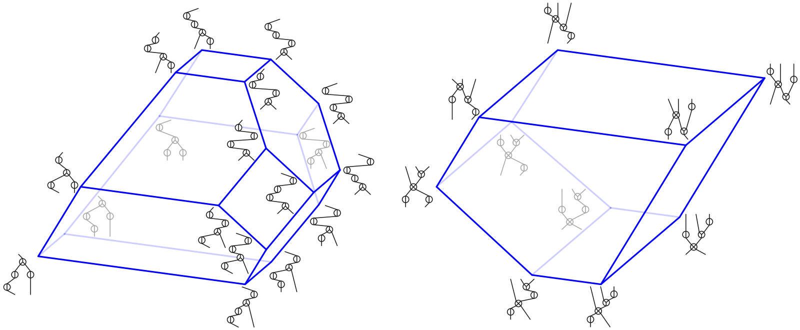
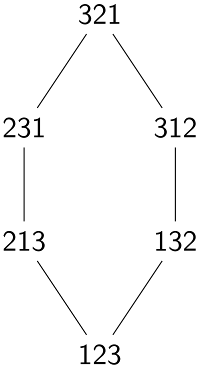
Permutrees

Vincent Pilaud – **Viviane Pons**
CNRS & Ecole Polytechnique – LRI, Univ. Paris-Saclay



Weak Order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213 132
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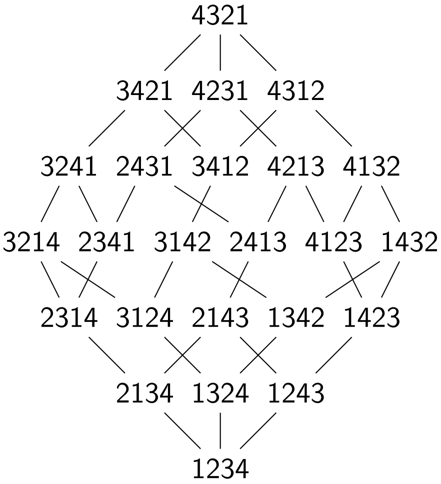
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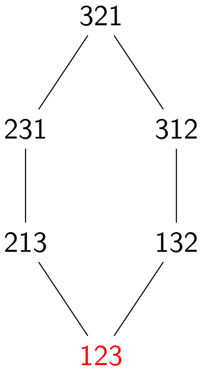
3241 2431 3412 4213 4132
3214 2341 3142 2413 4123 1432
2314 3124 2143 1342 1423
2134 1324 1243
1234

Weak Order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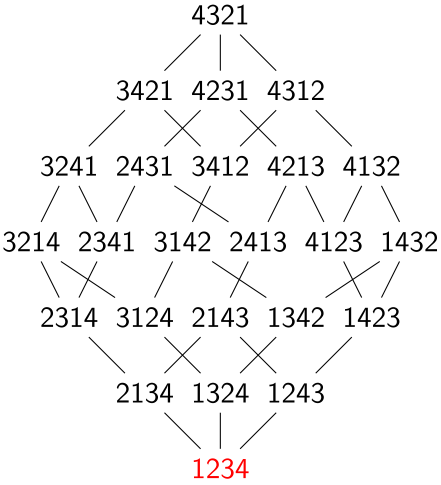
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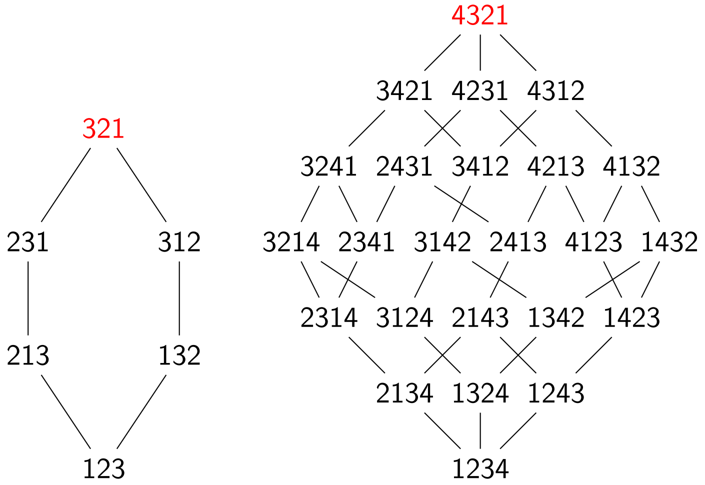
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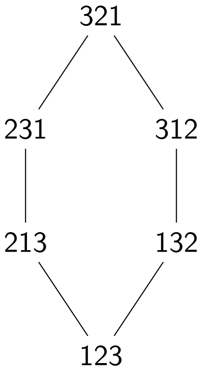
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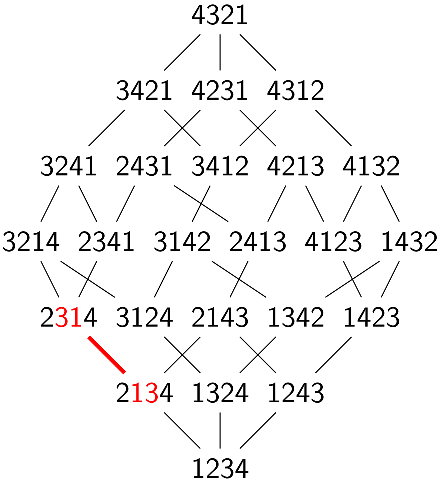
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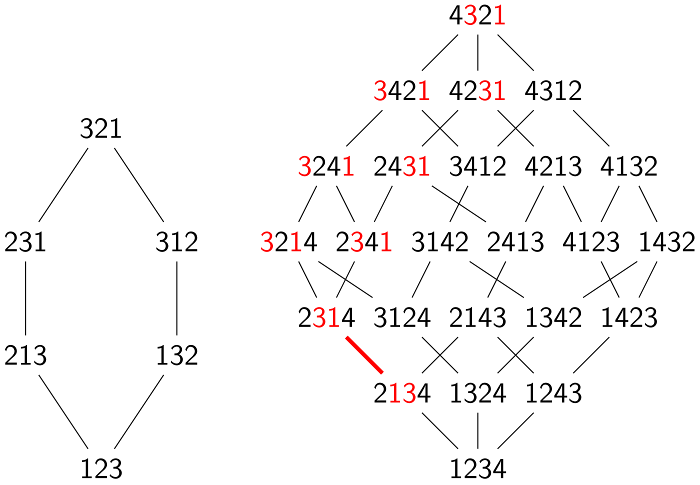
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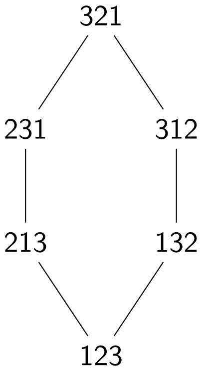
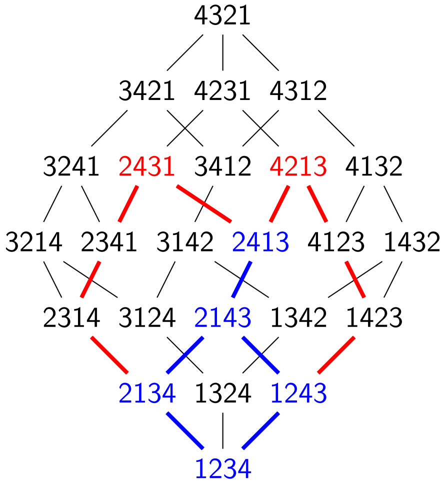
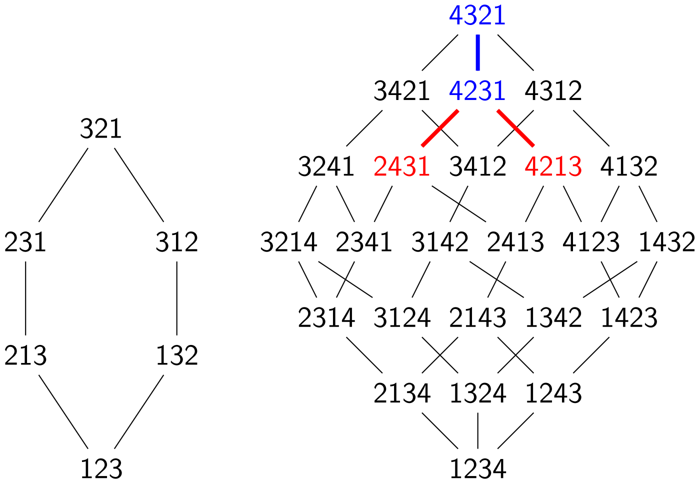
Weak Order





