On the Geometric Combinatorics of Steiner Points and a Conjecture of Grünbaum

Work in progress with Raman Sanyal

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May 20th 2021
Polytopes & f-vectors

The Steiner point

Generalized permutahedra

Tree Posets
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Polytopes

\[ f = (8, 12, 6, 1) \]

\[ 8 - 12 + 6 = 2 \]

\[ f = (24, 36, 14, 1) \]

\[ 24 - 36 + 14 = 2 \]
The f-vector of a polytope

**Definition**

For a $d$-polytope $P$ the **f-vector** $f(P) = (f_0, f_1, \ldots, f_d)$, is defined by

$$f_i = |\{ F \subseteq P \text{ face} : \dim(F) = i \}|.$$

**Theorem (Euler–Poincaré relation)**

The relation

$$f_0 - f_1 + \cdots + (-1)^d f_d = f_d$$

is the unique linear relation fulfilled by all f-vectors of $d$-polytopes.

Euler’s polyhedron formula: $f_0 - f_1 + f_2 = 2$
Simple polytopes

Definition

A $d$-polytope $P$ is simple if every vertex is incident to exactly $d$ edges. For a $d$-polytope $P$ the $h$-vector $h(P) = (h_0, h_1, \ldots, h_d)$ is defined by

$$
\sum_{i=0}^{d} h_i (t + 1)^i = \sum_{i=0}^{d} f_i t^i.
$$

Theorem (Dehn–Sommerville relations)

The \( \lfloor \frac{d+1}{2} \rfloor \) relations

$$
h_i = h_{d-i} \text{ for } 0 \leq i \leq \lfloor \frac{d}{2} \rfloor
$$

give a basis for all linear relations on $h$-vectors of simple $d$-polytopes.
Polytopes & f-vectors

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$h = (1, 3, 3, 1)$

$h = (1, 1, 1, 1, 1)$
An interpretation of \( h \)-vectors

Let \( P \) be a simple \( d \)-polytope. We call \( c \in \mathbb{R}^d \) \textit{generic}, if \( c^t u \neq c^t v \) for all \( u, v \in V(P) \).
\[ \dim (F(v_1c)) \]

\[ \sim (1, 3, 3, 1) \]

\[ \#v : \dim F(v_1c) = 1 \]
An interpretation of the $h$-vector

For a vertex $v \in V(P)$ let $F(v, c) = F \subseteq P$ be the inclusion-maximal face, such that

$$F^c := \{x \in F : c^t x \geq c^t y \text{ for all } y \in F\} = \{v\}.$$ 

**Proposition (McMullen)**

If $P$ is a simple $d$-polytope and $c \in \mathbb{R}^d$ generic, then

$$h_i = |\{(F, v) : \dim(F) = i \text{ and } F = F(v, c)\}|.$$
The Steiner point
History

• 1840 Steiner introduced the Steiner point (also: ”Krümmungsschwerpunkt”) within an extremal problem for plane convex regions.
• For a plane convex region $K$, he defined $s(K)$ as the centroid of its boundary.
• Later, Shephard defined it for bounded convex sets in $\mathbb{R}^d$ via an integral.
The Steiner point of a convex set

For a convex, bounded set $K \subseteq \mathbb{R}^d$ the **Steiner point** is defined by

$$s(K) = \frac{1}{\text{vol}_d(B_d)} \int_{c \in S^{d-1}} c \ h(c, K) d\omega,$$

where $h(c, K)$ is the **support function**

$$h(c, K) = \max_{x \in K} c^t x.$$
Uniqueness of the Steiner Point

Theorem (Shephard, Schneider)

Let $K, K_1, K_2 \subseteq \mathbb{R}^d$ be convex bounded sets. Then

(i) $s(K_1 + K_2) = s(K_1) + s(K_2)$,

(ii) $s(AK) = As(K)$ for rigid motions $A$ and

(iii) $s$ is continuous with respect to Hausdorff metric.

Moreover, if any map

$$f : \{K \subseteq \mathbb{R}^d \text{ convex, bounded set}\} \rightarrow \mathbb{R}^d$$

satisfies (i)-(iii), then

$$f = s.$$
The normal cone

**Definition**

For a vertex $v \in V(P)$ we define the normal cone of $P$ at $v$ by

$$N_v P = \{ c \in \mathbb{R}^d : c^t v \geq c^t x \text{ for all } x \in P \}.$$ 

**Definition (External angle)**

$$\alpha(v, P) = \frac{\text{vol}_{d-1}(N_v P \cap S^{d-1})}{\text{vol}_{d-1}(S^{d-1})}.$$
The Steiner Point of a polytope

\[ s(P) = \sum_{v \in V(P)} \alpha(v, P) \cdot v. \]
The Steiner Point of a polytope

As $\alpha(v, P) \geq 0$ and

$$\sum_{v \in V(P)} \alpha(v, P) = 1,$$

it follows that $s(P) \in P.$
An analogue of the Euler–Poincaré relation

Let $P$ be a $d$-polytope. We define

$$s_i(P) := \sum_{F \subseteq P: \dim(F) = i} s(F) = \sum_{v \in V(P)} A(v, i) \cdot v,$$

where $A(v, i) := \sum_{v \in F \subseteq P: \dim(F) = i} \alpha(v, F)$.

