Computing the Type Cone of Nestoedra

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Tubes in Graphs

Définition (Tube)

A *tube of a graph G* is a connected proper induced sub-graph of G.

Compatibility of tubes

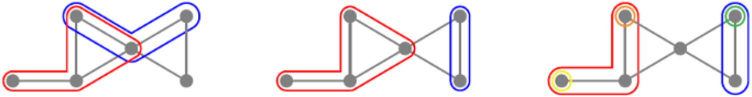
Two tubes t_1 and t_2 are *compatible* when they are :

OR nested (i.e. $t_1 \subsetneq t_2$ or $t_2 \subsetneq t_1$)

OR disjoint ($t_1 \cap t_2 = \emptyset$) and non-adjacent ($t_1 \cup t_2$ is not a tube)

Définition (Tubing)

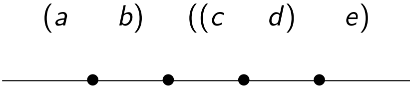
A *tubing* of a graph G is a set of pairwise compatible tubes.



Example

One can encode any Catalan family through tubings of the path.

$(ab)((cd)e)$



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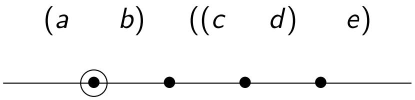

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$(a \quad b) \quad ((c \quad d) \quad e)$

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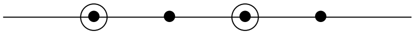

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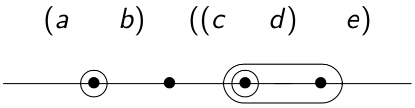

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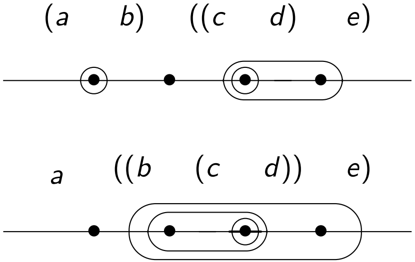

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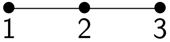
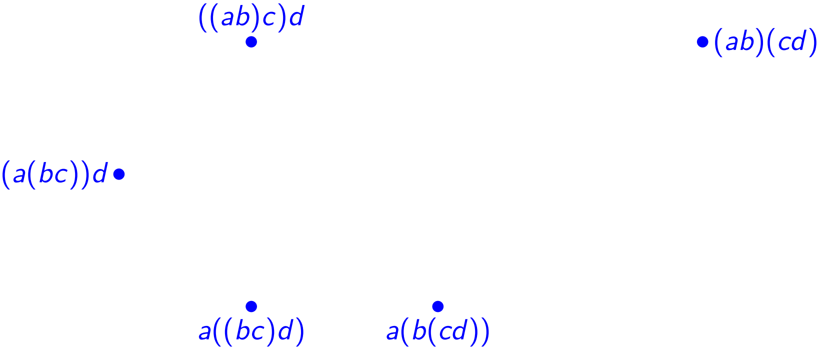
Bracketing on *abcd*, i.e. tubing on the graph 



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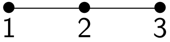
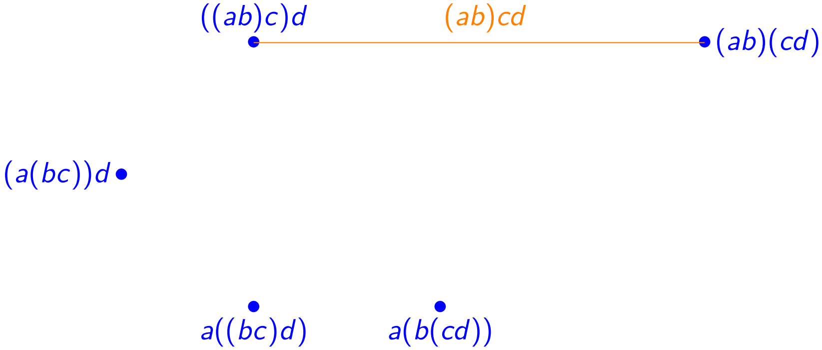
Bracketing on *abcd*, i.e. tubing on the graph 



