = \{\boldsymbol{t} \;:\; A \text{ captured by } P \in \mathbb{R}_d[X] \text{ on } \boldsymbol{t}\}$

Universal: $\mathcal{T}_{\mu(A)}^{\circ}(A) = \{t_1 < \dots < t_n\}$, *i.e.* if possible then everyone possible



$(2,3,8)$ is forward — $(7,8,9)$ is **not** forward

$(5,6,7)$ is backward — $(3,7,8)$ is **not** backward

Complete symmetric homogeneous poly. $h_\ell(X, Y, Z) = \sum_{p+q+s=\ell} X^p Y^q Z^s$

$\mathsf{P}_d^f(A, \boldsymbol{t}) = \mathrm{conv}\Big\{\big(h_\ell(t_i, t_j, t_k)\big)_{\ell \leq d-2}\Big\}_{\text{fwd}}$; $\mathsf{P}_d^b(A, \boldsymbol{t}) = \mathrm{conv}\Big\{\big(h_\ell(t_a, t_b, t_c)\big)_{\ell \leq d-2}\Big\}_{\text{bwd}}$

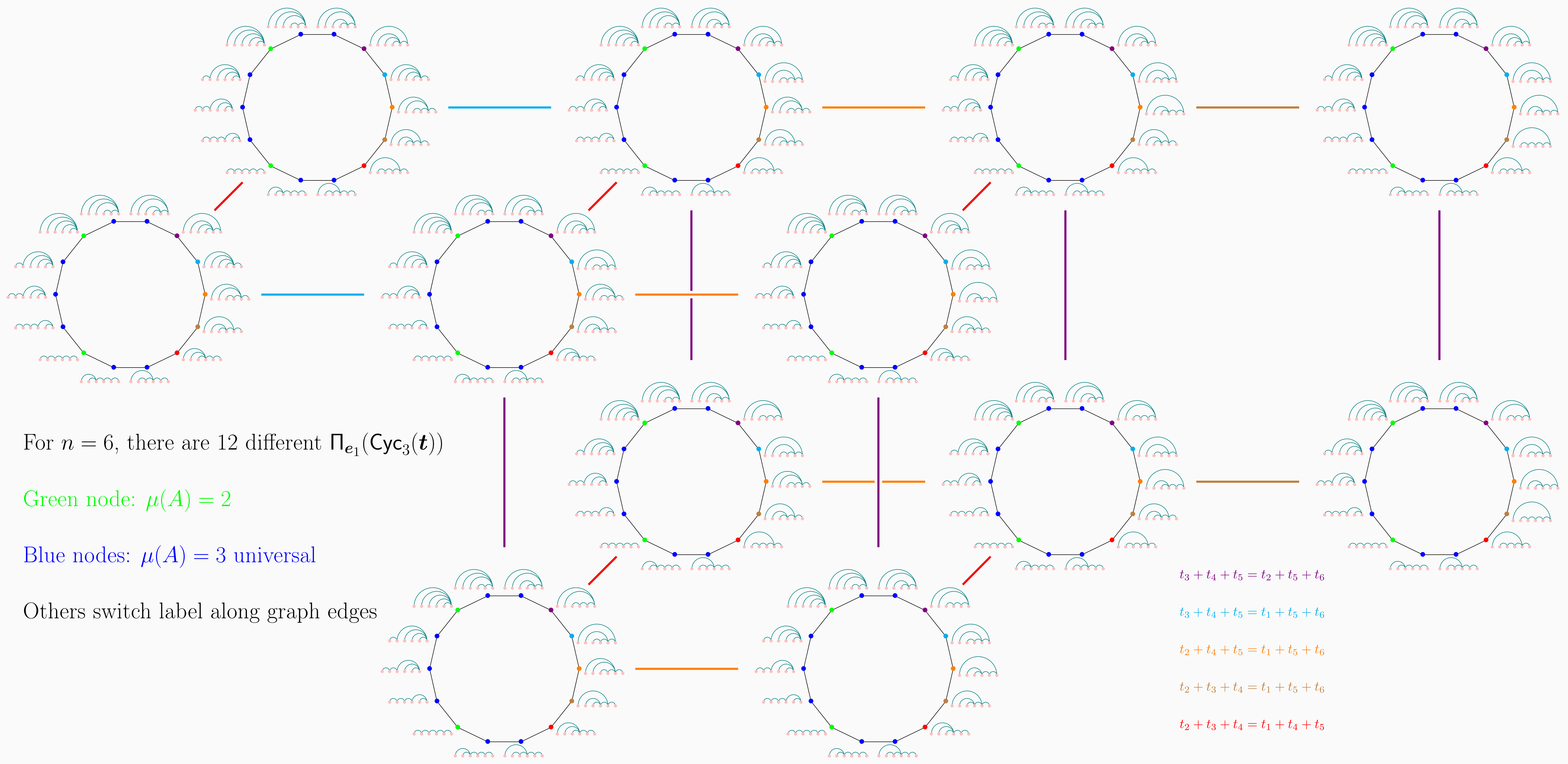
THM. $\boldsymbol{t} \in \mathcal{T}_d^{\circ}(A)$ iff $\mathsf{P}_d^f(A, \boldsymbol{t}) \cap \mathsf{P}_d^b(A, \boldsymbol{t}) = \emptyset$.

Full study for $d = 3$, *i.e.* 2-dimensional case, **but not** $\Pi_{e_1}(\mathsf{Cyc}_3(\boldsymbol{t}))$

$\mathcal{T}_3^{\circ}(A)$ are (open) polyhedral cones (we have facet description)

THM. For almost all $\boldsymbol{t}$, $\big|\{A \;:\; \boldsymbol{t} \in \mathcal{T}_3^{\circ}(A)\}\big| = \binom{n}{2} - 1$ (indep. $\boldsymbol{t}$)

Subdivision of the order cone



For $n = 6$, there are 12 different $\Pi_{e_1}(\mathsf{Cyc}_3(\boldsymbol{t}))$

Green node: $\mu(A) = 2$

Blue nodes: $\mu(A) = 3$ universal

Others switch label along graph edges