Concentration in order types of random point sets

arXiv:2003.08456 (prel. version in SoCG 2020)

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Emo Welzl ETH Zürich

DGeCo seminar, December 9th, 2021

Before we start, let me mention open problems
unrelated to this talk.

- ▷ smoothed complexity of 3D Delaunay triangulations
- ▷ topology of lines secant to convex sets
- ▷ sparse inclusion-exclusion formulas

The pitch:

- ▷ a (labeled) order type is a combinatorial object induced by a finite set of points in $\mathbb{R}^2$.
- ▷ I'd like to know to **generate randomly** such an object without **excessive bias**.
- ▷ Here, I'll discuss why sampling random point sets can be **inefficient**.

Order types
& labeled order types

"What is the simplest geometric object about which one can say something interesting? A single point did not quite do it, but a finite set of points—in general position in the plane, say—did: there was an essential difference between four points that formed the vertices of a quadrilateral and three points with a fourth inside their triangle."

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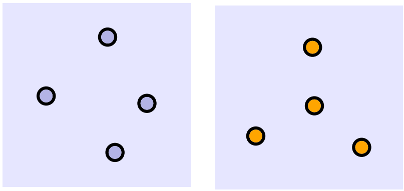

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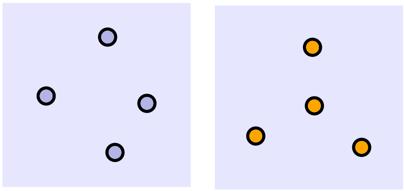

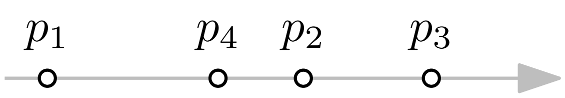

Take a **sequence** $p_1, p_2, \ldots, p_n$ of n points on the **real line** and assume you only have access to the information $\mathbf{p_i} \overset{?}{<} \mathbf{p_j}$.

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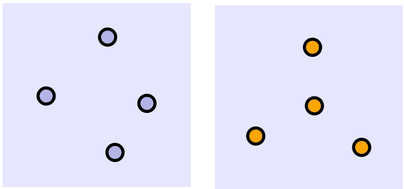

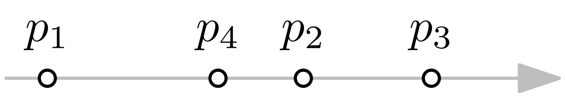

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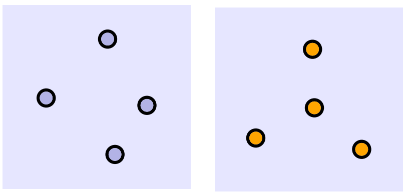

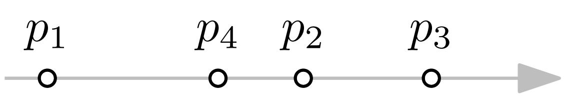

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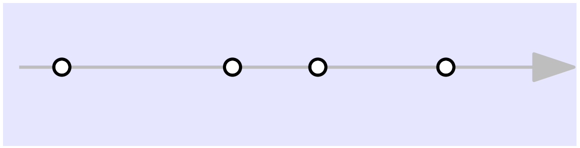

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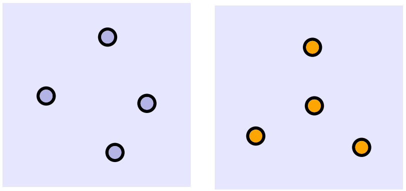

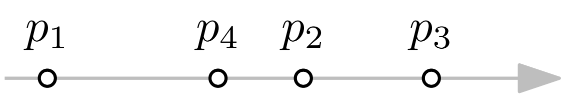

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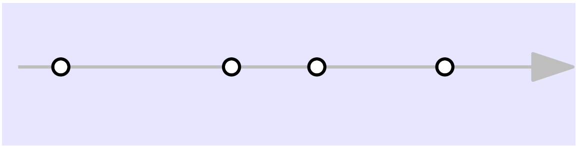

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As of dimension 2, the structure gets **richer** and **more intricate**.

Reduction of the **infinite** diversity of n-point sets in $\mathbb{R}^2$

to **finitely** many combinatorial types based on **orientation predicates**.

[Goodman-Pollack'83]

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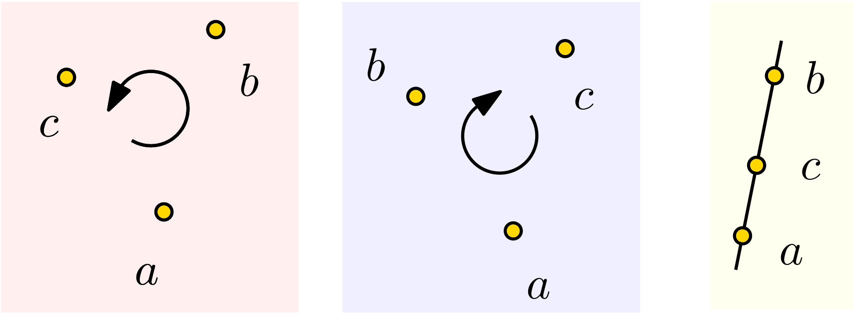

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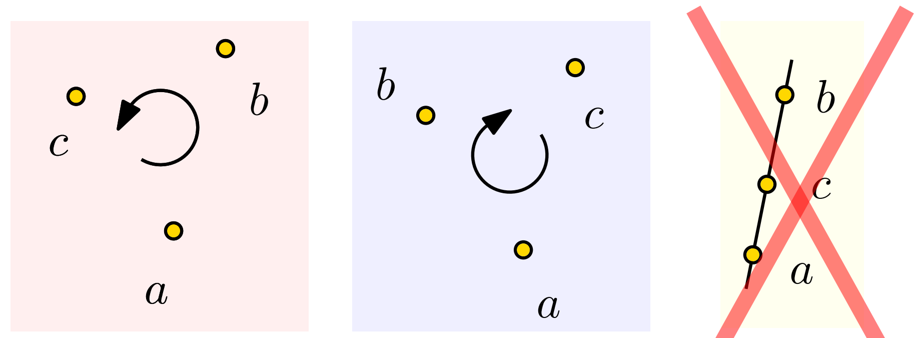

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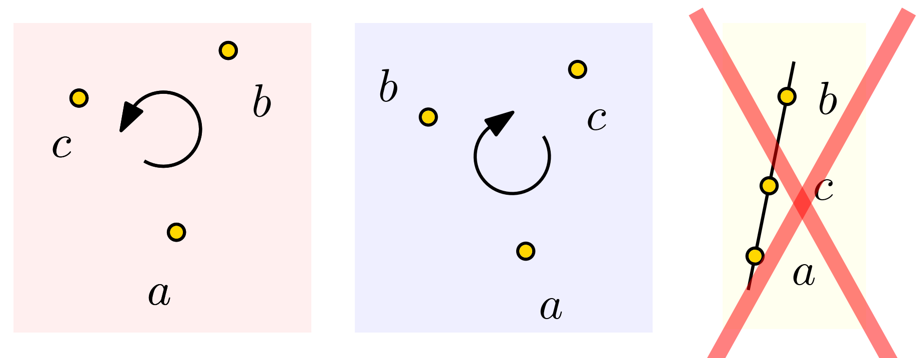

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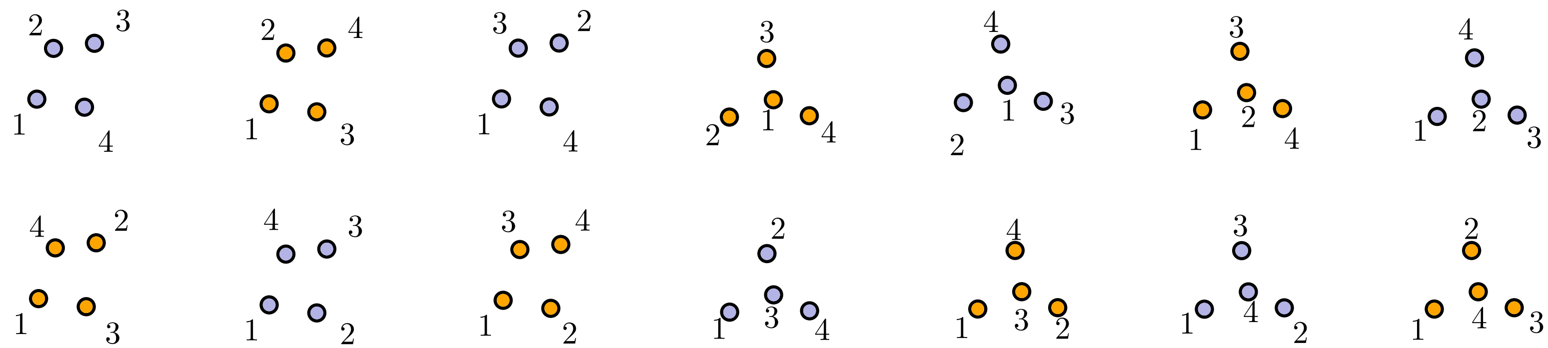
Two point **sequences** $p_1, p_2, \ldots, p_n$ and $q_1, q_2, \ldots, q_n$ such that

$$\forall 1 \leq i, j, k \leq n, \quad \chi(p_i, p_j, p_k) = \chi(q_i, q_j, q_k)$$

have the same **labeled order type**.

Equivalence relation, class $\simeq$ map: triples of indices $\to \{-1, 1\}$.

Example: representatives of the labeled order types of size 4.



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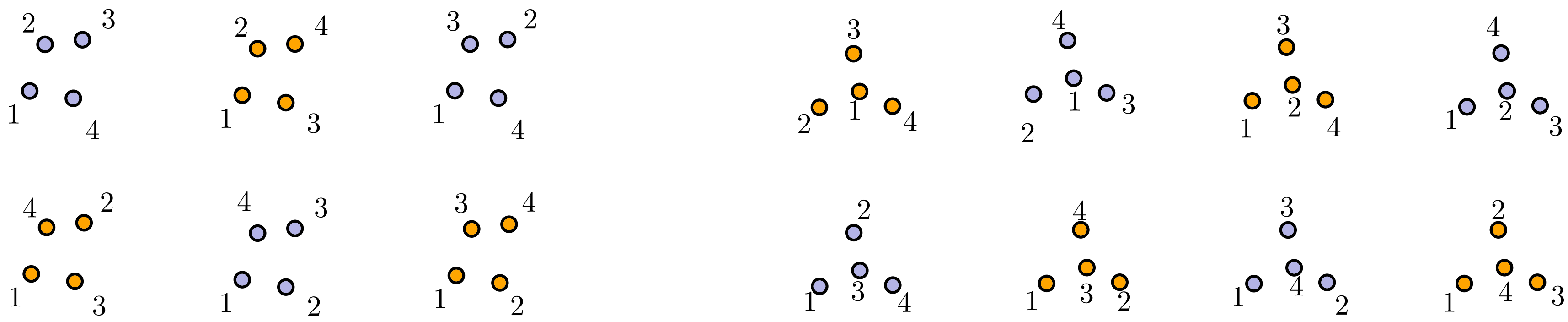

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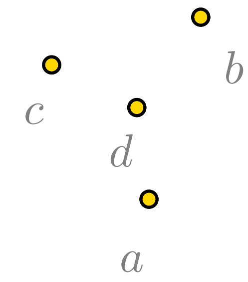

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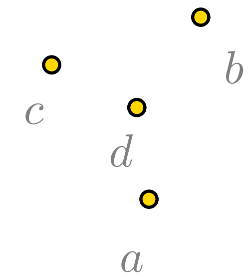

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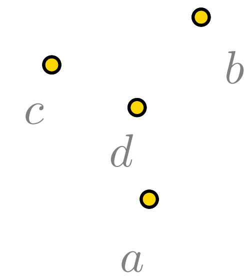

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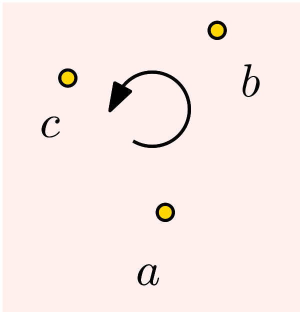

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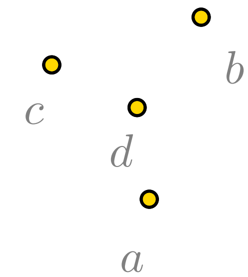

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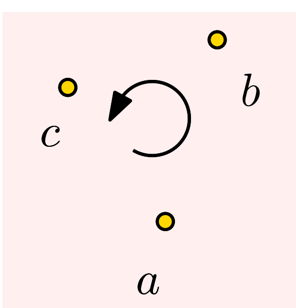

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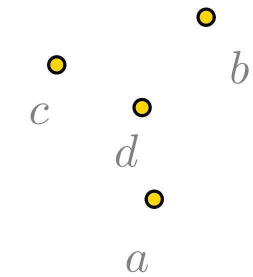

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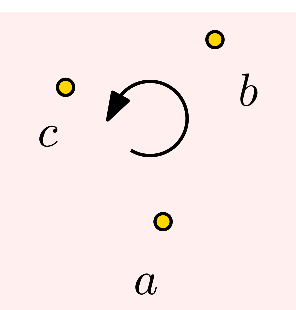

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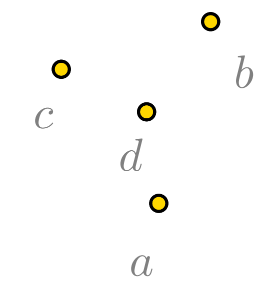

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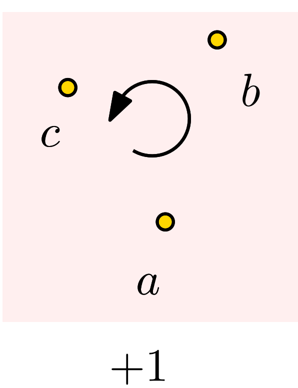

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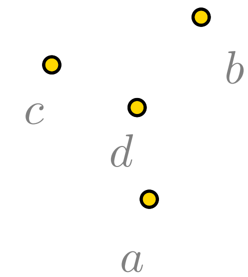

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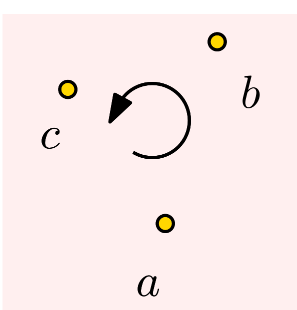

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$$\chi(a,b,c) = \text{sign of the determinant of } \begin{pmatrix} a_x & b_x & c_x \\ a_y & b_y & c_y \\ 1 & 1 & 1 \end{pmatrix}$$

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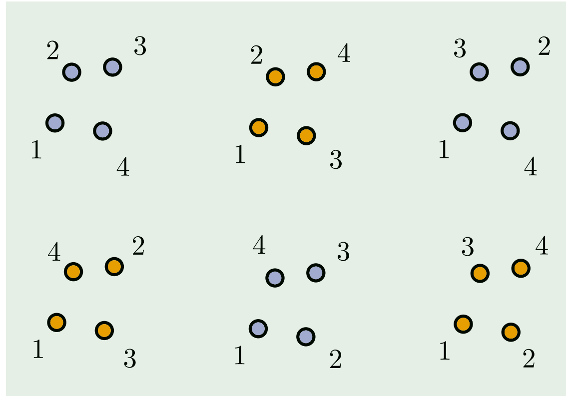

