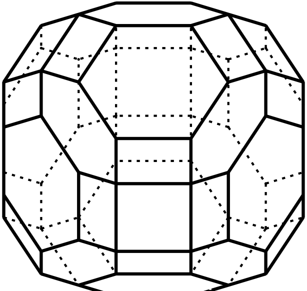
Sweep polytopes and sweep oriented matroids

Eva Philippe,
Joint work with Arnau Padrol

February 4, 2021



Sommaire

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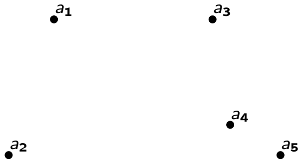

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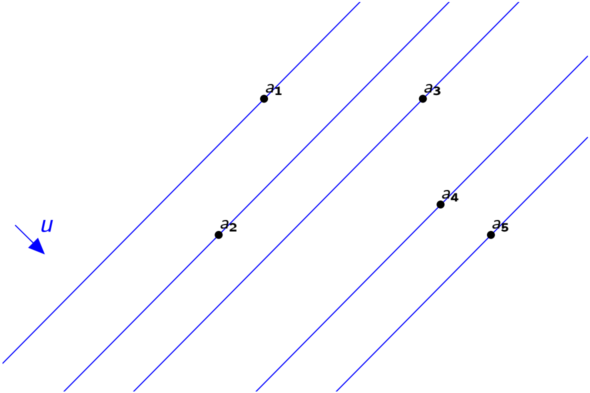

Sweep permutation $1, 2, 3, 4, 5$.

$\langle u, a_1 \rangle < \langle u, a_2 \rangle < \langle u, a_3 \rangle < \langle u, a_4 \rangle < \langle u, a_5 \rangle$.

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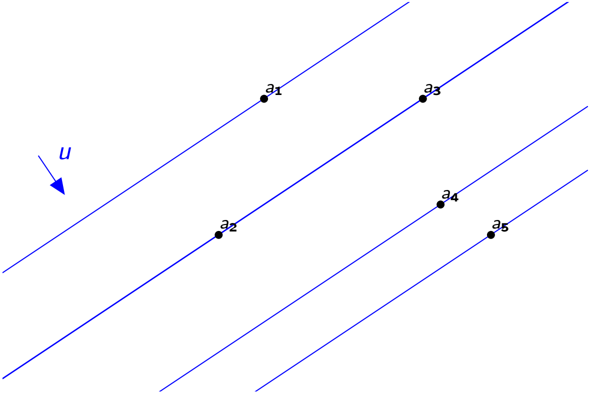

Sweep $1, 23, 4, 5.$

$$\langle u\,,\; a_1 \rangle < \langle u\,,\; a_2 \rangle = \langle u\,,\; a_3 \rangle < \langle u\,,\; a_4 \rangle < \langle u\,,\; a_5 \rangle\,.$$

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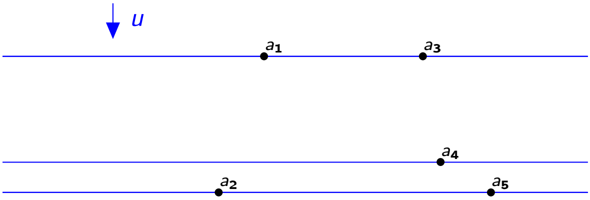

Sweep 13, 4, 25.

$$\langle u\,,\, a_1 \rangle = \langle u\,,\, a_3 \rangle < \langle u\,,\, a_4 \rangle < \langle u\,,\, a_2 \rangle = \langle u\,,\, a_5 \rangle\,.$$

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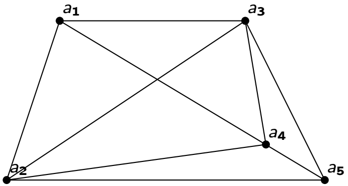
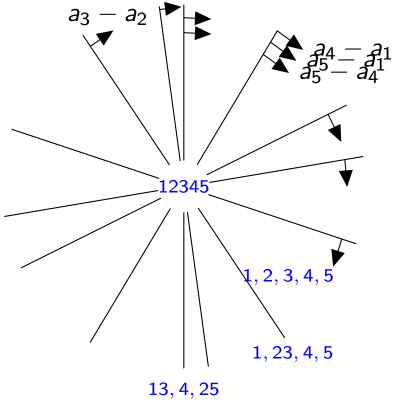


