Poset associahedra as sections of graph associahedra

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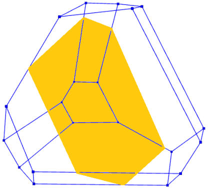
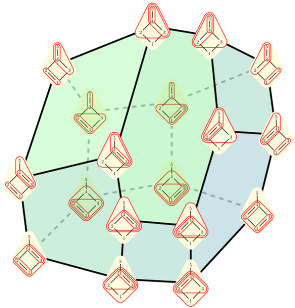
March 6, 2023



Poset associahedron:

→ combines the notions of graph associahedra and order polytopes



- Galashin, 2021
  - description of combinatorial structure
  - realization as stellar subdivision of order polytope

→ No explicit coordinates are provided

Graph associahedron: graph tubes and tubing

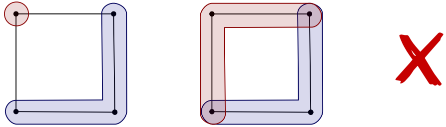
G finite connected graph

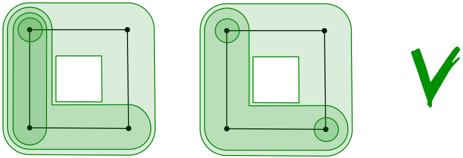
Tube: induced and connected subgraph;

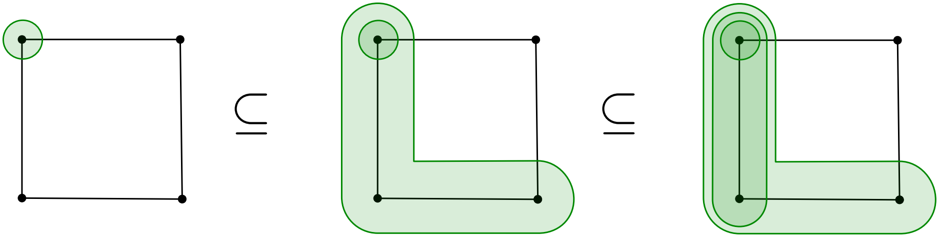
Compatible: pair of tubes σ, τ

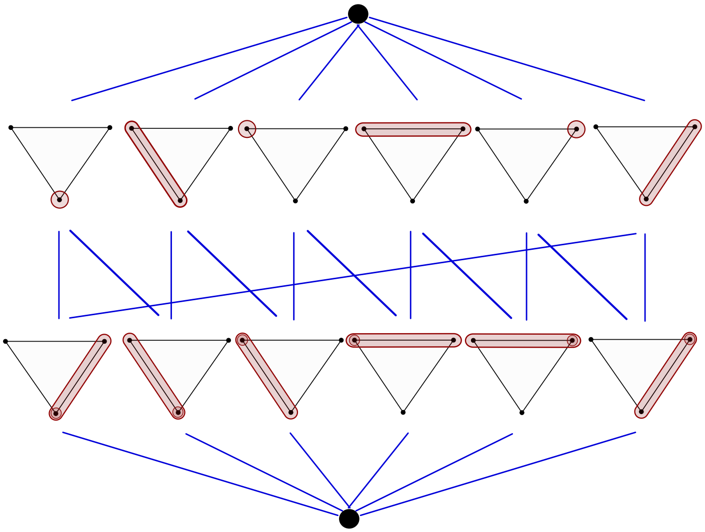
- nested ($\sigma \subseteq \tau$ or $\tau \subseteq \sigma$);
- disjoint and not adjacent ($\sigma \cup \tau$ not connected).

