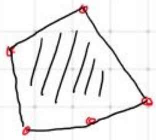
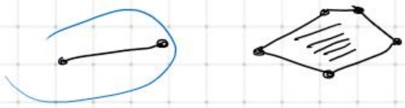
Polytopes

polytope: $\mathrm{conv}(\{p_1, \dots, p_n\})$, $\{p_1,\dots,p_n\} \subseteq \mathbb{R}^d$



Simplex: $\mathrm{conv}(\{p_1, \dots, p_n\})$, $\{p_1, \dots, p_n\}$ affinely independent in $\mathbb{R}^d$

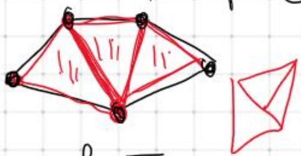
faces & dimension:



f-vector: $\underset{-1\ \ 0\ \ 1}{(1,2,1)}$ $\quad$ $\underset{-1\ \ 0}{(1,5,5,1)}$

triangulation of a polytope P: collection $\mathcal{T}$ of simplices s.t.

- $\bigcup_{S\in\mathcal{T}} S = P$
- faces of $S \in \mathcal{T}$ are in $\mathcal{T}$
- simplices of $\mathcal{T}$ intersect in faces



f-vector of $\mathcal{T}$: $\underset{-1\ \ 0}{(1,5,7,3)}$

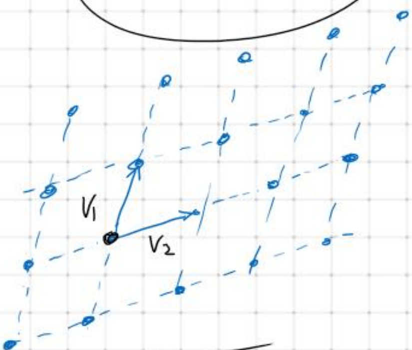
Lattices

$(\mathbb{Z}^d, +)$

Def: • Discrete additive subgroup of $\mathbb{R}^d$

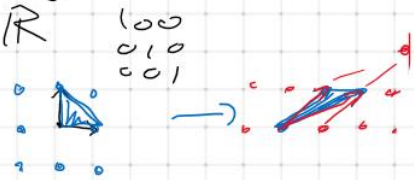
- Given $v_1, \dots, v_d \in \mathbb{R}^d$ lin independent,

$\Lambda(v_1, \dots, v_d) := \left\{ \sum_{i=1}^{d} a_i v_i \;\middle|\; a_i \in \mathbb{Z} \right\}$ is a lattice and $\{v_1, \dots, v_d\}$ is a basis of the lattice.

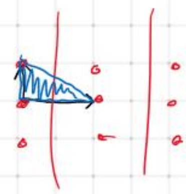


$e_1 \dots e_d$ standard basis of $\mathbb{R}^d$ $\begin{smallmatrix}1&0&0\\0&1&0\\0&0&1\end{smallmatrix}$

Ex: $\Lambda(e_1, e_2) = \mathbb{Z}^2$



$\Lambda(2e_1, e_2) =$



Unimodular triangulations, integer decomposition property and parallele pipeds

LATTICE POLYTOPES:

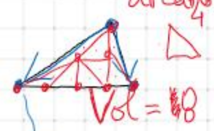
Fix a lattice Λ (usually we take $\Lambda = \mathbb{Z}^d$)

P is a Λ-polytope if $P = \mathrm{conv}(\underbrace{p_1, \dots, p_n}_{\in \Lambda})$ with

if the lattice is $\mathbb{Z}^d$, we often say P is a LATTICE polytope. $\mathbb{Z}^d$-poly

Lattice polytope

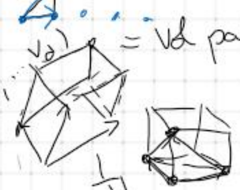
Lattice simplex



$\boxed{\text{unimodular simplex}}$ vertices of the simplex form an affine basis of the lattice.

$\Updownarrow$

$\det(v_1 \cdots v_d) = $ Vol parall



volume of the simplex is the smallest possible volume of a lattice polytope

$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\frac{1}{2}$ $\frac{1}{d!}$

$S = \mathrm{conv}(v_0 \dots v_d)$ $d! \, Vol(S) = \det(v_1 - v_0, \dots, v_d - v_0) \in \mathbb{Z}$

$|e_1, \dots, e_d| = 1$

- We obtained a triangulation into $\underbrace{\text{empty simplices}}_{\text{the only lattice pts in } S \text{ are the vertices}}$!

multiple of $1 = Vol$ (Unimod simp)

... but are empty simplices $\Leftarrow$ the same as unimodular simplices?

