On the Geometric Combinatorics of Steiner Points and a Conjecture of Grünbaum

Work in progress with Raman Sanyal

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Polytopes & f-vectors

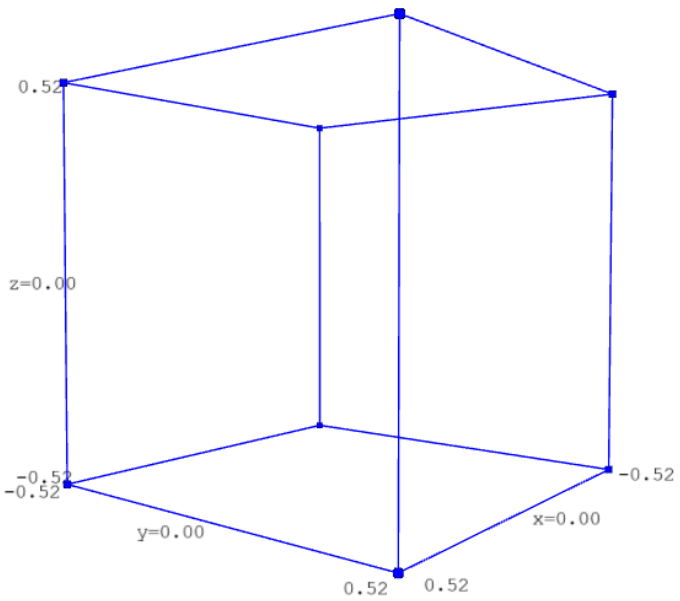
The Steiner point

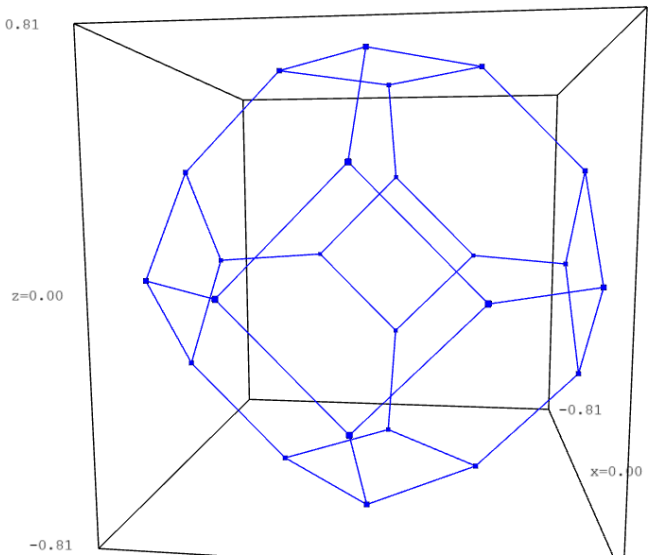
Generalized permutahedra

Tree Posets

Polytopes & f-vectors

Polytopes





$f = (8, 12, 6, 1)$

$8 - 12 + 6 = 2$

$f = (24, 36, 14, 1)$

$24 - 36 + 14 = 2$

The f-vector of a polytope

Definition

For a d-polytope P the **f-vector** $f(P) = (f_0, f_1, \ldots, f_d)$, is defined by

$$f_i = |\{F \subseteq P \text{ face } : \dim(F) = i\}|.$$

Theorem (Euler–Poincaré relation)

The relation

$$f_0 - f_1 + \cdots + (-1)^d f_d = f_d$$

is the unique linear relation fulfilled by all f-vectors of d-polytopes.

Euler's polyhedron formula: $f_0 - f_1 + f_2 = 2$

Simple polytopes

Definition

A d-polytope P is **simple** if every vertex is incident to exactly d edges. For a d-polytope P the **h-vector** $h(P) = (h_0, h_1, \ldots, h_d)$ is defined by