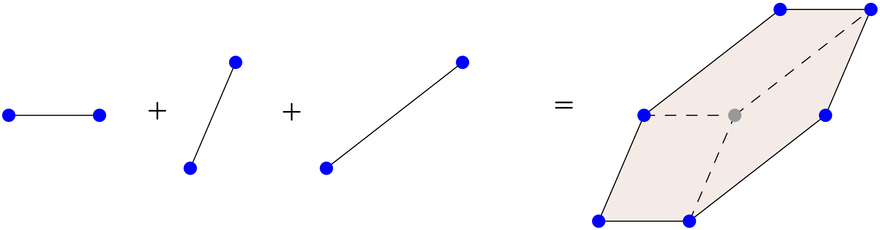


$\mathcal{SH}(A)$ = arrangement of the hyperplanes $\{u \in \mathbb{R}^d \mid \langle u\,,\, a_i\rangle = \langle u\,,\, a_j\rangle\}$, for all $(i,j) \in \binom{[n]}{2}$.
Its cones are in bijection with the sweeps of A.

Minkowski sum

Let $\boldsymbol{P}, \boldsymbol{Q}$ be two polytopes in $\mathbb{R}^d$.
Their *Minkowski sum* is the polytope :

$$\boldsymbol{P} + \boldsymbol{Q} = \{p + q \,|\, p \in \boldsymbol{P}, q \in \boldsymbol{Q}\}\,.$$



Sweep polytope

Definition

The *sweep polytope* $\boldsymbol{SP}(A)$ of the point configuration A is the *zonotope* :

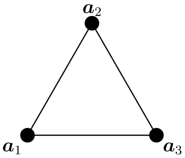
$$\boldsymbol{SP}(A) = \sum_{1 \leq i < j \leq n} \left[-\frac{a_i - a_j}{2}, \frac{a_i - a_j}{2} \right].$$

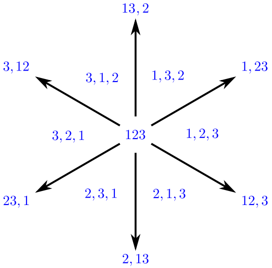
Its face poset is in bijection with the poset of sweeps of A.

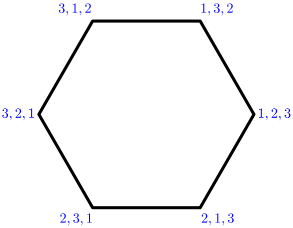
Sweeps of the simplex : the permutahedron

$A = \{e_1, \ldots, e_n\}$ in $\mathbb{R}^n$.

$$\boldsymbol{P}_n = \sum_{1 \leq i < j \leq n} \left[-\frac{e_i - e_j}{2}, \frac{e_i - e_j}{2} \right] = \text{conv}\left(\left\{ \sum_{i=1}^{n} \sigma(i) e_i \, , \, \sigma \in \mathfrak{S}_n \right\} \right) - \frac{n+1}{2} \sum_{i=1}^{n} e_i .$$







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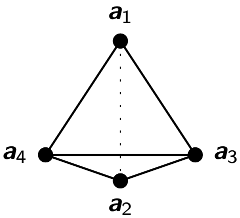
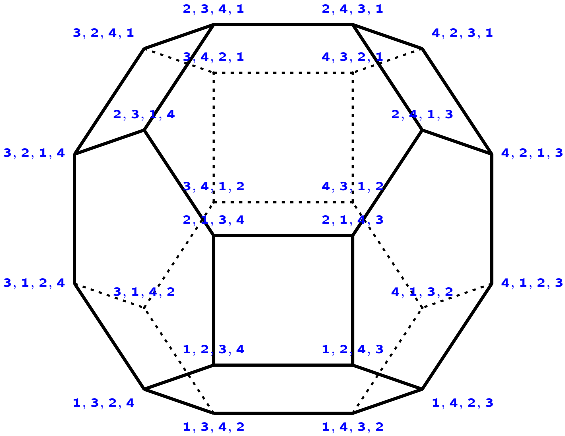
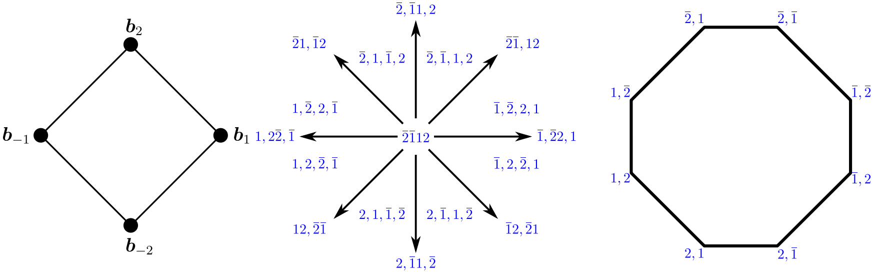




