Quotients of Uniform Positroids

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Positroids

Problem

Characterize combinatorically positroid flags.

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Outline

1. Matroids
2. Positroids
3. FSD-shifting

Matroids

Definition

A **matroid** M is an ordered pair $([n], \mathcal{B})$ with $\mathcal{B} \subseteq 2^{[n]}$ such that:

(B1) $\mathcal{B} \neq \emptyset$.

(B2) If $B_1 \neq B_2 \in \mathcal{B}$ and $x \in B_1 \setminus B_2$, then there exists $y \in B_2 \setminus B_1$ such that $(B_1 \setminus x) \cup y \in \mathcal{B}$.

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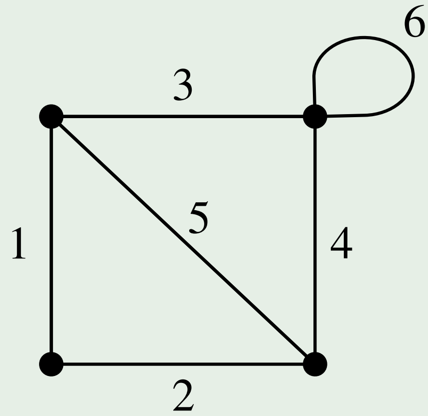
- If not, I is **dependent**.
- If I is a minimal dependent set, I is a **circuit**.

Matroids

Examples



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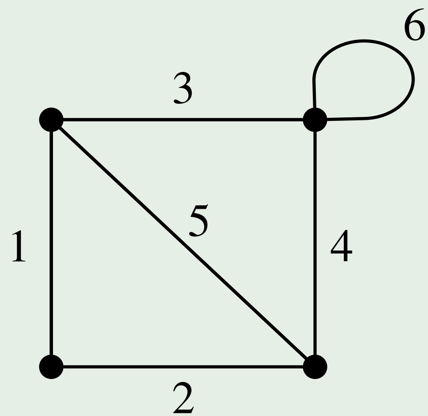

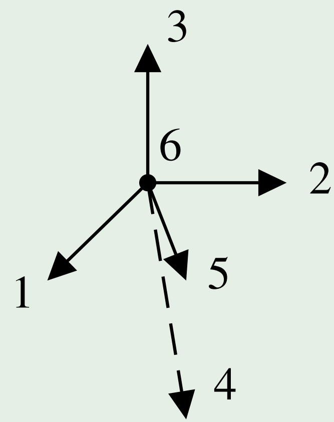

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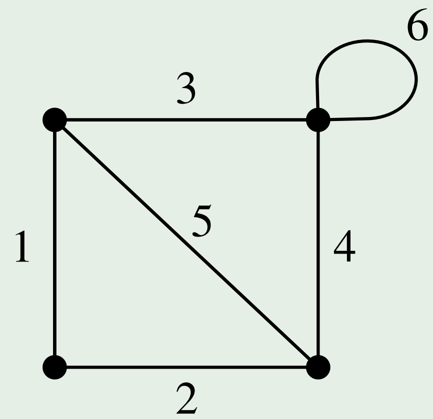

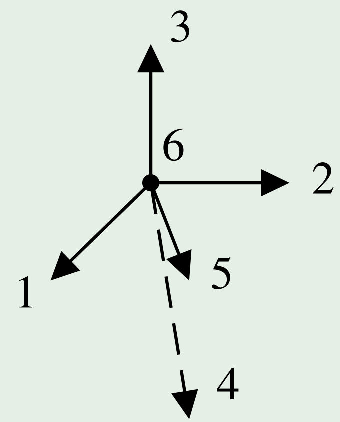

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Matroids

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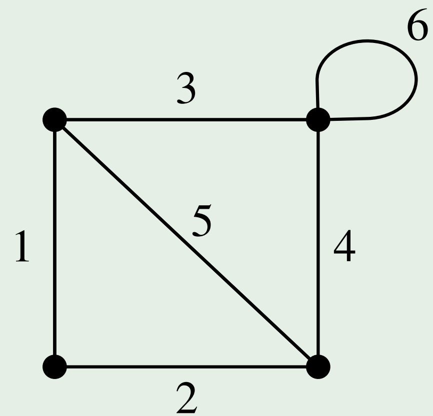