$2413 \wedge 4213 = 2413$ $2413 \vee 4213 = 4231$

Weak Order



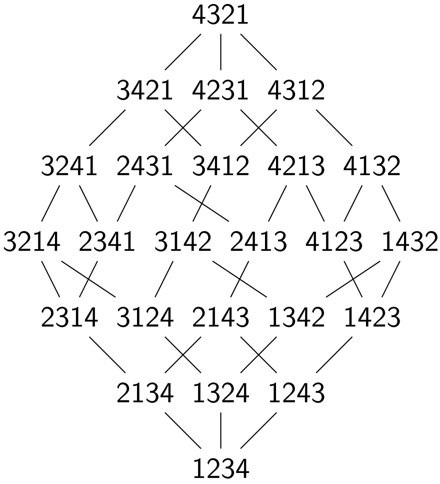
$2413 \wedge 4213 = 2413$

$2413 \vee 4213 = 4231$

Lattice congruence

The equivalence relation must be compatible with the lattice structure

$$\begin{array}{l} x_1 \equiv x_2 \\ y_1 \equiv y_2 \end{array} \Rightarrow \begin{array}{l} (x_1 \wedge y_1) \equiv (x_2 \wedge y_2) \\ (x_1 \vee y_1) \equiv (x_2 \vee y_2) \end{array}$$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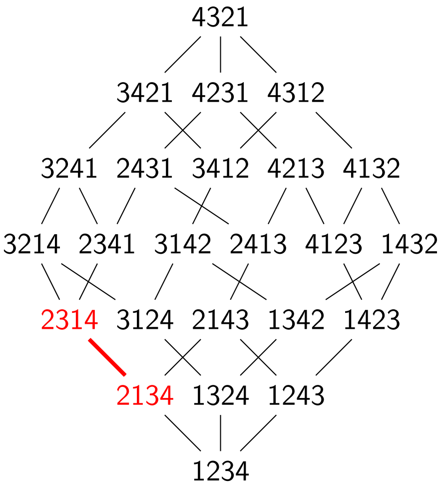
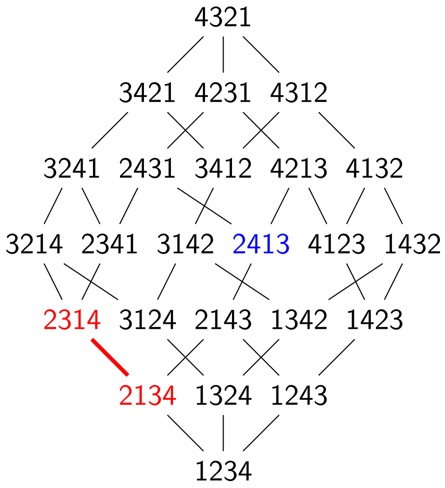
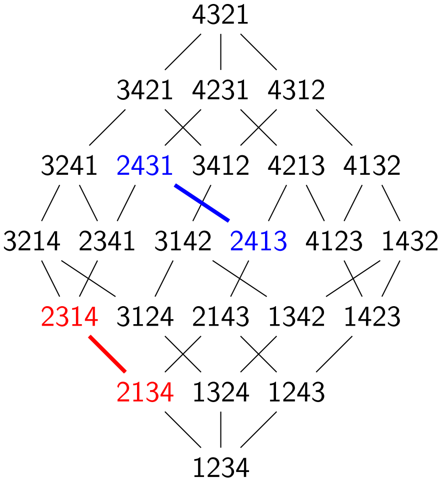


$2134 \equiv 2314$

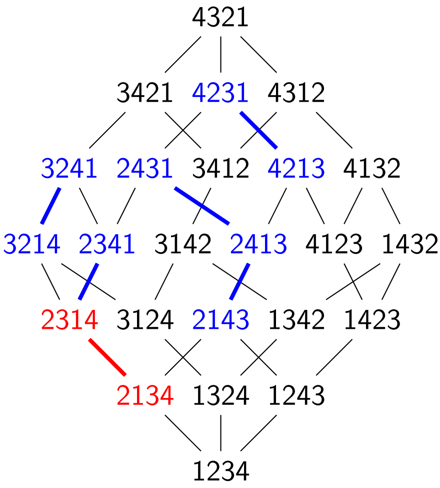


$$2134 \equiv 2314$$

$$2134 \vee 2413 \equiv 2314 \vee 2413$$



$$2134 \equiv 2314$$
$$2134 \vee 2413 \equiv 2314 \vee 2413$$
$$2431 \equiv 2413$$