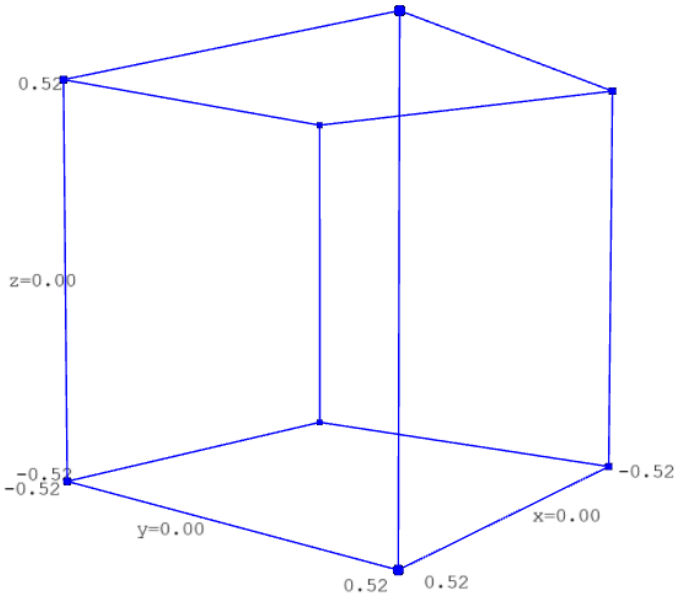
$$\sum_{i=0}^{d} h_i (t+1)^i = \sum_{i=0}^{d} f_i t^i.$$

Theorem (Dehn–Sommerville relations)

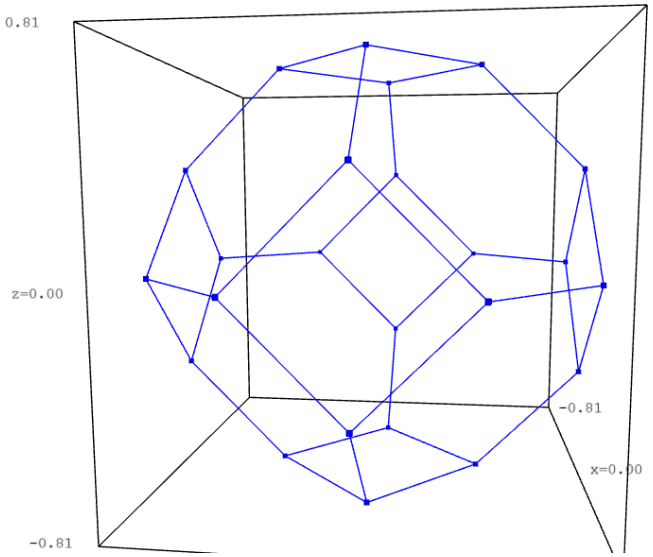
The $\lfloor \frac{d+1}{2} \rfloor$ *relations*

$$h_i = h_{d-i} \text{ for } 0 \leq i \leq \lfloor \tfrac{d}{2} \rfloor$$

give a basis for all linear relations on h-vectors of simple d-polytopes.