Tubing: set of pairwise compatible tubes



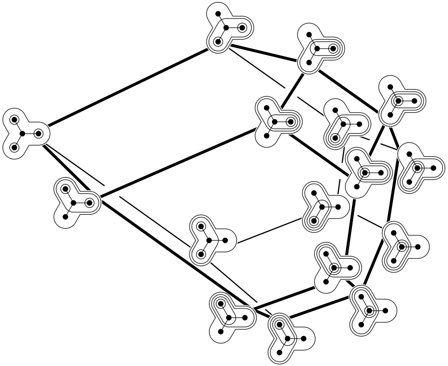






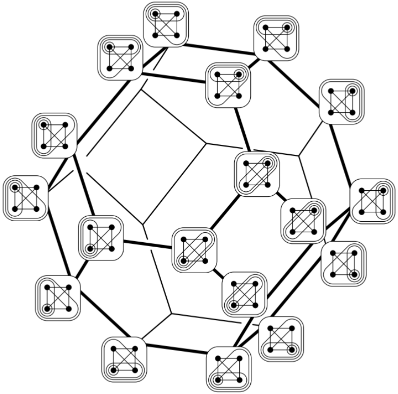
Graph associahedron: combinatorial structure

$\mathcal{P}(G)$: polytope whose face lattice is isomorphic to the set of tubings of G, ordered by reverse inclusion

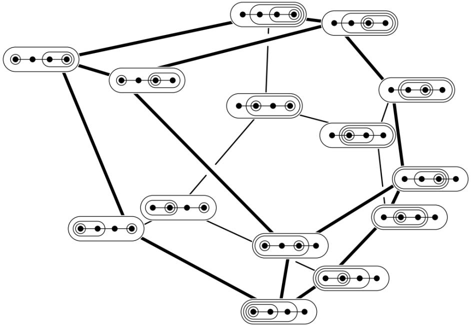


Vertices $\leftrightarrow$ Maximal tubings

Facets $\leftrightarrow$ Tubes



Complete graph → permutahedron



Path → associahedron

Graph associahedron: geometric realization

Theorem (Postnikov, 2009)

G graph with vertices $\{1, \dots, n\}$. For every choice of positive parameters $\{\lambda_\sigma\}_{\sigma \in B_G}$, the polytope

$$\mathcal{P}_G(\{\lambda_\sigma\}) = \sum_{\sigma \in B_G} \lambda_\sigma \Delta_\sigma$$

is a realization of the graph associahedron $\mathcal{P}(G)$ of G.

- $B_G \rightarrow$ set of tubes of G
- $\Delta_\sigma \rightarrow \mathrm{Conv}(e_i \mid i \in \sigma)$

Graph associahedron: vertices

G graph with vertices $\{1, \ldots, n\}$, T maximal tubing. There's a bijection

$$T \longleftrightarrow V(G)$$
$$\tau_i \longleftrightarrow i$$

$\tau_i \rightarrow$ minimal by inclusion tube of T that contains i.

Graph associahedron: vertices

Proposition

Let us define $C_i := \{\tau \in B_G \mid i \in \tau \subset \tau_i\}$. Then the vertex of $\mathcal{P}_G(\{\lambda_\sigma\})$ associated to the tubing T is

$$v_T := (t_1, \ldots, t_n), \quad \text{with} \quad t_i := \sum_{\sigma \in C_i} \lambda_\sigma$$

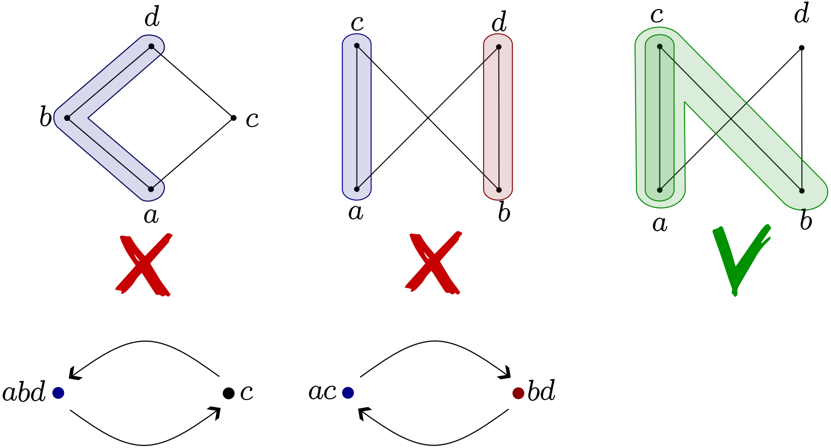
Poset associahedron: poset tubes and tubings