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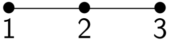
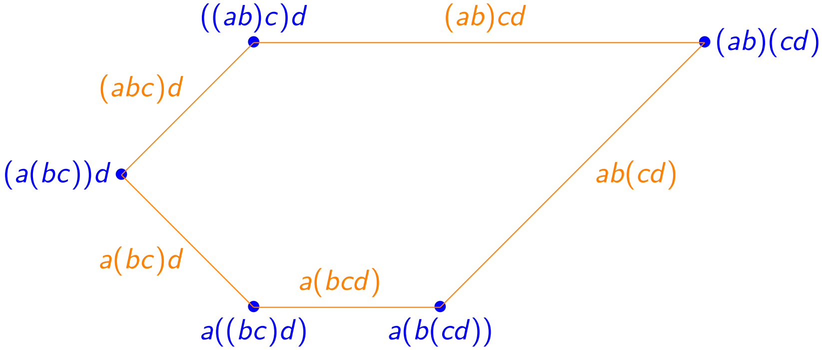
Bracketing on *abcd*, i.e. tubing on the graph 



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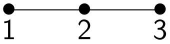
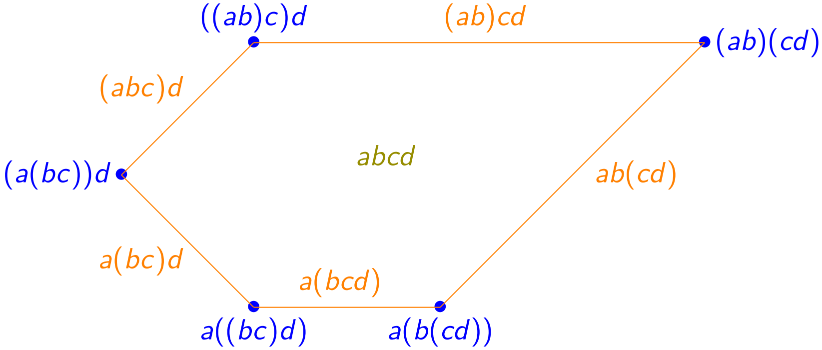
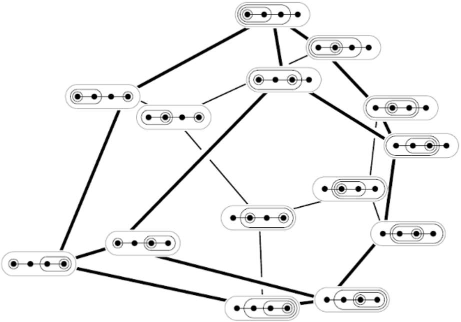
Bracketing on *abcd*, i.e. tubing on the graph 



Example

We can try to construct a polyhedron for a graph on 4 vertices, with :

- Vertices $\leftrightarrow$ 3-tubings
- Edges $\leftrightarrow$ 2-tubings
- Faces $\leftrightarrow$ 1-tubings
- Polyhedron $\leftrightarrow$ 0-tubing



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- Vertices ↔ 3-tubings
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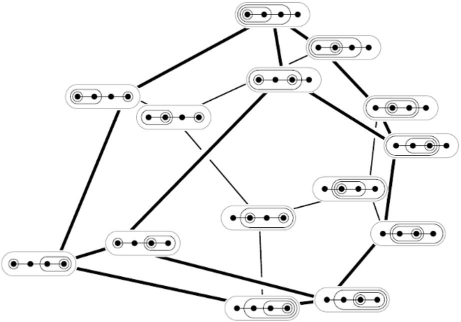
It seems that the structure of lattice of tubings gives rise to a polytope.
Does it generalize ?



