TROPICAL NASH EQUILIBRIA AND COMPLEMENTARITY PROBLEMS

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NASH EQUILIBRIA

Definition. In a game with N players, a *Nash equilibrium* is a tuple of strategies $(\sigma_1^*, \ldots, \sigma_N^*) \in S_1 \times \cdots \times S_N$ such that each strategy is a *best response* to the others:

$$u_i(\sigma_i^*, (\sigma_j^*)_{j \neq i}) \geqslant u_i(\sigma, (\sigma_j^*)_{j \neq i}) \quad \text{for all } \sigma \in S_i \text{ and } i \in [N].$$

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- Nash equilibrium $\equiv (x^*, y^*) \in \Delta_{m-1} \times \Delta_{n-1}$ such that for all $(x, y) \in \Delta_{m-1} \times \Delta_{n-1}$:

$$x^{*\top} A y^* \geqslant x^\top A y^* \qquad x^{*\top} B y^* \geqslant x^{*\top} B y$$

COMPLEXITY OF NASH EQUILIBRIA

Theorem (Chen, Deng, and Teng, 2009; Daskalakis, Goldberg, and Papadimitriou, 2009). *Computing a Nash equilibrium is* PPAD*-complete.*

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The complexity class PPAD (Papadimitriou, 1994) deals with *search problems*, i.e., that are known to have a solution.

Example. Problem EndOfTheLine:

- a (large) oriented graph with in-/out-degree $\leqslant 1$, specified by polynomial-time predecessor/successor functions
- a non-isolated vertex s with no predecessor
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PPAD $\equiv$ set of search problems that reduce to EndOfTheLine in polynomial time

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Hierarchy. PPAD is an intermediate class between FP and FNP.

Remark. FP = FNP if and only if P = NP

Notation. Tropical (max-plus) semifield $\mathbb{T} := \mathbb{R} \cup \{-\infty\}$ where

- $x \oplus y := \max(x, y)$ is the addition
- $x \odot y := x + y$ is the multiplication

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Remarks.

- $-\infty$ and 0 are the zero and unit elements;
- the inverse of x w.r.t. $\odot$ is $x^{\odot(-1)} := -x$;
- operations extend to matrices and vectors, e.g.,

$$x^\top \odot A \odot y = \bigoplus_{i,j} x_i \odot A_{ij} \odot y_j = \max_{i,j}(x_i + A_{ij} + y_j)\,;$$

- $\mathbb{T}$ is ordered by $\leqslant$, and every scalar $z \in \mathbb{T}$ is "nonnegative": $z \geqslant -\infty$.

TROPICAL NASH EQUILIBRIA (2)

Nash equilibria. Pair $(x^*, y^*) \in \Delta_{m-1} \times \Delta_{n-1}$ verifying

$$\forall (x, y) \in \Delta_{m-1} \times \Delta_{n-1}, \quad \begin{cases} x^{*\top} A y^* \geqslant x^\top A y^* \\ x^{*\top} B y^* \geqslant x^{*\top} B y \end{cases}$$

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Remark. W.l.o.g., we can assume that that the payoffs A_{ij}, B_{ij} are all nonnegative:

$$A_{ij} \leftarrow A_{ij} + \alpha \qquad B_{ij} \leftarrow B_{ij} + \beta$$

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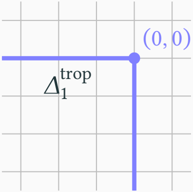
Tropical Nash equilibria should involve "tropical measures" $(x, y) \in \Delta_{m-1}^{\text{trop}} \times \Delta_{n-1}^{\text{trop}}$, where

$$\Delta_p^{\text{trop}} := \left\{ z \in \mathbb{T}^{p+1} : \bigoplus_{k=1}^{p+1} z_k = 0 \right\}$$ ≡ tropical standard p-simplex



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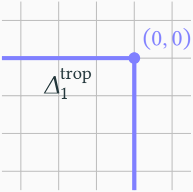

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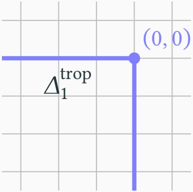

