(q,t)-symmetry in triangular partitions

Loïc Le Mogne, Viviane Pons

Galac, LISN

14/02/2023

Combinatorics and combinatorial families

In combinatorics, we are maily counting families of objects.
What is a family?
A family is a set E and a degree function $f : E \to \mathbb{N}$ such that $f^{-1}(n)$ is finite for any $n \in \mathbb{N}$.

Catalan families

We will call Catalan families all families (E, f) such that for any $n \in \mathbb{N}$,

$$|f^{-1}(n)| = C_n$$

the $n-th$ Catalan number.
One famous example of Catalan family is the family of Dyck paths:

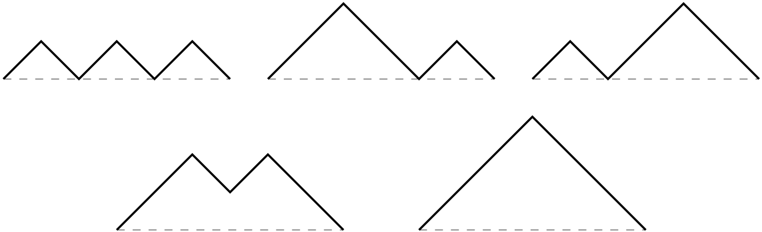


Figure 1: All Dyck paths of length 3

A bit of context

- First step: counting with integers
- Problem: $6 = 2 + 4$? $6 = 2 \times 3$? $6 = 3!$

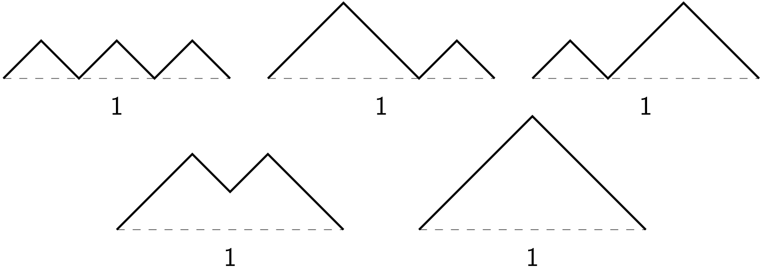


Figure 2: The 5 (3rd Catalan number) Dyck paths of length 3

A bit of context

Second step: counting with polynomials by creating a *statistic*

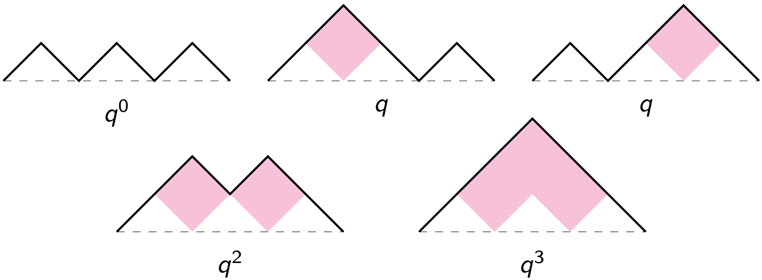


Figure 3: The 5 Dyck paths of length 3, counted by their area

For the figure 3, we get $F_3(q) = 1 + 2q + q^2 + q^3$

A bit of context

Last step: counting with symmetric function

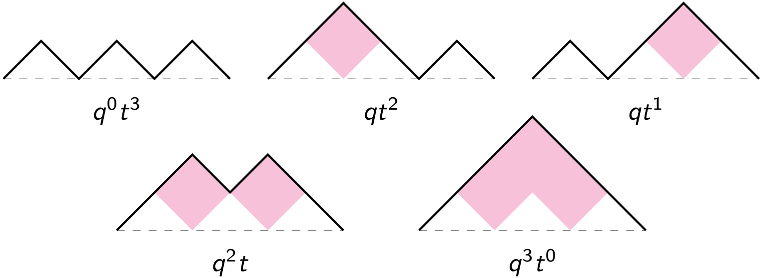


Figure 4: The 5 Dyck paths of length 3, counted with their area and dinv. We get $F_3(q,t) = q^0t^3 + qt^2 + qt^1 + q^2t + q^3t^0 = s_3(q,t) + s_{1,1}(q,t)$

This dinv statistic was introduced by Haiman. The polynomial obtained are proven to be symmetric thanks to representation theory.

(q,t)-Enumeration

- Other statistic was first introduced by James Haglund in works like "*The q,t-Catalan numbers and the space of diagonal harmonics*".
- In fact, this idea of counting with (q,t)-monomial isn't a novel idea, researchers such as Garsia already introduced it some times ago.