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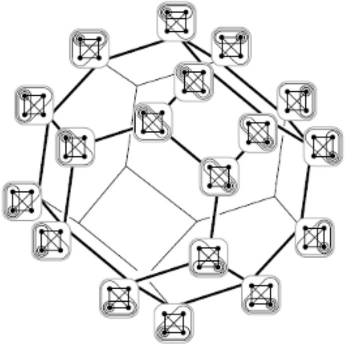
Does it generalize ?

For $\mathcal{K}_n$, tubings are ordered partitions (permutations).

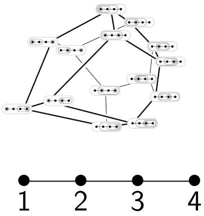


Graph associahedron

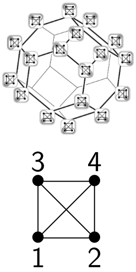
Définition (Graph associahedron)

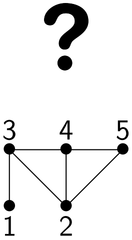
For a graph G, a *G-associahedron* is a polytope which lattice of faces is the (reverse) lattice of tubings of G.

Our problem : Does it exist ?
If yes how to construct it (explicitly) ?



classical associahedron



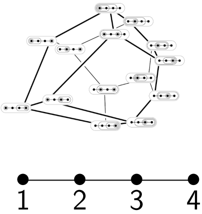


Nestohedron

Définition (Nestohedron)

For an hypergraph H, an *H-nestohedron* is a polytope which lattice of faces is the (reverse) lattice of nested sets of H.

Our problem : Does it exist ?
If yes how to construct it (explicitly) ?



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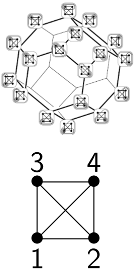
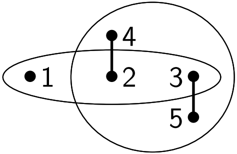


?



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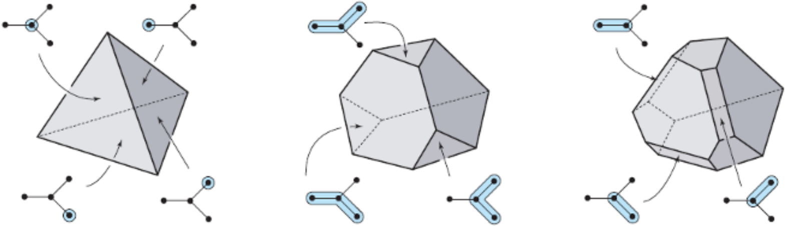
Facets truncations

Carr & Devadoss construction

For a graph G, the G-associahedron exists and we can find an explicit realization.

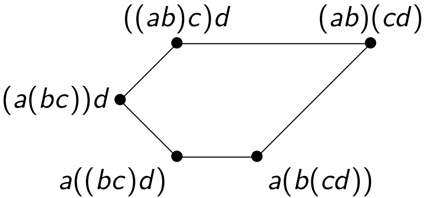


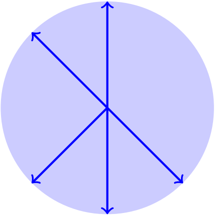
S. Devadoss

Normal fan

Définition (Normal fan)

For a polytope P and a face $F \subseteq P$, let C_F be the cone of all the directions maximized on F.
The *normal fan of P* is the fan formed by the cones C_F.