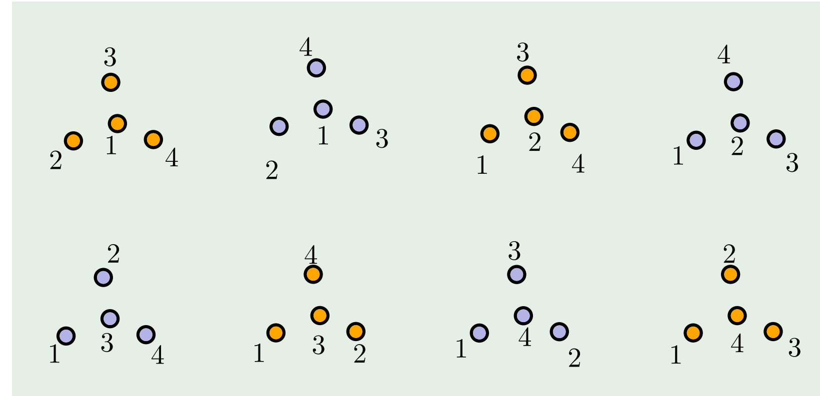

Discrete geometers

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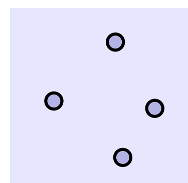

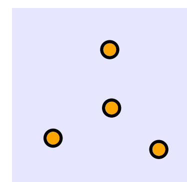

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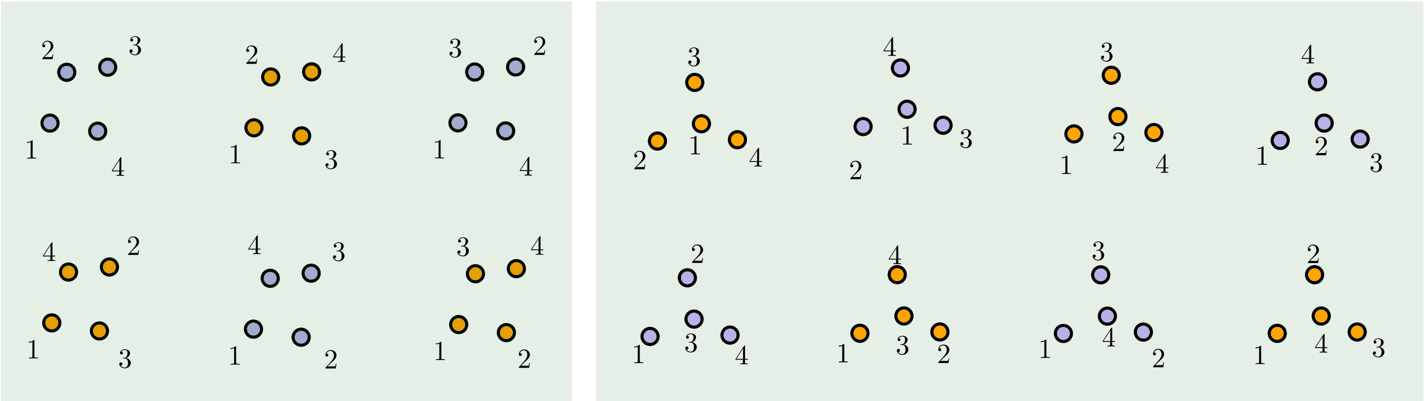

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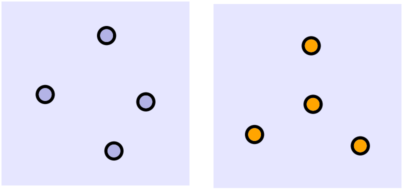


Two point **sets** P and Q have **the same order type** if there exists an **orientation-preserving** bijection between them.

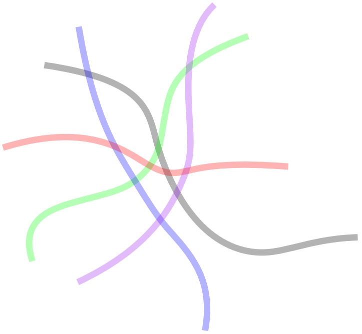
Equivalence relation, order type $\stackrel{\text{def}}{=}$ an equivalence class.

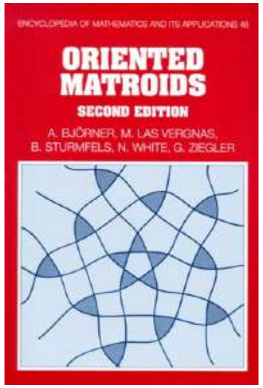
Common properties for convex hull, segments intersections, ...

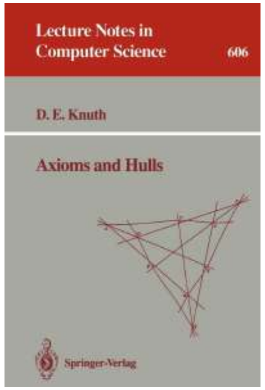
Can be defined **abstractly** in **topological affine planes**.

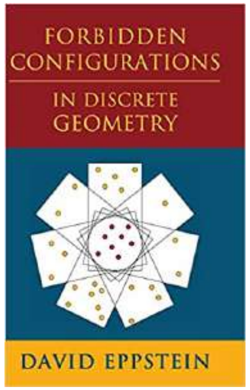
Labeled abstract order types = **acyclic uniform oriented matroids**.

[Salzmann'67][Folkman-Lawrence'78]









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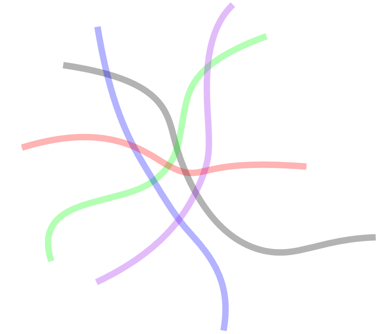


Labeled abstract order types = **acyclic uniform oriented matroids**.

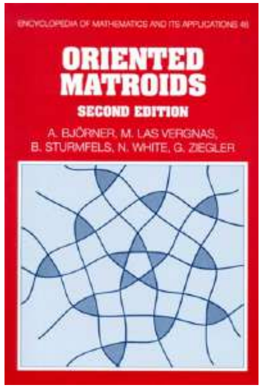
[Salzmann'67][Folkman-Lawrence'78]

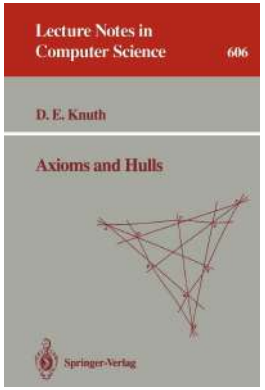
Computationally, abstract OT are simple

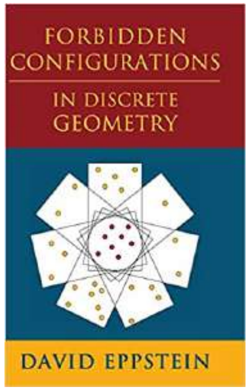
characterized by a few axioms on ≤ 5 points.

but realizable OT are complicated.

Universality [Mnëv'86], $\exists\mathbb{R}$-completeness [Shor'92]







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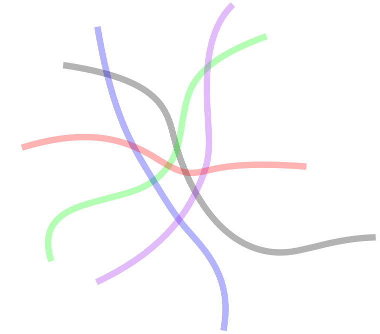

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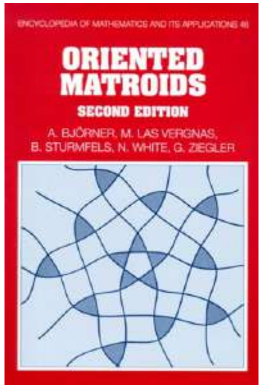

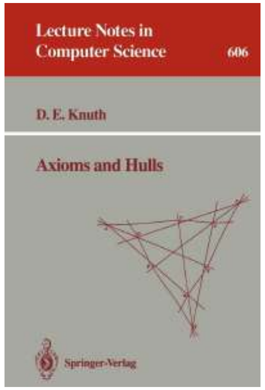

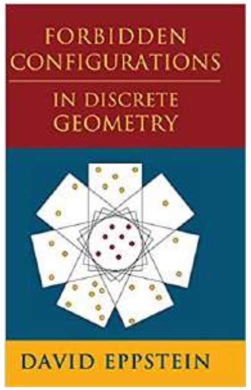

Realizable OT were **enumerated** up to size 11.

3	4	5	6	7	8	9	10	11
1	2	3	16	135	3 315	158 817	14 309 547	2 334 512 907

(mirror images are identified)

[Aichholzer-Aurenhammer-Krasser'02]

but even the **number** of (abstract) n-point order types is unknown.

Sampling order types

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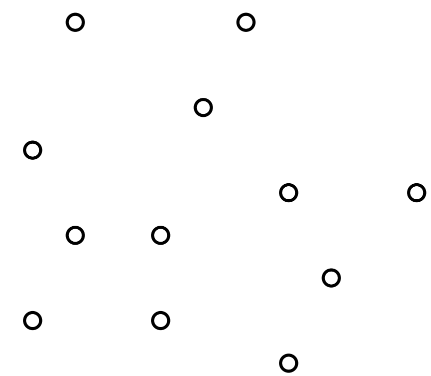

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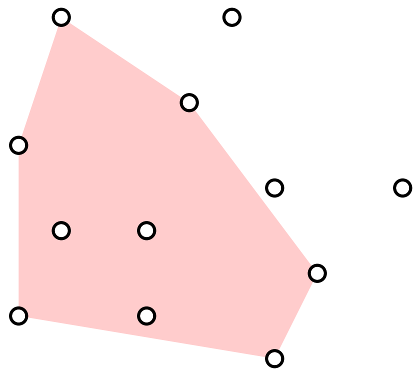

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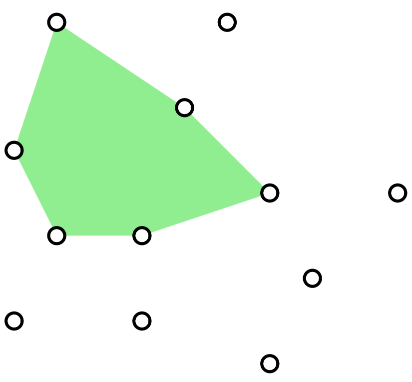

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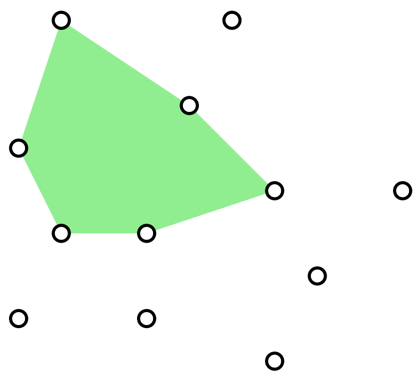

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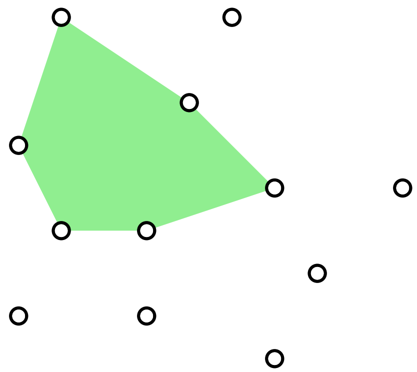

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▷ Testing geometric algorithms (that use only orientation predicates).

No need to test the same trace twice.

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Our "concentration" result:

There is a **vanishingly small** subset A_n of the n-point order types such that a random n-point i.i.d. sample of a square has order type in A_n **with probability** $\to \mathbf{1}$.

Random polygons
based on order types

Random polytopes 101:

How many extreme points in a random **point set** sampled **independently** from some measure μ over $\mathbb{R}^d$?

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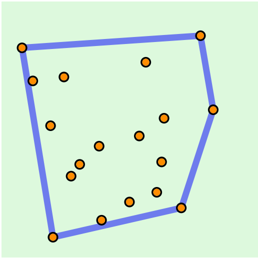

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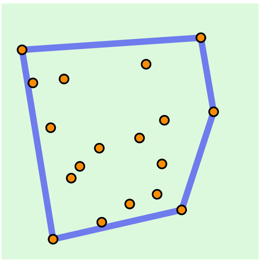

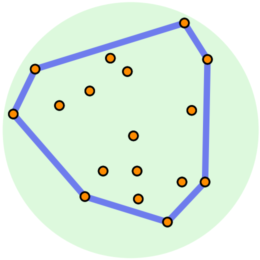

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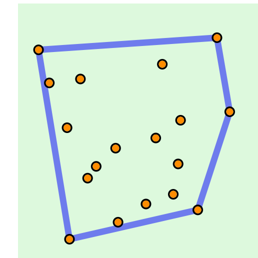

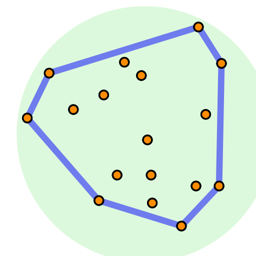

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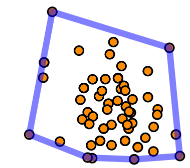

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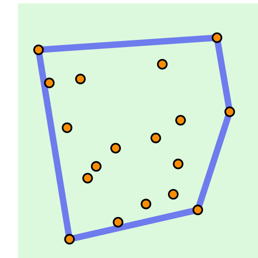

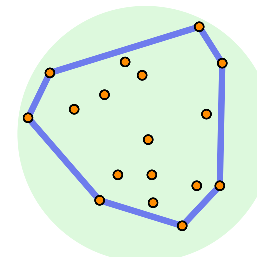

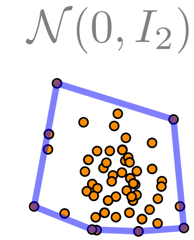

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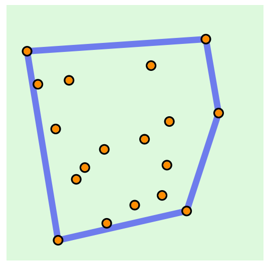

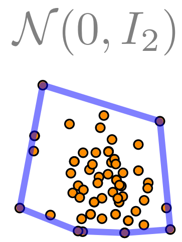

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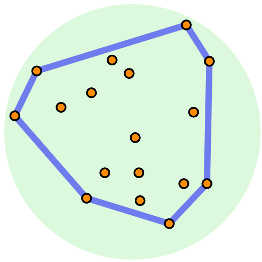