**Theorem (Grünbaum)**

*The linear relation*

$$s_0(P) - s_1(P) + \cdots + (-1)^d s_d(P) = s_d(P)$$

*is the unique linear relation that $s_0(P), s_1(P), \ldots, s_d(P)$ fulfil for all $d$-polytopes $P$.***
A theorem and a conjecture of Grünbaum

**Theorem (Grünbaum)**

If $P \subseteq \mathbb{R}^d$ is a simple $d$-polytope, then

$$s_i(P) = \sum_{k=0}^{i} (-1)^k \binom{d}{i-k} s_k(P) \text{ for } 0 \leq i \leq d$$

and for $i = 1, 3, \ldots, m$ the linear relations are linearly independent, where $m$ is the largest odd integer not exceeding $d$.

Moreover, Grünbaum conjectures that the relations form a basis for all linear relations on Steiner points of simple $d$-polytopes and thus are an analogue of the Dehn–Sommerville relations.
This motivated us to define $t_0(P), t_1(P), \ldots, t_d(P) \in \mathbb{R}^d$ by

$$
\sum_{i=0}^{d} t_i(P)(x+1)^i = \sum_{i=0}^{d} s_i(P)x^i.
$$

so we can restate the relations as

$$
t_i(P) = t_{d-i}(P) \quad \text{for } 0 \leq i \leq d.
$$

Obviously for $0 \leq i \leq \left\lfloor \frac{d}{2} \right\rfloor$ they are linearly independent and the remaining question is, if they form a basis for all linear relations on $t_0(P), \ldots, t_d(P)$ for simple $d$-polytopes $P$. 
Let $P$ be a simple $d$-polytope. Similar to $s_i(P)$ we want to write $t_i(P)$ as a weighted sum over the vertices:

$$t_i(P) = \sum_{v \in V(P)} B(v, i) \cdot v.$$

Is $B(v, i) \geq 0$ and how can the weights $B(v, i)$ be interpreted?
Let $P$ be a simple $d$-polytope, $v \in V(P)$ a vertex and $F \subseteq P$ a face of $P$ with $v \in F$.

**Definition (Pure normal cone of $v$ at $F$)**

$$\hat{N}_v(F) := N_v F \setminus \bigcup_{F \subseteq G \subseteq P} N_v G.$$ 

The pure normal cone of $v$ at $F$ is the half-open cone of those $c \in \mathbb{R}^d$, such that $F = F(v, c)$. 

**Example:**
The pure external angles

Definition (Pure external angle of \( v \) at \( F \))

\[
\beta(v, F) := \frac{\text{vol}_{d-1}(\hat{N}_v F \cap S^{d-1})}{\text{vol}_{d-1}(S^{d-1})}.
\]

Proposition (B., Sanyal)

Let \( P \) be a simple \( d \)-polytope and \( v \in V(P) \) a vertex. Then

\[
B(v, i) = \sum_{v \in F \subseteq P: \dim(F) = i} \beta(v, F)
\]

and thus \( B(v, i) \geq 0 \).
A connection between the $h$-vector and pure external angles

**Proposition (B., Sanyal)**

*If $P$ is a simple $d$-polytope, then $\sum_{v \in V(P)} B(v, i) = h_i(P)$.*

Suppose there are $a_0, \ldots, a_d, C \in \mathbb{R}$, such that for any simple $d$-polytope $P$ we have $\sum_{i=0}^{d} a_i t_i(P) = C$. Then for all $x \in \mathbb{R}^d$

$$C = \sum_{i=0}^{d} a_i t_i(P + x) = \sum_{i=0}^{d} a_i \sum_{v \in V(P)} B(v, i)(v + x)$$

$$= C + x \sum_{i=0}^{d} a_i \sum_{v \in V(P)} B(v, i) = C + x \sum_{i=0}^{d} a_i h_i$$

$\Rightarrow \sum_{i=0}^{d} a_i h_i(P) = 0 \Rightarrow$ The relation is a linear combination of the Dehn-Sommerville relations.
Grünbaums conjecture is true!

Theorem (B., Sanyal)

The \( \left\lfloor \frac{d+1}{2} \right\rfloor \) relations

\[ t_i = t_{d-i} \text{ for } 0 \leq i \leq \left\lfloor \frac{d}{2} \right\rfloor \]

form a basis for all linear relations on \( t_0, \ldots, t_d \) of simple \( d \)-polytopes.