Sweeps of the cross-polytope

$A = \{-e_n, \ldots, -e_1, e_1, \ldots, e_n\}$ in $\mathbb{R}^n$.



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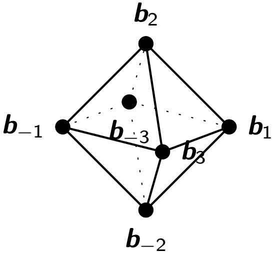

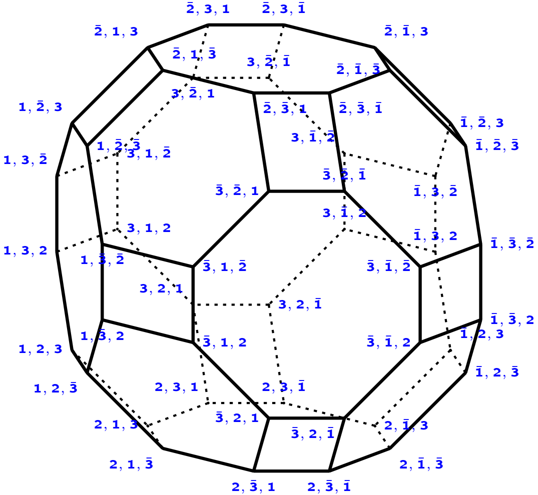

Projection of the permutahedron

$A = \{a_1, \ldots, a_n\}$ a configuration of n points in $\mathbb{R}^d$.
We consider the projection

$$\begin{aligned} M_A : \mathbb{R}^n &\to \mathbb{R}^d \\ e_i &\mapsto a_i. \end{aligned}$$

Then $\boldsymbol{SP}(A) = M_A(\boldsymbol{P}_n)$.

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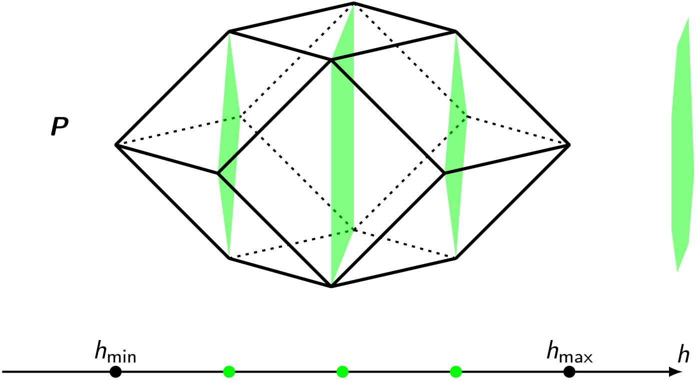
Then $\boldsymbol{SP}(A) = M_A(\boldsymbol{P}_n)$.

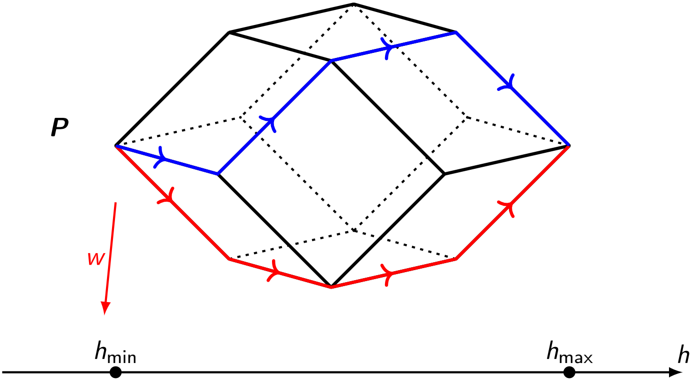
Conversely, if $M : \mathbb{R}^n \to \mathbb{R}^d$ is a linear map, $M(\boldsymbol{P}_n)$ is a sweep polytope.

