Réalisations géométriques du s-permutoèdre

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s-decreasing trees (Ceballos-Pons '20)

Let s be a weak composition (i.e. $s_i \geq 0$).

An *s-decreasing tree* is a planar rooted tree on n internal vertices (called nodes), labeled on $[n]$.

Each node labeled i has $s_i + 1$ children and any descendant j of i satisfies $j < i$.

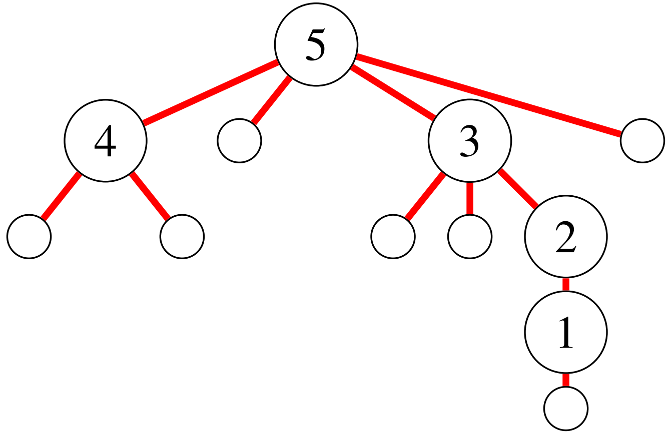


Figure 1: An $(0, 0, 2, 1, 3)$-decrasing tree.

s-weak order

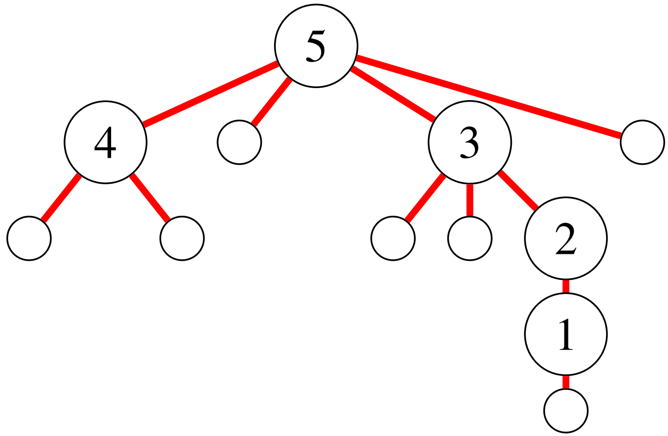
If T is a s-decreasing tree, $\mathrm{inv}(T)$ is the multi-set of *tree-inversions* of T formed by pairs $(y < x)$ with multiplicity (also called cardinality)

$$\#(y,x) = \begin{cases} 0, & \text{if } x \text{ is left of } y, \\ i, & \text{if } x \in T_i^y, \\ s_y, & \text{if } x \text{ is right of } y. \end{cases}$$

If R, T are s-decreasing trees, the *s-weak order* $\trianglelefteq$ is given by $R \trianglelefteq T$ iff $\mathrm{inv}(R) \subseteq \mathrm{inv}(T)$.

Tree inversions

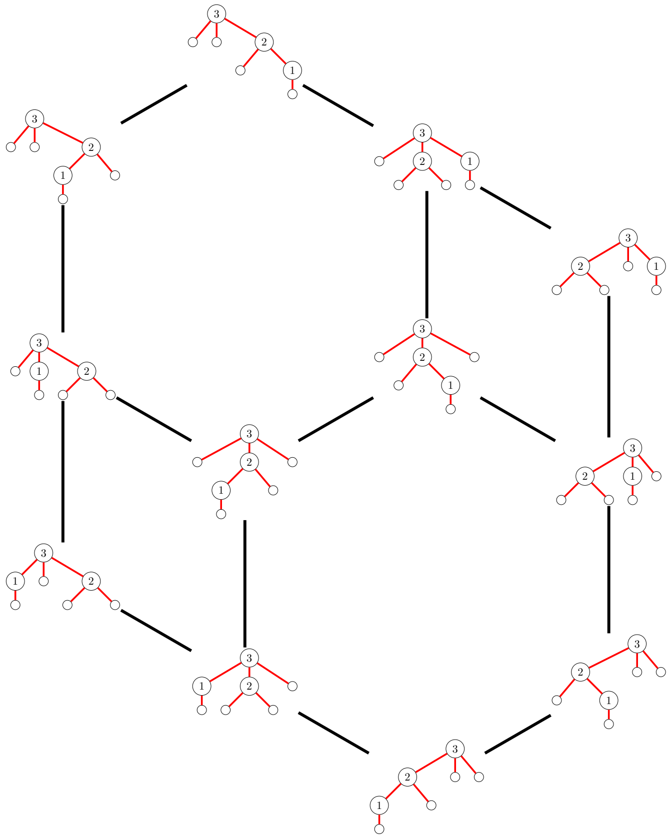
$$\#(y,x) = \begin{cases} 0, & \text{if } x \text{ is left of } y, \\ i, & \text{if } x \in T_i^y, \\ s_y, & \text{if } x \text{ is right of } y. \end{cases}$$