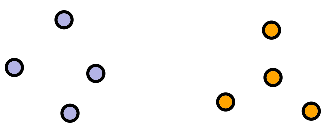

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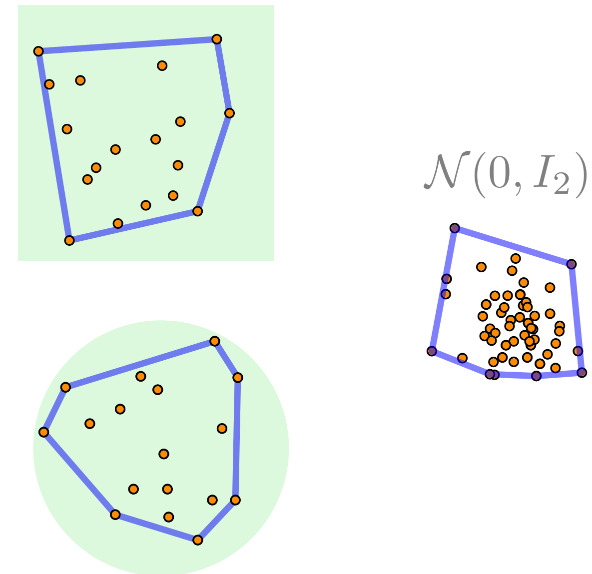

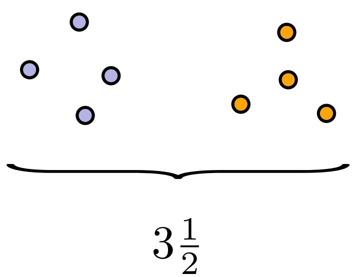

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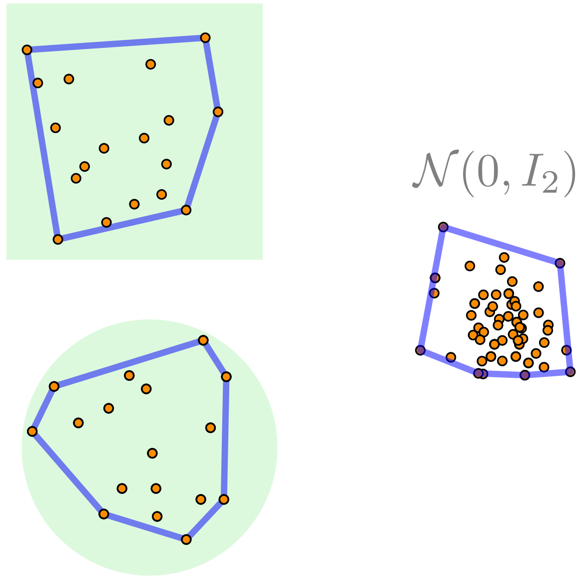

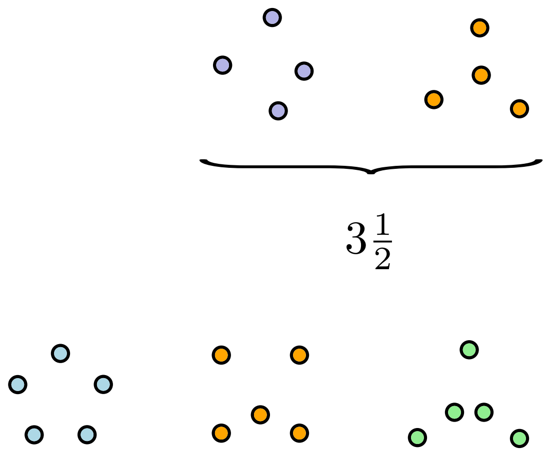

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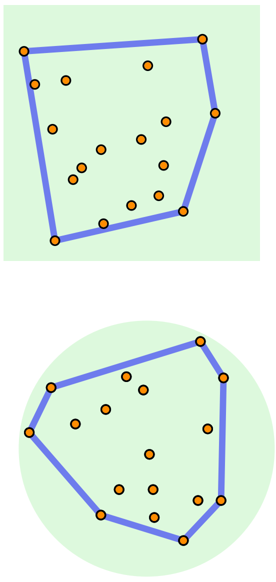

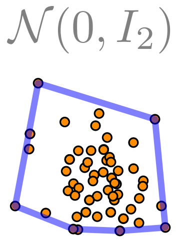

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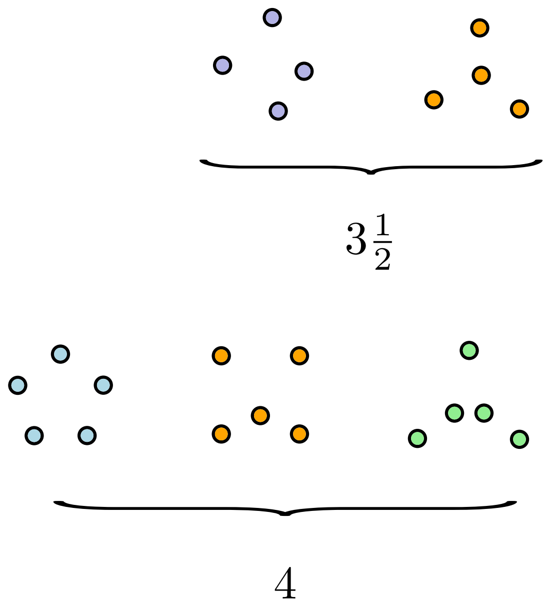

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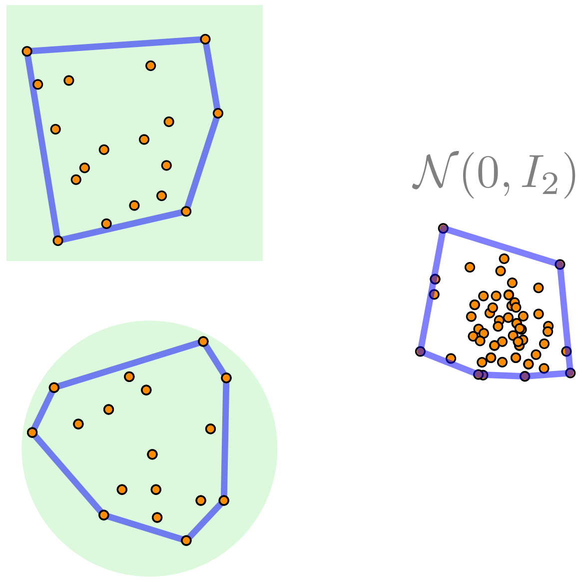

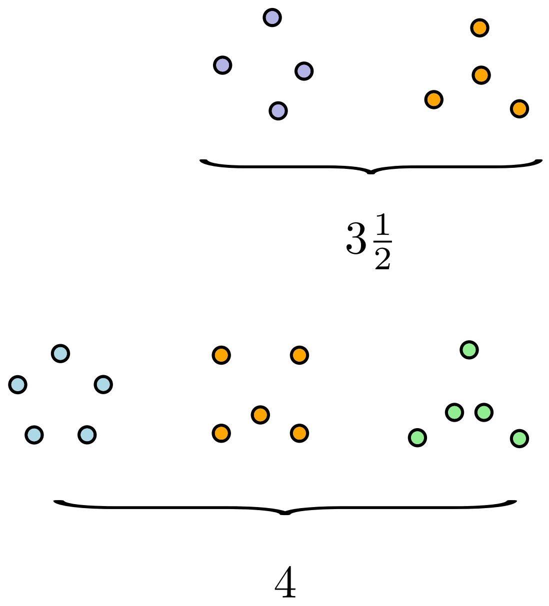

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Answer: ~ 4 as $\mathbf{n} \to \infty$.

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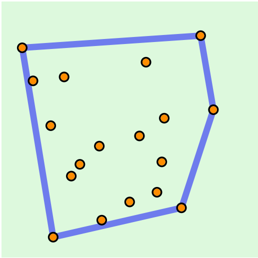

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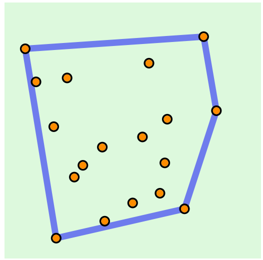

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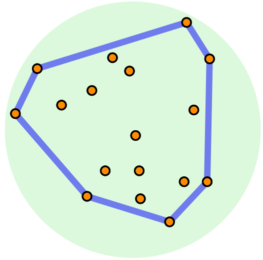

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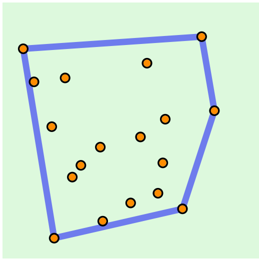

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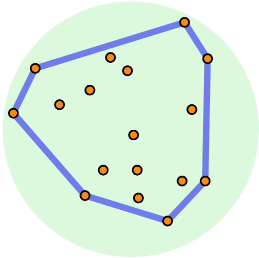

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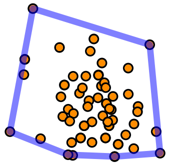

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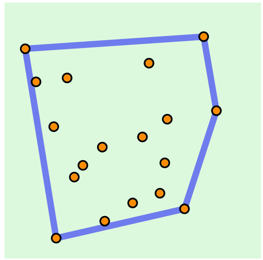

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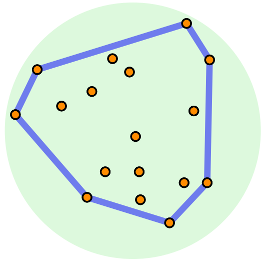

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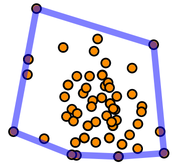

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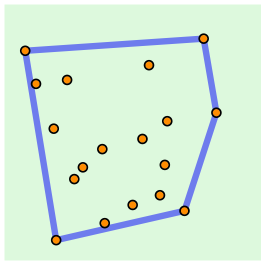

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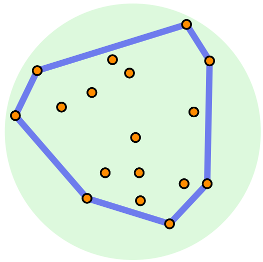

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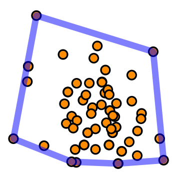

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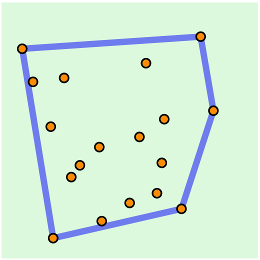

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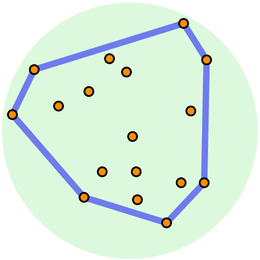

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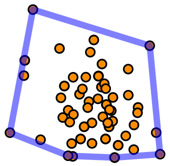

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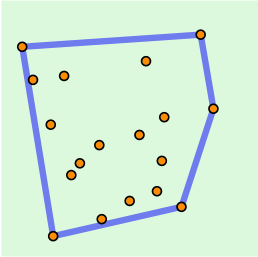

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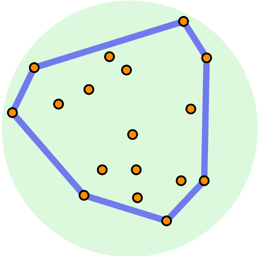

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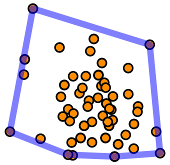

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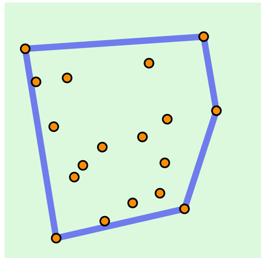

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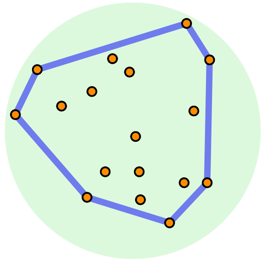

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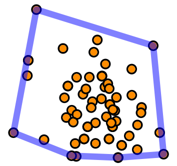

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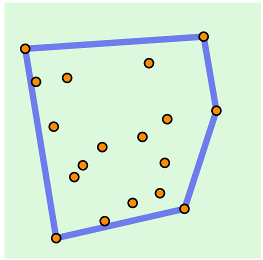

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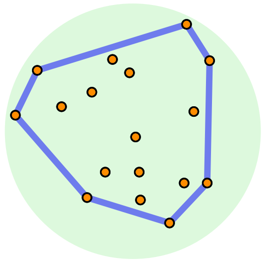

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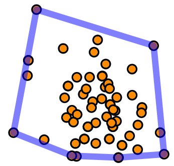

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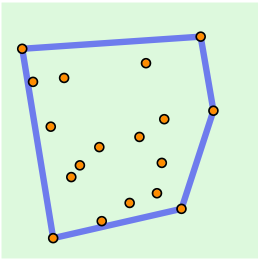

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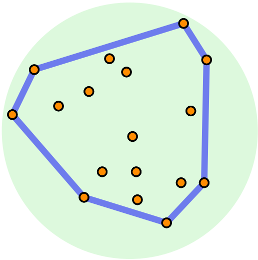

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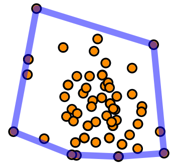

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Our main results

Theorem. For $n \geq 3$, the number of extreme points in a random simple labeled order type chosen uniformly among the simple, labeled order types of size n in the plane has average $4 - \frac{8}{n^2-n+2}$ and variance less than 3.

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Averaging on subsets

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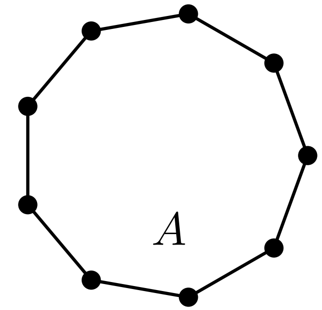

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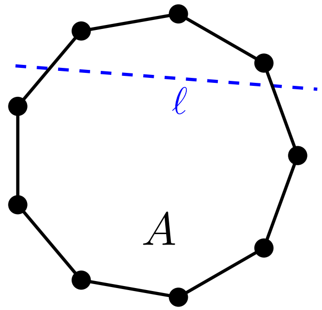

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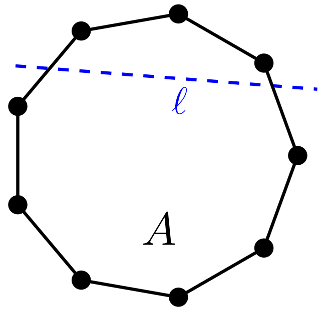

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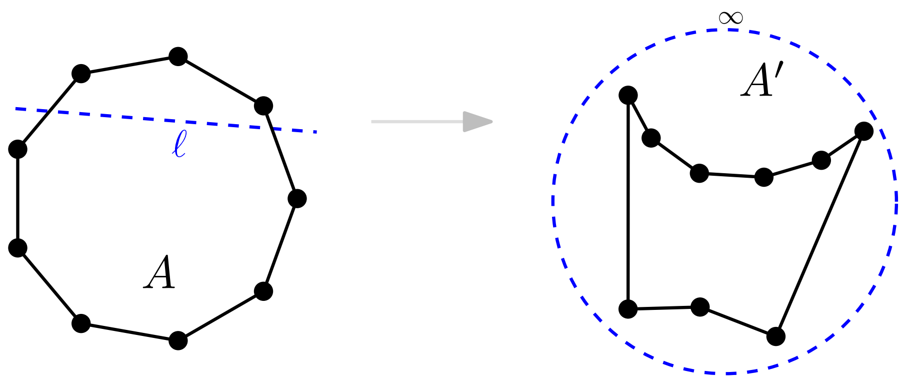