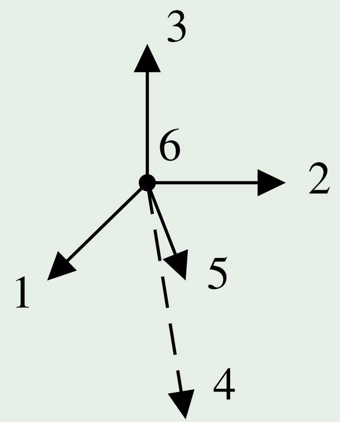

$([6], \{123, 124, 134, 145, 234, 235\})$

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \end{bmatrix}$$

Linear matroids

Theorem [Oxley]

Let $A \in M_{k\times n}(\mathbb{F})$ be a full rank matrix, $\Delta_I(A)$ the minor of size $k \times k$ of A indexed by $I \in \binom{[n]}{k}$ and

$$\mathcal{B}_A = \left\{ I \in \binom{[n]}{k} : \Delta_I(A) \neq 0 \right\}.$$

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Let $\mathbb{F}_p$ be the field with p elements. The matroid $M = ([4], 12, 13, 14, 23, 24, 34)$ is not representable over $\mathbb{F}_2$

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$$A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \end{bmatrix}.$$

$$\begin{array}{lll} \Delta_{12}(A) = 1 & \Delta_{13}(A) = 1 & \Delta_{14}(A) = 2 \\ \Delta_{23}(A) = -1 & \Delta_{24}(A) = -1 & \Delta_{34}(A) = 1 \end{array}$$

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Definition

Let N and M be matroids over $[n]$. N is a **quotient** of M if every circuit of M is the union of circuits of N. In such case N, M is a matroid flag.

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Examples

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- $N = ([5], \{13, 23, 14, 24, 34, 15, 25, 35, 45\})$
  $\mathcal{C}(N) = \{12, 134, 234, 145, 245, 345, 135, 235\}$

Uniform Matroids

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Example

$U_{2,4} = ([4], \{12, 13, 14, 23, 24, 34\})$

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix}.$$

Positroid

Definition

A matroid $M = ([n], \mathcal{B})$ is a **positroid** if there exists a matrix $A \in M_{k\times n}(\mathbb{R})$ such that

- $\Delta_I(A) > 0$ if $I \in \mathcal{B}$,
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$$\Delta_{123}(A) = 1 \quad \Delta_{124}(A) = 0 \quad \Delta_{234}(A) = 1$$

Grassmannians

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Theorem [Postnikov 06,18]

If P is a positroid of rank k and

$$S_P = \{A \in Gr_{k\times n} \,:\, \Delta_I(A) > 0 \text{ iff } I \in \mathcal{B}(P)\}$$

then

$$Gr^{\geq 0}_{k\times n} = \bigsqcup_P S^{\geq 0}_P.$$

Positroids

Decorated Permutations D_n

A **decorated permutation** is a permutation $\sigma \in S_n$ whose fixed points are decorated as $\sigma(i) = \underline{i}$ or $\sigma(i) = \overline{i}$. We say that

- if $\sigma(i) = \underline{i}$, i is a **loop**,
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If $\sigma(i) = \overline{i}$ or $i < \sigma(i)$ we say that i is a **weak excedance** of σ.

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Example

1. $\sigma_1 = 76184235$
2. $\sigma_2 = 24\overline{3}561\underline{7}\overline{8}$

Positroids

Definition

A **Grassmann necklace of type** (k, n) is a sequence $\mathcal{I} = (I_1, \ldots, I_n)$ of subsets $I_i \in \binom{[n]}{k}$ such that for $i \in [n]$,

- if $i \in I_i$, then $I_{i+1} = (I_i \setminus i) \cup j$ for some $j \in [n]$,
- if $i \notin I_i$, then $I_{i+1} = I_i$,

and $I_{n+1} := I_1$.

Positroids

Theorem [Postnikov, 2006] [Oh, 2011]

Positroids on $[n]$ of rank k are in bijection with Grassmann necklaces of type (k,n) and decorated permutations with k weak excedances.