The algebraic formula

The formula for the symmetric function of Catalan objects is the following:

$$a_\tau(q,t) = \sum_{\mu \vdash l(v_\tau)} \sum_{\theta \in \mathrm{SYT}(\mu)} \Omega_\mu(\theta) \prod_{(i,j)\mu} (q^i t^j)^{v_\tau(\theta(i,j))}.$$

with $v_\tau = (0, \tau_1 - \tau_2, \dots, \tau_i - \tau_{i+1}, \dots)$, and

$$\Omega(\theta) := \prod_{\theta(a,b) > \theta(i,j)} \frac{(q^a t^b - q^i t^j)^* (q^a t^b - q^{i+1} t^{j+1})^*}{(q^a t^b - q^{i+1} t^j)^* (q^a t^b - q^i t^{j+1})^*} *$$

$$\prod_{\theta(a,b) = \theta(i,j)+1} \frac{q^a t^b}{q^a t^b - q^{i+1} t^{j+1}} \prod_{\theta(i,j) \neq 1} \frac{1}{q^i t^j - 1},$$

with $(A)^*$ equal 1 if A is null, and A otherwise.

Uses of the symmetric function

The symmetric function we get can be used in multiple ways:

- if we put $t = 1$, we get the previous polynomial, it represents the recursive formula of Catalan numbers, obtained with the first return
- if we put $t = 1/q$, we get the direct formula of Catalan numbers $(\frac{1}{2n+1}\binom{2n+1}{n})$

Symmetric functions

A polynomial function p over n variables is defined as symmetric if it is invariant by the action of the symmetric group $\mathfrak{S}_n$ over it's variable.

$$\forall \sigma \in \Sigma_n, p(x_1, ..., x_n) = p(x_{\sigma(1)}, ..., x_{\sigma(n)})$$

Most symmetric functions are seen as functions over infinitely many variables.
An important set of symmetric functions are :

$$h_i(x_1, x_2, ...) = \sum_{k_1 \leq k_2 \leq ... \leq k_i} x_{k_1} x_{k_2} ... x_{k_i}$$

it is the sum of all monomials of total degree i.

Schur fonctions

We define of the Schur function $s_{(}i_1, i_2, ..., i_n)$ with $i_1 \geq i_2 \geq ... \geq i_n$ as the determinant of the following matrix:

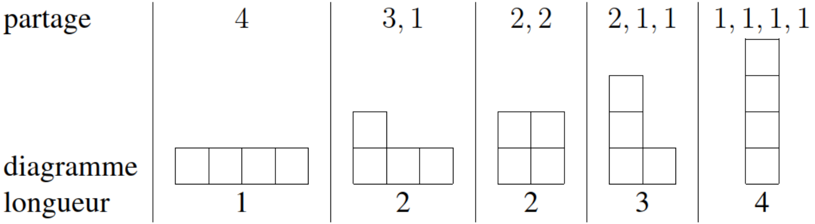
$$\begin{pmatrix} h_{i_1} & h_{i_1+1} & h_{i_1+2} & \cdots & h_{i_1+n-1} \\ h_{i_2-1} & h_{i_2} & h_{i_2+1} & \cdots & h_{i_2+n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ h_{i_n-n+1} & h_{i_n-n+2} & h_{i_n-n+3} & \cdots & h_{i_n} \end{pmatrix}$$

(q,t)-symmetry

- Symmetry between 2 statistics (area and dinv) on objects of a Catalan family
- Proof of this symmetry thanks to representation theory
- No combinatorial proof!

Partition

- A partition of n is a way to represent n as a sum of positive integers (the number of zero being irrelevant)
- We write them as k-uplet $(i_1, i_2, ..., i_k)$ with $i_1 \geq i_2 \geq ... \geq i_k$
- An other representation would be with Ferrer diagrams



Triangular partitions: definition

- A partition is said to be triangular if it is the biggest partition under a line going from $(r, 0)$ to $(0, s)$ with r and s two real numbers

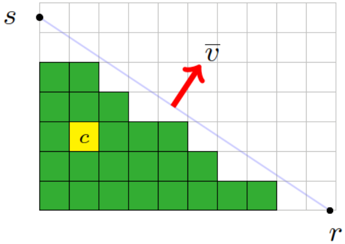


Figure 5: A triangular partition, figure made by François Bergeron

Two key notions

- A *subpartition* μ of a partition λ is a partition such that $\forall i, \mu_i \leq \lambda_i$. We then not $\mu \subset \lambda$
- A triangular partition is said to be a *step-partition* if it's the biggest partition under a line of slope -1

