Shard polytopes

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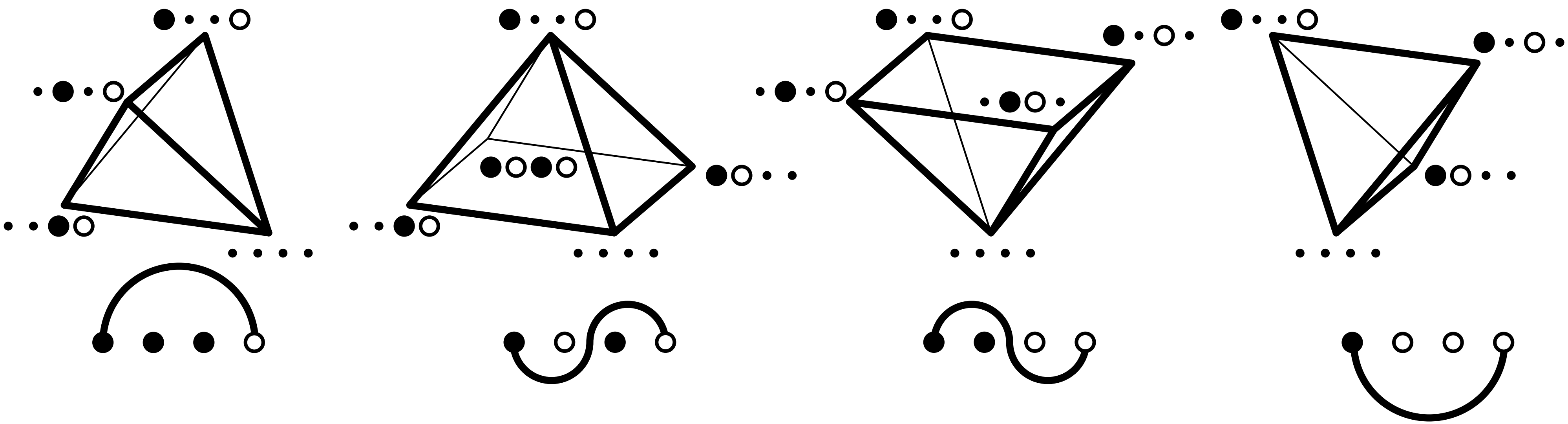
(CNRS & École Polytechnique)