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Example

$$\sigma = 351\underline{4}2\overline{6}$$

$$I_\sigma = (126, 236, 356, 561, 561, 612)$$

$$P_\sigma = ([6], \{126, 136, 236, 156, 256, 356)\})$$

Positroids

Problem

If Q and P are positroids over $[n]$ such that Q is quotient of P, how do their corresponding decorated permutations interact?

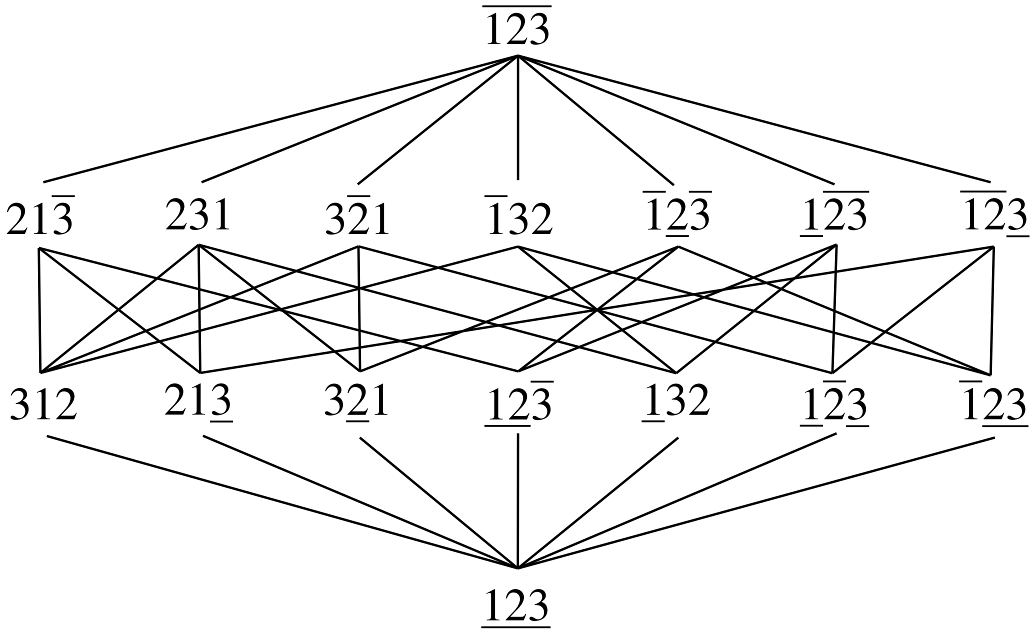

Figure 1: The **poset of positroid quotients** $\mathbb{P}_3$ with decorated permutations.

FDS-Shifting

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Example

Let $\pi = 245631$. Then

$$\overleftarrow{\rho_\emptyset}(\pi) = 456312 \qquad \overleftarrow{\rho_{146}}(\pi) = 54\underline{3}621$$

FDS-Shifting

Teorema [Benedetti, Chavez, T.]

Let $A \subseteq [n]$ with $|A| < k$ and $\sigma = \overleftarrow{\rho_A}(\pi_{k,n})$. Then $rank(P_\sigma) = k - 1$ and P_σ is quotient of $U_{k,n}$.