$\#(5,4) = 0$ $\#(5,3) = 2$ $\#(5,2) = 2$ $\#(5,1) = 2$

$\#(4,3) = 1$ $\#(4,2) = 1$ $\#(4,1) = 1$

$\#(3,2) = 2$ $\#(3,1) = 2$ $\#(2,1) = 0$



Credit: Ceballos, Pons.

Figure 2: The lattice of $(0, 1, 2)$-decreasing trees.

s-Stirling permutations

Let s be free of zeroes and T an s-decreasing tree. The *s-Stirling permutation* $\sigma(T)$ (also called *121-avoiding s-permutations*) is the word obtained by taking the in-order T and reading the nodes in each chamber.

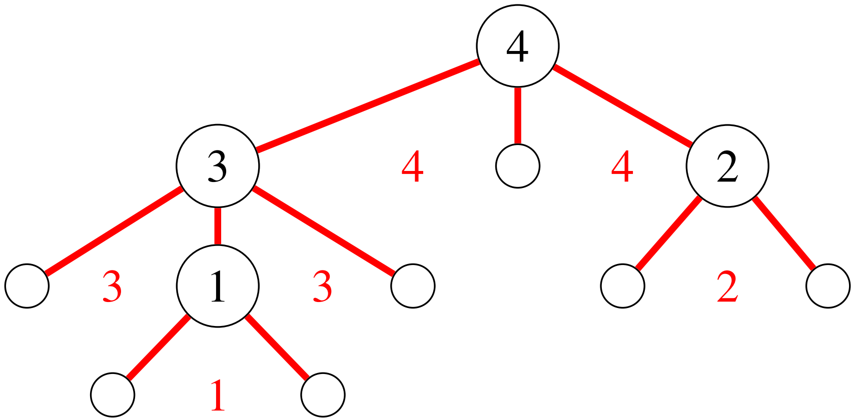
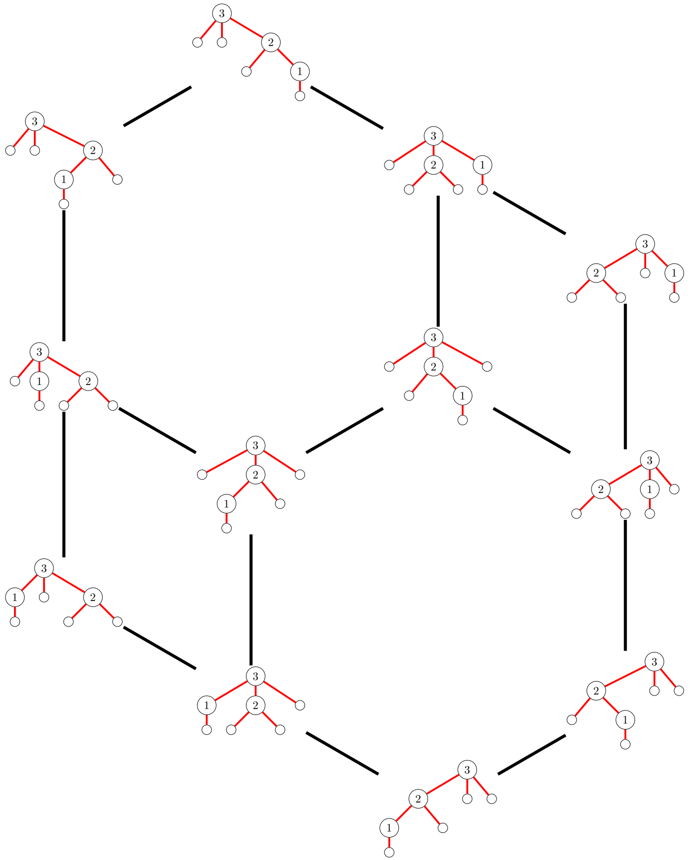
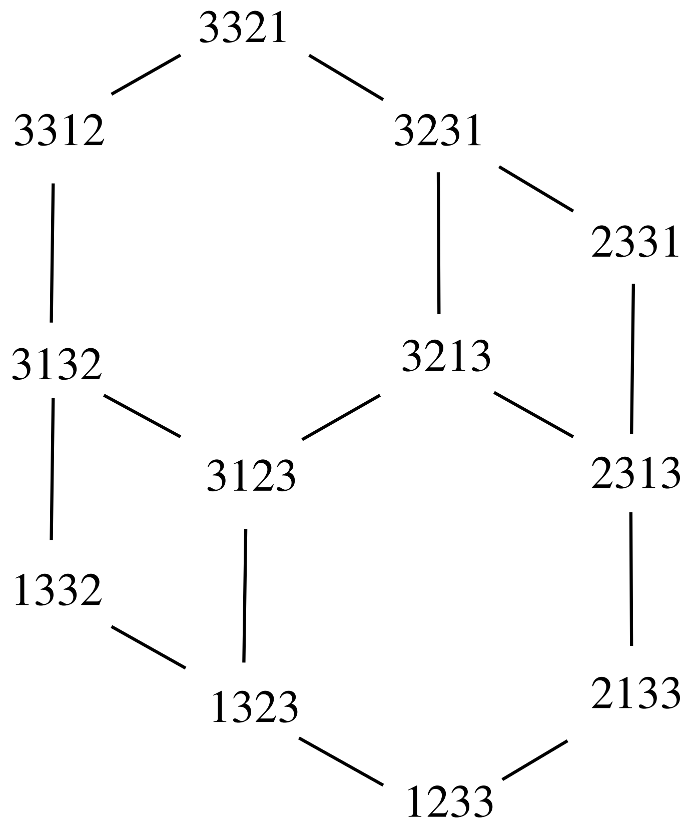