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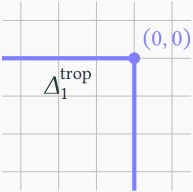
Problem. $(x^*, y^*) = (0,0)$ is always a solution:

$$\forall (x,y) \in \Delta_{m-1}^{\text{trop}} \times \Delta_{n-1}^{\text{trop}}, \quad \begin{cases} \max_{i,j} A_{ij} \geqslant \max_{i,j}(x_i + A_{ij}) \\ \max_{i,j} B_{ij} \geqslant \max_{i,j}(B_{ij} + y_j) \end{cases}$$



TROPICAL NASH EQUILIBRIA — SECOND ATTEMPT

Proposition. $(x^*, y^*) \in \Delta_{m-1} \times \Delta_{n-1}$ *is a Nash equilibrium if and only if*

$$\forall i \in [m], \quad x_i^* > 0 \implies (Ay^*)_i = \max_{k \in [m]} (Ay^*)_k ,$$
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Remark. Compare with

$$\forall (x,y) \in \Delta_{m-1}^{\text{trop}} \times \Delta_{n-1}^{\text{trop}}, \ \begin{cases} x^{*\top} \odot A \odot y^* \geqslant x^\top \odot A \odot y^* \\ x^{*\top} \odot B \odot y^* \geqslant x^{*\top} \odot B \odot y \end{cases} \tag{2}$$

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In a linear program, a face is optimal if and only if all its vertices are optimal

is not a valid statement in the tropical setting!

Set $N := m + n$.

Equivalent formulation. Tropical complementarity problem over $w, z \in \mathbb{T}^N$

$$\begin{cases} w \oplus \begin{bmatrix} -\infty & A \\ B^\top & -\infty \end{bmatrix} \odot z = 0 \\ w^\top \odot z = -\infty \\ z \neq -\infty \end{cases} \tag{3}$$

$z \equiv$ homogenized version of $(x, y) \in \Delta_{m-1}^{\text{trop}} \times \Delta_{n-1}^{\text{trop}}$ + $w \equiv$ slack variables

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normal form of $z = (u, v) \in \mathbb{T}^{m+n}$ $:=$ $(\alpha^{\odot(-1)} \odot u, \beta^{\odot(-1)} \odot v) \in \Delta_{m-1}^{\text{trop}} \times \Delta_{n-1}^{\text{trop}}$

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Tropical Nash Equilibrium Complementary Problem (TNECP).
Find $(w, z) \in \mathbb{T}^N \times \mathbb{T}^N$ such that:

$$\begin{cases} w \oplus M \odot z = q \\ w^\top \odot z = -\infty \\ z \neq -\infty \end{cases} \qquad (M \in \mathbb{T}^{N \times N}, \quad q \in \mathbb{T}^N)$$

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Corollary. *Nash equilibria of bimatrix games satisfying the dominance condition can be found in polynomial time.*

OUTLINE OF THE TALK

Tropical Nash Equilibria in Polynomial Time

Correspondence with Classical Nash Equilibria

Concluding Remarks

TROPICAL NASH EQUILIBRIA IN POLYNOMIAL TIME

TNECP. Find $(w, z) \in \mathbb{T}^N \times \mathbb{T}^N$ verifying:

$$\begin{cases} w \oplus M \odot z = q \\ \quad w^\top \odot z = -\infty \\ \qquad\quad z \neq -\infty \end{cases} \qquad (M \in \mathbb{T}^{N\times N}, \;\; q \in \mathbb{T}^N)$$

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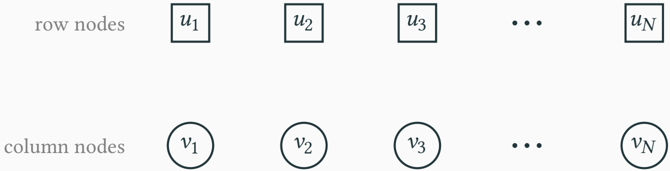
Associated bipartite graph.