Monotone path polytope



$$\Sigma(\boldsymbol{P}, h) = \frac{1}{h_{\max} - h_{\min}} \int_{h_{\min}}^{h_{\max}} h^{-1}(y)\, dy.$$

Monotone path polytope

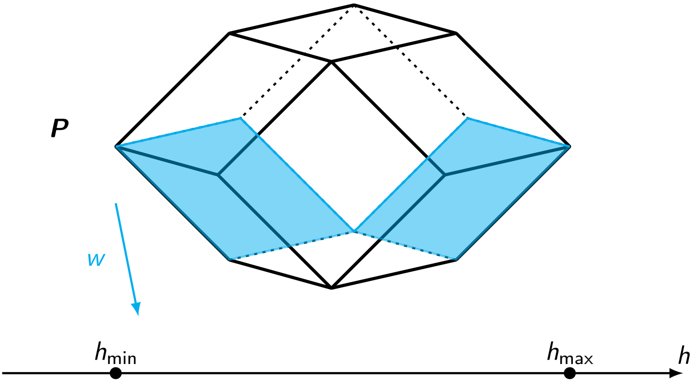


$$\Sigma(\boldsymbol{P}, h) = \left\{ \frac{1}{h_{\max} - h_{\min}} \int_{h_{\min}}^{h_{\max}} \gamma(y)\, dy \,\middle|\, \gamma \text{ is a section of } h \right\}.$$

Theorem (Billera-Sturmfels, 1992)

The vertices of $\Sigma(\boldsymbol{P}, h)$ are in bijection with the coherent h-monotone paths of $\boldsymbol{P}$.

Monotone path polytopes

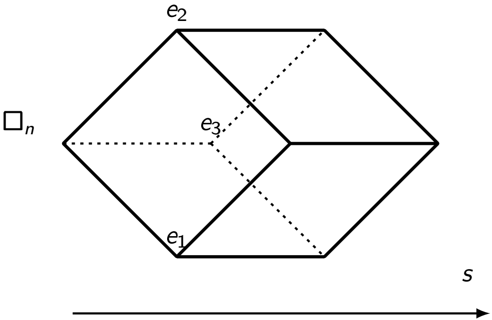


Theorem (Billera-Sturmfels, 1992)

The face poset of $\Sigma(\boldsymbol{P}, h)$ is isomorphic to the poset of coherent h-cellular strings of $\boldsymbol{P}$.

The permutahedron, monotone path polytope of the cube

$\square_n = [-1, 1]^n$, $s : x \in \mathbb{R}^n \mapsto \sum_{i=1}^n x_i$.



$$\Sigma(\square_n, s) = \frac{2}{n} \boldsymbol{P}_n.$$

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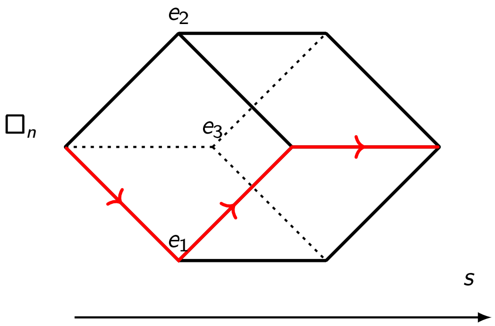
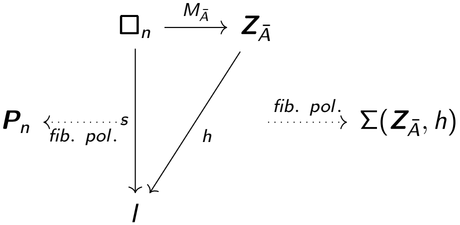
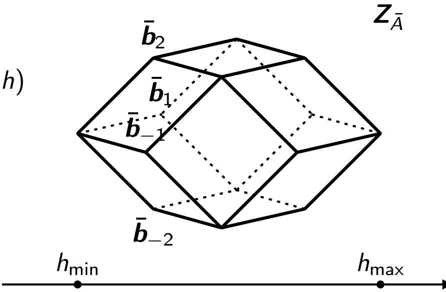


$$\Sigma(\square_n, s) = \frac{2}{n} \boldsymbol{P}_n.$$

$\bar{A} = \{(a_1, 1), \ldots, (a_n, 1)\}$ a homogenized configuration of n points in $\mathbb{R}^{d+1}$.