Figure 3: An $(1, 1, 2, 2)$-decreasing tree with $\sigma(T) = 313442$.

s-weak order





Credit: Ceballos, Pons.

Flows

Given a graph G on $[n]$ and vector $\mathbf{a} = (a_0, a_1 \ldots, a_{n-1}, a_n)$ with $\sum_i a_i = 0$, a *flow* of G with *netflow* $\mathbf{a}$ is a vector $(f_e)_{e \in E} \in (\mathbb{R}_{\geq 0})^E$ such that for all inner vertices:

$$\sum_{e \in \mathrm{in}(i)} f_e + a_i = \sum_{e \in \mathrm{out}(i)} f_e.$$

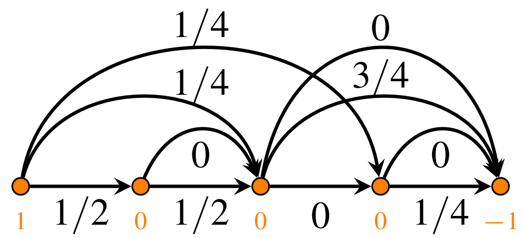


Figure 5: A directed graph with a flow for $\mathbf{a} = (1, 0, 0, 0, -1)$.

If all f_e are integers we get *integer flows*. We denote by $\mathcal{F}_G^{\mathbb{Z}}(\mathbf{a})$ the set of integer flows of G with netflow $\mathbf{a}$.

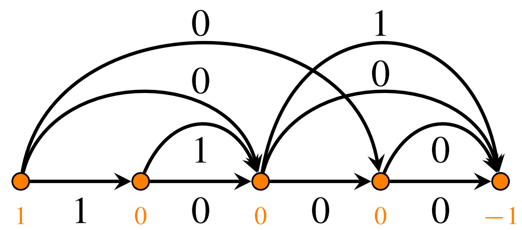


Figure 6: A directed graph with an integer flow.

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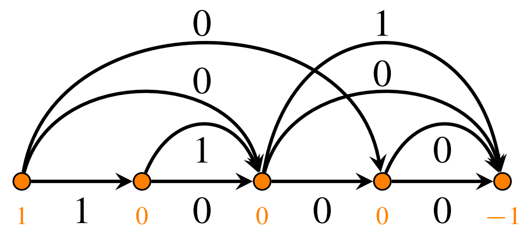


Figure 6: A directed graph with an integer flow.

Theorem (Corollary of the Lidskii formula, Baldoni-Vergne '08, Postnikov-Stanley '00)

The volume of the flow polytope $\mathcal{F}_G(1, 0, \ldots, 0, -1)$ *is equal to the number of integer flows of* $\mathcal{F}_G^{\mathbb{Z}}(\mathbf{d})$ *where* $d_i = \mathrm{indeg}_G(v_i) - 1$.