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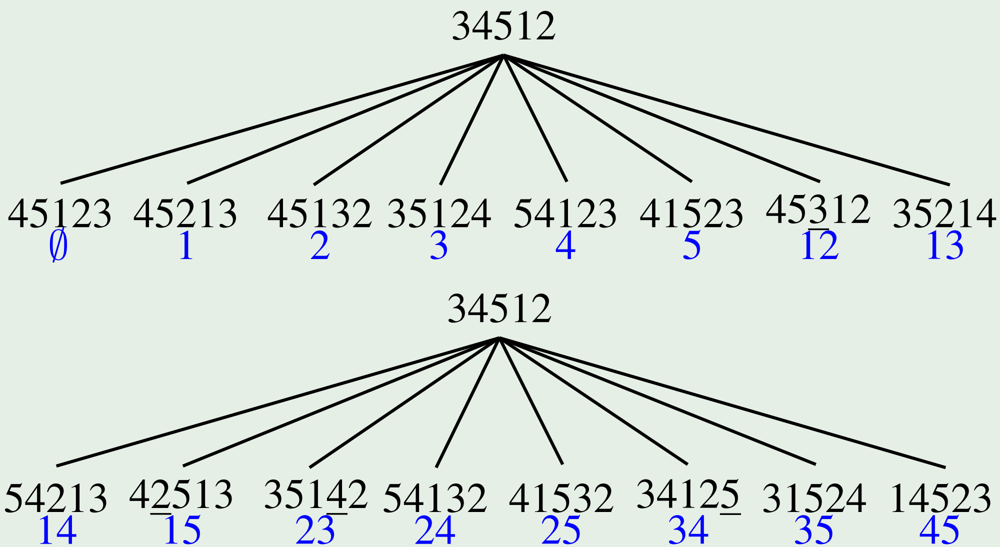
Example k=3 n=5



Let $\pi_{5,7} = 3456712$ y $A = \{1, 4, 7\}$. Then $\sigma := \overleftarrow{\rho_{147}}(\pi_{5,7}) = 5462713$ and the circuits of P_σ are characterized as

$$\mathcal{C}(P_\sigma) = \{[2,4],[5,1]\} \cup \left\{ C \in \binom{[n]}{k} \;:\; [2,4],[5,1] \not\subseteq C \right\}$$

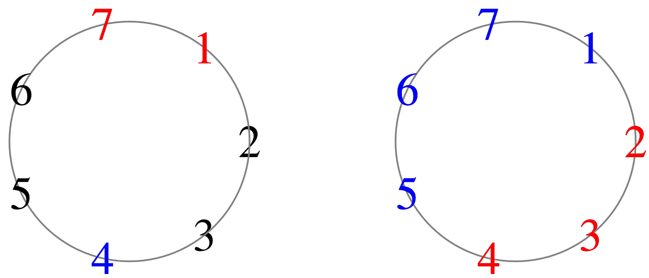


Figure 2: The frozen subset A=$[4] \cup [7,1]$ (left) and the corresponding circuits (right).

FDS-Shifting

n	k	Quotient Positroids	Characterized	Missing
6	3	24	22	2
7	3	36	29	7
	4	71	64	7
8	3	55	37	18
	4	119	93	26
	5	179	163	16
9	3	85	46	39
	4	202	130	72
	5	322	256	66
	6	412	382	30

FDS-Shifting

Conjeture

Let $\pi, \sigma \in D_n$ with respective positroids P_π and P_σ and ranks $k-1$ and k. Then P_σ is quotient of P_π if and only if $\sigma = \overleftarrow{\rho_A}(\pi)$ for some $A \subset [k]$.

FDS-Shifting

Take $\pi_{4,9} = 678912345$ and let $A = \{1,3,5,6,8\} = \{1\} \cup \{3\} \cup [5,6] \cup \{8\}$ such that $\sigma := \rho_A(\pi) = 698214375$.

Calculating the circuits of P_σ gives

$$\begin{aligned}\mathcal{C}_A =& \{2,3,4\} \cup \{4,5,6\} \cup \{7,8\} \cup \{1,2,9\} \\ & \cup \left\{ C \in \binom{[9]}{4} \ : \ [2,4],[4,6],[7,8],[9,1] \not\subset C \right\}.\end{aligned}$$

With SageMath we see that $\sigma \lessdot \pi_{4,9}$.

FDS-Shifting

Let $A = \{2, 4, 7, 8, 9\} = [2] \cup [4] \cup [7, 8, 9]$ and consider the positroid P_σ with decorated permutation $\sigma = 698214375$.

$\tau := \rho_A(\sigma) = \underline{1}98234576$ and the circuits of P_τ are

$$\mathcal{C}_A = \{1\} \cup \{7, 8\} \cup \{2, 9\} \cup \left\{ C \in \binom{[9]}{3} \ : \ [1], [7, 8], \{2, 9\} \not\subset C \right\}.$$

Again, using SageMath we see that $\tau \lessdot \sigma$.

References

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