P finite connected poset, $|P| \geq 2$, H_P Hasse diagram

Tubing: set T of connected subgraphs of H_P:

- pairwise nested ($\sigma \subseteq \tau$ or $\tau \subseteq \sigma$) or disjoint
- there exist no subsets T' of T such that the graph obtained from the Hasse diagram H_P of P by contracting every $\tau_i \in T'$ to a vertex v_i has a directed cycle

Proper tubing: $2 \leq |\tau| \leq |P| - 1$ for all $\tau \in T$.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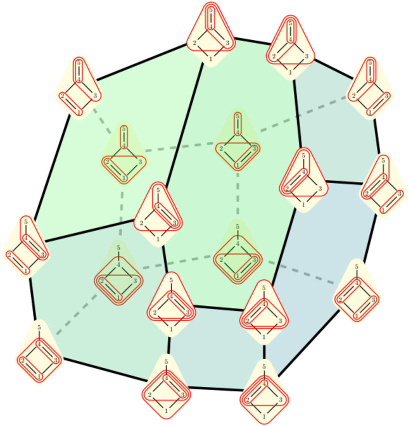
bd

Poset associahedron: combinatorial structure

$\mathcal{A}(P)$: polytope whose face lattice is isomorphic to the set of proper tubings of P, ordered by reverse inclusion



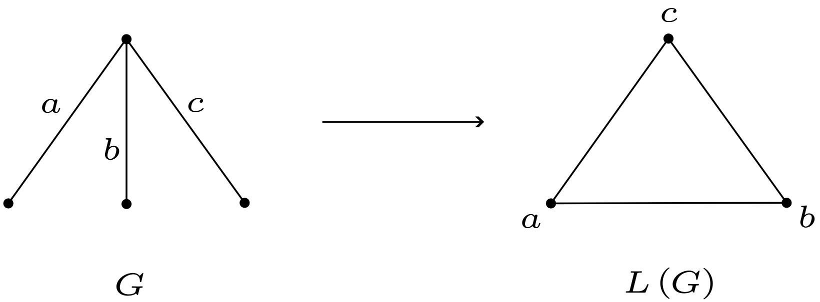
Vertices $\leftrightarrow$ Maximal tubings

Facets $\leftrightarrow$ Tubes

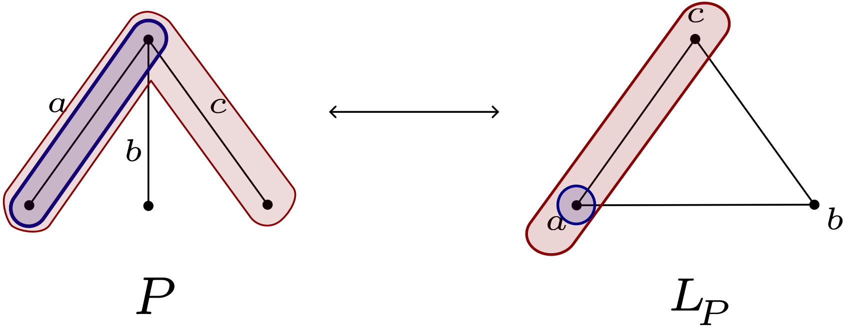
Poset associahedron: our realization

Line graph: graph $L(G)$ with:

- a vertex for every edge of G
- an edge for every incidence in G



Motivating example: Hasse diagram with no cycles



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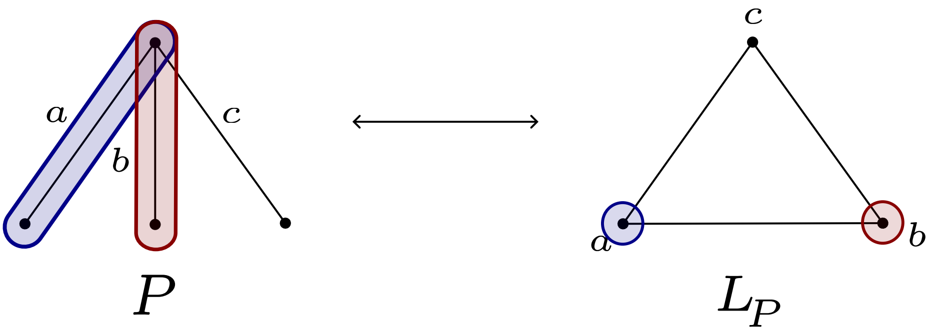

Motivating example: Hasse diagram with no cycles

$\rightarrow$ bijection between proper poset tubings of P and graph tubings of the line graph.