Connection

Consider the graph G_s with vertex set $\{0, \ldots, n\}$ such that there are

- 2 edges between i and $i+1$ and
- $s_{n+1-i} - 1$ edges between 0 and i.

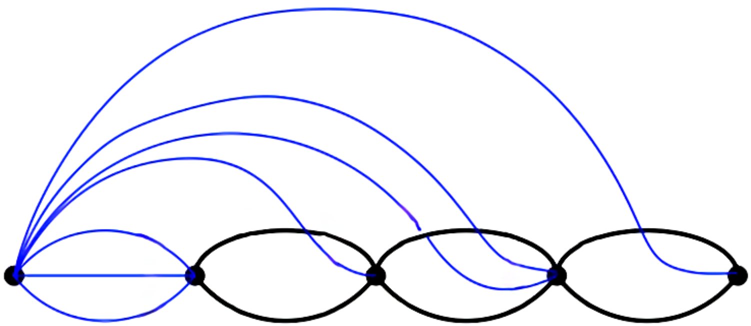
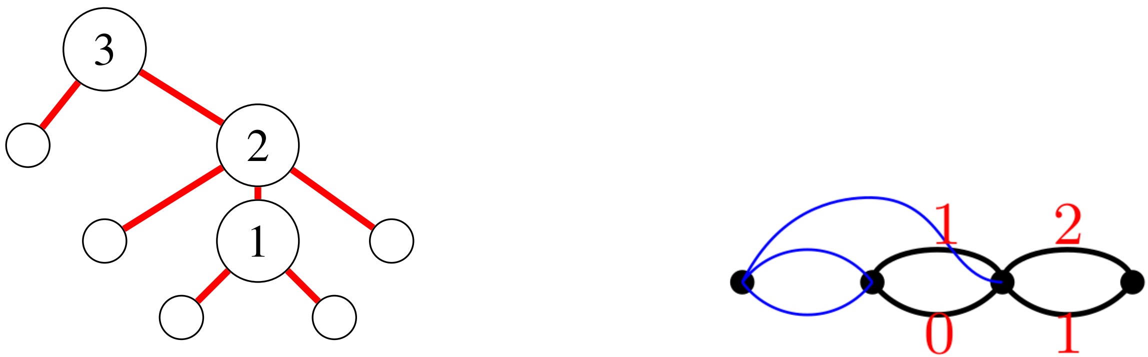


Figure 7: The graph $G_{(2,3,2,2)}$.

Trees and Flows

These graphs allows us to get bijections between *s*-decreasing trees and integer flows as follows:



$$\#(3,2) = 1$$
$$\#(3,1) = 1$$
$$\#(2,1) = 1$$

Figure *: An 121-decreasing tree and an integer flow of $G_{(1,2,1)}$.

Coherent routes

A *framing* of G is a choice of linear orders $\preceq_{\mathcal{I}_i}$ and $\preceq_{\mathcal{O}_i}$ on the sets of incoming and outgoing edges for each inner vertex v_i.

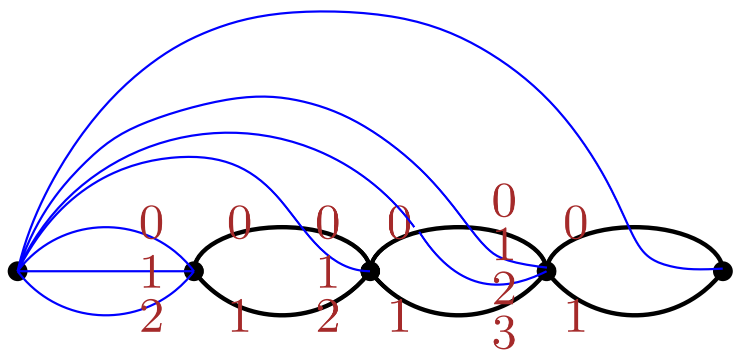


Figure 8: A graph with framing in red.

We say that 2 routes (paths from v_0 to v_n) P, Q of G are *coherent* if for every common vertex v_i, the prefixes Pv_i and Qv_i have the same order as the suffixes v_iP and v_iQ.

Maximal cliques of coherent routes

A maximal set of coherent routes gives us *maximal cliques*.