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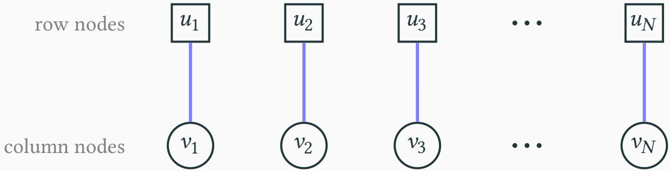

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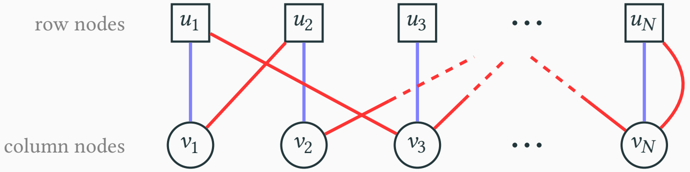

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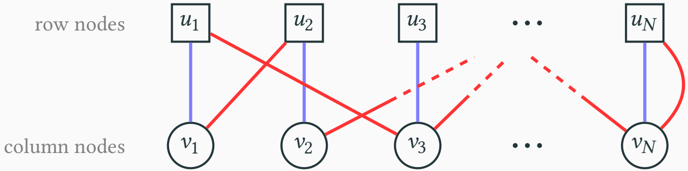

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Fact. 1. Every column node is incident to at least one red edge.

2. The blue edges form a perfect matching.

TROPICAL NASH EQUILIBRIA VS PERFECT MATCHINGS (2)

Given a subset F of edges, we build the point $(w, z) \in \mathbb{T}^N \times \mathbb{T}^N$, where

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Remark. There is a one-to-one correspondence between non-fully blue perfect matchings and tropical Nash equilibria when the instance (M, q) is *nondegenerate*, i.e.,

$$\forall j, \; \min_{k \in [N]} (q_k - M_{kj}) \; \text{ is attained only once,}$$

or, equivalently, every column node is incident to exactly one red edge.

POLYNOMIAL TIME ALGORITHM

Theorem. *TNECP always admits a solution and such a solution can be computed in polynomial time.*

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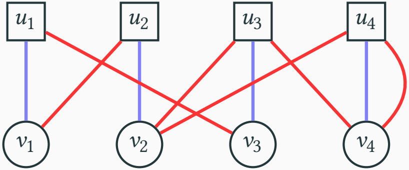
Proof. It suffices to find a non-fully blue perfect matching!



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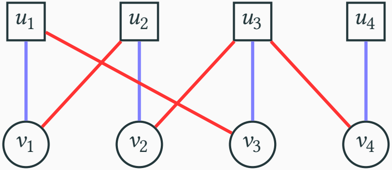

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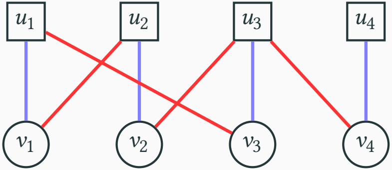

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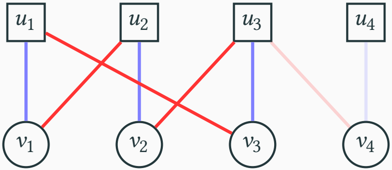

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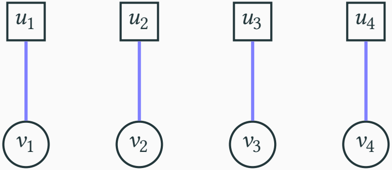

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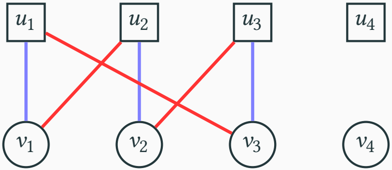

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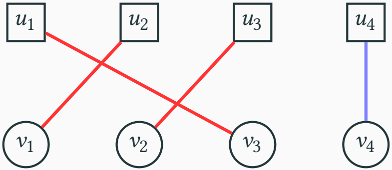

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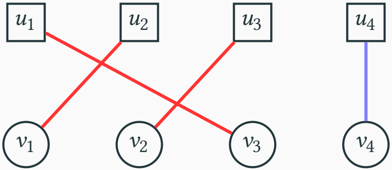

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All the ingredients of the proof (e.g., bipartite graph, cycle C) can be computed in polynomial time.

HOW MANY?

Theorem.

- *The number of solutions of TNECP is at least* $2^{\kappa} - 1$ *where* κ *is the number of connected components of the bipartite graph.*
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Remark.