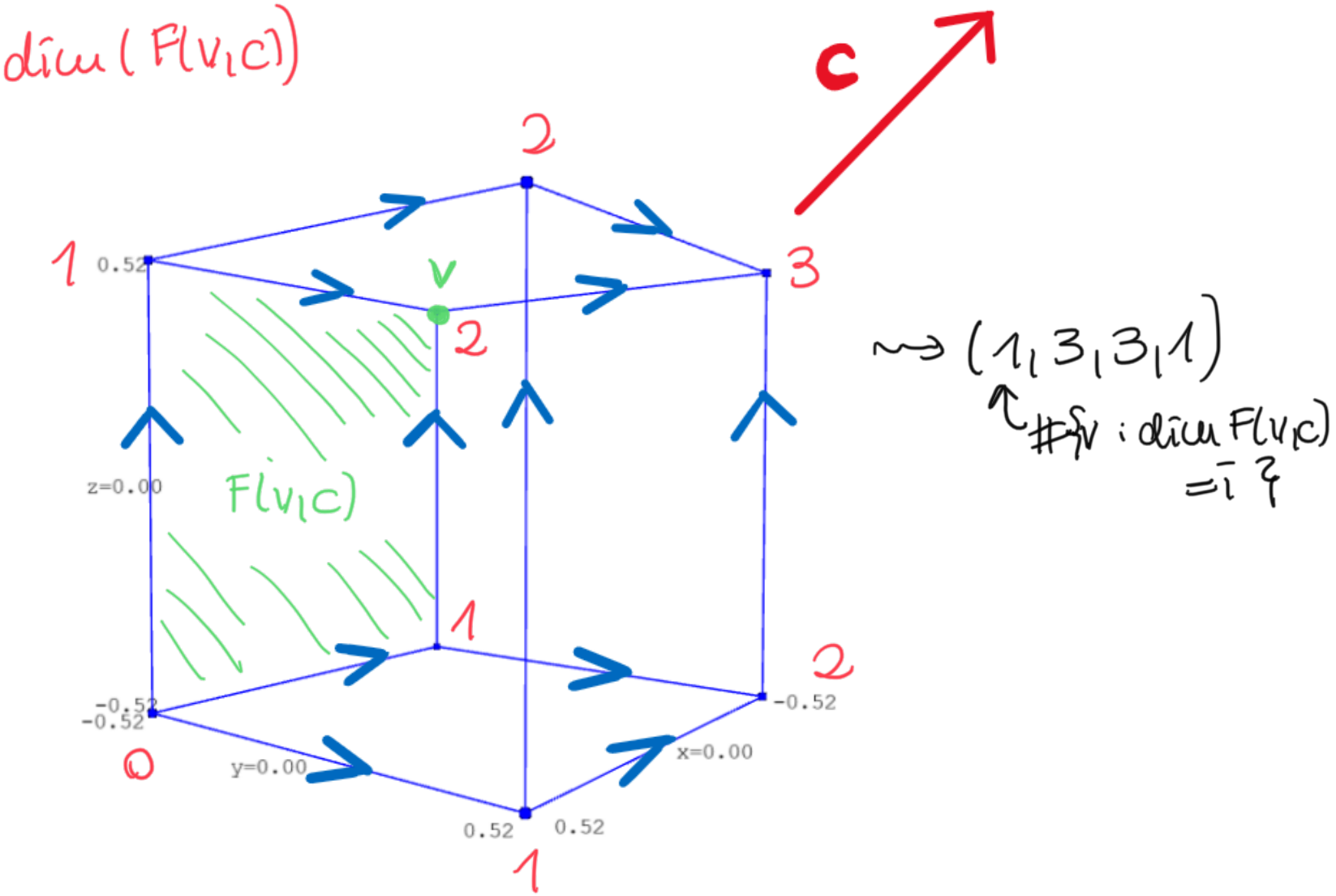
$h = (1, 3, 3, 1)$



$h = (1, 11, 11, 1)$

An interpretation of h-vectors

Let P be a simple d-polytope. We call $c \in \mathbb{R}^d$ **generic**, if $c^t u \neq c^t v$ for all $u, v \in V(P)$.


dim(F(v,c))
c
2
1
0.52
v
3
2
z=0.00
F(v,c)
1
2
-0.52
-0.52
-0.52
0
y=0.00
x=0.00
0.52
0.52
1
⇝ (1,3,3,1)
#{v : dim F(v,c) = i}

An interpretation of the h-vector

For a vertex $v \in V(P)$ let $F(v,c) = F \subseteq P$ be the inclusion-maximal face, such that

$$F^c := \{x \in F : c^t x \geq c^t y \text{ for all } y \in F\} = \{v\}.$$

Proposition (McMullen)

If P is a simple d-polytope and $c \in \mathbb{R}^d$ generic, then

$$h_i = |\{(F,v) : \dim(F) = i \text{ and } F = F(v,c)\}|.$$

The Steiner point

History

- 1840 Steiner introduced the Steiner point (also: "Krümmungsschwerpunkt") within an extremal problem for plane convex regions.
- For a plane convex region K, he defined $s(K)$ as a the centroid of its boundary.
- Later, Shephard defined it for bounded convex sets in $\mathbb{R}^d$ via an integral.