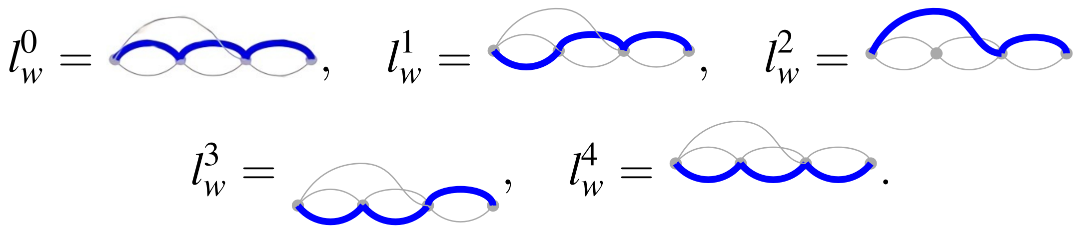


Figure 9: The maximal clique $\{l_w^0, \dots, l_w^4\}$.

Theorem (Mészáros, Morales, Striker, 19')

Given a framed graph $(G, \preceq)$, its maximal cliques are in bijection with the integer flows of $\mathcal{F}_G^{\mathbb{Z}}(\mathbf{d})$ where $d_i = \mathrm{indeg}_G(v_i) - 1$.

Permutations and cliques

Cliques are naturally connected with *s*-Stirling permutations.

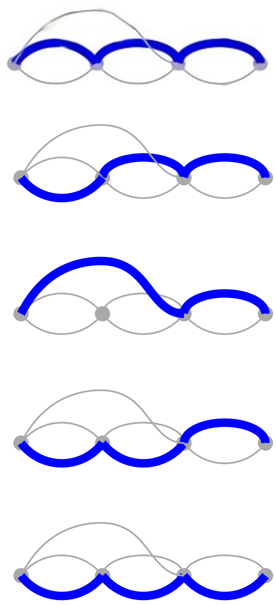


Figure 10: The maximal clique $\{l_w^0, \ldots, l_w^4\}$ corresponding to the $(1,2,1)$-Stirling permutation $w = 3221$.

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Flow polytope

The *flow polytope* of G is

$$\mathcal{F}_G(\mathbf{a}) = \left\{ (f_e)_{e\in E} \text{ flow of } G \text{ with netflow } \mathbf{a} \right\} \subset \mathbb{R}^E.$$

It is a polytope of dimension $|E| - |V| + 1$.
In the case $\mathbf{a} = (1, 0, \ldots, 0, -1)$, the vertices of $\mathcal{F}_G$ are the indicator vectors of the routes of G.

DKK Triangulations (Danilov-Karzanov-Koshevoy, '12)

For $C \in \mathcal{C}^{\max}$ (set of maximal cliques), Δ_C denotes the simplex with vertices the indicator vectors of the routes in C.

Theorem (DKK, 12)

The simplices $\{\Delta_C \mid C \in \mathcal{C}^{\max}(G, \preceq)\}$ form a (regular) triangulation of $\mathcal{F}_G$, called the DKK triangulation of $\mathcal{F}_G$ with respect to the framing $\preceq$.

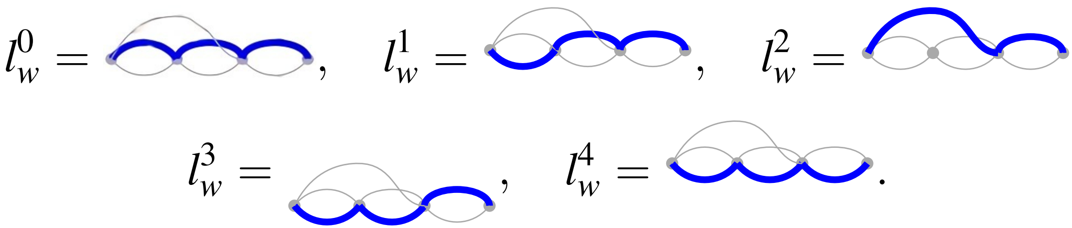

Figure 11: The maximal clique $\{l_w^0, \ldots, l_w^4\}$ corresponding to the $(1, 2, 1)$-Stirling permutation $w = 3221$.

Theorem (GMPTVY, 22')

The s-decreasing trees are in bijection with the simplices of the DKK triangulation of $(\mathcal{F}_{G_s}, \preceq)$.
Moreover, two simplices are adjacent if and only if there is a cover relation in the s-weak order for the corresponding s-decreasing trees.