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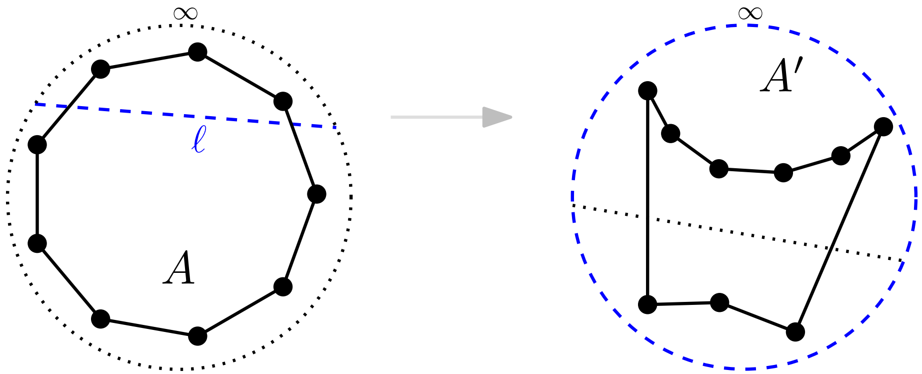

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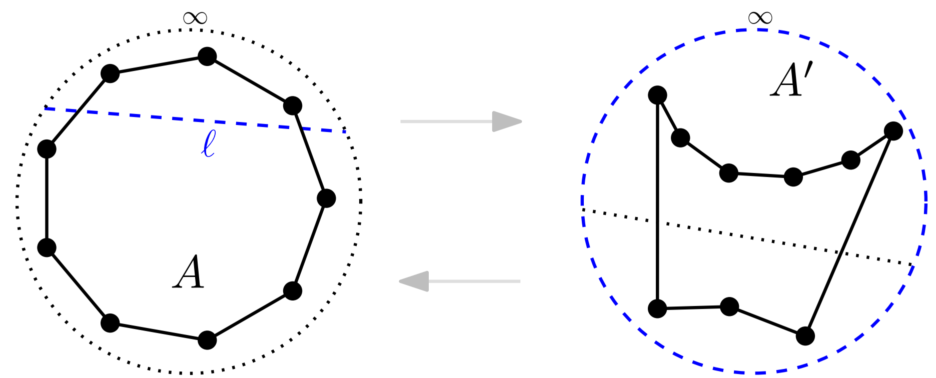

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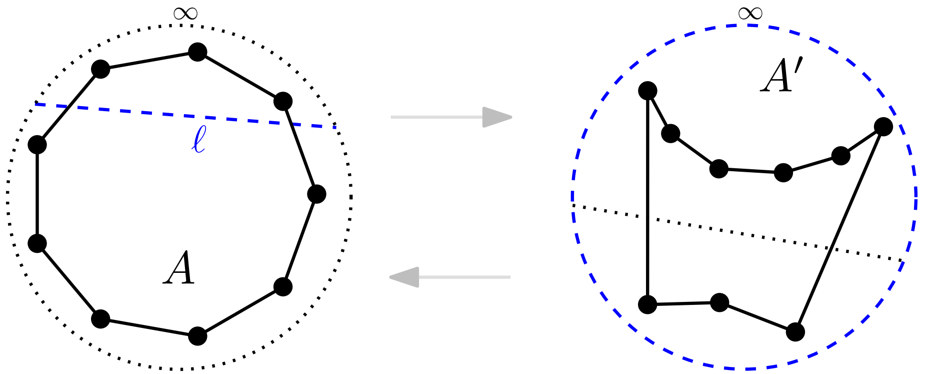

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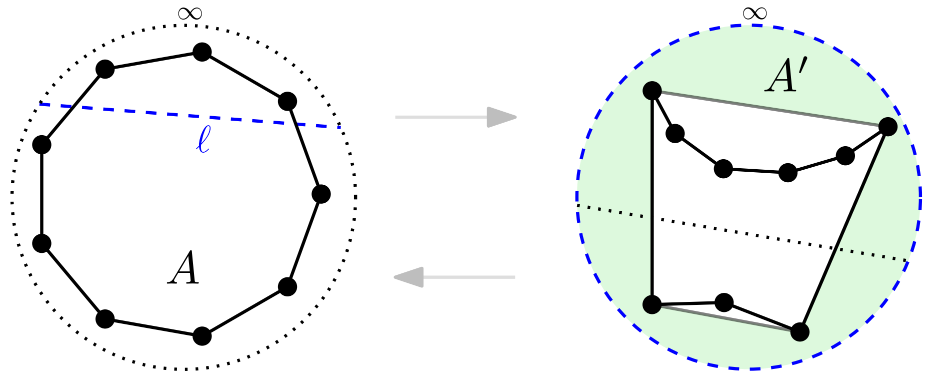

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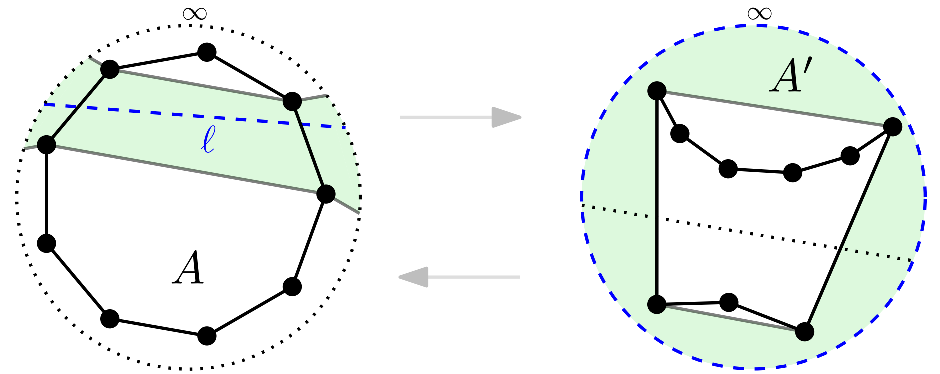

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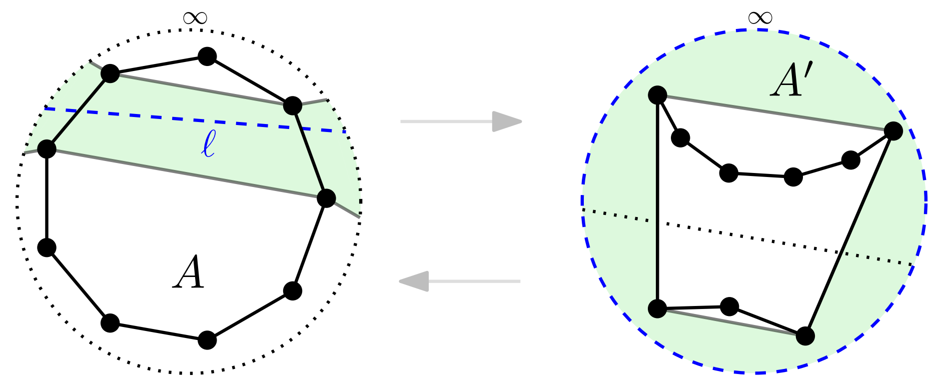

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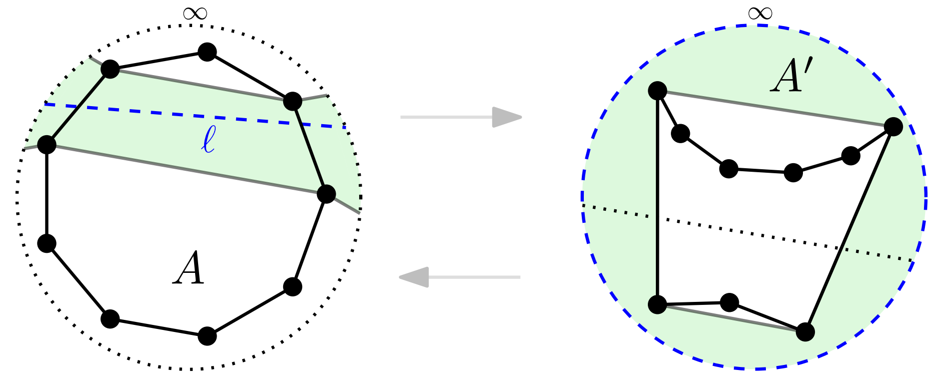

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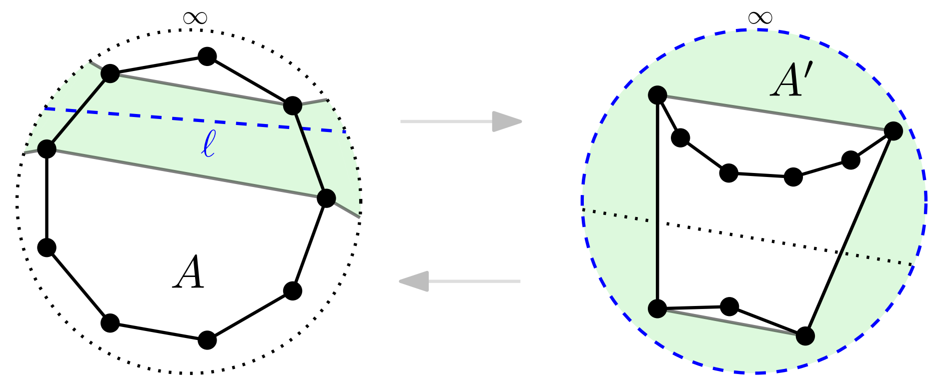

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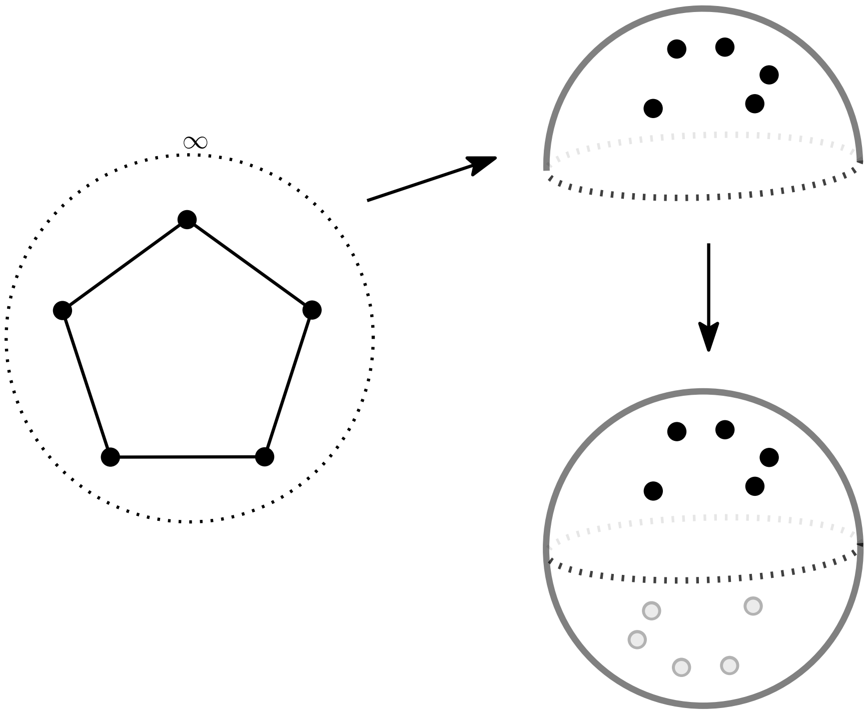
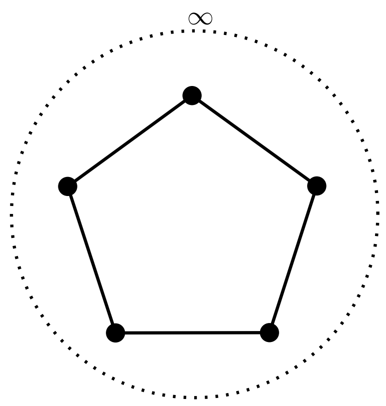
We further **divide up** order types into classes of order types equal **up to projective transforms**.

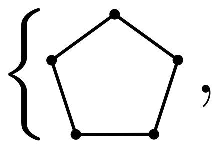
Then, we average the number of extreme points within each class.

Projective order types



What does a "class of order types equal up to projective transforms" look like?





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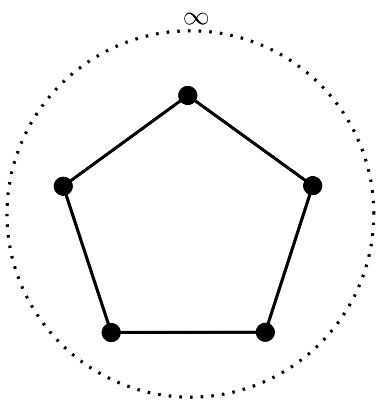

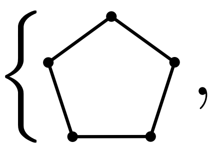

Invariance under **affine transforms**
$\Rightarrow$ the only choice is which line is sent to infinity.

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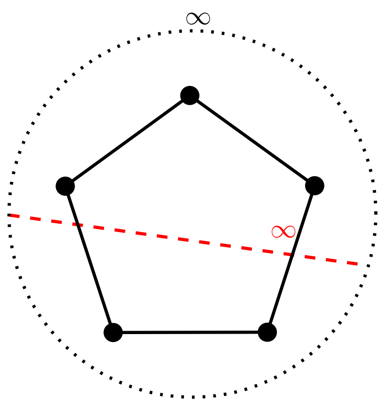

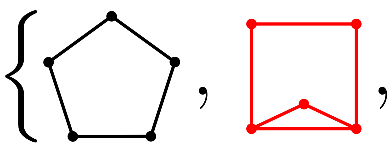

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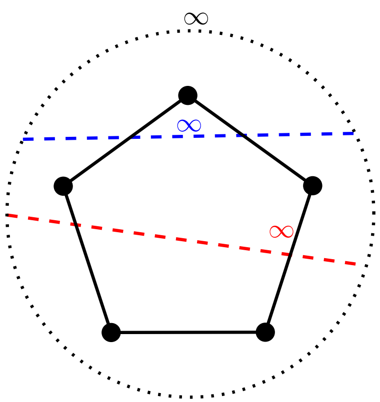

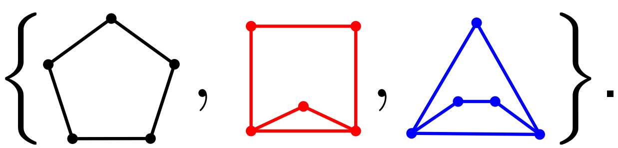

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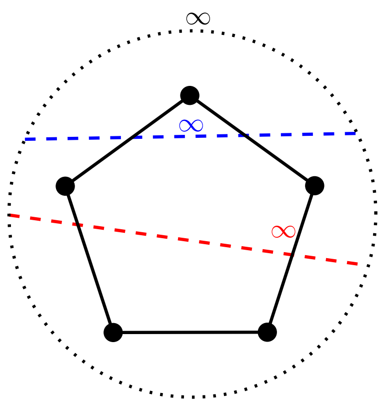
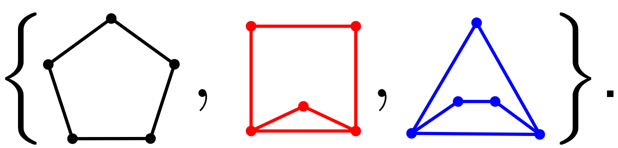
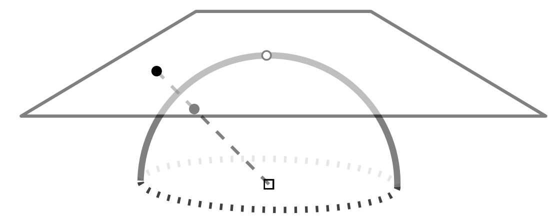
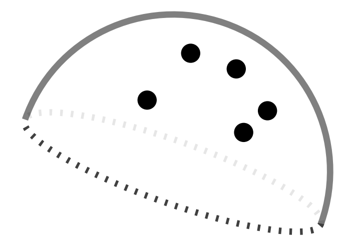






$$\begin{pmatrix}x\\y\\1\end{pmatrix} \mapsto \frac{1}{\sqrt{x^2+y^2+1}}\begin{pmatrix}x\\y\\1\end{pmatrix}$$