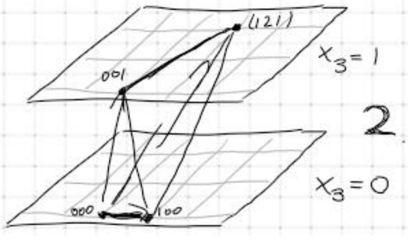
Dim 2 yes! (Pick's theorem: $A = i + \frac{b}{2} - 1$)

Dim ≥ 3 No!

Thm (White): All empty simplices are (unimodularly equivalent to)

$$\Delta_{(p,q)} = \mathrm{conv}\left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} p \\ q \\ 1 \end{pmatrix} \right)$$

for $p, q \in \mathbb{Z}$ with $\gcd(p,q) = 1$.



$p = 1$, $q = 2$

empty! $Vol(\Delta_{(p,q)}) = \left| \begin{bmatrix} 1 & 0 & p \\ 0 & 0 & q \\ 0 & 1 & 1 \end{bmatrix} \right| = q$

2

$\exists$ empty triangulation ✓

$\exists$ unimodular triangulation ? — NO —

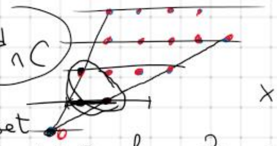
↳ strong requirement to ask of a polytope!

When a polytope does have a unimodular triangulation, great things happen...

... in enumerative combinatorics, integer linear programming, DETOUR INTO EHRHART THEORY?

toric geometry, commutative algebra ...

(affine) monoids or semigroups : $\mathbb{Z}^d \cap C$



$x = (x,y) + (z)$

Hilbert basis: inclusion minimal set $\{p_1, \dots, p_\ell\} \subseteq M$ s.t. $\forall p \in M$, $p = a_1 p_1 + \dots + a_\ell p_\ell$ for some $a_i \in \mathbb{Z}_+$

Ehrhart theory in brief:

$\boxed{|tP \cap \mathbb{Z}^d| = ?}$ $t \in \mathbb{Z}_{\geq 0}$ $= L_P(t)$. $\rightsquigarrow$ $Ehr_P(z) = \sum_{t \in \mathbb{Z}_{\geq 0}} |tP \cap \mathbb{Z}^d| z^t$

$$= \frac{h^*_P(z)}{(1-z)^{d+1}}, \quad h^*_P(z) =: h^*_0 + h^*_1 z + \dots + h^*_d z^d$$

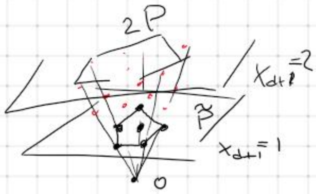
if T unim triang of P,

- $h^*_P(z) = \boxed{h_T(z)}$ manipulation of $\boxed{f_T(z)}$,

Polytopal semigroups:

$P \subseteq \mathbb{R}^d$ lattice polytope, $\tilde{P} = P \times \{1\} \subseteq \mathbb{R}^{d+1}$

$M_P := \operatorname{cone}(\tilde{P}) \cap \mathbb{Z}^{d+1}$ ... Hilbert basis?



Integer decomposition property (IDP): the lattice points of $\tilde{P}$

- form a Hilbert basis of M_P, that is, $\mathbb{Z}_+(\tilde{P} \cap \mathbb{Z}^d) = \operatorname{cone}(\tilde{P}) \cap \mathbb{Z}^{d+1} = M_P$

...equivalently: $\forall k \in \mathbb{Z}_+$, $\forall p \in kP \cap \mathbb{Z}^d$, $p = p_1 + \dots + p_k$ with $p_i \in P \cap \mathbb{Z}^d$.

WARNING !! IDP is sometimes also called integrally closed

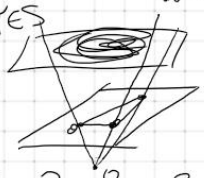
... but integrally closed in some sources indicates a weaker property, which we call normality → $\mathbb{Z}_+(\tilde{P} \cap \mathbb{Z}^d) = \operatorname{cone}(\tilde{P}) \cap \mathbb{Z}(\tilde{P} \cap \mathbb{Z}^d)$ ← Ex: $\operatorname{conv}\left(\begin{pmatrix}0\\0\\0\end{pmatrix}, \begin{pmatrix}2\\0\\0\end{pmatrix}, \begin{pmatrix}0\\3\\0\end{pmatrix}, \begin{pmatrix}0\\0\\5\end{pmatrix}\right)$