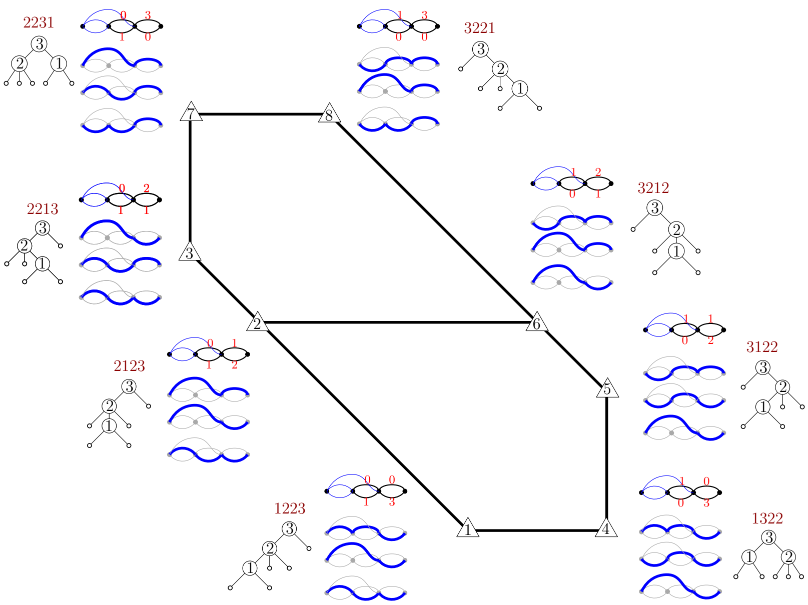

Figure 12: Dual of the DKK triangulation for $s = (1, 2, 1)$.

Minkowski sums

- Given polytopes $P_1, \ldots, P_k$ in $\mathbb{R}^n$, their *Minkowski sum* is the polytope $P_1 + \ldots + P_k := \{\sum x_i \,|\, x_i \in P_i\}$.

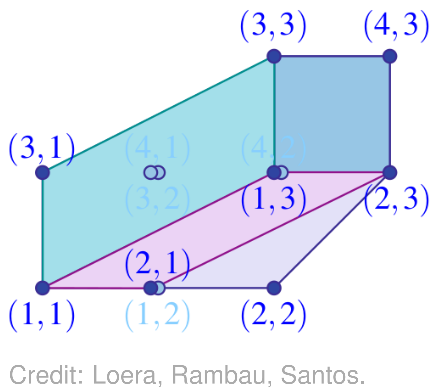

Figure 13: A (non fine) mixed subdivision of a sum of a square and a triangle.

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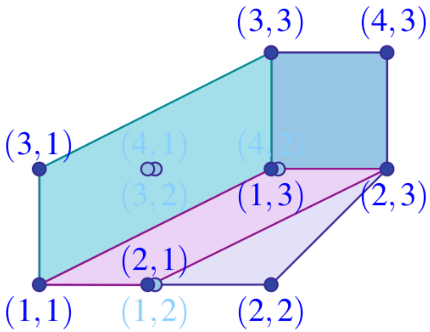
- The *Minkowski cells* of the sum are $\sum B_i$ where B_i is the convex hull of a subset of vertices of P_i.



Credit: Loera, Rambau, Santos.

Figure 13: A (non fine) mixed subdivision of a sum of a square and a triangle.

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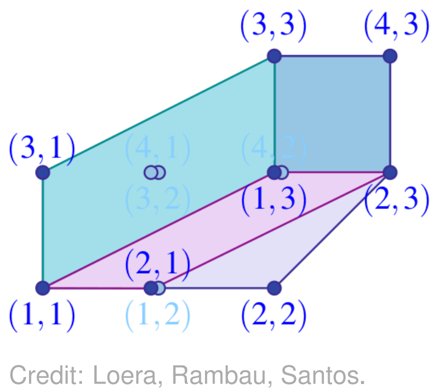
- The *Minkowski cells* of the sum are $\sum B_i$ where B_i is the convex hull of a subset of vertices of P_i.
- A *mixed subdivision* of a Minkowski sum is a collection of Minkowski cells such that their union covers the Minkowski sum and they intersect properly.



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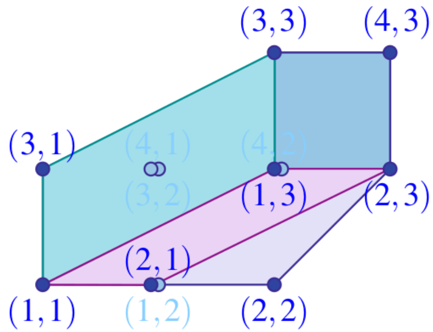
- A *mixed subdivision* of a Minkowski sum is a collection of Minkowski cells such that their union covers the Minkowski sum and they intersect properly.
- A *fine mixed subdivision* is a minimal mixed subdivision via containment.