$\boldsymbol{Z}_{\bar{A}} = \sum_{i=1}^{n} [-(a_i, 1), (a_i, 1)] = M_{\bar{A}}(\square_n)$, $h : x \in \mathbb{R}^{d+1} \mapsto x_{d+1}$.



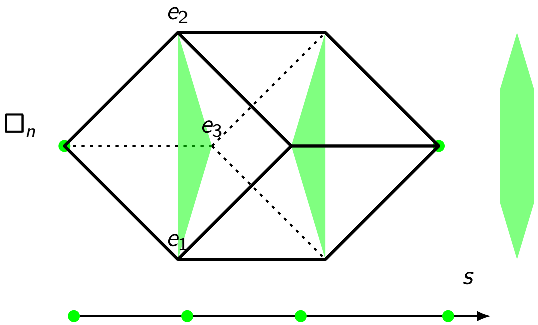


$\Sigma(\boldsymbol{Z}_{\bar{A}}, h) = M_{\bar{A}}(\boldsymbol{P}_n) = \boldsymbol{SP}(A)$

Sum of k-set polytopes

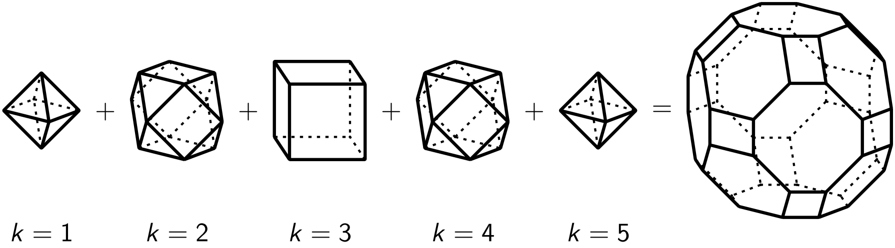
$\boldsymbol{P}_n + \frac{n+1}{2}\sum_{i=1}^{n} e_i = \sum_{k=0}^{n} \boldsymbol{\Delta}_{n,k}$, where $\boldsymbol{\Delta}_{n,k}$ is the *hypersimplex*

$$\boldsymbol{\Delta}_{n,k} = \text{conv}\left(\left\{\sum_{i\in I} e_i \mid I \subset n,\ |I| = k\right\}\right).$$



$A = \{a_1, \ldots, a_n\}$ a configuration of n points in $\mathbb{R}^d$.
$\boldsymbol{SP}(A)$ is the Minkowski sum of the *k-set polytopes* of A, for $k \in \{0, \ldots, n\}$:

$$\operatorname{conv}\left(\left\{\sum_{i \in I} a_i \mid I \subset n,\ |I| = k\right\}\right).$$



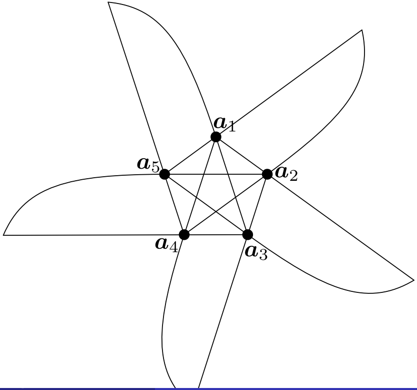
Sommaire

In dim 2, Goodman and Pollack's allowable sequences

An *allowable sequence of permutations* is a sequence of permutations in $\mathfrak{S}_n$ from the identity $1, 2, \ldots, n$ to its reverse $n, n-1, \ldots, 1$ such that :

- the move from a permutation to the next one consists of reversing one or more disjoint substrings ;
- each pair i, j with $1 \leq i < j \leq n$ is reversed in exactly one move along the sequence.