J. RITTER
(École Polytechnique)

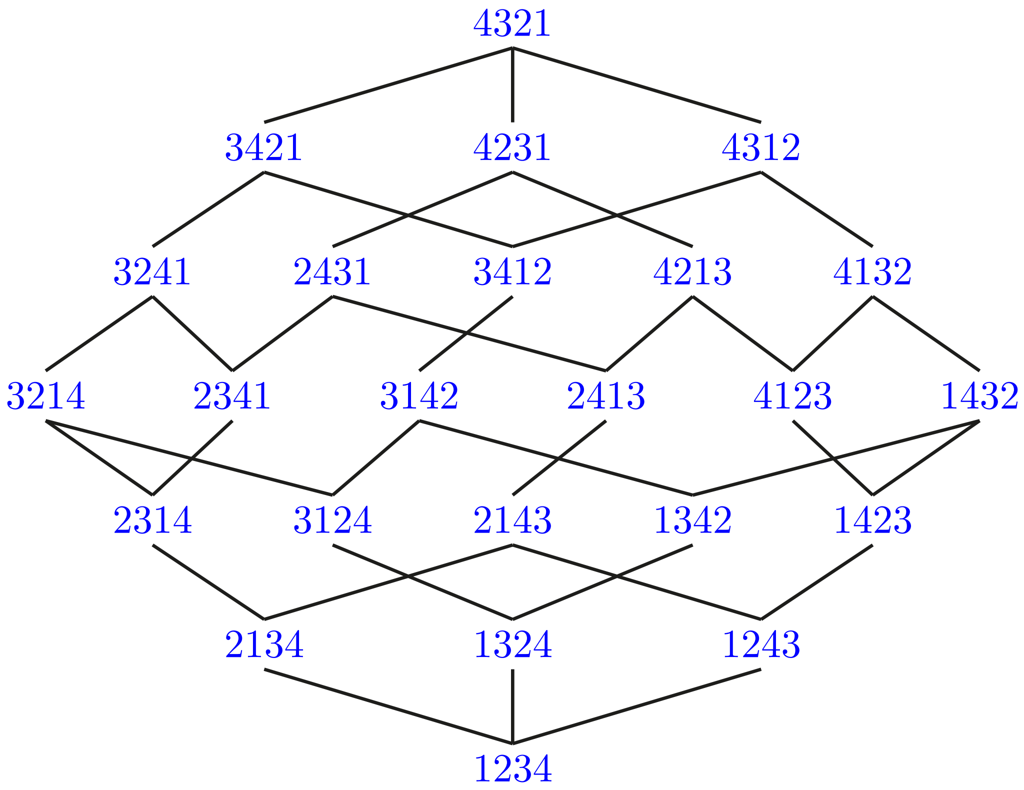


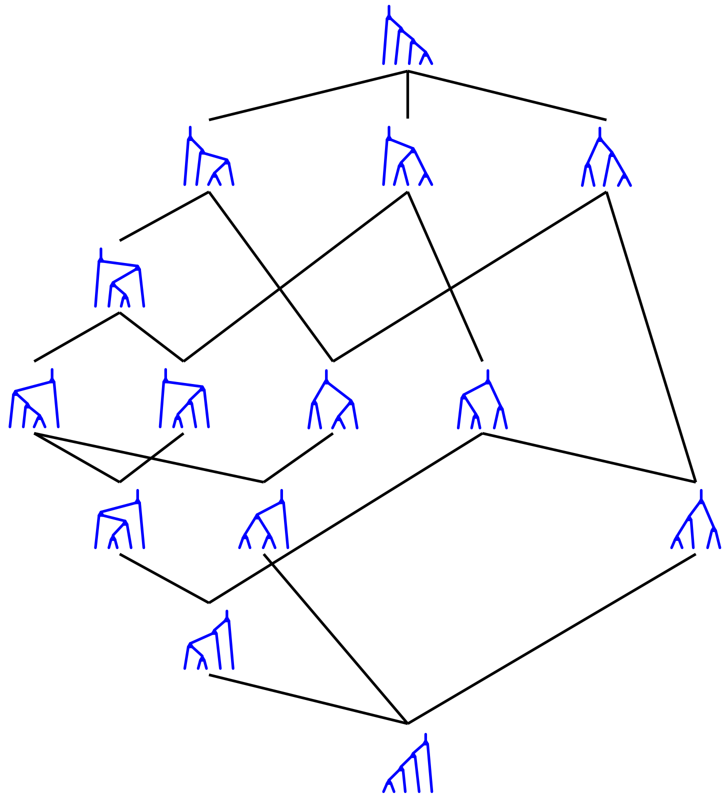
Séminaire DGeCo
14.01.2021

slides adapted from a presentation by Vincent Pilaud:
http://www.lix.polytechnique.fr/~pilaud/documents/presentations/shardPolytopes.pdf
article preprint available at: http://www.arXiv.org/abs/2007.01008

TWO CLASSICAL LATTICES AND POLYTOPES

LATTICES: WEAK ORDER AND TAMARI LATTICE





weak order $=$ permutations of $\mathfrak{S}_n$
ordered by inclusion of inversion sets