Credit: Loera, Rambau, Santos.

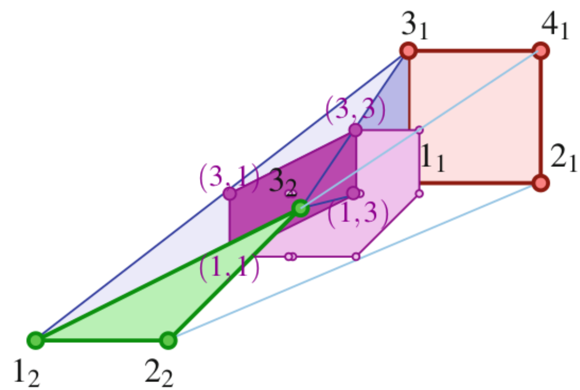
Figure 13: A (non fine) mixed subdivision of a sum of a square and a triangle.

Cayley Trick

$\mathcal{C}(P_1, \ldots, P_k) := \text{conv}(\{e_1\} \times P_1, \ldots, \{e_k\} \times P_k) \subset \mathbb{R}^k \times \mathbb{R}^n$ is the *Cayley embedding* of $P_1, \ldots, P_k$.

Proposition (The Cayley trick)

The (regular) polytopal subdivisions (resp. triangulations) of $\mathcal{C}(P_1, \ldots, P_k)$ are in bijection with the (coherent) mixed subdivisions (resp. fine mixed subdivisions) of $P_1 + \ldots + P_k$.



Credit: Loera, Rambau, Santos.

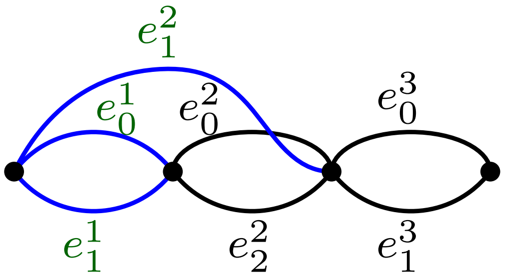
Flow polytopes are Cayley embeddings

Theorem (GMPTVY, 22')

The s-decreasing trees are in bijection with the maximal cells of a fine mixed subdivision of the Minkowski sum of hypercubes in $\mathbb{R}^{n-1}$ given by

$$(s_n + 1)\square_{n-1} + \sum_{i=1}^{n-1}(s_i - 1)\square_{i-1}.$$

Proof : The flow polytope of G_s is a Cayley embedding of hypercubes.