Q: Do unimodular simplices have IDP ? YES

$\forall k \in \mathbb{Z}$ $p \in kS \cap \mathbb{Z}^d \Rightarrow \exists p_1 \dots p_k \in S$

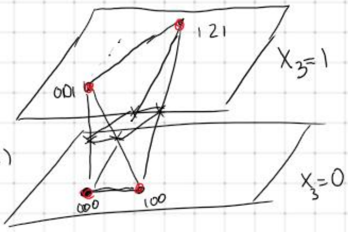
$p = p_1 + \dots + p_k$

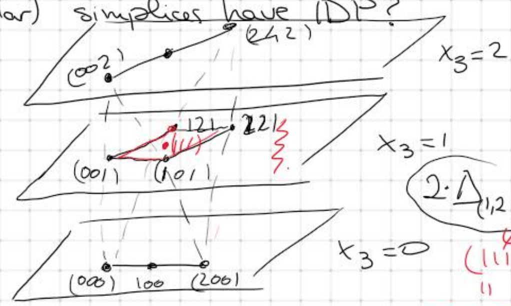
$= \underbrace{2p_1 + p_3}_{k} = p_1 + p_1 + p_3$



Q: Do empty (non-unimodular) simplices have IDP?

$\Delta_{(1,2)}$





$2 \cdot \Delta_{(1,2)} \cap \mathbb{Z}^3$

(111)

$\neq$

$p + p_2$

$\exists$ regular unimodular triangulation of P $\Rightarrow$ $\exists$ unimodular triangulation of P $\underset{\Leftarrow}{\Longrightarrow}$ P has IDP $\Longrightarrow$ P is normal

$\Updownarrow$ $\exists$ unimod cover of P $\Rightarrow$

$\Uparrow$

P totally unimodular

Pf: $p \in k\cdot P \cap \mathbb{Z}^d \Rightarrow p \in k\cdot S \cap \mathbb{Z}^d$ $\quad p = p_1 + \dots + p_k$

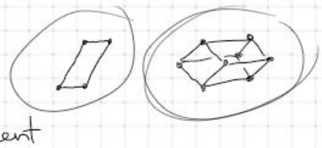
$P \supseteq S$ unim simplex $\quad S \cap \mathbb{Z}^d \subseteq P \cap \mathbb{Z}^d$

In fact, in the proof we only used that $\forall p \in P\ \exists \Delta \subseteq P$ unimod simplex s.t. $p \in \Delta$ ...

collection of unimod simplices $\mathcal{S}$ s.t. $\bigcup_{S\in\mathcal{S}} S = P$

Conjecture (Oda): Every smooth lattice polytope has IDP

LATTICE PARALLELOTOPES:



$\bar{v}_1, \dots, \bar{v}_d \in \mathbb{Z}^d$ lin independent

$P(\bar{v}_1 \dots \bar{v}_d) = \left\{ \sum_{i=1}^{d} \alpha_i \bar{v}_i \;\middle|\; \alpha_i \in [0,1]\ \forall i \right\}$ is a parallelotope.

Fact: All parallelotopes have IDP $\Leftarrow$ Exercise

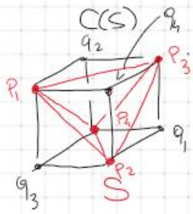
Q: Do they have unimodular triangulations? covers?

Thm (C-Santos) 3-dim parallelepipeds have unimodular covers. (triangulation) $\Longrightarrow$ 3-dim centrally symm pol has unim cover

Proof ideas. Triangulate into empty simplices.

If all unimodular ✓

If not suppose $S \in$ triangulation of P not unimodular of Vol = V.



Given $S = \mathrm{conv}(p_1, p_2, p_3, p_4)$ non-unimod empty simplex, we define

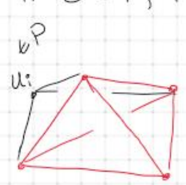
$C(S) = \mathrm{conv}(p_1, p_2, p_3, p_4, q_1, q_2, q_3, q_4)$, $q_i := \frac{1}{2}\sum p_j - p_i$

circumscribed parallelepiped.

Lemma 1: Each $T_i := \mathrm{conv}(\{p_j, j \neq i\}, q_i)$ contains a lattice point u_i distinct from its vertices

Lemma 2: If $S \subseteq P$, P parallelepiped $\Rightarrow \exists i$ s.t. $T_i \subseteq P$

To conclude:



... what about a unimodular triangulation ???