- in the nondegenerate case, we recover the upper bound conjectured by Quint and Shubik (1997) (later disproved by von Stengel, 1999)
- the result applies to classical Nash equilibria, under the dominance condition

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Simplex-like algorithm, deals with (classical) NECP: find $(w, z) \in \mathbb{R}^N_{\geqslant 0} \times \mathbb{R}^N_{\geqslant 0}$ such that

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```
B ← {1, ..., N};  γ ← 1
loop
    B′ ← unique basis ⊂ B ∪ {γ} and distinct B
    if B′ is fully labeled then STOP
    γ ← twin of the unique element in B \ B′
    B ← B′
end loop
```

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For a tropical nondegenerate instance,

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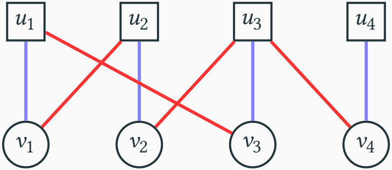

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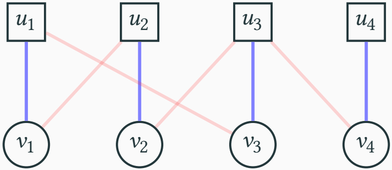

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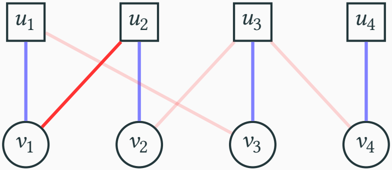

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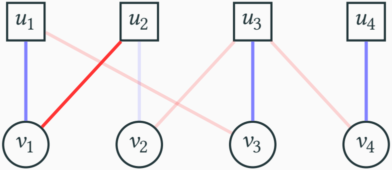

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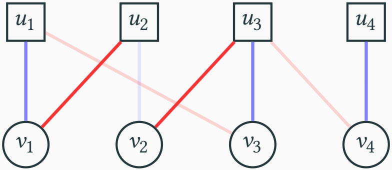

LEMKE–HOWSON ALGORITHM RELOADED

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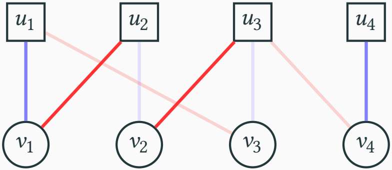

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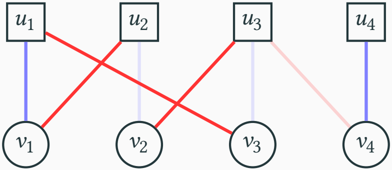

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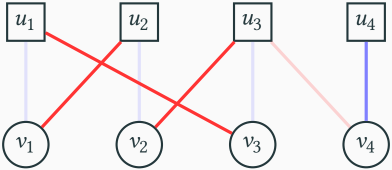

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Proof (cont'd). Two possible cases:

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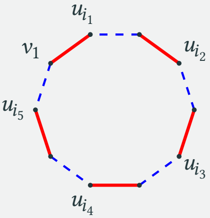

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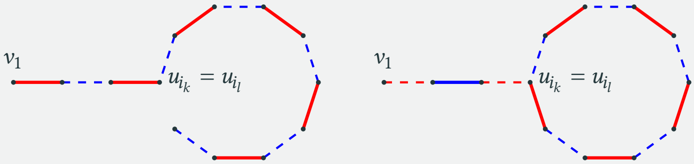

CORRESPONDENCE WITH CLASSICAL NASH EQUILIBRIA

We consider a linear feasibility system

$$A \odot x = b\,, \quad x \in \mathbb{T}^n$$

where $A \in \mathbb{T}^{m \times n}$ and $b \in \mathbb{T}^m$.

Definition. A *tropical basis* is a subset $B \subset [n]$ of cardinality m such that there exists a bijection $\phi\colon\ [m] \to B$ satisfying

$$\forall i \in [m]\,,\ \ b_i - A_{i\phi(i)} = \min_k (b_k - A_{k\phi(i)}) < \infty\,. \qquad (\star)$$

The basis is *nondegenerate* when the minimum in $(\star)$ is uniquely attained.

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Tropical bases "behave" like classical (feasible) bases:

Proposition. *Let B be a tropical basis, and $j \notin B$ such that the jth column of A is distinct from $-\infty$. Then:*

- *there exists a basis B' included in $B \cup \{j\}$ and distinct from B;*
- *this basis is unique if the instance is nondegenerate.*

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Comparison with the previous setting.

Tropical bases correspond to matchings covering all row nodes in an unfolded bipartite graph.

$\Longrightarrow$ This covers the subsets of edges considered so far.

DOMINANCE CONDITION

We consider linear feasibility systems

$$Ax = b\,, \quad x \in \mathbb{R}^n_{\geqslant 0}$$

where $A \in \mathbb{R}^{m \times n}$ is nonnegative and $b > 0$.

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Definition. A columnwise normal matrix satisfies the *dominance condition* if:

1. there exists an $n \times n$ submatrix covering a permutation matrix;
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$$\begin{bmatrix} 1 & 0.5 & 0.4 & 1 \\ 0 & 1 & 0.2 & 0.6 \\ 0.6 & 0.3 & 1 & 0.5 \end{bmatrix}$$