$2134 \equiv 2314 \equiv 2341$

$2143 \equiv 2413 \equiv 2431$

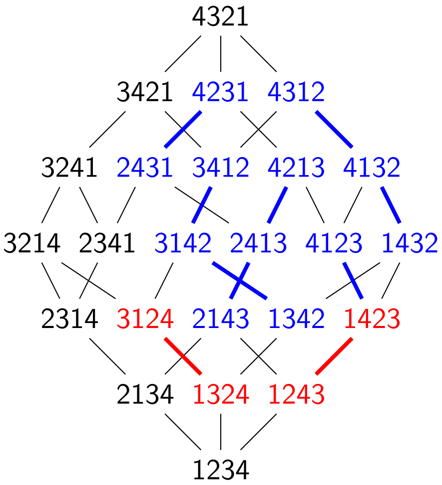
$3214 \equiv 3241$

$4213 \equiv 4231$

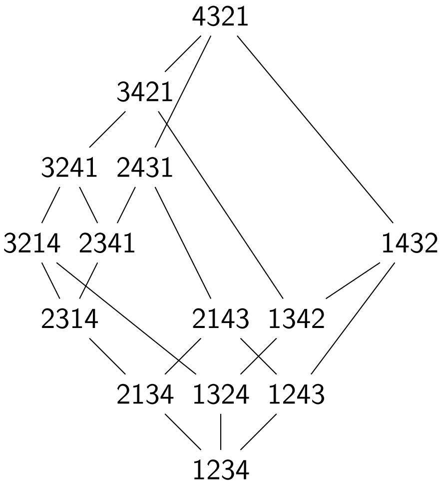
What are all the lattice congruence of the Weak order?

Lattice Congruences of the Weak Order, Nathan Reading, *Order*, 2005

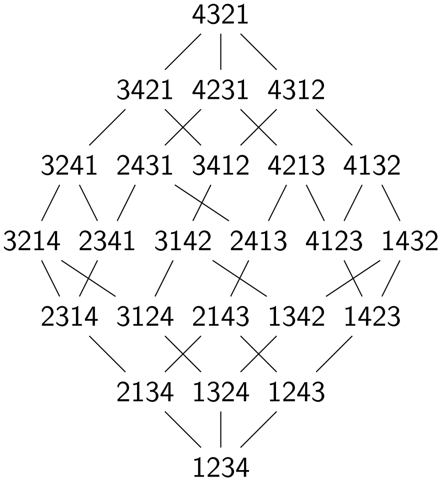
the sylvester congruence and the Tamari lattice



the sylvester congruence and the Tamari lattice

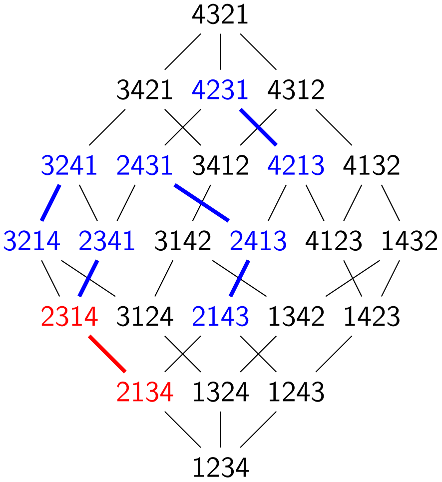


The Permutree congruences

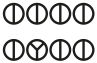


→ ⦶⦶⦶⦶ 24 classes

The Permutree congruences



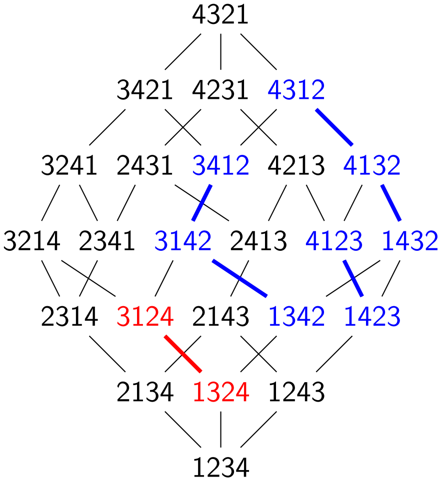
$\rightarrow$



24 classes

18 classes

The Permutree congruences



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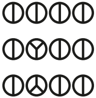
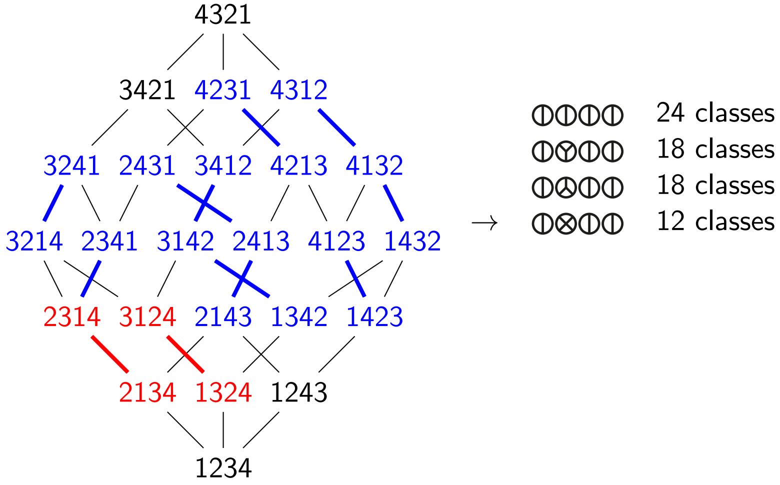
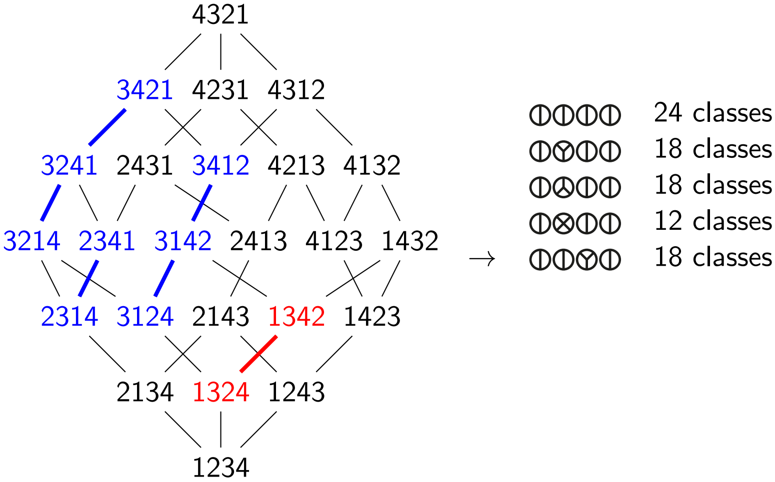