Theorem

Let P be a finite poset such that its Hasse diagram H_P has no cycles. Let L_P be the line graph of H_P. Then the graph associahedron $\mathcal{P}(L_P)$ is combinatorially equivalent to the poset associahedron $\mathcal{A}(P)$ of P.

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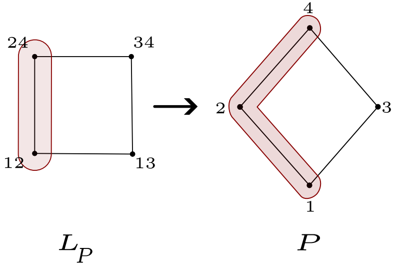
General case: Hasse diagram with cycles



Problem: there are tubings of L_P that do not correspond to tubings of P

- Forbidden tubing: tubing of L_P that doesn't correspond to a tubing of P
- Allowed tubing: tubing of L_P that corresponds to a tubing of P

Idea: section of the graph associahedron of L_P with a subspace that intersects all and only the faces corresponding to allowed tubings

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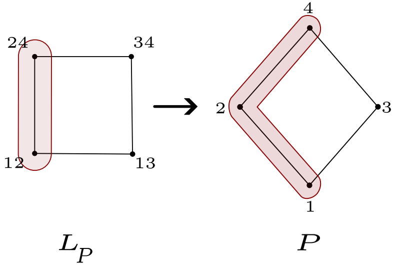

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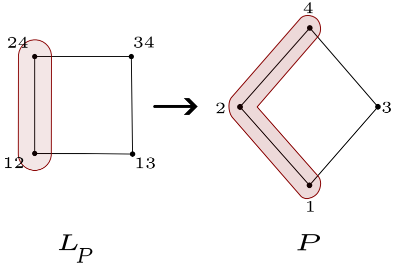

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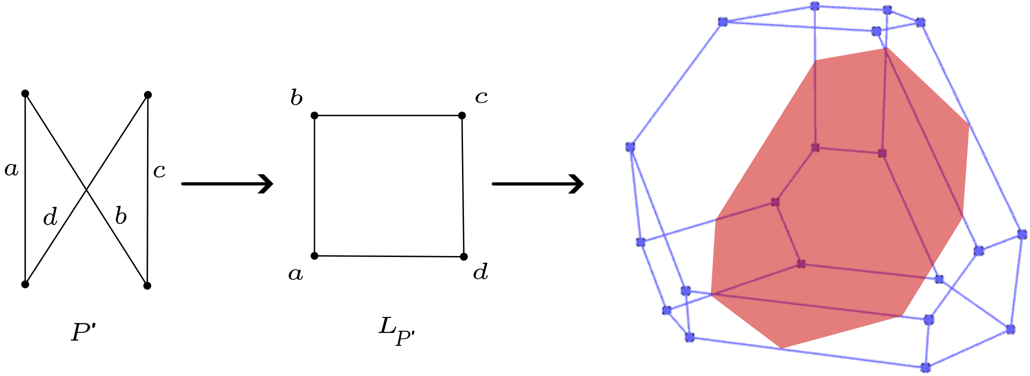
a
d
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P'
b
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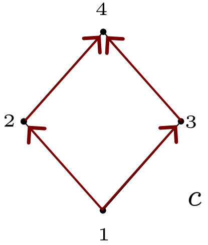
L_{P'}

c cycle in H_P.