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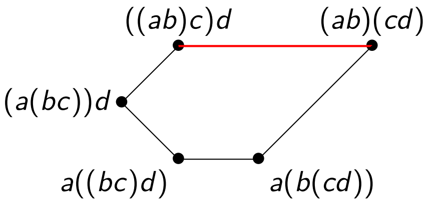

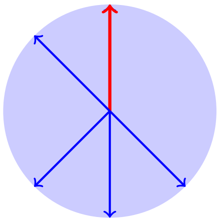

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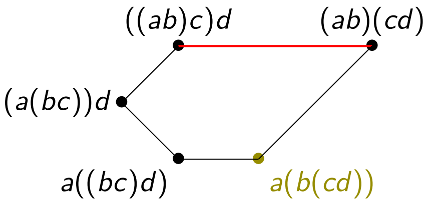
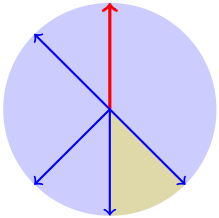
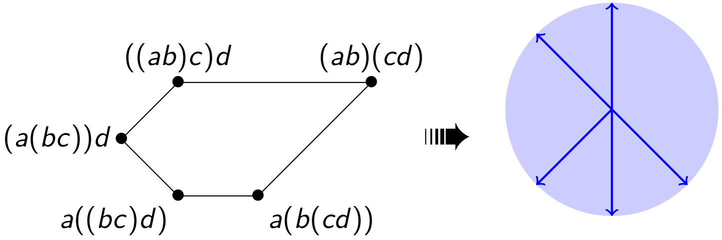




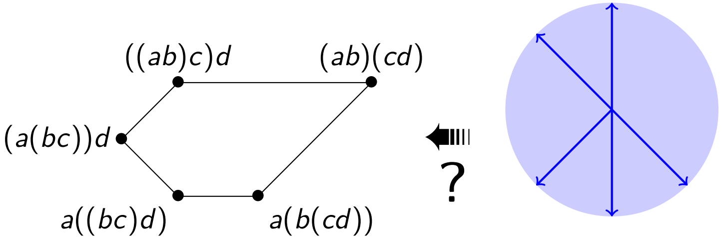
Normal fan

From a polytope, you get a normal fan.



Normal fan

From a polytope, you get a normal fan.
From a fan, you can try to get a polytope (which normal fan is the chosen one).



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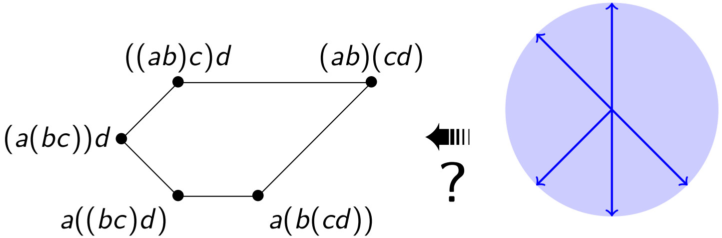

To do this, you need to prescribe *height* on each ray.

Nested fan and height function

Step 1 : Find a suitable fan that encode the lattice of tubings.

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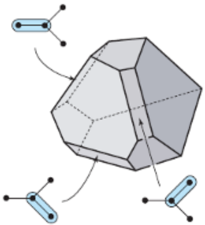
Nested fan

The normal vector associated to a tube t is $\mathbb{1}_t = \sum_{v \in t} e_v$.
The normal fan $\mathcal{F}_G$ is formed by the cones $\text{Cone}\{\mathbb{1}_t \,;\, t \in T\}$ for all tubing T of G.

Height functions

Explicit height functions exist. Carr & Devadoss' height function is given by :

$$h_t = -3^{|t|-2}$$



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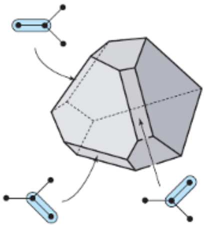
Height functions

Explicit height functions exist :

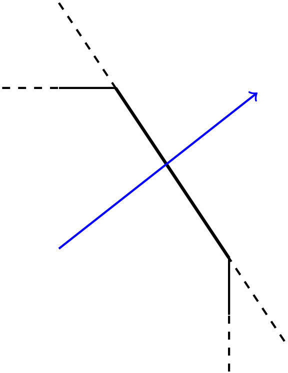
- Carr & Devadoss : $h_t = -3^{|t|-2}$
- Postnikov : $h_t = -|\{s \ ; \ s \subseteq t\}|$

They can be adapted for hypergraphs.