24 classes
18 classes
18 classes

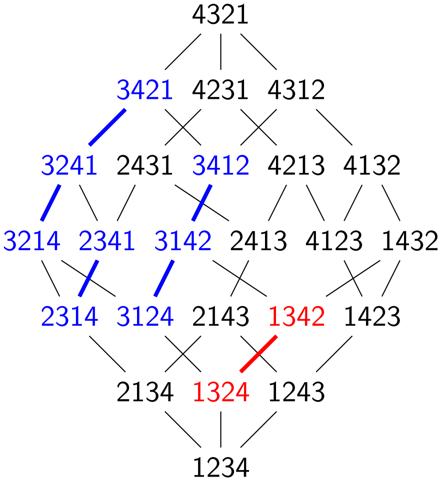
The Permutree congruences



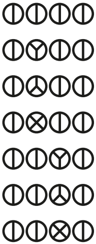
The Permutree congruences



The Permutree congruences

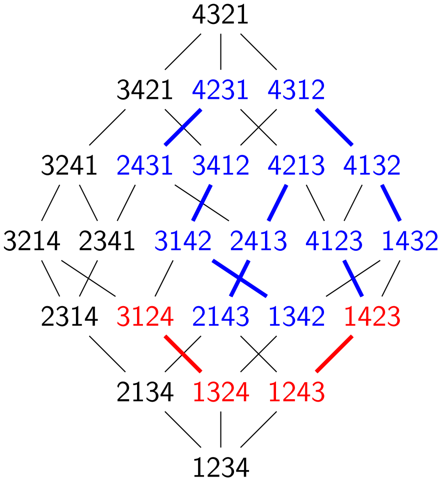


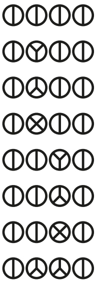
→



24 classes
18 classes
18 classes
12 classes
18 classes
18 classes
12 classes

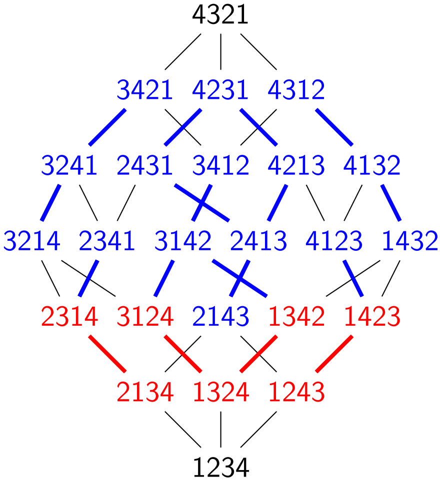
The Permutree congruences

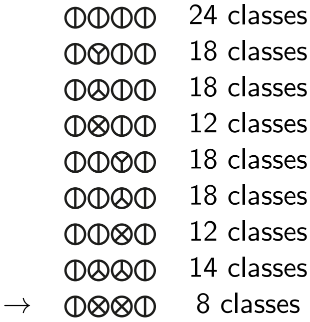




24 classes
18 classes
18 classes
12 classes
18 classes
18 classes
12 classes
→ 14 classes

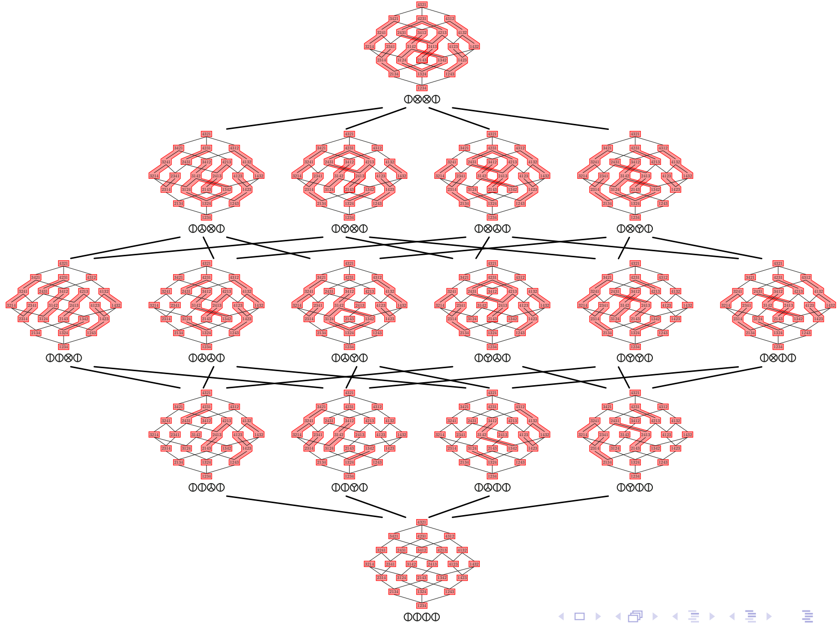
The Permutree congruences





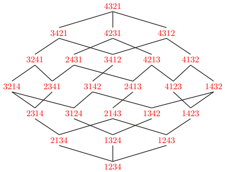
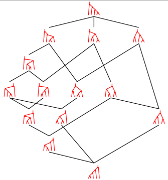
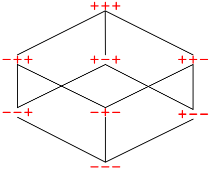
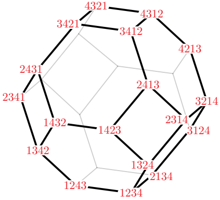
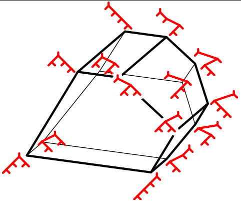
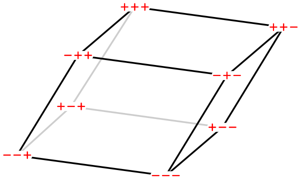
Numerology

⦶⊗⊗⦶ 8					
⦶☮⊗⦶ 10	⦶Ⓨ⊗⦶ 10	⦶⊗☮⦶ 10	⦶⊗Ⓨ⦶ 10		
⦶⦶⊗⦶ 12	⦶☮☮⦶ 14	⦶☮Ⓨ⦶ 14	⦶Ⓨ☮⦶ 14	⦶ⓎⓎ⦶ 14	⦶⊗⦶⦶ 12
⦶⦶☮⦶ 18	⦶⦶Ⓨ⦶ 18	⦶☮⦶⦶ 18	⦶Ⓨ⦶⦶ 18		
⦶⦶⦶⦶ 24					



Permutrees

The permutrees are combinatorial objects which encode those lattice congruences.

<table>
<tr><th></th><th>permutations
⦶⦶⦶⦶</th><th>binary trees
☮☮☮☮</th><th>binary sequences
⊗⊗⊗⊗</th></tr>
<tr><td>Combinatorics</td><td></td><td></td><td></td></tr>
<tr><td>Geometry</td><td></td><td></td><td></td></tr>
<tr><td>Algebra</td><td>Malvenuto-Reutenauer algebra
$\mathsf{FQSym} = \mathrm{vect}\langle \mathbb{F}_\tau \mid \tau \in \mathfrak{S} \rangle$
$\mathbb{F}_\tau \cdot \mathbb{F}_{\tau'} = \sum_{\sigma \in \tau \,\bar{\sqcup}\, \tau'} \mathbb{F}_\sigma$
$\triangle \mathbb{F}_\sigma = \sum_{\sigma \in \tau \star \tau'} \mathbb{F}_\tau \otimes \mathbb{F}_{\tau'}$</td><td>Loday-Ronco algebra
$\mathsf{PBT} = \mathrm{vect}\langle \mathbb{P}_\mathrm{T} \mid \mathrm{T} \in \mathcal{BT} \rangle$
$\mathbb{P}_T \cdot \mathbb{P}_{T'} = \sum_{{}^{T}\nearrow{}^{T'} \le T'' \le {}_{T}\nwarrow{}_{T'}} \mathbb{P}_{T''}$
$\triangle \mathbb{F}_\gamma = \sum_{\gamma \text{ cut}} B(T,\gamma) \otimes A(T,\gamma)$</td><td>Solomon algebra
$\mathsf{Rec} = \mathrm{vect}\langle \mathbb{X}_\eta \mid \eta \in \pm^* \rangle$
$\mathbb{X}_\eta \cdot \mathbb{X}_{\eta'} = \mathbb{X}_{\eta+\eta'} + \mathbb{X}_{\eta-\eta'}$
$\triangle \mathbb{X}_\eta = \sum_{\gamma \text{ cut}} B(\eta,\gamma) \otimes A(\eta,\gamma)$</td></tr>
</table>