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CORRESPONDENCE BETWEEN CLASSICAL AND TROPICAL BASES

Take a classical system satisfying the dominance condition:

$$Ax = e\,,\ x \in \mathbb{R}^n_{\geqslant 0}\,. \tag{4}$$

Theorem. *The feasible bases of* (4) *are the same as the tropical bases of the tropical system*

$$(\log A) \odot x = 0\,, \quad x \in \mathbb{T}^n\,. \tag{5}$$

Moreover, both systems are nondegenerate.

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Proof sketch. Let $B \subset [n]$ be a tropical basis. There exists a permutation matrix P such that

$$A_B = P + F$$

where F is nonnegative and the sum of each row is < 1.

Lemma. *The matrix $P + F$ is nonsingular, and the vector $(P + F)^{-1}e$ has positive entries.*

$\Longrightarrow$ B is a nondegenerate feasible basis of (4).

NECP. Find $(w, z) \in \mathbb{R}^N_{\geqslant 0} \times \mathbb{R}^N_{\geqslant 0}$ such that

$$\begin{cases} w + Mz = q \\ w^\top z = 0 \\ z \neq 0 \end{cases} \qquad \left(M \in \mathbb{R}^{N\times N}_{\geqslant 0},\ q \in \mathbb{R}^N_{>0} \right)$$

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Theorem. *Under the dominance condition, the sequence of bases over which the Lemke–Howson algorithm iterates does not depend on whether it is applied to the classical instance or the tropical one.*

In particular, it solves NECP within at most $2N - 1$ iterations.

COMPLEXITY OF NASH EQUILIBRIA UNDER DOMINANCE CONDITION (2)

In practice, we get a new class of polynomial-time bimatrix games, e.g., In practice, we get a new class of polynomial-time bimatrix games, e.g.,

Corollary. *The computation of Nash equilibria for bimatrix games can be done in polynomial time when the payoff matrices $A, B \in \mathbb{R}^{m\times n}$ are nonnegative and*

(i) *every column of A has an entry that is $m-1$ larger than any other entry in the column;*

(ii) *every row of B has an entry that is $n-1$ larger than any other entry in the row.*

Lemma. *Deciding if a system satisfies the dominance condition can be done in polynomial time.*

CONCLUDING REMARKS

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Summary. Tropical Nash equilibria are both easy and useful:

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- linear bound on the number of LH iterations;
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We have considered tropical Nash equilibria with "nonnegative" matrices:

$$\begin{aligned}
\forall i \in [m], \quad x_i^* > -\infty \implies (A \odot y^*)_i = \max_{k \in [r]} (A \odot y^*)_k \,, \\
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where $A, B \in \mathbb{T}^{m \times n}$

Remark. In the tropical setting, the signed case do not reduce to the unsigned (nonnegative) case:

adding a big constant to the payoffs do not preserve Nash equilibria.

Signed tropical numbers. We duplicate the semifield $\mathbb{T}$ into $\mathbb{T}_\pm := \mathbb{T}_+ \cup \mathbb{T}_-$ where

$$\mathbb{T}_+ := \mathbb{T}\,, \qquad \mathbb{T}_- := \{\ominus a :\ a \in \mathbb{T}\}$$

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TropNash. Given $A, B \in \mathbb{T}_\pm^{m\times n}$, find $(x^*, y^*) \in \Delta_m^{\text{trop}} \times \Delta_n^{\text{trop}}$ such that

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Questions.

- is TropNash in PPAD? is it PPAD-hard?
- complexity of the zero-sum case, i.e., "$A = \ominus B$"? Relation with tropical linear programming?

THANK YOU!

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