Mixed subdivision of hypercubes

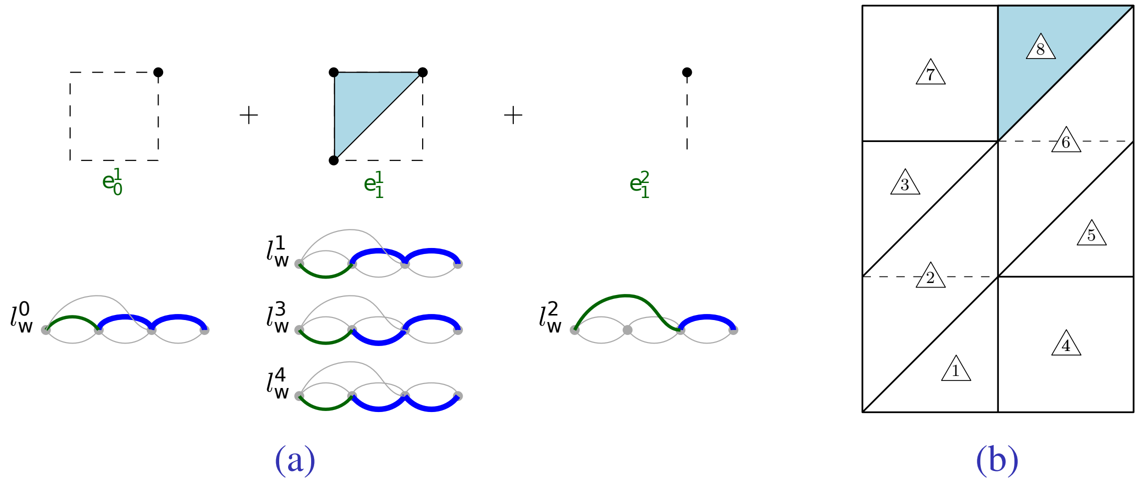
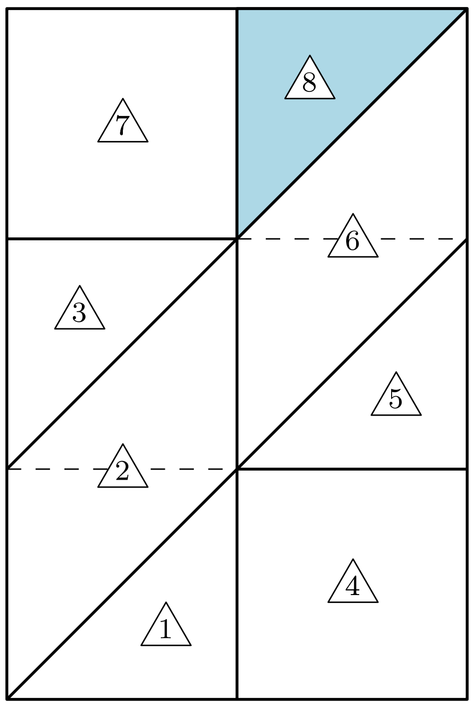
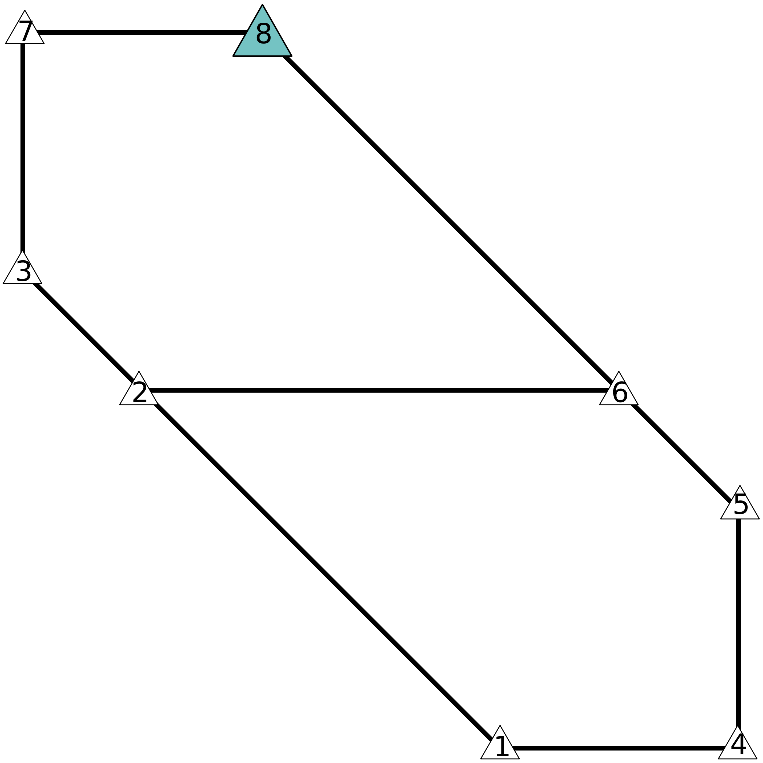


Figure 14: (a) Summands of the Minkowski cell corresponding to $w = 3221$.
(b) Mixed subdivision of $2\square_2 + \square_1$ realizing the $(1, 2, 1)$-permutahedron.

From the mixed subdivision to a dual polyhedral complex

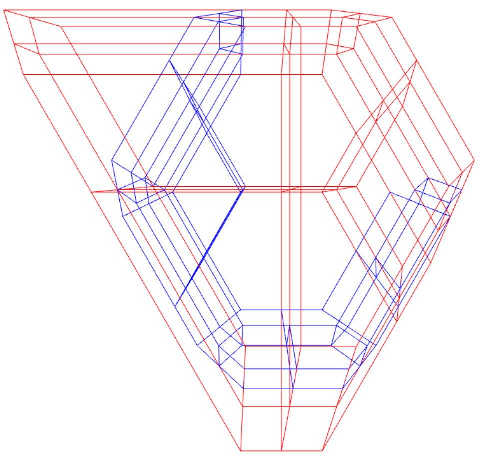




Outline

Future work

- Nicer coordinates?
- Relation with the s-associahedron
- Case where s has zeros.



Credit: Ceballos, Pons.

Figure 15: 0222-permutahedron (blue) and 0222-associahedron (red).

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