The Steiner point of a convex set

For a convex, bounded set $K \subseteq \mathbb{R}^d$ the **Steiner point** is defined by

element of surface area

$$s(K) = \frac{1}{\mathrm{vol}_d(B_d)} \int_{c \in S^{d-1}} c \; h(c, K) d\omega,$$

d-dim. unit ball

(d-1)-dim. unit sphere

where $h(c, K)$ is the **support function**

$$h(c, K) = \max_{x \in K} c^t x.$$

Uniqueness of the Steiner Point

Theorem (Shephard, Schneider)

Let $K, K_1, K_2 \subseteq \mathbb{R}^d$ be convex bounded sets. Then

(i) $s(K_1 + K_2) = s(K_1) + s(K_2)$,

(ii) $s(AK) = As(K)$ *for rigid motions* A *and*

(iii) s *is continuous with respect to Hausdorff metric.*

Moreover, if any map

$$f : \{K \subseteq \mathbb{R}^d \text{ convex, bounded set}\} \to \mathbb{R}^d$$

satisfies (i)-(iii), then

$$f = s.$$

The normal cone

Definition

For a vertex $v \in V(P)$ we define the normal cone of P at v by

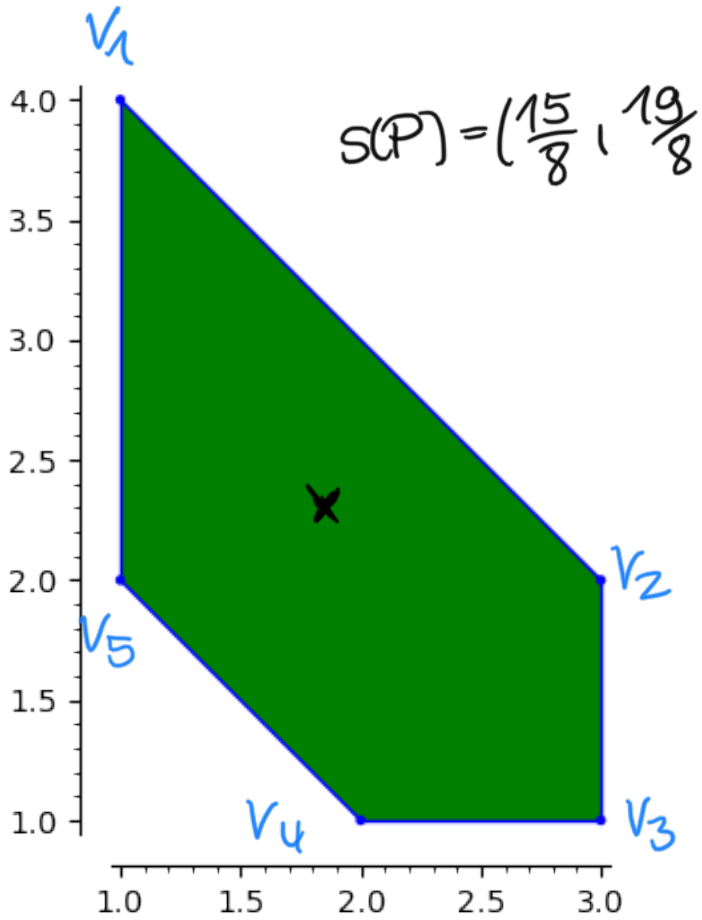
$$N_v P = \{c \in \mathbb{R}^d : c^t v \geq c^t x \text{ for all } x \in P\}.$$

Definition (External angle)