A non-realizable allowable sequence :
$(1,2,5,3,4) \to (2,1,5,3,4) \to$
$(2,1,3,5,4) \to (2,1,3,4,5) \to$
$(2,3,1,4,5) \to (3,2,1,4,5) \to$
$(3,2,4,1,5) \to (3,2,4,5,1) \to$
$(3,4,2,5,1) \to (4,3,2,5,1) \to$
$(4,3,5,2,1)$.



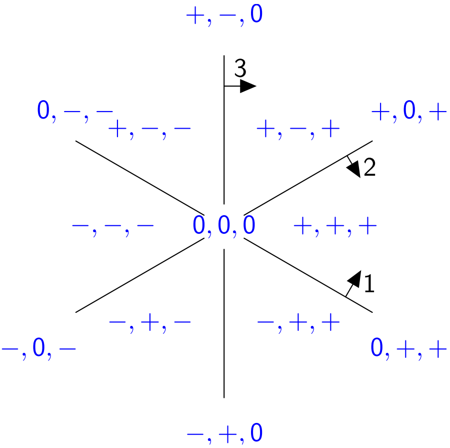
An *oriented matroid* $\mathcal{M}$ on ground set E can be described as a set of elements in $\{+,-,0\}^E$, called its *covectors*, that satisfies :

(V0) $\mathbf{0} \in \mathcal{M}$,

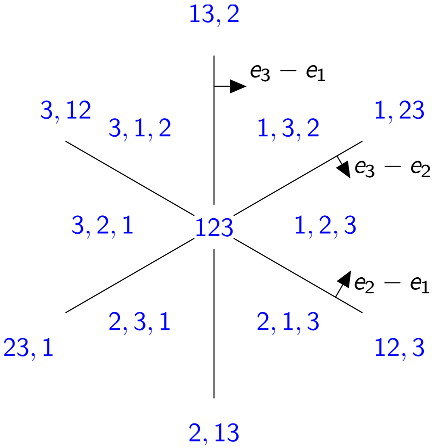
(V1) $X \in \mathcal{M}$ implies $-X \in \mathcal{M}$,

(V2) $X, Y \in \mathcal{M}$ implies $X \circ Y \in \mathcal{M}$,

(V3) if $X, Y \in \mathcal{M}$ and $e \in E$ is such that $(X_e, Y_e) \in \{(+,-),(-,+)\}$ then there exists $Z \in \mathcal{M}$ such that $Z_e = 0$ and $Z_f = (X \circ Y)_f$ for all $f \in E$ such that $(X_f, Y_f) \notin \{(+,-),(-,+)\}$.



Sweep oriented matroids



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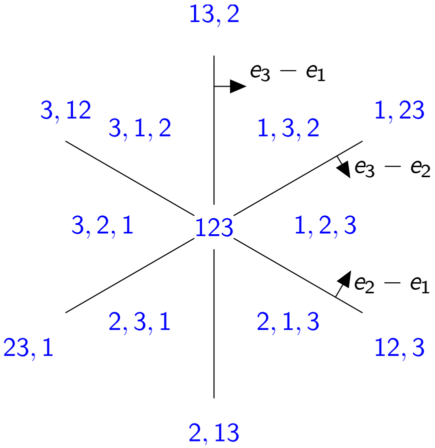
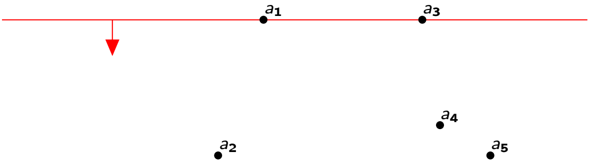


A *sweep oriented matroid* is an oriented matroid on ground set $\binom{[n]}{2}$ that is a *strong map* of the braid arrangement (all its covectors correspond to ordered partitions).