The Permutree Recipe

- Take a word in $\{\text{⦶}, \text{☮}, \text{Ⓨ}, \otimes\}^n$
- Take a permutation
- Do the insertion: get a Leveled Permutree (bijection)
- Remove the levels: get a Permutree (surjection)

The Permutree Recipe

- Take a word in $\{⦶, ☮, Ⓨ, ⊗\}^n$
- Take a permutation
- Do the insertion: get a Leveled Permutree (bijection)
- Remove the levels: get a Permutree (surjection)

Example

$⦶^n$	$\longleftrightarrow$	permutations of $[n]$
$☮^n$	$\longleftrightarrow$	standard binary search trees
$\{☮, Ⓨ\}^n$	$\longleftrightarrow$	Cambrian trees (Reading)
$⊗^n$	$\longleftrightarrow$	binary sequences

The Permutree insertion

Permutation: 2751346

Decoration: ☮Ⓨ⦶⊗⦶☮Ⓨ

The Permutree insertion

Permutation: 2751346

Decoration: ☮Ⓨ⦶⊗⦶☮Ⓨ

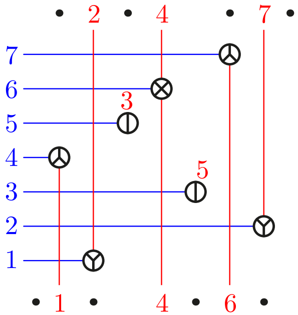
Decorated permutation: $\overline{27}5\underline{1}3\underline{\overline{4}6}$

The Permutree insertion

Permutation: 2751346

Decoration: ☮Ⓨ⦶⊗⦶☮Ⓨ

Decorated permutation: $\overline{2}\overline{7}5\underline{1}3\underline{\overline{4}}\underline{6}$



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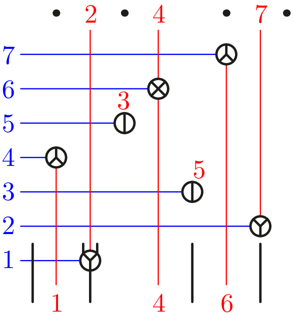
Permutation: 2751346

Decoration: ☮Ⓨ⦶⊗⦶☮Ⓨ

Decorated permutation: $\overline{27}5\underline{1}3\overline{\underline{4}}\underline{6}$

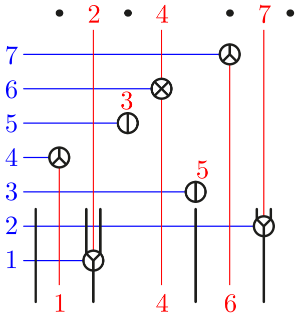


The Permutree insertion

Permutation: 2751346

Decoration: ☮ Ⓨ ⦶ ⊗ ⦶ ☮ Ⓨ

Decorated permutation: $\overline{2}\overline{7}5\underline{1}3\overline{\underline{4}}\underline{6}$

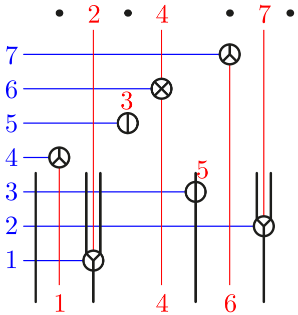


The Permutree insertion

Permutation: 2751346

Decoration: ☮Ⓨ⦶⊗⦶☮Ⓨ

Decorated permutation: $\overline{2}\,\overline{7}\,5\,\underline{1}\,3\,\underline{\overline{4}}\,\underline{6}$

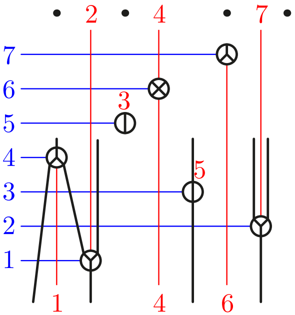


The Permutree insertion

Permutation: 2751346

Decoration: ☮ⓎⓘⓍⓘ☮Ⓨ

Decorated permutation: $\overline{2}\overline{7}5\underline{1}3\overline{\underline{4}}\underline{6}$

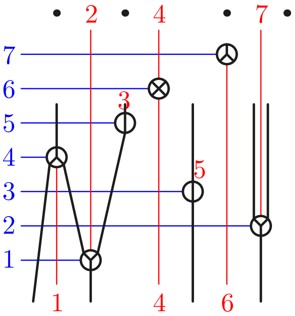


The Permutree insertion

Permutation: 2751346

Decoration: ☮ Ⓨ ⦶ ⊗ ⦶ ☮ Ⓨ

Decorated permutation: $\overline{2}\overline{7}5\underline{1}3\overline{\underline{4}}\underline{6}$



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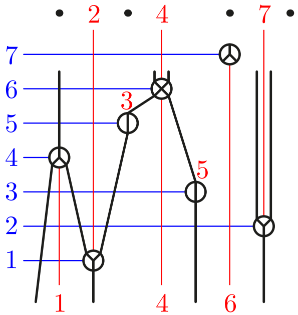
Permutation: 2751346