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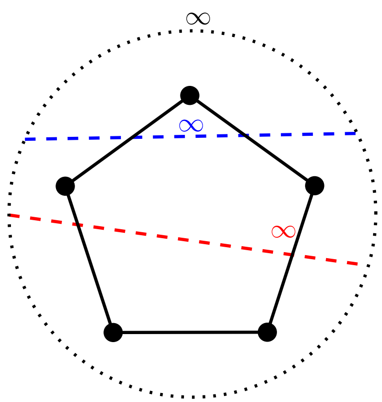

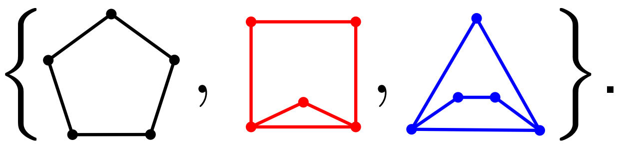

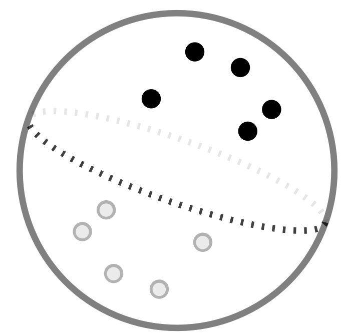

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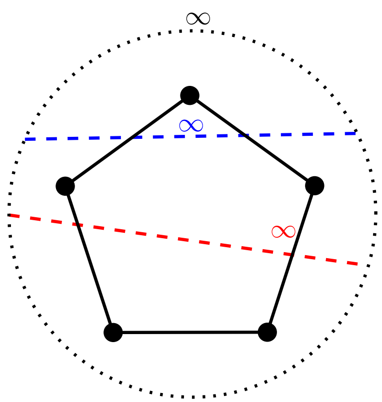

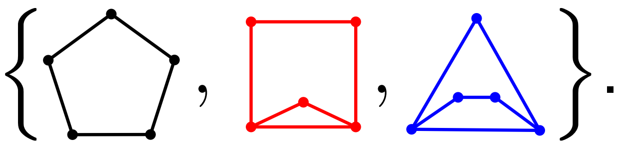

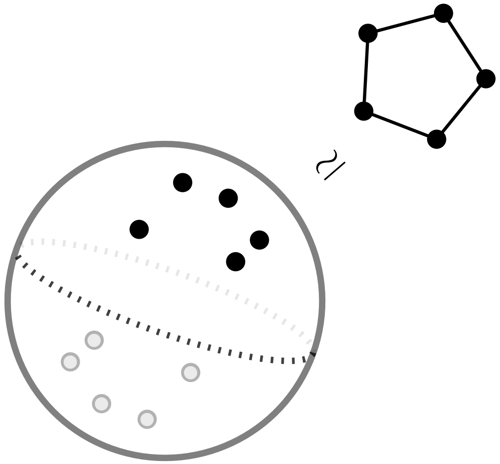

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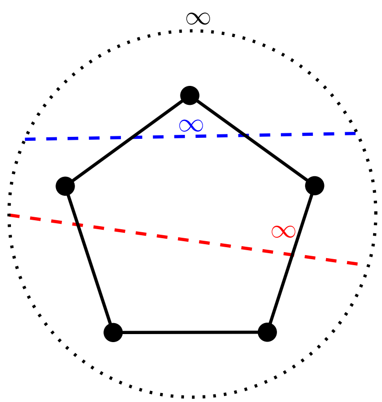
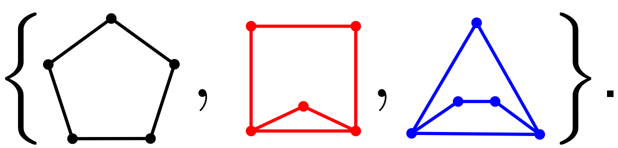
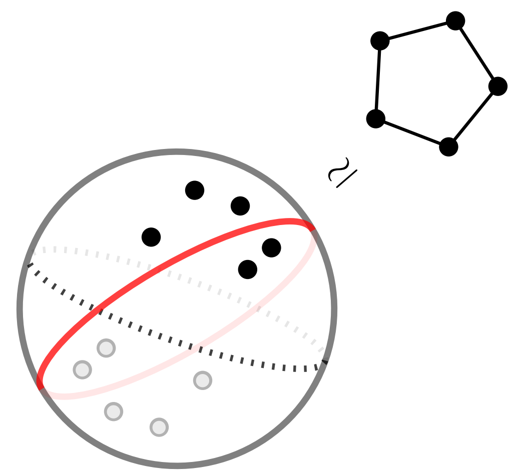
⇒ the only choice is which line is sent to infinity.

What does a "class of order types equal up to projective transforms" look like?







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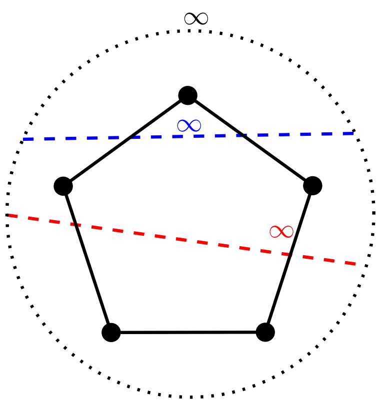

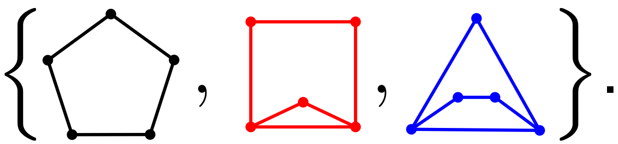

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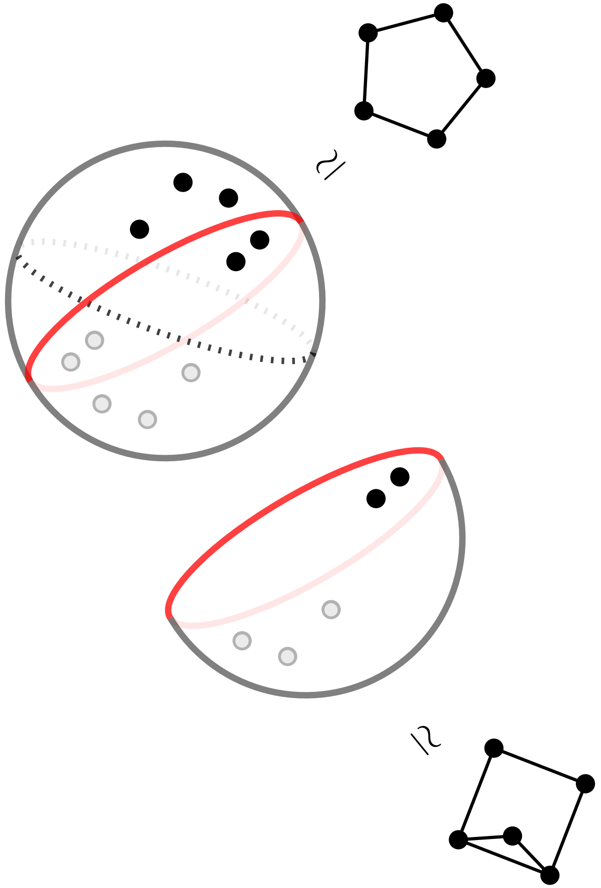

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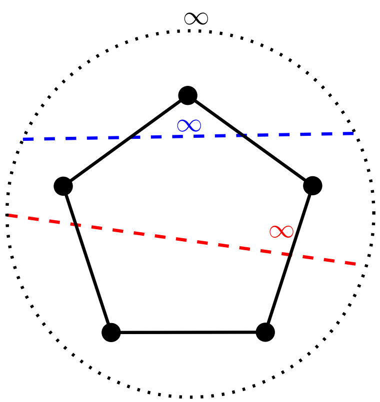

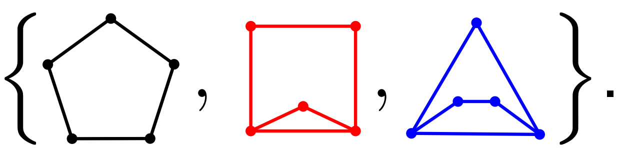

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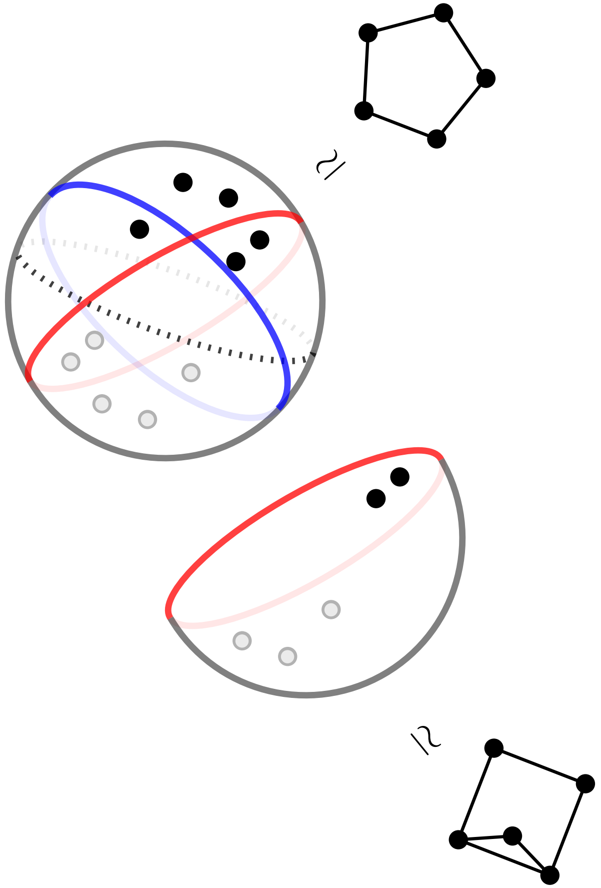

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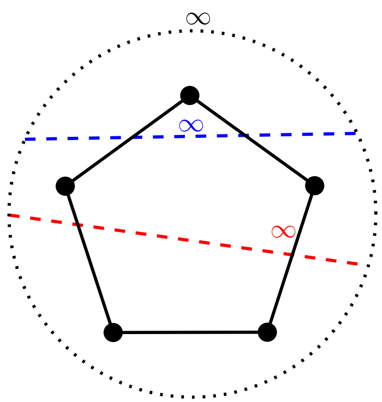

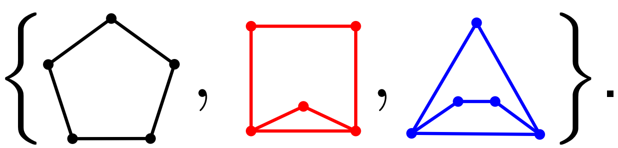

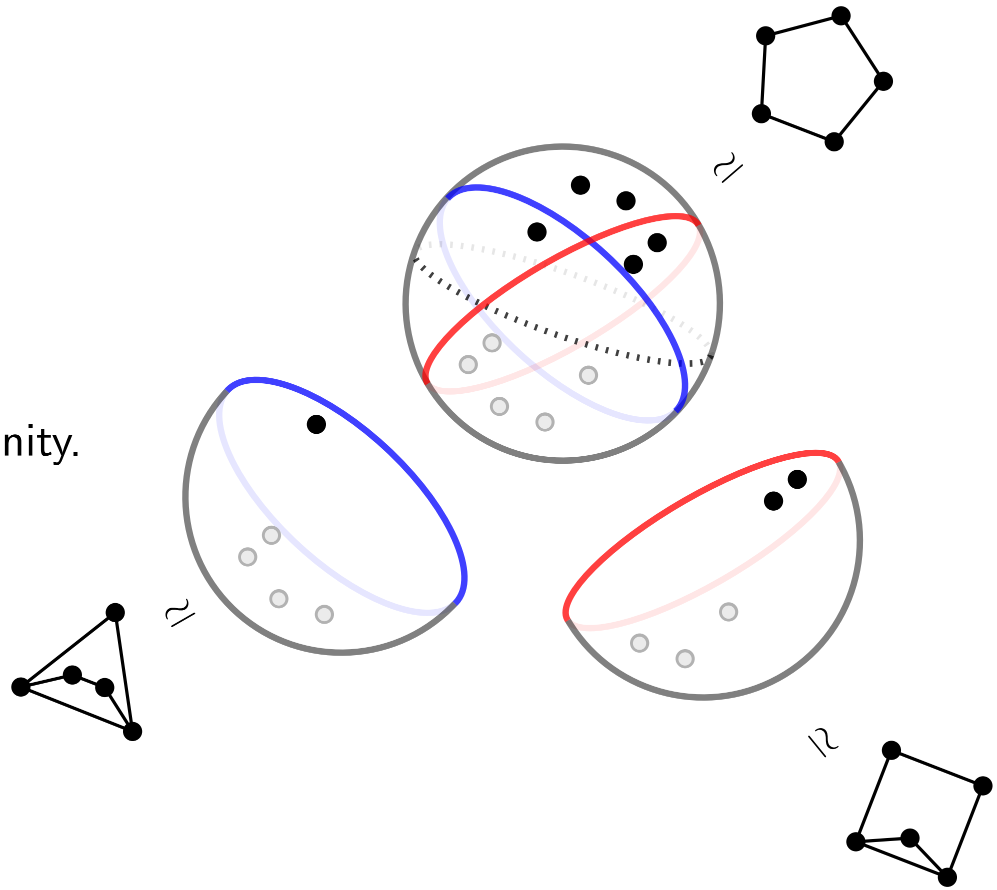

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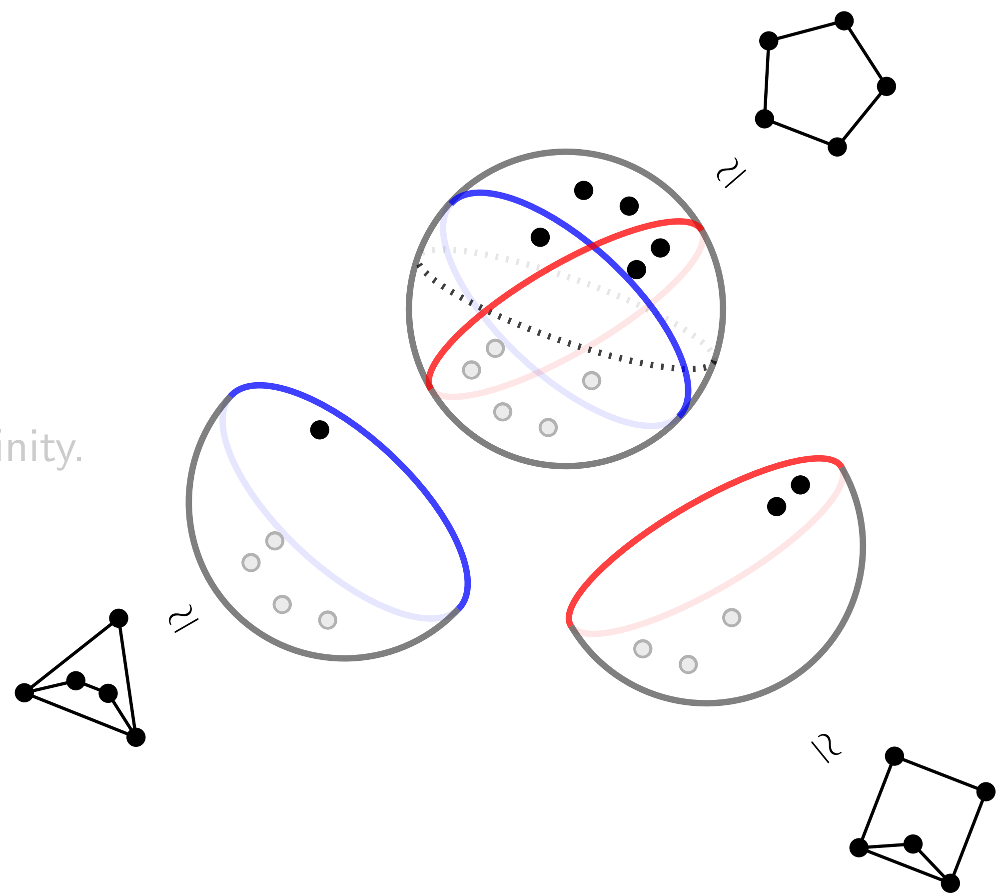
Represent a "class of order types of $\mathbb{R}^2$ equal up to projective transforms" by a "symmetric order type" on $\mathbb{S}^2$