Wall crossing inequalities

To prove a height function works, you need to show it respects *wall crossing inequalities*.



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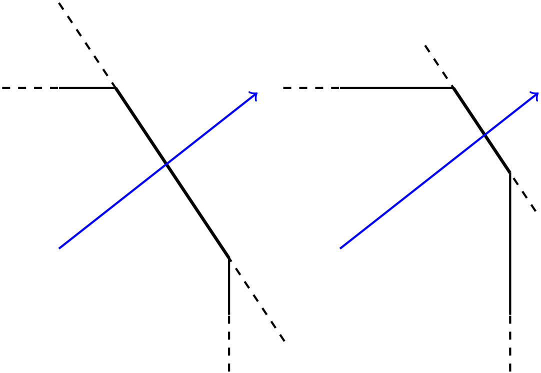

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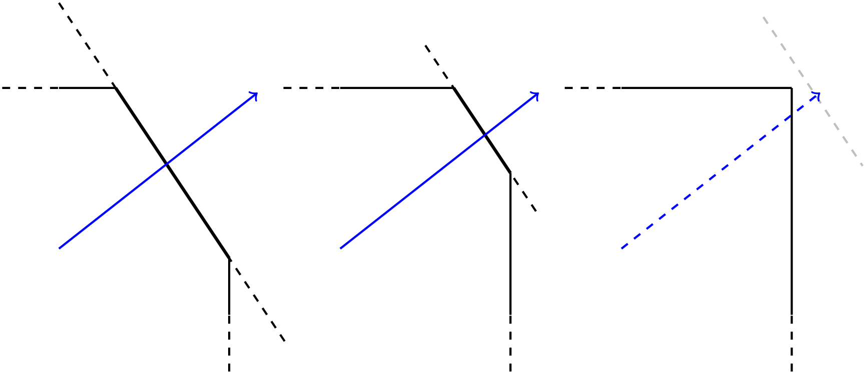

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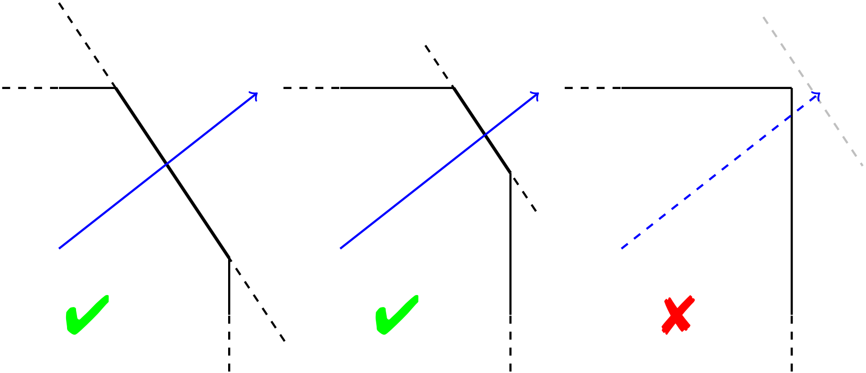
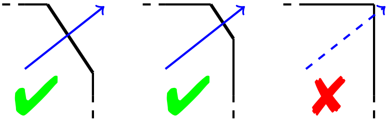


Type cone

Définition (Type cone (McMullen-'73))

Let $\mathcal{F}$ be a fan, with matrix of rays M.
For $\vec{h}$, one notes $P_{\vec{h}} = \left\{\vec{x}\ ;\ M\vec{x} \leq \vec{h}\right\}$.
The *type cone of $\mathcal{F}$* is $\mathbb{TC}(\mathcal{F}) = \left\{\vec{h}\ ;\ P_{\vec{h}} \text{ has for fan } \mathcal{F}\right\}$.