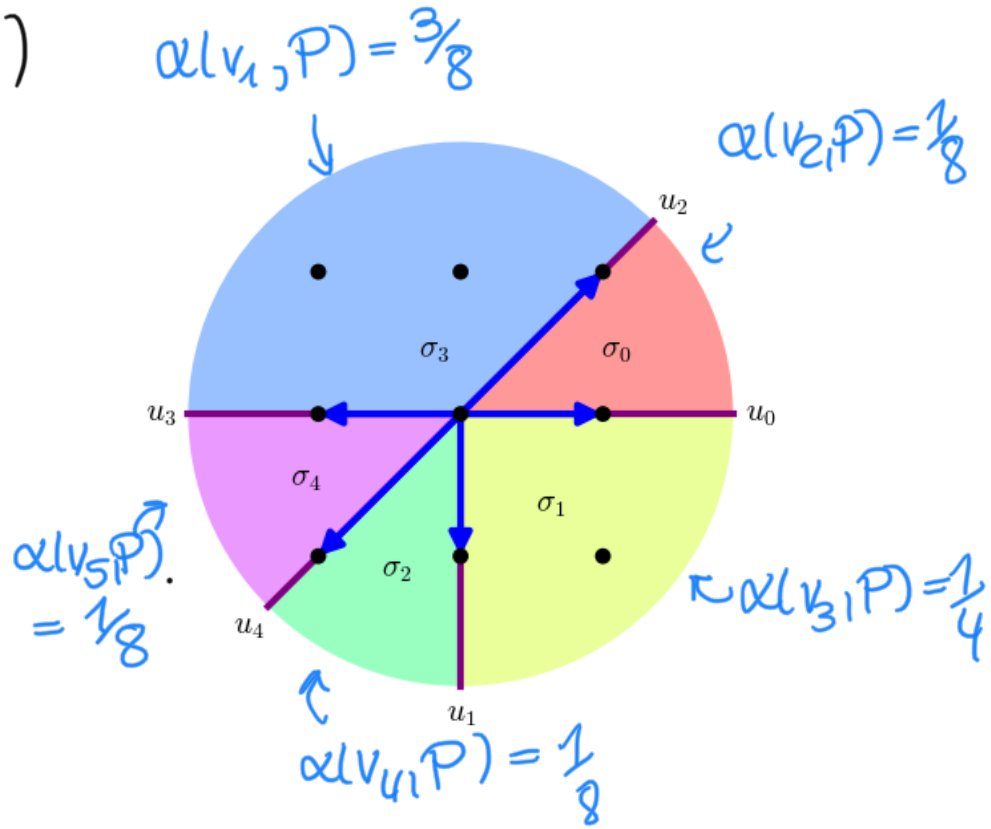
$$\alpha(v, P) = \frac{\mathrm{vol}_{d-1}(N_v P \cap S^{d-1})}{\mathrm{vol}_{d-1}(S^{d-1})}.$$

The Steiner Point of a polytope

$$s(P) = \sum_{v \in V(P)} \alpha(v, P)\ v.$$



$s(P) = \left(\frac{15}{8}, \frac{19}{8}\right)$



The Steiner Point of a polytope

As $\alpha(v, P) \geq 0$ and

$$\sum_{v \in V(P)} \alpha(v, P) = 1,$$

it follows that $s(P) \in P$.

An analogue of the Euler–Poincaré relation

Let P be a d-polytope. We define

$$s_i(P) := \sum_{\substack{F \subseteq P: \\ \dim(F)=i}} s(F) = \sum_{v \in V(P)} A(v,i)\ v,$$

where $A(v,i) := \sum_{\substack{v \in F \subseteq P: \\ \dim(F)=i}} \alpha(v,F)$.

Theorem (Grünbaum)

The linear relation

$$s_0(P) - s_1(P) + \cdots + (-1)^d s_d(P) = s_d(P)$$

is the unique linear relation that $s_0(P), s_1(P), \ldots, s_d(P)$ fulfil for all d-polytopes P.

A theorem and a conjecture of Grünbaum

Theorem (Grünbaum)

If $P \subseteq \mathbb{R}^d$ is a simple d-polytope, then

$$s_i(P) = \sum_{k=0}^{i} (-1)^k \binom{d}{i-k} s_k(P) \text{ for } 0 \leq i \leq d$$

and for $i = 1, 3, \ldots, m$ the linear relations are linearly independent, where m is the largest odd integer not exceeding d.

Moreover, Grünbaum conjectures that the relations form a basis for all linear relations on Steiner points of simple d-polytopes and thus are an analogue of the Dehn–Sommerville relations.

This motivated us to define $t_0(P), t_1(P), \ldots, t_d(P) \in \mathbb{R}^d$ by

$$\sum_{i=0}^{d} t_i(P)(x+1)^i = \sum_{i=0}^{d} s_i(P)x^i.$$

so we can restate the relations as

$$t_i(P) = t_{d-i}(P) \quad \text{for } 0 \leq i \leq d.$$

Obviously for $0 \leq i \leq \lfloor \frac{d}{2} \rfloor$ they are linearly independent and the remaining question is, if they form a basis for all linear relations on $t_0(P), \ldots, t_d(P)$ for simple d-polytopes P.

$t_0(P), t_1(P), \ldots, t_d(P)$ as weighted sums over the vertices

Let P be a simple d-polytope. Similar to $s_i(P)$ we want to write $t_i(P)$ as a weighted sum over the vertices:

$$t_i(P) = \sum_{v \in V(P)} B(v, i)\ v.$$

Is $B(v, i) \geq 0$ and how can the weights $B(v, i)$ be interpreted?