Decoration: ☮Ⓨ⦶⊗⦶☮Ⓨ

Decorated permutation: $\overline{27}5\underline{1}3\underline{\overline{4}6}$

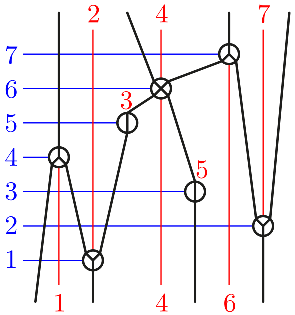


The Permutree insertion

Permutation: 2751346

Decoration: ⓋⓎ⦶⊗⦶ⓋⓎ

Decorated permutation: $\overline{27}5\underline{1}3\underline{\overline{4}6}$

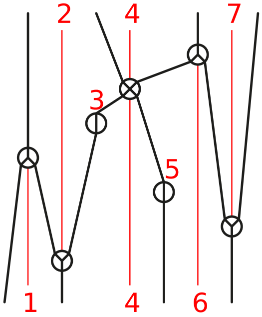


The Permutree insertion

Permutation: 2751346

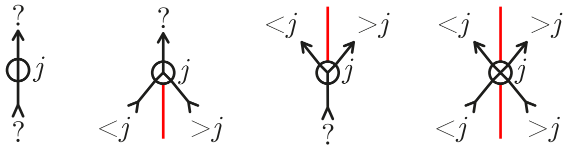
Decoration: ☮Ⓨ⦶⊗⦶☮Ⓨ

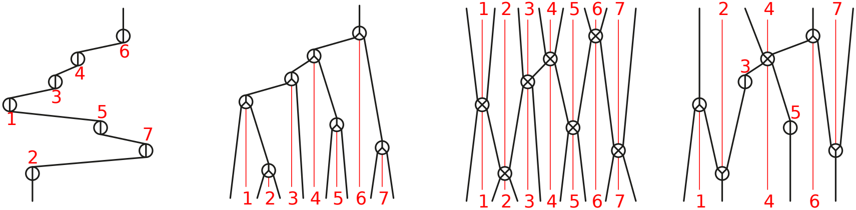
Decorated permutation: $\overline{2}\overline{7}5\underline{1}3\overline{\underline{4}}\underline{6}$



Definition of a permutree

directed (bottom to top) and labeled (bijectively by $[n]$) tree such that





The Permutree insertion

Permutation : 2751346

Decorations:

Decorated Permutations:

2751346 $\quad \underline{2751346} \quad \overline{\underline{2751346}} \quad \overline{2}\overline{7}5\underline{1}3\overline{\underline{4}}\underline{6}$

Leveled permutrees:

The Permutree insertion

Permutation : 2751346

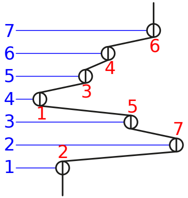
Decorations:

⦶⦶⦶⦶⦶⦶⦶ ☮☮☮☮☮☮☮ ⊗⊗⊗⊗⊗⊗⊗ ☮Ⓨ⦶⊗⦶☮Ⓨ

Decorated Permutations:

2751346 $\underline{2751346}$ $\overline{\underline{2751346}}$ $\overline{27}5\underline{1}3\underline{\overline{4}6}$

Leveled permutrees:



The Permutree insertion

Permutation : 2751346

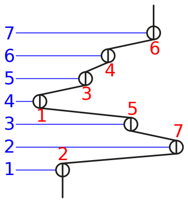
Decorations:

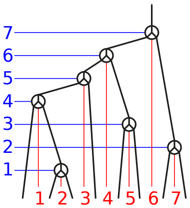
⦶⦶⦶⦶⦶⦶⦶ ☮☮☮☮☮☮☮ ⊗⊗⊗⊗⊗⊗⊗ ☮Ⓨ⦶⊗⦶☮Ⓨ

Decorated Permutations:

2751346 $\underline{2751346}$ $\overline{\underline{2751346}}$ $\overline{2}\overline{7}5\underline{1}3\overline{\underline{4}}\underline{6}$

Leveled permutrees:





The Permutree insertion

Permutation : 2751346

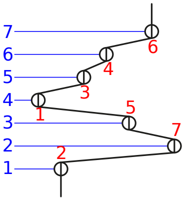
Decorations:

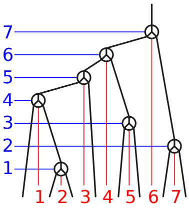
⦶⦶⦶⦶⦶⦶⦶ ☮☮☮☮☮☮☮ ⊗⊗⊗⊗⊗⊗⊗ ☮Ⓨ⦶⊗⦶☮Ⓨ

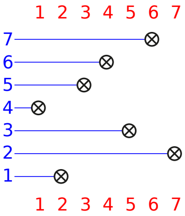
Decorated Permutations:

2751346 $\underline{2751346}$ $\underline{\overline{2751346}}$ $\overline{2}\,\overline{7}5\underline{1}3\underline{\overline{4}}\underline{6}$

Leveled permutrees:







The Permutree insertion

Permutation : 2751346

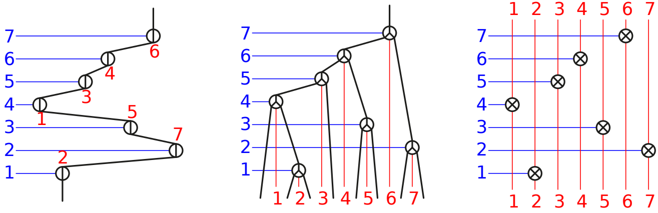
Decorations:

⦶⦶⦶⦶⦶⦶⦶ ☮☮☮☮☮☮☮ ⊗⊗⊗⊗⊗⊗⊗ ☮Ⓨ⦶⊗⦶☮Ⓨ

Decorated Permutations:

2751346 $\underline{2751346}$ $\underline{\overline{2751346}}$ $\overline{2}\overline{7}5\underline{1}3\underline{\overline{4}}\underline{6}$

Leveled permutrees:



The Permutree insertion

Permutation : 2751346

Decorations:

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Decorated Permutations:

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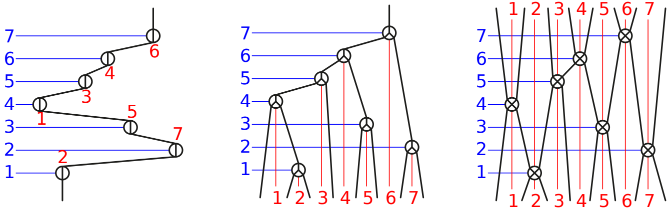
Leveled permutrees:



The Permutree insertion

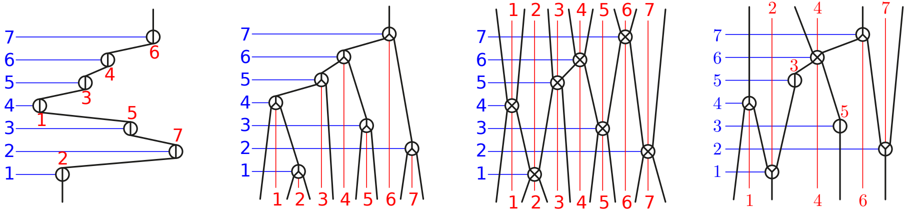
Permutation : 2751346

Decorations:

Decorated Permutations:

2751346 $\underline{2751346}$ $\overline{\underline{2751346}}$ $\overline{27}5\underline{1}3\overline{\underline{4}}\underline{6}$

Leveled permutrees:



The Permutree insertion

Permutation : 2751346

Decorations:

⦶⦶⦶⦶⦶⦶⦶ ☮☮☮☮☮☮☮ ⊗⊗⊗⊗⊗⊗⊗ ☮Ⓨ⦶⊗⦶☮Ⓨ

Decorated Permutations:

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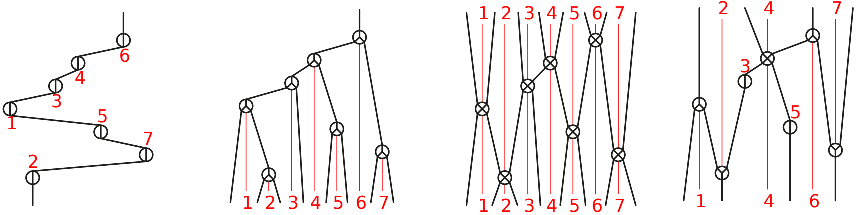
~~Leveled~~ permutrees:



Insertion

$$\begin{aligned} \mathfrak{S}_n \times \{\text{⦶}, \text{☮}, \text{Ⓨ}, \otimes\}^n &\longrightarrow \text{Permutrees} \\ (\sigma, \delta) &\longrightarrow \mathbf{P}_\delta(\sigma) \end{aligned}$$

Insertion

$$\mathfrak{S}_n \times \{\text{⦶}, \text{☮}, \text{Ⓨ}, \otimes\}^n \longrightarrow \text{Permutrees}$$
$$(\sigma, \delta) \longrightarrow \mathbf{P}_\delta(\sigma)$$

Congruence

$$\dots ac \dots \underline{b} \dots \equiv_\delta \dots ca \dots \underline{b} \dots \qquad (\delta_b \in \{\text{☮}, \otimes\})$$
$$\dots \overline{b} \dots ac \dots \equiv_\delta \dots \overline{b} \dots ca \dots \qquad (\delta_b \in \{\text{Ⓨ}, \otimes\})$$

Insertion

$$\begin{aligned} \mathfrak{S}_n \times \{⦶, ☮, Ⓨ, \otimes\}^n &\longrightarrow \text{Permutrees} \\ (\sigma, \delta) &\longrightarrow \mathbf{P}_\delta(\sigma) \end{aligned}$$

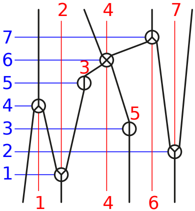
Congruence

$$\dots ac \dots \underline{b} \dots \equiv_\delta \dots ca \dots \underline{b} \dots \qquad (\delta_b \in \{☮, \otimes\})$$

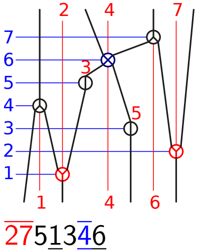
$$\dots \overline{b} \dots ac \dots \equiv_\delta \dots \overline{b} \dots ca \dots \qquad (\delta_b \in \{Ⓨ, \otimes\})$$

Property

$$\sigma \equiv_\delta \tau \Leftrightarrow \mathbf{P}_\delta(\sigma) = \mathbf{P}_\delta(\tau)$$



$\overline{27}5\underline{1}3\underline{\overline{4}6}$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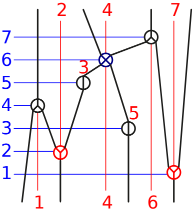
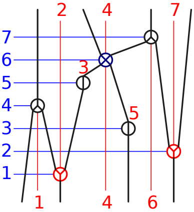
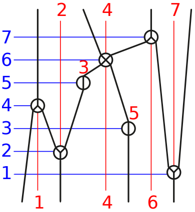


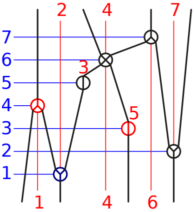
$\overline{72}51\underline{}3\overline{4}6$ —



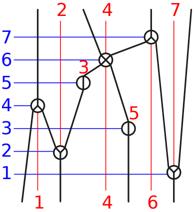
$\overline{27}51\underline{}3\overline{4}6$



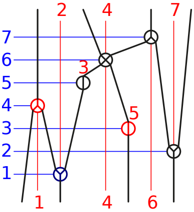
$\overline{7}\overline{2}5\underline{1}3\overline{\underline{4}}\underline{6} \equiv$



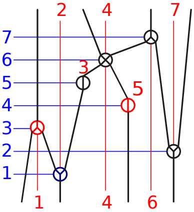
$\overline{2}75\underline{1}3\overline{\underline{4}}\underline{6}$



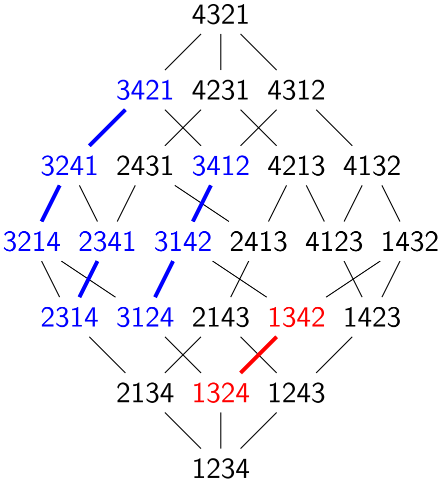
$\overline{7}\overline{2}5\underline{1}3\underline{\overline{4}}\underline{6} \equiv$



$\overline{2}\overline{7}5\underline{1}3\underline{\overline{4}}\underline{6}$



$\equiv \overline{2}\overline{7}\underline{1}53\underline{\overline{4}}\underline{6}$



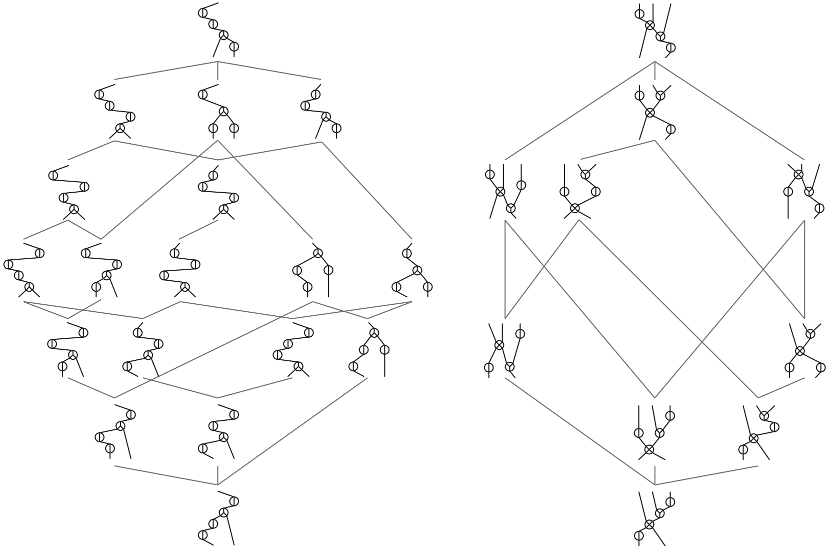
Decoration: ⦶⦶Ⓨ⦶

$\overline{3}214 \equiv \overline{3}241 \equiv \overline{3}421$

$2\overline{3}14 \equiv 2\overline{3}41$

$\overline{3}124 \equiv \overline{3}142 \equiv \overline{3}412$

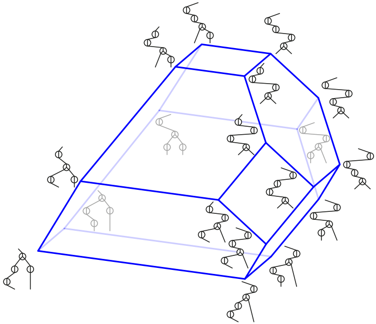
$1\overline{3}24 \equiv 1\overline{3}42$