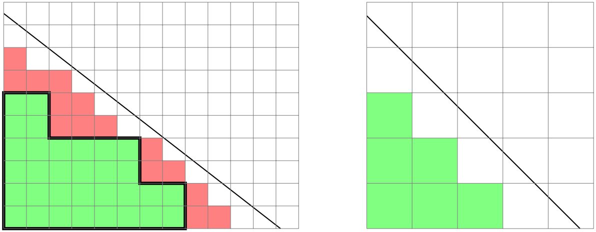


Figure 6: On the left side, a subpartition (green) of the partition in red and green. On the right sidfe, the step-partition of size 4

Frame of work

When we study the symmetry, we fix a partition λ and only study its subpartitions.

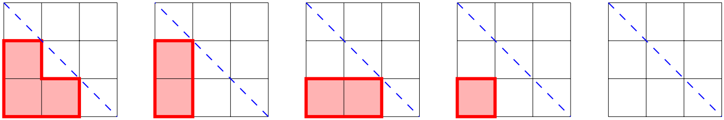


Figure 7: All the subpartitions of (2,1)

Subpartitions of a step-partition

Subpartitions of a step-partition can easily be seen as Dyck paths of the same size than than step-partition.

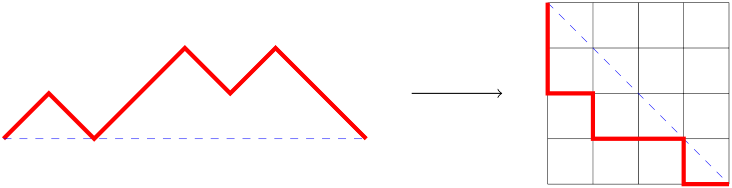


Figure 8: On the left, a Dyck path of size 3, on the right, the subpartition of the step-partition of size 3 associated

The history of generalization of step-partition

Step-partition and Dyck paths have seen many generalization along their study.

- First was m-Catalan where partitions under the line going from $(0, n)$ to $(nm, 0)$ were studied
- Then came *Rational* Catalan, with partitions under a line going from $(r, 0)$ to $(0, s)$ with r and s co-prime
- Finally *Rectangular* Catalan with r and s being any integer

So why triangular partitions?

The next step in this generalization process is naturally to drop the integral condition.
The bigger partition get, the more preponderant non-integral triangular partitions became within the set of all triangular partition:
in size 50, more than 87, 5% triangular partition are non-integral!

New questions coming with triangular partitions

Because it generalizes step-partitions, this bring forth questions such as:

- Can we generalize the dinv and the area on these objects
- Will we still have the symmetry? And more importantly, can it helps us to prove this combinatorially?

Generalization of the statistics

- The area can easily be generalized
- Recent works by François Bergeron, "*Combinatorics of Triangular partitions*", introduced the similarity index, noted *sim*, which is equal to the sim on step-partition

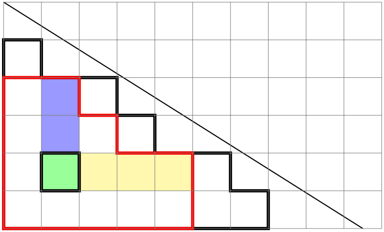


Figure 9: In red, a subpartition. The green cells of said subpartition is *similar* if the average slope of the partition is a slope of the hook (shape formed by the blue and yellow cells).

Example of a (q,t) enumeration on (3,2)

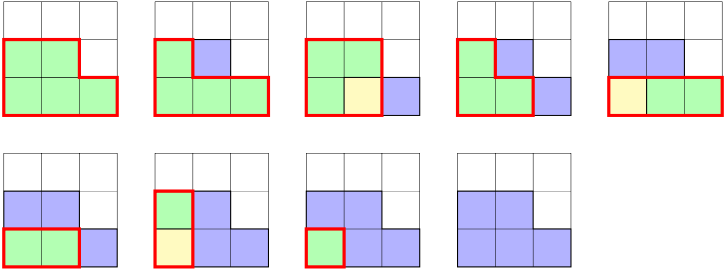


Figure 10: All subpartitions of (3,2) with, in green, the similar cells of the subpartitions, in yellow, cells of the subpartitions that aren't similar and in blue, cells that aren't in the subpartition but are in the fixed partition.