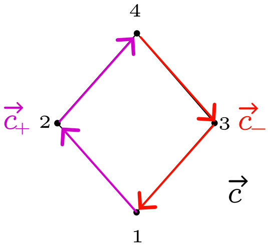
Orientation: one of the two ways of turning the edges of c into arcs to get a directed cycle $\vec{c}$



Oriented cycle: cycle with an orientation

Positive part: $\vec{c}_+ := A(\vec{c}) \cap A(H_P)$

$\rightarrow$ arcs that have the same direction in H_P and in $\vec{c}$



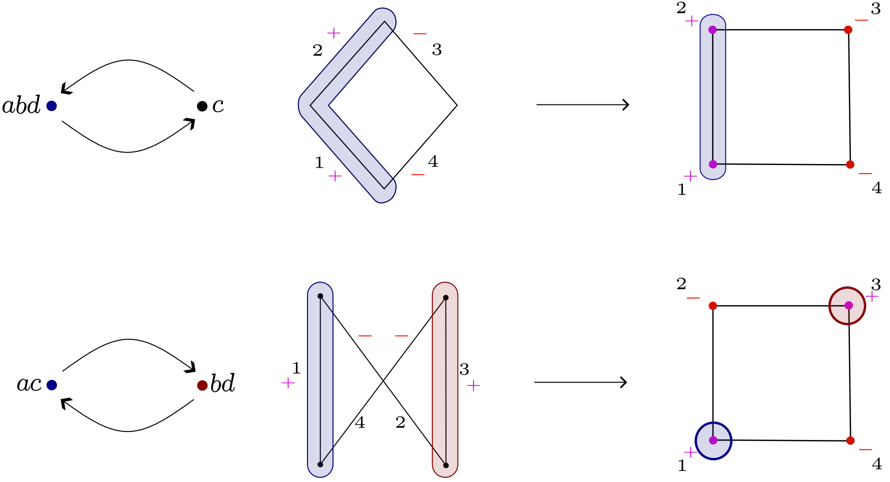
Negative part: $\vec{c}_- := A(\vec{c}) \setminus A(H_P)$

$\rightarrow$ arcs that have opposite directions in H_P and in $\vec{c}$

Characterisation of forbidden tubings

Proposition

Let T be a tubing of L_P. Then T is forbidden if and only if there exist a subset T' of T and an oriented cycle $\vec{c}$ in P such that $\vec{c}_+ \subseteq \bigcup_{\sigma \in T'} \sigma$ and $\vec{c}_- \nsubseteq \bigcup_{\sigma \in T'} \sigma$ (or viceversa).



abd
c
2
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4
ac
bd

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Forbidden by $\vec{c}_+$ $\rightarrow \vec{c}_+ \subseteq \bigcup_{\sigma \in T'} \sigma$ and $\vec{c}_- \nsubseteq \bigcup_{\sigma \in T'} \sigma$

Forbidden by $\vec{c}_-$ $\rightarrow \vec{c}_- \subseteq \bigcup_{\sigma \in T'} \sigma$ and $\vec{c}_+ \nsubseteq \bigcup_{\sigma \in T'} \sigma$

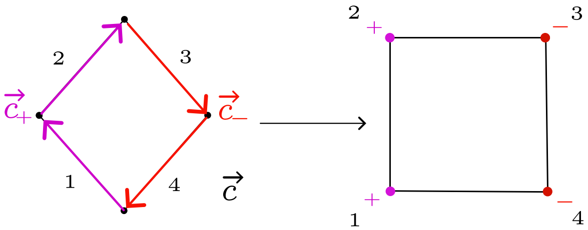
Definition

Let $\vec{c}$ be an oriented cycle in H_P. We define the hyperplane

$$h_{\vec{c}} := \left\{ x \in \mathbb{R}^n \mid \sum_{i \in \vec{c}_+} x_i - \sum_{j \in \vec{c}_-} x_j = 0 \right\}$$

Let $\mathcal{C}_{\mathcal{P}}$ be a basis of the cycle space of H_P. Chosen an orientation $\vec{c}$ for every element c of $\mathcal{C}_{\mathcal{P}}$, we define:

$$\mathcal{S} := \bigcap_{\vec{c} \in \mathcal{C}_{\mathcal{P}}} h_{\vec{c}}$$



$$\mathcal{S} = h_{\vec{c}} = \left\{x \in \mathbb{R}^4 \mid x_1 + x_2 - x_3 - x_4 = 0\right\}$$