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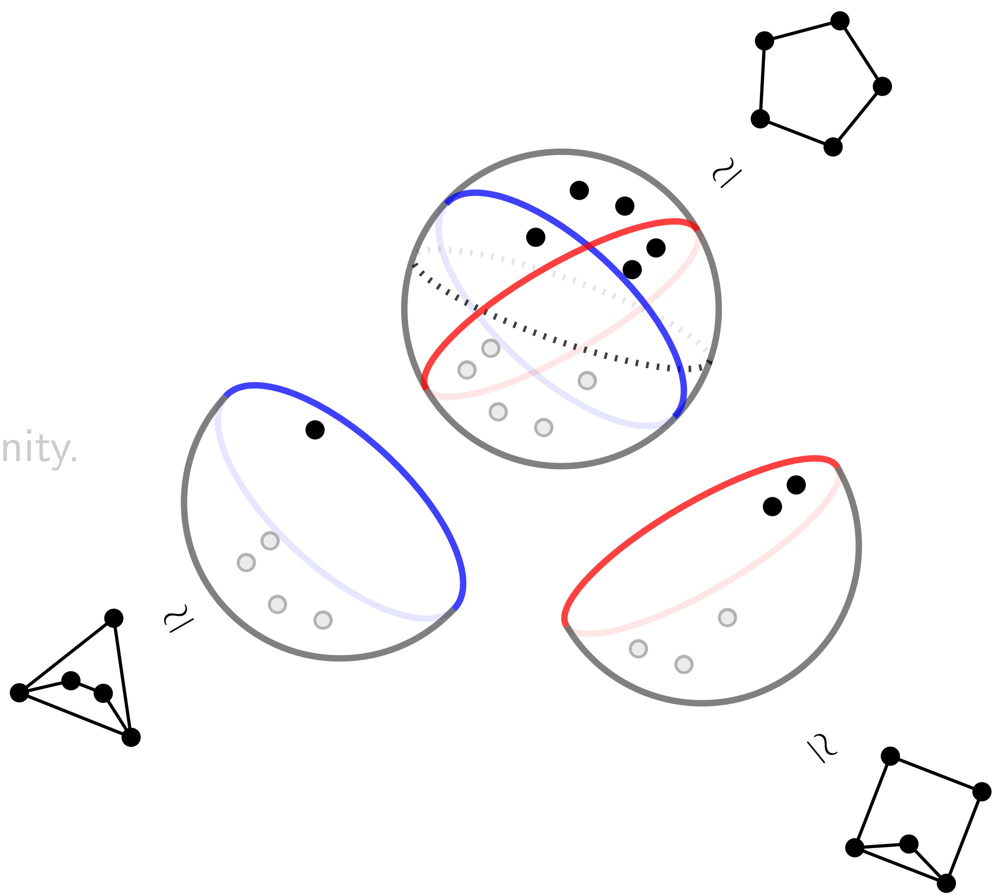

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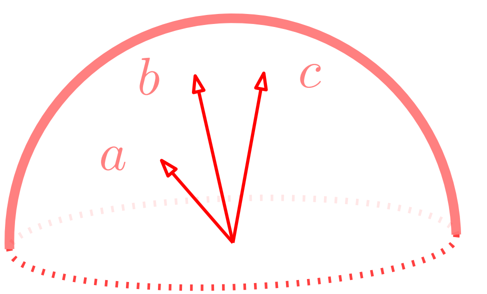
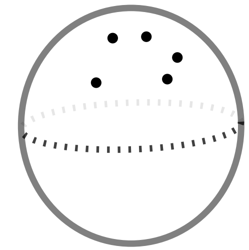


 $\chi(a,b,c) \overset{\text{def}}{=} \text{sign}\det(a,b,c)$

An affine point set.



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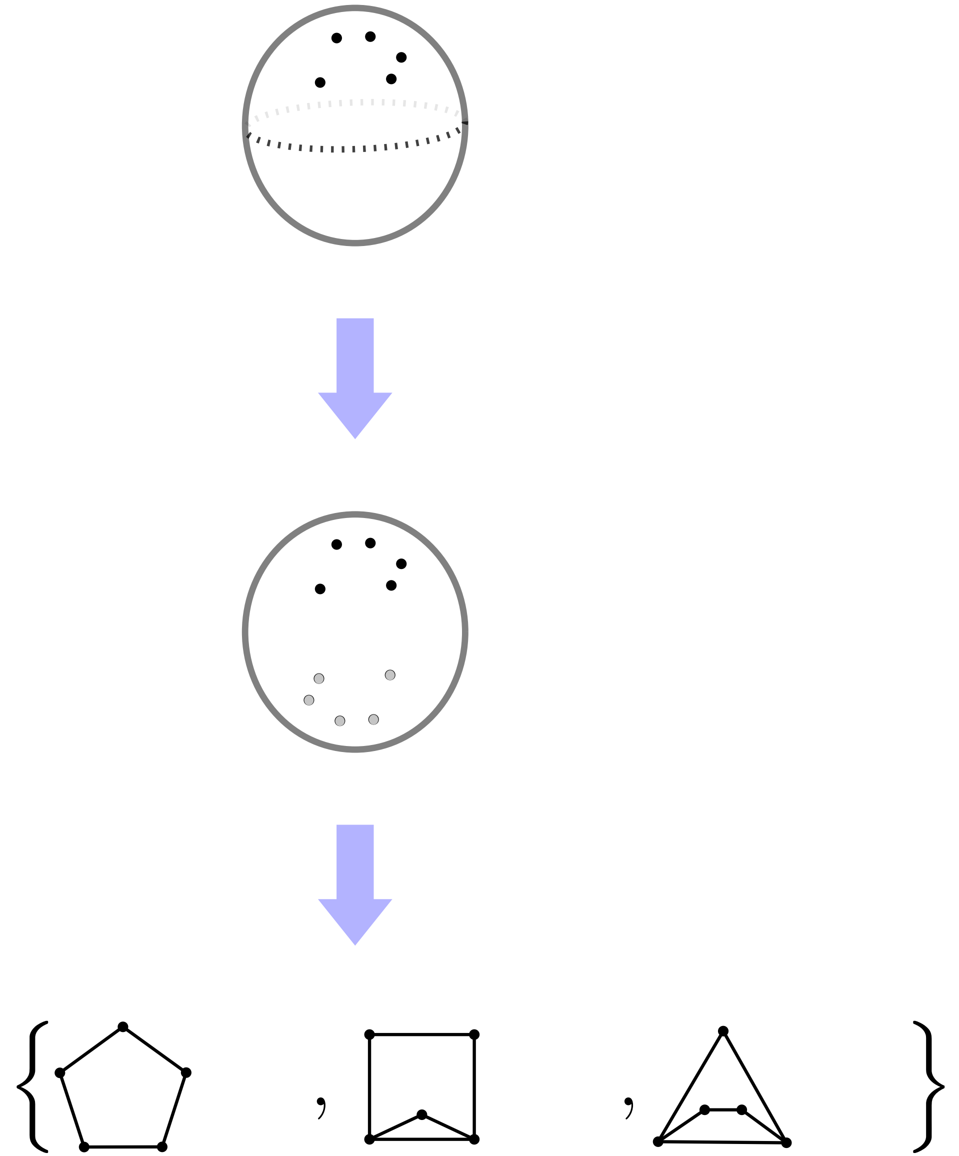

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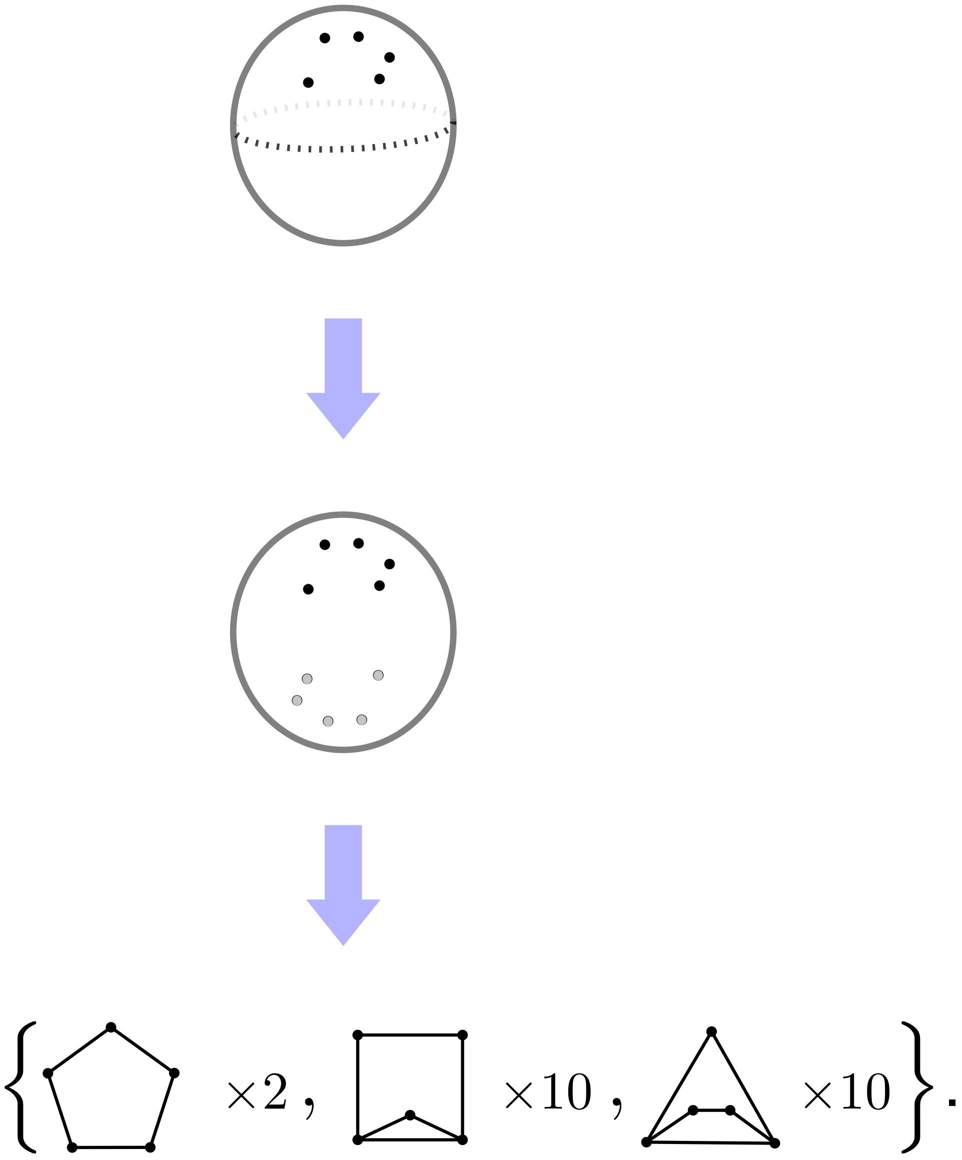
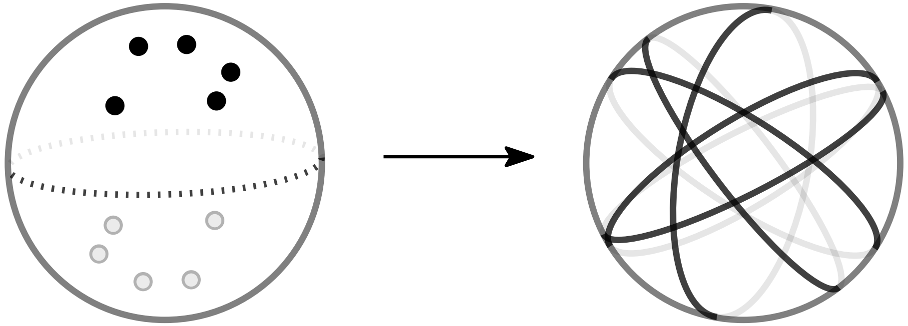
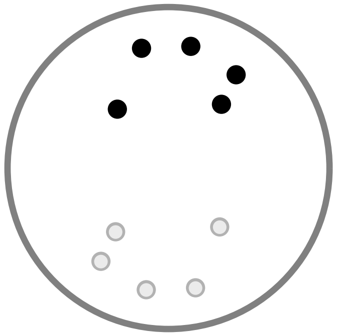


Duality



P a projective $2n$-point set.