The pure normal cone

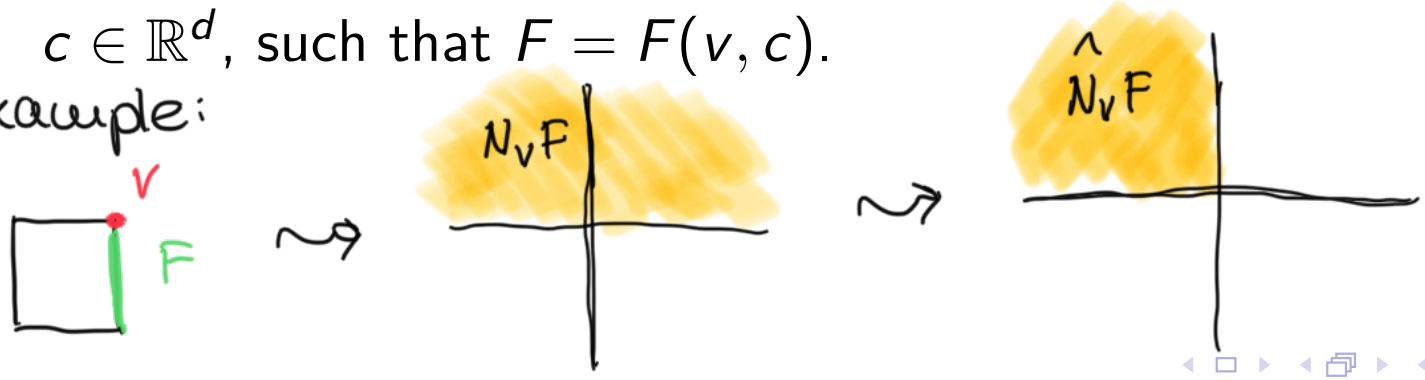
Let P be a simple d-polytope, $v \in V(P)$ a vertex and $F \subseteq P$ a face of P with $v \in F$.

Definition (Pure normal cone of v at F)

$$\hat{N}_v(F) := N_v F \setminus \bigcup_{F \subseteq G \subseteq P} N_v G.$$

The pure normal cone of v at F is the half-open cone of those $c \in \mathbb{R}^d$, such that $F = F(v, c)$.

Example:



The pure external angles

Definition (Pure external angle of v at F)

$$\beta(v,F) := \frac{\mathrm{vol}_{d-1}(\hat{N}_v F \cap S^{d-1})}{\mathrm{vol}_{d-1}(S^{d-1})}.$$

Proposition (B.,Sanyal)

Let P be a simple d-polytope and $v \in V(P)$ a vertex. Then

$$B(v,i) = \sum_{\substack{v \in F \subseteq P: \\ \dim(F)=i}} \beta(v,F)$$

and thus $B(v,i) \geq 0$.

A connection between the h-vector and pure external angles

Proposition (B.,Sanyal)

If P is a simple d-polytope, then $\sum_{v \in V(P)} B(v,i) = h_i(P)$.

Suppose there are $a_0, \ldots, a_d, C \in \mathbb{R}$, such that for any simple d-polytope P we have $\sum_{i=0}^{d} a_i t_i(P) = C$. Then for all $x \in \mathbb{R}^d$

$$C = \sum_{i=0}^{d} a_i t_i(P+x) = \sum_{i=0}^{d} a_i \sum_{v \in V(P)} B(v,i)(v+x)$$

$$= C + x \sum_{i=0}^{d} a_i \sum_{v \in V(P)} B(v,i) = C + x \sum_{i=0}^{d} a_i h_i$$

$\Rightarrow \sum_{i=0}^{d} a_i h_i(P) = 0 \Rightarrow$ The relation is a linear combination of the Dehn-Sommerville relations.

Grünbaums conjecture is true!

Theorem (B., Sanyal)

The $\lfloor \frac{d+1}{2} \rfloor$ *relations*

$$t_i = t_{d-i} \text{ for } 0 \leq i \leq \lfloor \tfrac{d}{2} \rfloor$$

form a basis for all linear relations on $t_0, \ldots, t_d$ *of simple* d*-polytopes.*