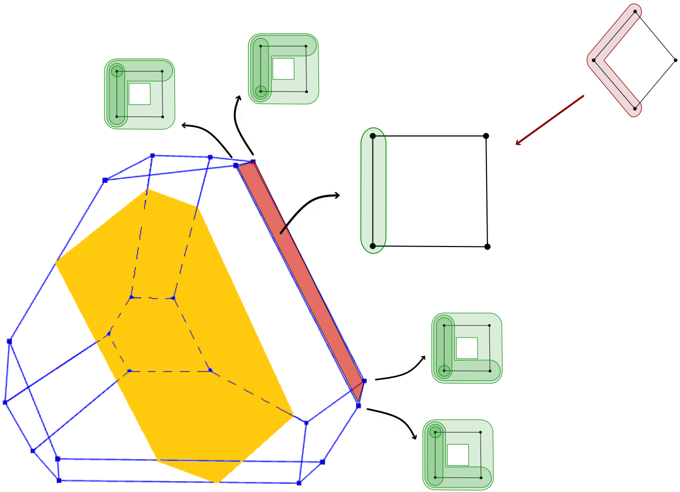
Theorem

Let us consider the realization $\mathcal{P}_{L_P}(\{\lambda_\sigma\})$ of the graph associahedron of L_P, with $\lambda_\sigma := |B_{L_P}|^{|\sigma|}$. Let $\vec{c}$ be an oriented cycle in H_P and let T be a tubing of L_P forbidden by $\vec{c}_+$. Let f_T be the face of $\mathcal{P}_{L_P}(\{\lambda_\sigma\})$ corresponding to the tubing T.

Then $h_{\vec{c}} \cap f_T = \emptyset$.

More precisely, we have $\sum_{i \in \vec{c}_+} x_i - \sum_{j \in \vec{c}_-} x_j < 0$ for every point $x = (x_1, \ldots, x_n) \in f_T$.

Idea of the proof



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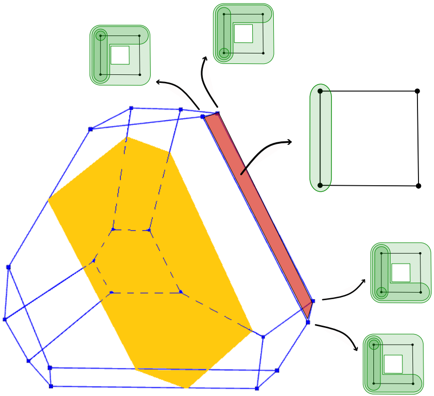

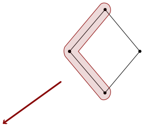

face $f_T \leftrightarrow$ forbidden tubing T

vertices $\leftrightarrow$ maximal tubings that contain T

Idea of the proof

T forbidden by $\vec{c}_+$, T_M maximal tubing that contains T

- $C_i = \{\sigma \in B_{L_P} \mid i \in \sigma \subseteq \tau_i\}$
- $v_{T_M} = (t_1, \ldots, t_n)$, with $t_i = \sum_{\sigma \in C_i} \lambda_\sigma$
- $C_i \cap C_j = \emptyset$ if $i \neq j$

$\rightarrow$ we show that v_{T_M} is on the negative side of $h_{\vec{c}}$:

$$\sum_{i \in \vec{c}_+} t_i - \sum_{j \in \vec{c}_-} t_j < 0$$

Idea of the proof

Lemma

There exists a tube $\widehat{\sigma} \in \bigsqcup_{j \in \vec{c}_-} C_j$ such that $|\widehat{\sigma}| > |\sigma|$ for all $\sigma \in \bigsqcup_{i \in \vec{c}_+} C_i$

Idea:

$T' \subseteq T$ minimal such that $\vec{c}_+ \subseteq \bigcup_{\sigma \in T'} \sigma$ and $\vec{c}_- \nsubseteq \bigcup_{\sigma \in T'} \sigma$.