For (3,2), we will get the following polynomial

$$F(q,t) = t^5 + t^4q + t^3q + t^3q^2 + t^2q^2 + t^2q^3 + tq^3 + tq^4 + q^5$$

standard and semi-standard Young tableau

A **Young tableau** is a Ferrer diagram in which each cell is associated with a number. It is said to be **semi-standard** if these numbers are strictly increasing along the column and largely along the row. It is **standard** if the number are increasing strictly along both the columns and the rows and no two cells share the same number.

2	2	3
1	3	4

2	4	4
1	2	2

2	5	6
1	3	4

Figure 11: 3 Young tableaux, the first is not semi-standard, the second is semi-standard only, the last is standard (and as such, also semi-standard).

A new definition of Schur functions

Young tableaux allow us to give a new definition, more combinatorial, of schur functions. For λ a partition, the schur function s_λ is

$$s_\lambda(x_1, x_2, ..., x_n, ...) = \sum_{\theta \in SSYT(\lambda)} \prod_{i \geq 1} x_i^{|\theta|_i}$$

with $SSYT(\lambda)$ the set of all semi-standard Young tableaux on λ and $|\theta|_i$ the number of i in the tableau θ.

Open questions that are a motive for our work

- Finding a combinatorial proof of the sim between the area and the sim/dinv
- Associating a lattice to triangular partitions with some expected properties

Similar sub-partition

Definition *Similar subpartition*: A subpartition in which each cells are similar.
For example in (3,2), the similar subpartitions were:
$(3,2),(3,1),(2,1),(2),(1)and()$.
Proposition: There is exactly one such sub-partition for each size smaller than the original partition.

qt-symmetry in triangular partitions

Results

V. PONS, L.L.M. Creation of the triangular Young tableau: a tool to easily obtained the sim

Created with the similar sub-partitions of the studied triangular partition

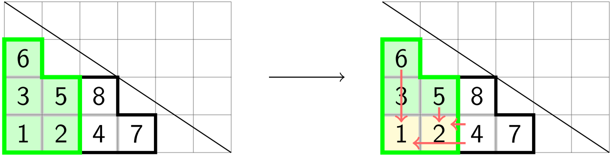


Figure 12: On the left, the partition (4,3,1) with it's triangular Young tableau and it's subpartition (2,2,1). On the right, how to get the sim of (2,2,1) with the tableau: 2 deficit cells, so a sim of 3 and an area of 3.

qt-symmetry in triangular partitions

Results

For a triangular 2-partition $\tau = mn$, the associated symmetric function is :

$$a_\tau(q,t) = \sum_{i=0}^{n} s_{(m+n-2i,i)}$$

- It's a result first conjectured by François Bergeron a few months before
- The proof being purely combinatorial, it gives use a combinatorial proof of the (q,t)-symmetry in 2-partitions.
- it is, as such, an answer to one of the open questions, in an infinity of particular cases.

qt-Symmetry in triangular Young tableau

Results

Characterisation of standard Young tableau giving the expected symmetric function when we compute the sim using said tableau.

We observe the need to alternate between the two parts of the 2-partitions

6	8				
1	2	3	4	5	7

5	7				
1	2	3	4	6	8

4	6				
1	2	3	5	7	8

3	5				
1	2	4	6	7	8

Figure 13: The 4 different symmetric tableau on (6,2).

Definition: poset, lattice

- Un Poset (Partially Orderded Set) est la donnée d'une ensemble E et d'un ordre partiel sur cette ensemble
- Un Treillis est un Poset P tel que:
  Pour tout $i, j \in P$ l'ensemble des éléments plus petits que i et j possède un max et l'ensemble des éléments plus grand un min

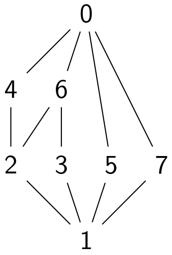


Figure 14: Treillis des entiers de 0 à 7 ordonnés par divisibilité

Definition: Le treillis de Tamari

- Treillis sur les objets d'une famille de Catalan
- Peut être construit à partir de plusieurs de ces objets (arbre binaire, mot de Dyck,...)
- lien avec les fonctions symétriques décrites précédemment (conjectures décrites dans les travaux de Préville-Ratelle)

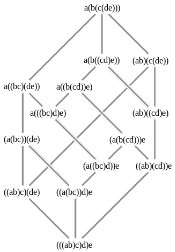


Figure 15: Treillis de Tamari sur les bons parenthésages de taille 4, figure de wikipédia

context: Treillis sur les partages triangulaire

On peut étendre les fonctions symétriques précédentes sur 3 variables.