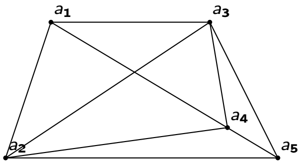
Little oriented matroid of a point configuration

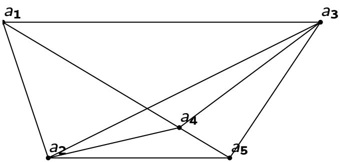
$A = \{a_1, \ldots, a_n\}$ a configuration of n points in $\mathbb{R}^d$.
The *little oriented matroid* of A is defined on ground set $[n]$ and corresponds to the hyperplane arrangement $\{(a_i, 1)^\perp, i \in [n]\}$ in $\mathbb{R}^{d+1}$.



Covector $(0, +, 0, +, +)$

From the sweep oriented matroid of a point configuration we can recover its little oriented matroid, but not the converse.





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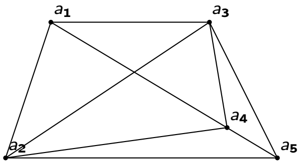
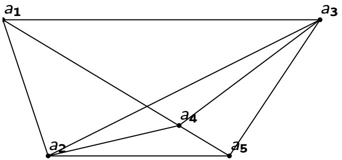
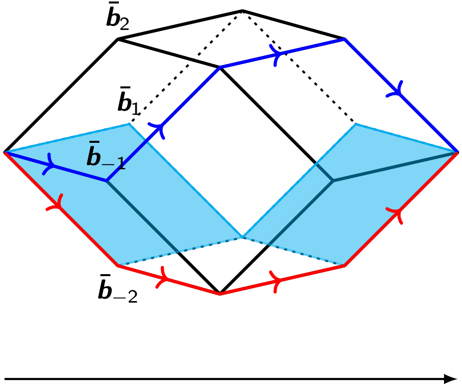


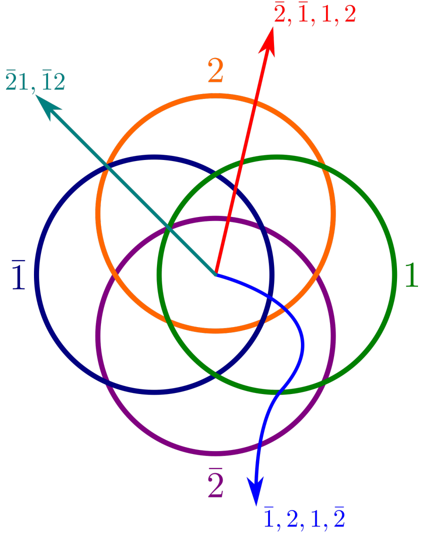


Not every oriented matroids is the little oriented matroid of a sweep oriented matroid.

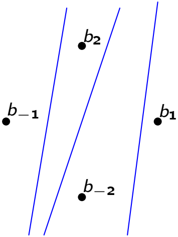
Sommaire

Realizable case : coherent and non-coherent cellular strings



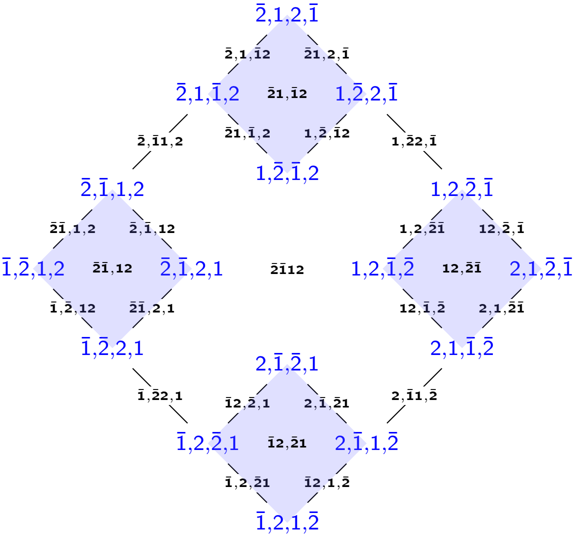


Sweeps and pseudo-sweeps



Pseudo-sweep $\bar{1}, 2, 1, \bar{2}$.