The Permutreehedron

Thm (Pilaud, P.) for every $\delta \in \{⦶, ⦸, ⦸, \otimes\}^n$, there is an explicit construction of a

- a complete simplicial fan, the δ-permutree fan $\mathcal{F}(\delta)$;
- a polytope, the Permutreehedron $\mathbb{PT}(\delta)$, whose normal fan is $\mathcal{F}(\delta)$.



The Permutreehedron

Thm (Pilaud, P.) for every $\delta \in \{\textcircled{\tiny |}, \textcircled{\tiny \wedge}, \textcircled{\tiny \vee}, \otimes\}^n$, there is an explicit construction of a

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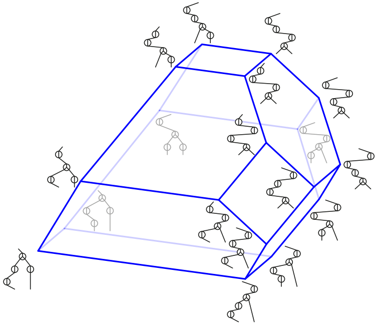

The Permutreehedron can be constructed by convex hull or hyperplane intersection.

The Permutreehedron

Thm (Pilaud, P.) for every $\delta \in \{⦶, ☮, ⛒, \otimes\}^n$, there is an explicit construction of a

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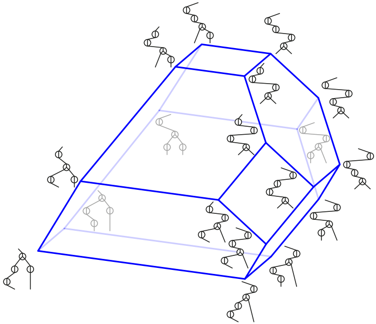

The vertices of $\mathbb{PT}(\delta)$ are the δ-permutrees.

The Permutreehedron

Thm (Pilaud, P.) for every $\delta \in \{⦶, ☮, ⦸, \otimes\}^n$, there is an explicit construction of a

- a complete simplicial fan, the δ-permutree fan $\mathcal{F}(\delta)$;
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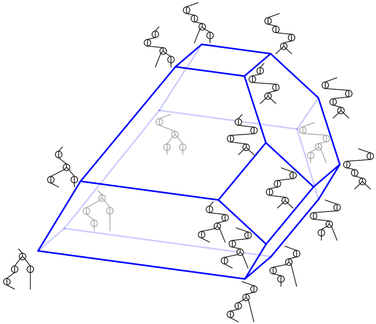

The oriented graph of $\mathbb{PT}(\delta)$ is the Hasse diagram of the δ-permutree lattice.

The Permutreehedron

Thm (Pilaud, P.) for every $\delta \in \{\text{⦶}, \text{☮}, \text{⦷}, \otimes\}^n$, there is an explicit construction of a

- a complete simplicial fan, the δ-permutree fan $\mathcal{F}(\delta)$;
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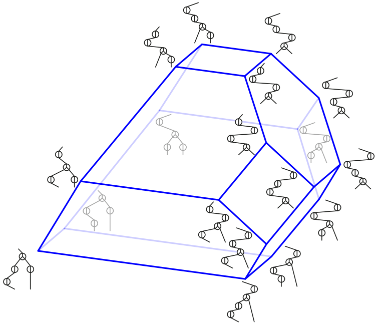
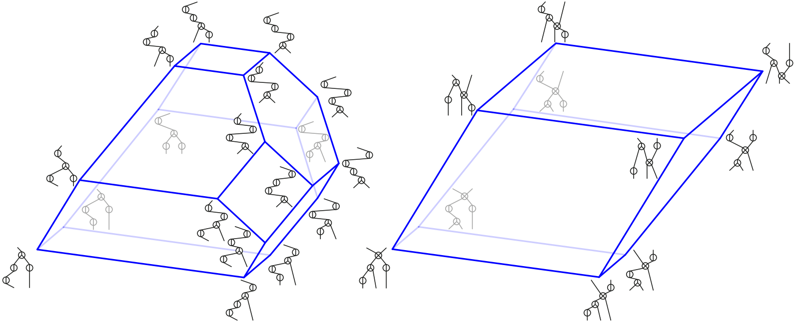
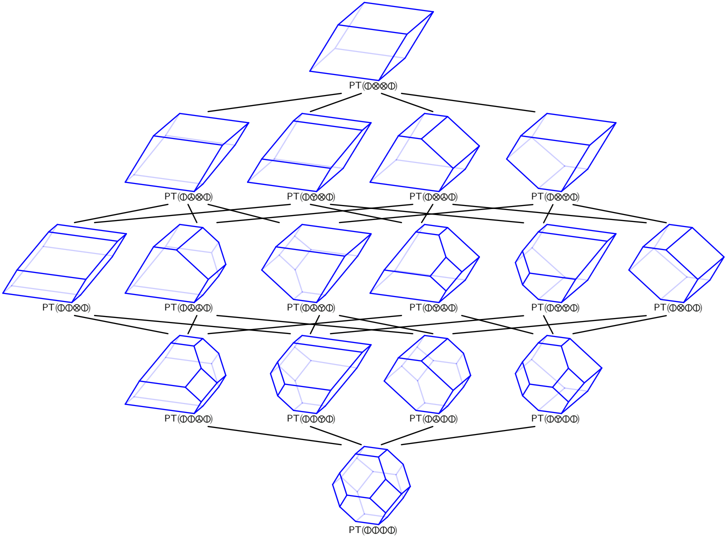


Combinatorics of the faces: Schröder permutrees.

Matriochka Permutreehedra

refinement $\delta \preccurlyeq \delta'$ $\implies$ inclusion $\mathbb{PT}(\delta) \subset \mathbb{PT}(\delta')$





Main paper

V. Pilaud, V. Pons. Permutrees. *Algebraic Combinatorics* (2018).

Some background

N. Reading. Cambrian lattices. *Advances in Mathematics* (2006)
G. Châtel, V. Pilaud, Cambiran Hopf algebras, *Advances in Mathematics* (2017)

New work

V. Pilaud, V. Pons, and D. Tamayo Jiménez. Permutree sorting. (2020)

Next step

Other types