Wall crossing inequalities are linear $\Rightarrow \mathbb{TC}(\mathcal{F})$ is a polyedral cone.

Our problem : For a graph G and its nested fan $\mathcal{F}_G$, what is the type cone $\mathbb{TC}(\mathcal{F}_G)$?

Computing the Type Cone of Nestoedra

Extremal wall crossing inequalities

Problem of extremality

Take this three inequalities :

$$a > c \qquad b > c \qquad a + b > 2c$$

Knowing the first two, the third one is useless : it is *redundant*.

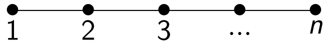
Among wall crossing inequalities, we are looking for non-redundant ones : *extremal wall crossing inequalities*.

Computing extremal wall crossing inequalities will provide the *facet description* of the type cone $\mathbb{TC}(\mathcal{F}_G)$.

Example : associahedron of a path

Associahedron of a path

For $G = P_n$ the path on n vertices, there are $\binom{n+2}{4}$ wall crossing inequalities.
Only $\binom{n}{2}$ are extremal.



Extremal wall crossing inequalities of a path

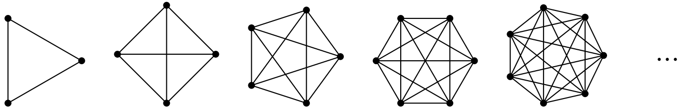
A height vector $\vec{h} \in \mathbb{R}^{\mathcal{T}}$ is in $\mathbb{TC}(\mathcal{F}_{P_n})$ iff for $1 \leq i < j \leq n$:

$$h_{[i,j-1]} + h_{[i+1,j]} > h_{[i,j]} + h_{[i+1,j-1]}$$

Example : associahedron of a complete graph

Associahedron of a complete graph

For $G = \mathcal{K}_n$ the complete graph on n vertices, there are $2^{n-2}\binom{n}{2}$ wall crossing inequalities.
All wall crossing inequalities are extremal.



Extremal wall crossing inequalities of a complete graph

A height vector $\vec{h} \in \mathbb{R}^{\mathcal{T}}$ is in $\mathbb{TC}(\mathcal{F}_{\mathcal{K}})$ iff it is a sub-modular function, i.e. for $R \subsetneq [1, n]$ and $v, v' \notin R$:

$$h_{R\cup\{v\}} + h_{R\cup\{v'\}} > h_R + h_{R\cup\{v,v'\}}$$

Type cone of nestohedra

Type cone of a graph associahedron [Padrol-Pilaud-P.-'21+]

For a graph G, a height vector $\vec{h} \in \mathbb{R}^{\mathcal{T}}$ is in $\mathbb{TC}(\mathcal{F}_G)$ iff for all $s \in \mathcal{T}$ and $v, v' \in s$ non-disconnecting s :

$$h_{s\setminus\{v\}} + h_{s\setminus\{v'\}} > h_s + h_{s\setminus\{v,v'\}}$$

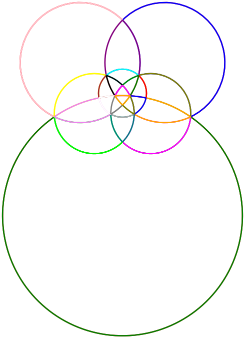
Type cone of a nestohedron [Padrol-Pilaud-P.-'21+]

In general, let $H = (V, E)$ be an hypergraph. Then a height vector $\vec{h} \in \mathbb{R}^E$ is in $\mathbb{TC}(\mathcal{F}_H)$ iff for all $s \in E$ and $a, b \subsetneq s$ maximal in s :

$$h_a + h_b + \sum_{w \in \kappa(s\setminus(a\cup b))} h_w > h_s + \sum_{f \in \kappa(a\cap b)} h_f$$

where $\kappa(e)$ denote the connected components of e in V.

Thank you for your attention !
¡ Gracias por su atención !
Merci pour votre attention !



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