Realizable case : coherent and non-coherent cellular strings



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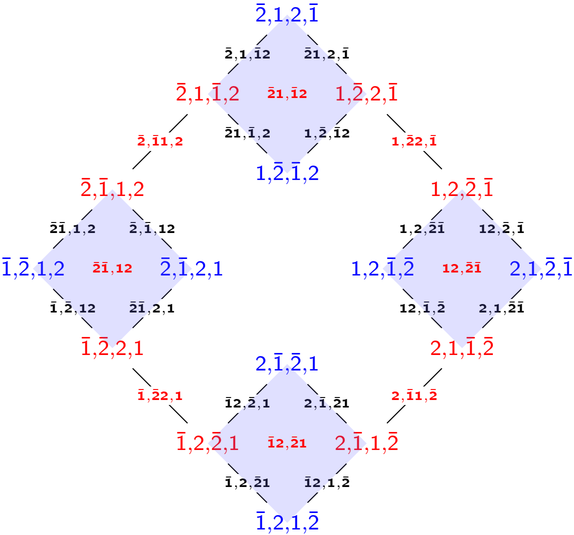
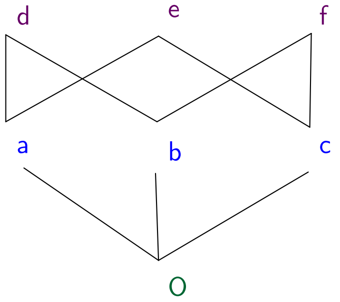
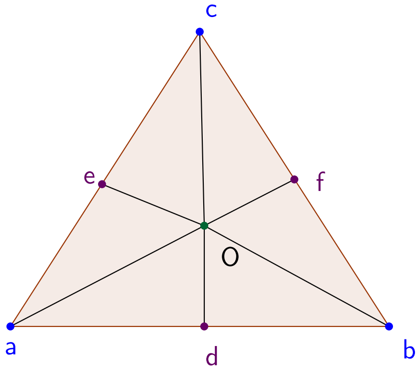


The *order complex* of a poset $\mathcal{P}$ is the simplicial complex of its chains.





Generalized Baues Problem for cellular strings

$\boldsymbol{P}$ a d-polytope, h a linear functional in $\mathbb{R}^d$,
$\omega(\boldsymbol{P}, h)$ the order complex of the non trivial cellular strings,
$\omega_{coh}(\boldsymbol{P}, h)$ the order complex of the non trivial coherent cellular strings.

Weak GBP. Is $\omega(\boldsymbol{P}, h)$ homotopy equivalent to a $(d-2)$-sphere ?
Strong GBP. Is $\omega_{coh}(\boldsymbol{P}, h)$ a deformation retract of $\omega(\boldsymbol{P}, h)$?

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Weak GBP. Is $\omega(\boldsymbol{P}, h)$ homotopy equivalent to a $(d-2)$-sphere ?
Strong GBP. Is $\omega_{coh}(\boldsymbol{P}, h)$ a deformation retract of $\omega(\boldsymbol{P}, h)$?

Answer : YES (Billera-Kapranov-Sturmfels, 1992)

Cellular strings of oriented matroids

Let $\mathcal{M}$ be an oriented matroid of rank r, and T one of its tope.
A *cellular string* of $(\mathcal{M}, T)$ is a sequence of covectors $(X_1, \ldots, X_m)$ that are not topes such that $X_1 \circ T = T$, $X_m \circ -T = -T$ and $X_i \circ (-T) = X_{i+1} \circ T$ for all i.

Theorem (Björner, 1992)

The poset $\omega(\mathcal{M}, T)$ of non-trivial cellular strings of $(\mathcal{M}, T)$ is homotopy equivalent to the $(r-2)$-sphere.

Strong GBP for cellular strings of little oriented matroids

Let $\mathcal{M}$ be an oriented matroid of rank r, that is the little oriented matroid of a sweep oriented matroid $\mathcal{M}^{\mathsf{sw}}$.
Then, $\Pi(\mathcal{M}^{\mathsf{sw}})$ the poset of sweeps of $\mathcal{M}^{\mathsf{sw}}$ is a subposet of $\omega(\mathcal{M},+)$, and it is a cell decomposition of the $(r-2)$-sphere.

Theorem (Padrol-P., 2021)

$\Pi(\mathcal{M}^{\mathsf{sw}})$ is a deformation retract of $\omega(\mathcal{M},+)$.

What I did not talk about

- modular hyperplane
- Dilworth truncation
- allowable graphs of permutations
- partial sweep polytopes

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Thank you for your attention !