$\rightarrow$ we can take $\widehat{\sigma} := \vec{c} \cup \bigcup_{\tau \in T'} \tau$

Idea of the proof

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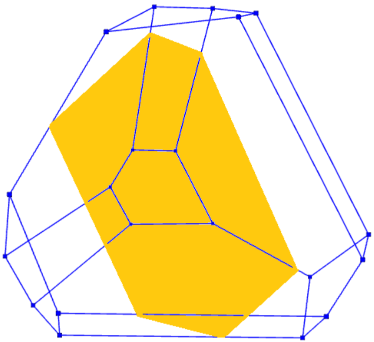
$\rightarrow$ This implies:

$$\sum_{i \in \vec{c}_+} t_i = \sum_{\sigma \in \bigsqcup_{i \in \vec{c}_+} C_i} |B_{L_P}|^{|\sigma|} < |B_{L_P}|^{|\widehat{\sigma}|} < \sum_{\tilde{\sigma} \in \bigsqcup_{j \in \vec{c}_-} C_j} |B_{L_P}|^{|\tilde{\sigma}|} = \sum_{j \in \vec{c}_-} t_j$$

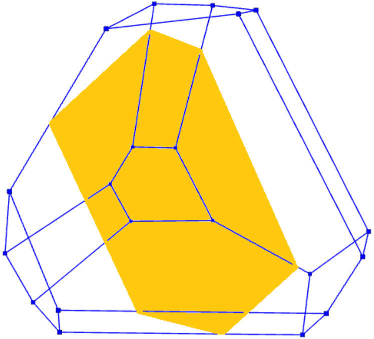
$$\Longrightarrow \sum_{i \in \vec{c}_+} t_i - \sum_{j \in \vec{c}_-} t_j < 0$$



→ forbidden faces are not intersected by $\mathcal{S}$

→ allowed faces are all intersected by $\mathcal{S}$
(topological argument)

→ Face lattice of $\mathcal{P}_{L_P}(\lambda_\sigma) \cap \mathcal{S}$ isomorphic to the lattice of proper tubings of P



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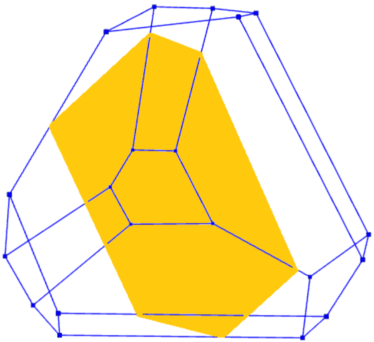

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$\rightarrow$ Face lattice of $\mathcal{P}_{L_P}(\lambda_\sigma) \cap \mathcal{S}$ isomorphic to the lattice of proper tubings of P

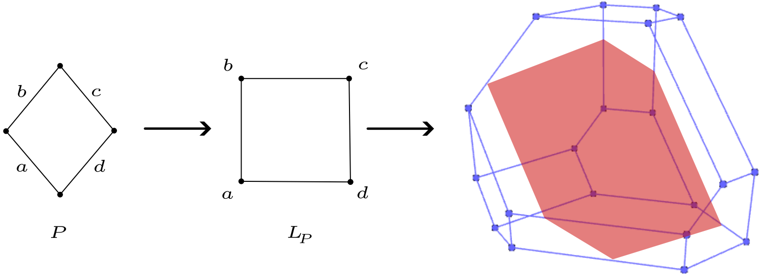
Theorem

Let P be a finite connected poset such that $|P| \geq 2$. Let $\mathcal{P}_{L_P}(\lambda_\sigma)$ be the realization of the graph associahedron of the line graph L_P of P, with parameters $\lambda_\sigma := |B_{L_P}|^{|\sigma|}$.

Then:

$$\mathcal{A}_P := \mathcal{P}_{L_P}(\lambda_\sigma) \cap \mathcal{S}$$

is a realization of the poset associahedron $\mathcal{A}(P)$ of P.



Thanks for your attention!