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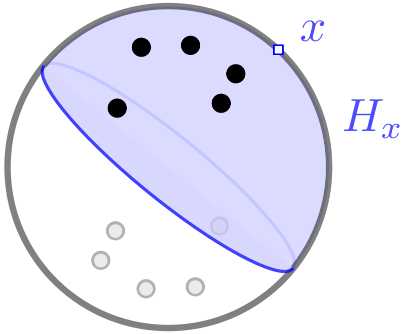

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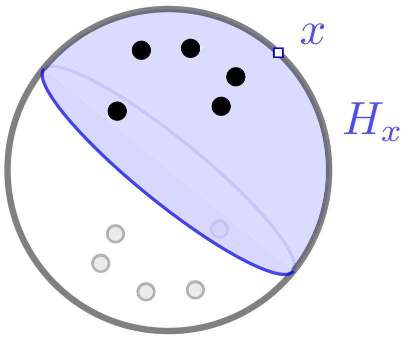

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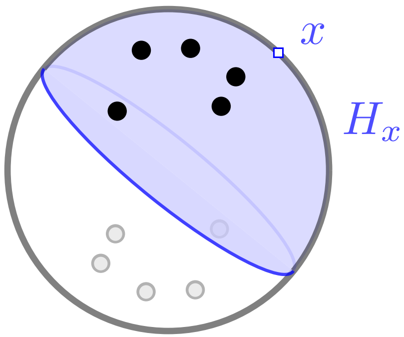

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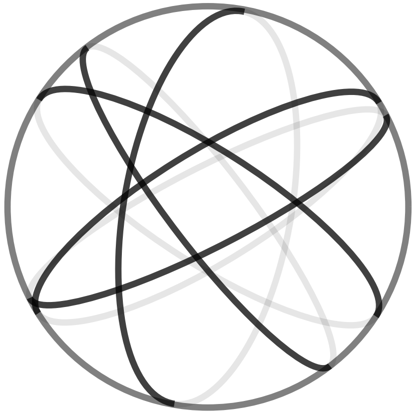

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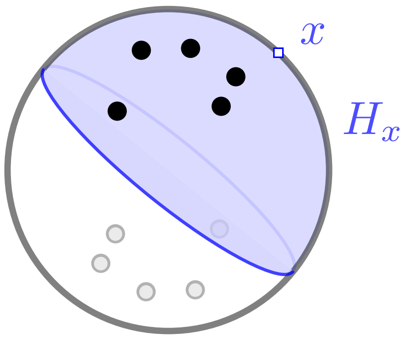

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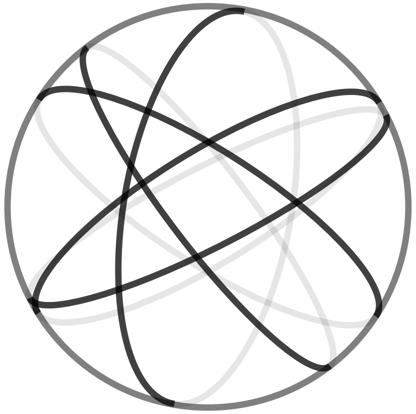

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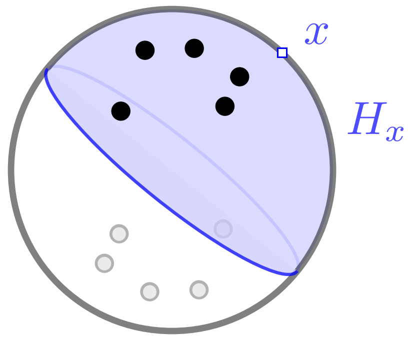

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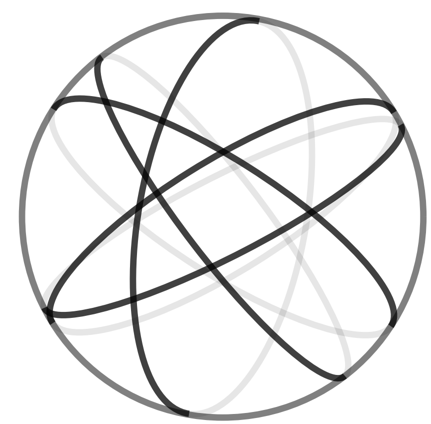

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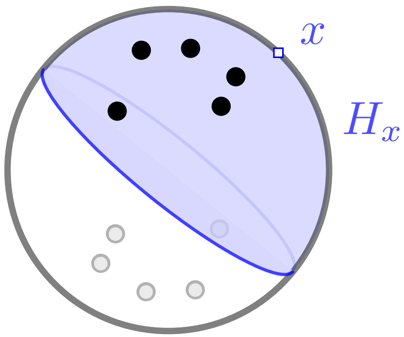

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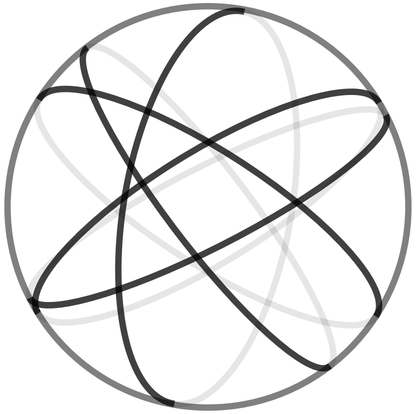

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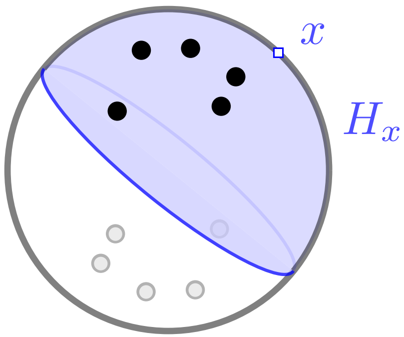

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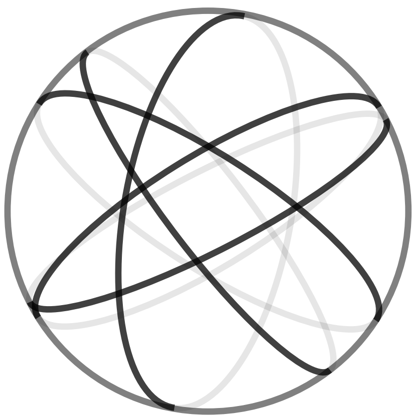

Affine hemisets of P
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How can we control the multiplicities?

Symmetries

and labeled order types

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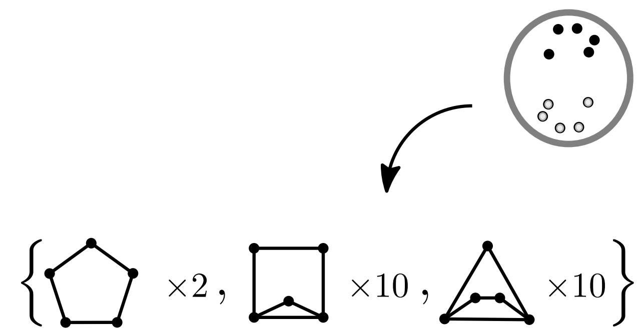

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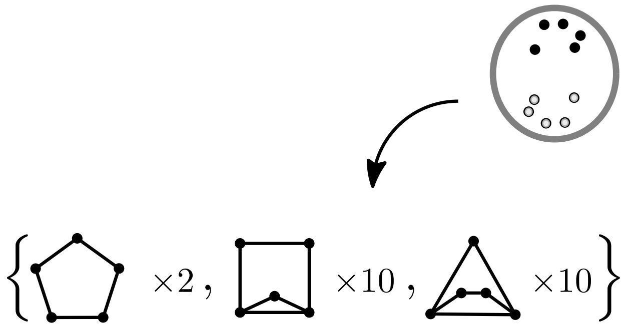

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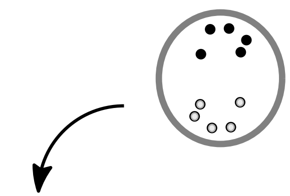

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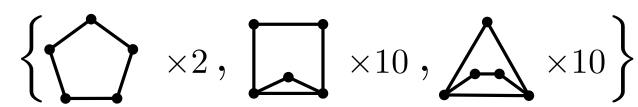

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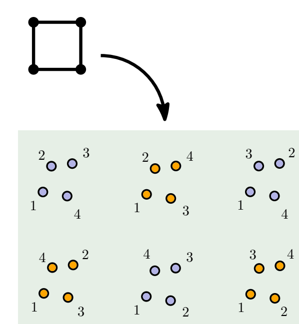

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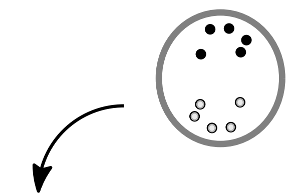

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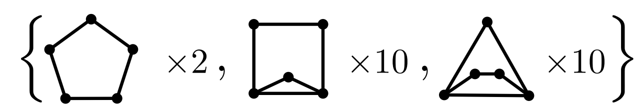

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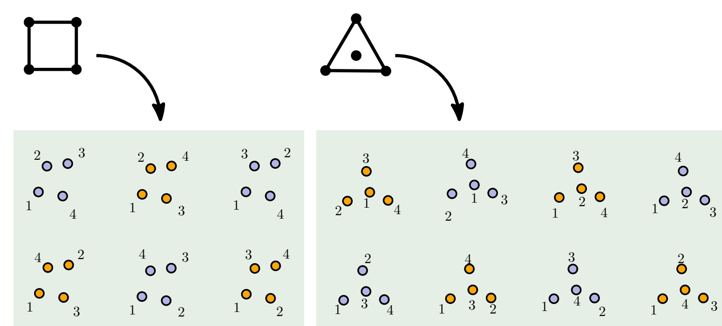

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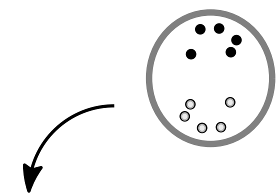
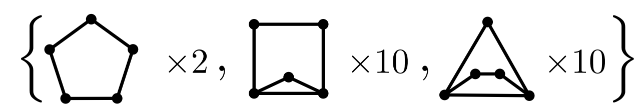
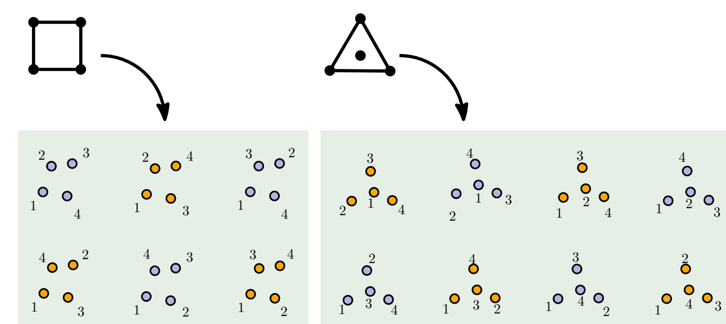


Two sources of bias:

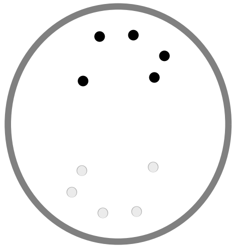
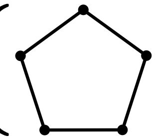
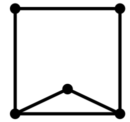
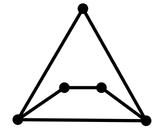
The order type of A is not uniformly distributed.

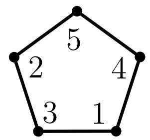


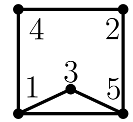
Some **unlabeled** order types may decompose into more **labeled** order types than others.



These two biases cancel each other out!

 $\leftrightarrow$ $\Big\{$  $\times 2$,  $\times 10$,  $\times 10$ $\Big\}$.

Let us compare the probabilities of 

and 

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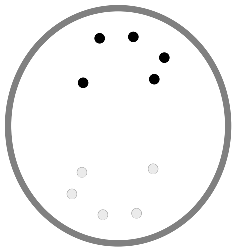
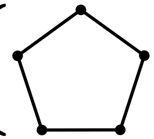
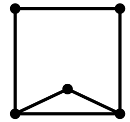
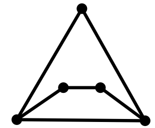
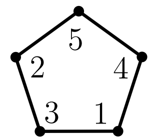
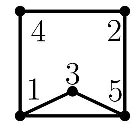
 $\leftrightarrow$ $\Bigg\{$  $\times 2$,  $\times 10$,  $\times 10 \Bigg\}$.

Let us compare the probabilities of 

and 

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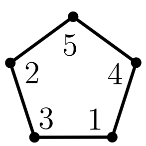
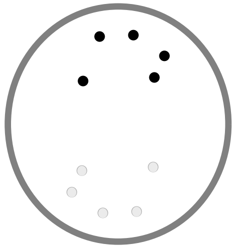
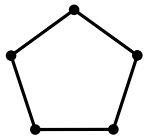
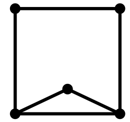
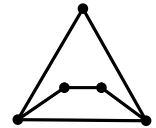
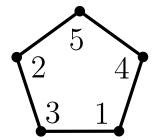
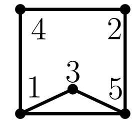
$\Pr\Big(\bar{A} \simeq$ 

$\Big) =$



$\leftrightarrow$ $\Big\{$  $\times 2$,  $\times 10$,  $\times 10$ $\Big\}$.

Let us compare the probabilities of 

and 

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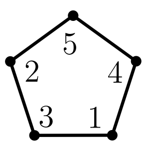
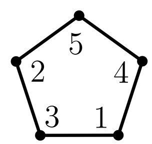
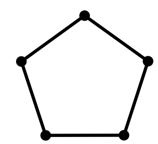
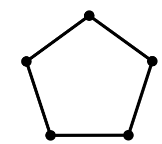
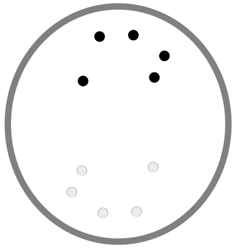
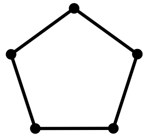
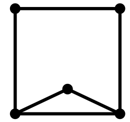
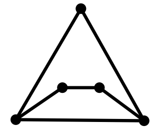
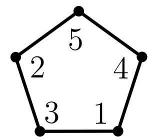
$\Pr\Big(\bar{A} \simeq$ 

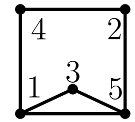
$\Big) = \Pr\Big(\bar{A} \simeq$ 

$\Big|\, A \simeq$  $\Big) \times \Pr\Big(A \simeq$  $\Big)$



$\leftrightarrow$ $\Big\{$  $\times 2$,  $\times 10$,  $\times 10 \Big\}$.

Let us compare the probabilities of 

and 

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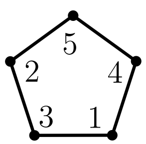
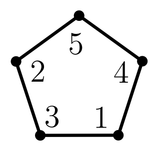
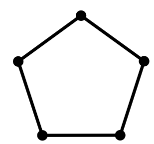
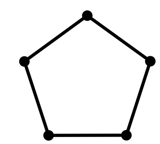
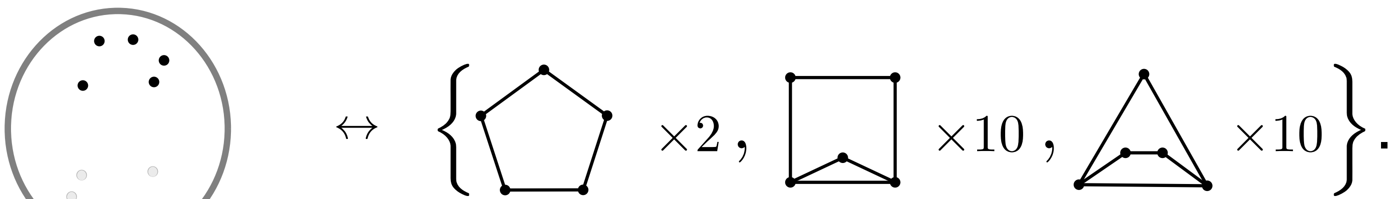
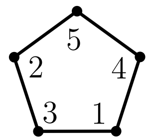
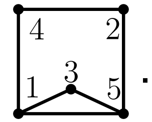
$\Pr\Big(\bar{A} \simeq$ 

$\Big) = \Pr\Big(\bar{A} \simeq$ 

$\Big|\, A \simeq$  $\Big) \times \Pr\Big(A \simeq$  $\Big)$

$$= \frac{5}{120} \times \frac{2}{22}$$



Let us compare the probabilities of  and .