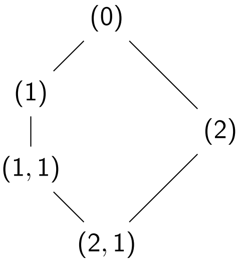


Figure 16: Treillis de Tamari sur les sous-partitions de taille 3

Sur 3 variables, $F_3(q,t,r) = s_3(q,t,r) + s_{1,1}(q,t,r)$. Donc $F_3(1,1,1) = 13$, comme le nombre d'intervalle du treillis de la figure 16.

context: Treillis sur les 2-partages

- Une des généralisation du treillis de Tamari est le treillis Préville-Ratelle Viennot
- Il a en générale des problèmes par rapport à nos attentes, mais pas sur les 2-partages

context: Exemple de treillis sur un 2-partage

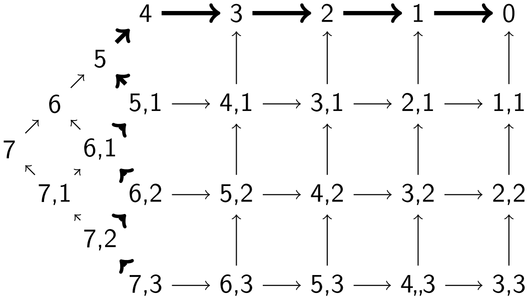


Figure 17: Le treillis de PRV sur (7,3).

On définit le tableau PRV à partir de la chaîne maximale mise en exergue. Ce tableau est bien un tableau symétrique et il permettra de mieux comprendre le treillis.

Exemple de treillis sur un 2-partage

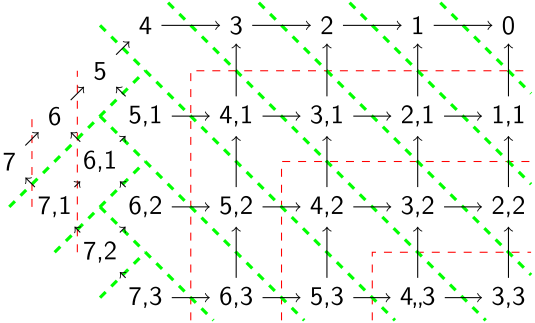


Figure 18: Le treillis de PRV, avec la représentation de certaines caractéristiques.

Symmetry in the lattice of 2-partitions

Results

On a fixed partition $\lambda = (m, n)$*, the distance and the simmax are symmetric on the lattice of* λ*. More precisely, if we consider* $R(\lambda)$ *the set of relations in the lattice of* λ*, we have:*
$\sum_{(\alpha,\beta)\in R(\lambda)} q^{dist(\alpha,\beta)} t^{sim(\beta)} = \sum_{k=0}^{min(n,m-n)} s_{(m+n-2k,k)}(q, t, 1)$

Nature sturmienne des frontières des partages triangulaires

Les partages peuvent aussi être représentés par un "mot-frontière", dont la i-ème lettre correspond à la différence entre la i-ème part et la $i+1$-ème. Les travaux de Lothaire montre que ces mots sont liés à des mots sturmien.

- Un potentiel moyen d'exprimer les fonctions de partages triangulaires grâces aux fonctions des partages escaliers qui la compose
- Décrire un système de réécriture sur les mots-frontières des partages triangulaires

Sur les partages escaliers uniquement

Pour un partage escalier fixé, j'ai obtenu une opération qui fixe l'aire et augmente le déficit de 1. Cette opération permet donc d'obtenir un PoSet sur les sous-partitions d'un partage escalier fixé et d'une aire donnée. Plusieurs questions se posent la dessus:

- Peut-on trouver une opératio symmétrique (qui fixe le sim)?
- Cette opération induit-elle le même poset (à isomorphisme près)?
- Peut-on déduire de ces familles de poset qu'elles sont les fonctions de schur qui apparaissent? (Oui)
- Peut-on le faire facilement?

Merci pour votre attention!

Récapitulatif des contributions: Tableau de Young triangulaire, preuve d'une conjecture de Bergeron, caractérisation des tableaux symétriques, études des treillis de PRV sur les 2-partages et interprétation de la 3ème statistique sur une partie d'une treillis.