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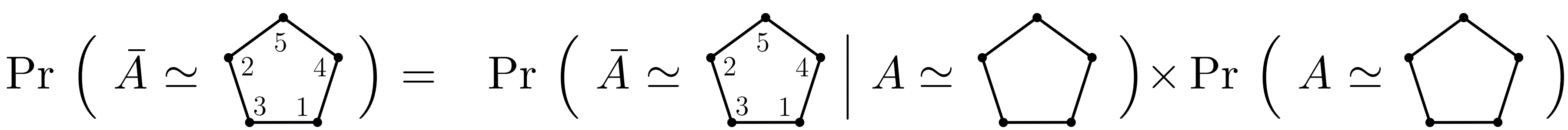

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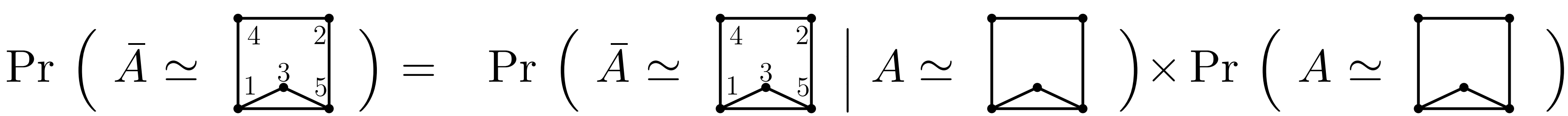
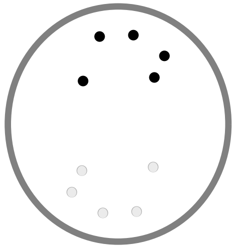
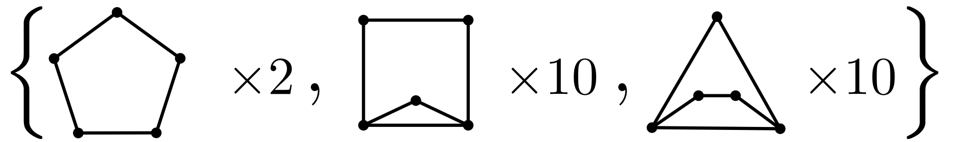
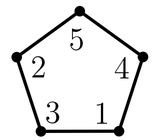
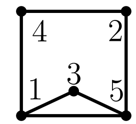


 $\leftrightarrow$ 

Let us compare the probabilities of 

and 

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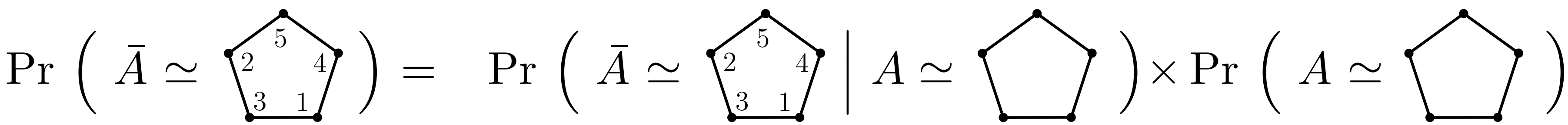

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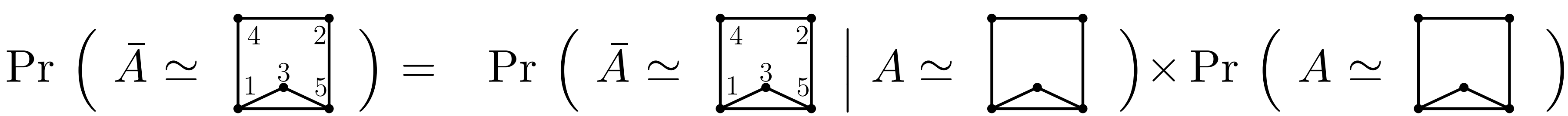

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Symmetry of $S \subset \mathbb{S}^2 \stackrel{\text{def}}{=}$ orientation-preserving bijection $S \to S$.

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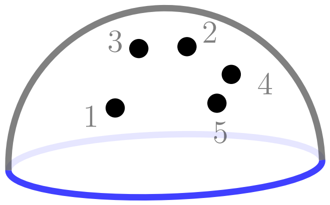

Cyclic group $\mathbb{Z}_5$

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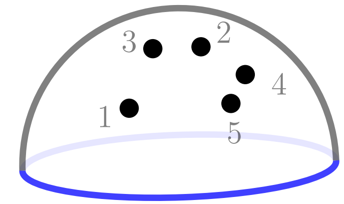

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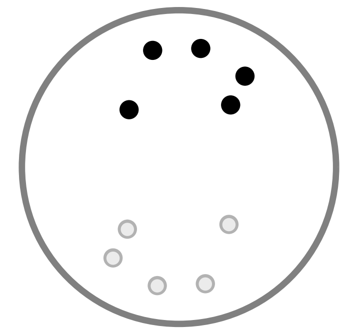

Dihedral group D_5

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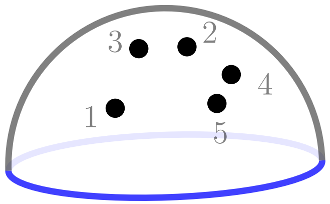

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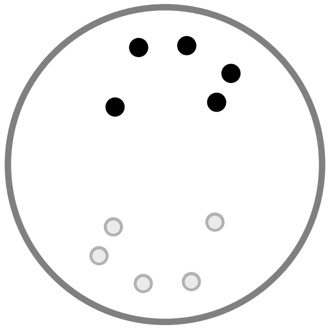

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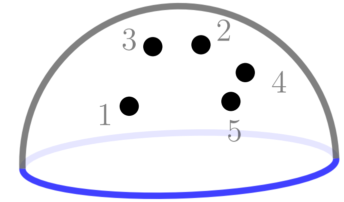

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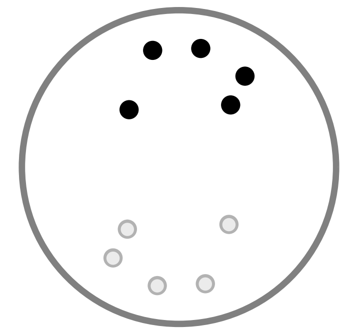

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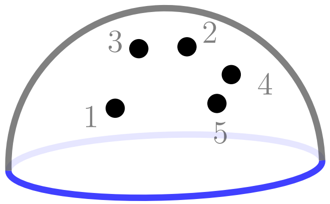

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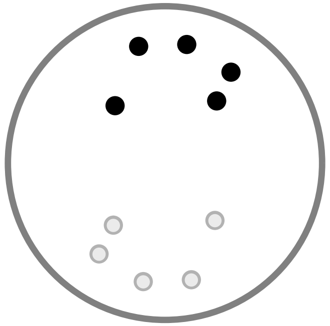

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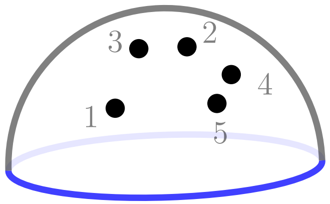

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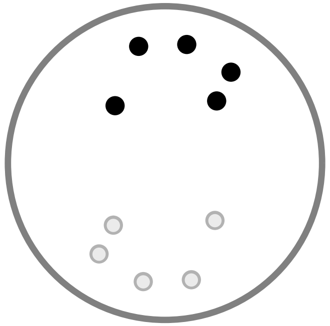

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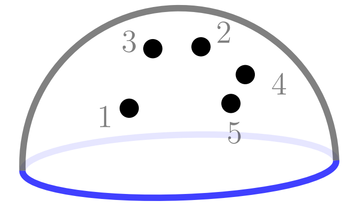

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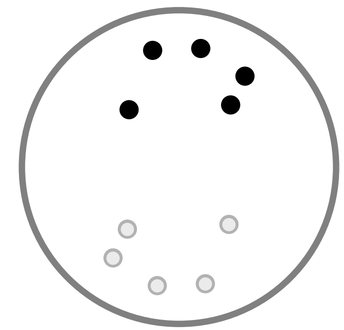

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1st source of bias

2nd source of bias

Altogether...

Theorem. For $n \geq 3$, the number of extreme points in a random simple labeled order type chosen uniformly among the simple, labeled order types of size n in the plane has average $4 - \frac{8}{n^2-n+2}$ and variance at most 3.

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average complexity of a 2-cell
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the zone theorem

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Altogether...

Theorem. For $n \geq 3$, the number of extreme points in a random simple labeled order type chosen uniformly among the simple, labeled order types of size n in the plane has average $4 - \frac{8}{n^2-n+2}$ and variance at most 3.

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Time to wrap-up!

Conjecture. Order types of n i.i.d. points whose number of extreme points goes to ∞ as $n \to \infty$ exhibit concentration.

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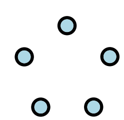
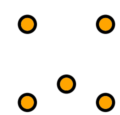
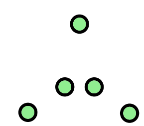
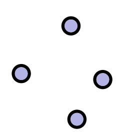
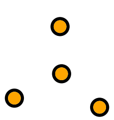
Sample uniformly in    then drop a random point

yields  with proba 0.6, and  with proba 0.4

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Polynomial time per sample + not too biased.

Our suggestion: sample **projective** order types first!

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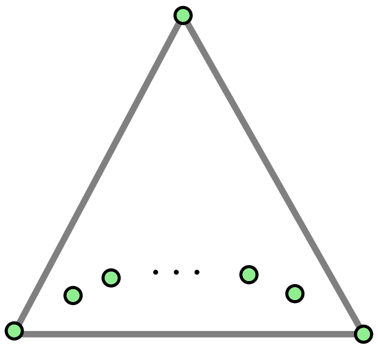

Thank you for your attention!

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P a projective set and A an affine hemiset of P.

There are $\frac{|\mathsf{Sym}(P)|}{|\mathsf{Sym}(A)|}$ affine hemisets with same order type as A.

$n \stackrel{\text{def}}{=} |A|$ and $\mathsf{Sym}(X) \stackrel{\text{def}}{=}$ the symmetry group of X.

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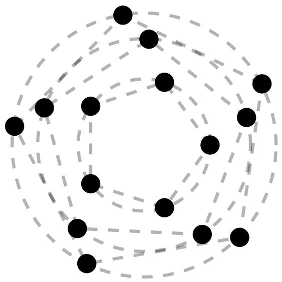

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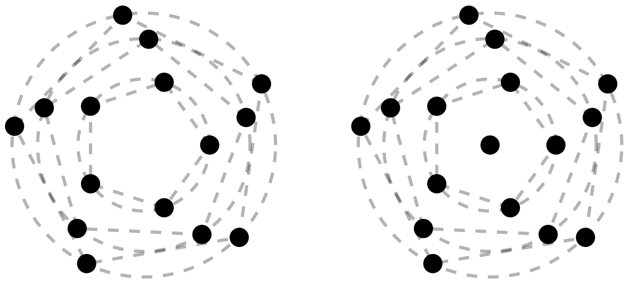

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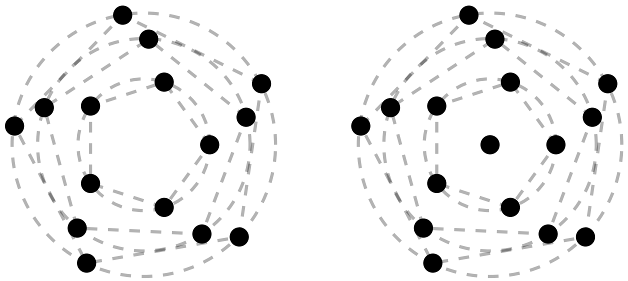

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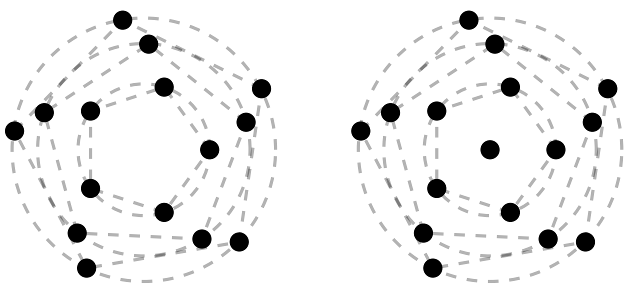

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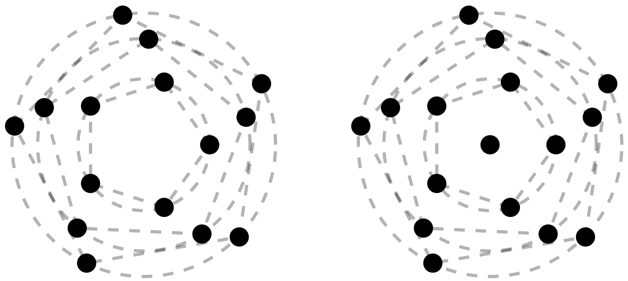

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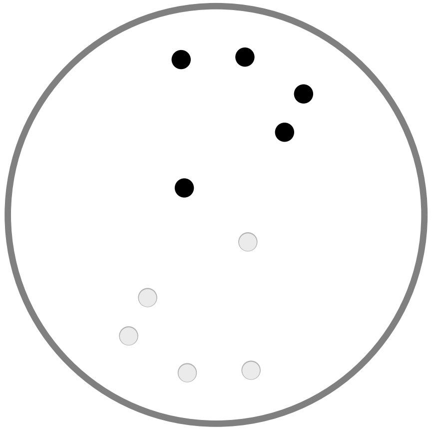

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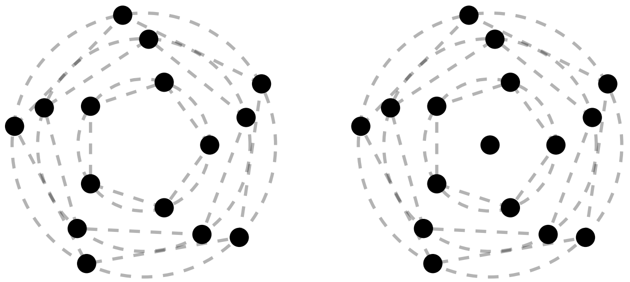

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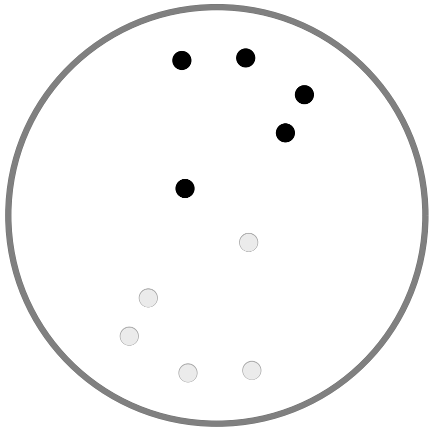

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Orbit type $[\mu_1,\mu_2,\ldots,\mu_k]$ means k orbits, of respective size μ_1, μ_2, $\ldots$, μ_k.

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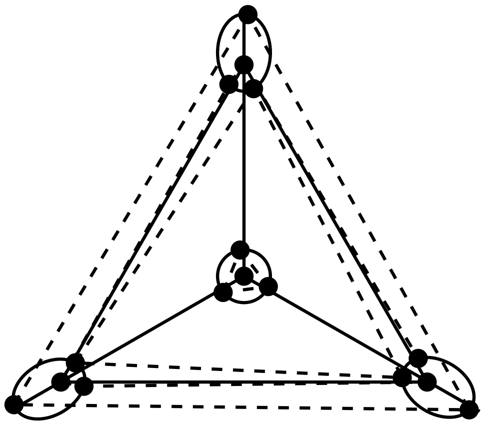

Each of these groups occurs as symmetry group of some projective order type.