Tamari lattice $=$ binary trees on $[n]$
ordered by paths of right rotations

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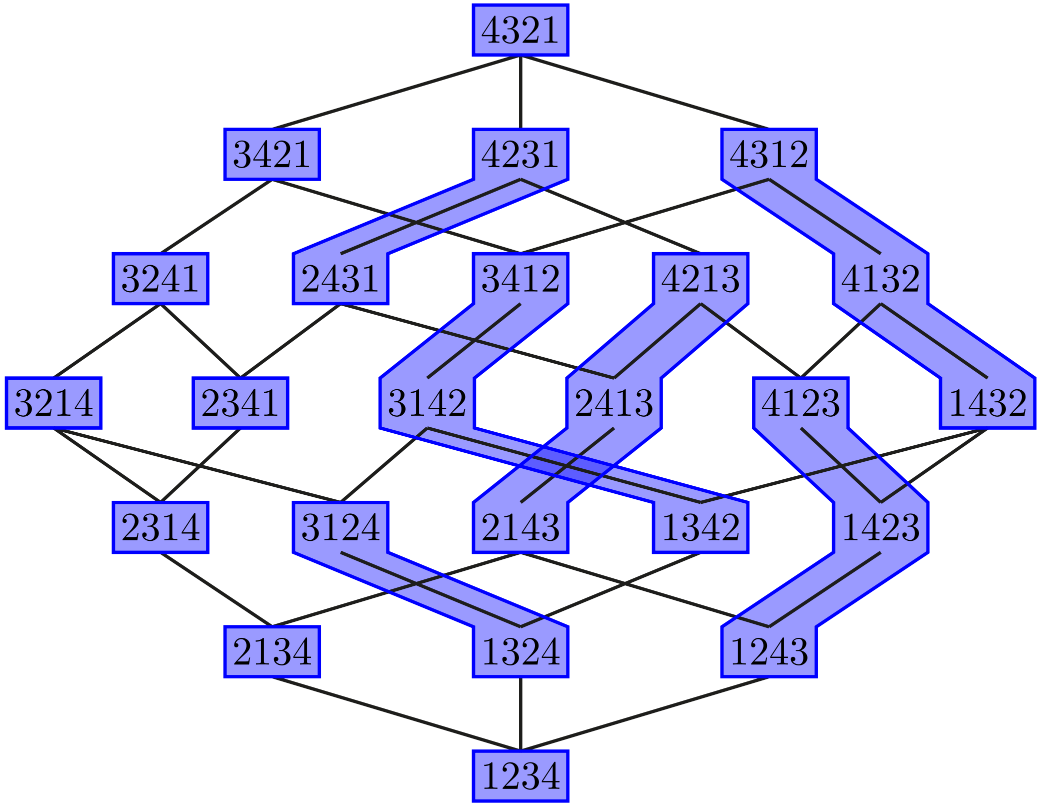

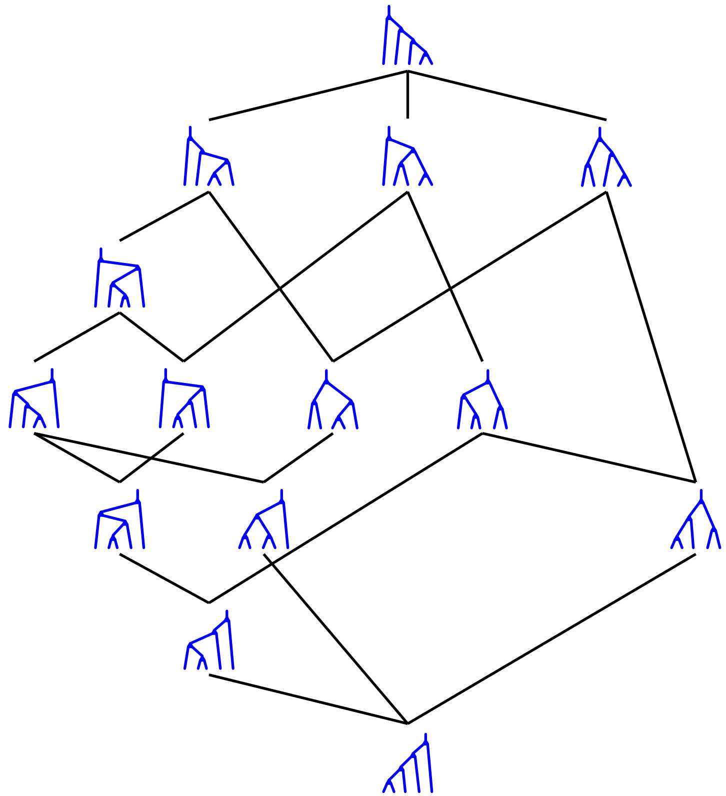

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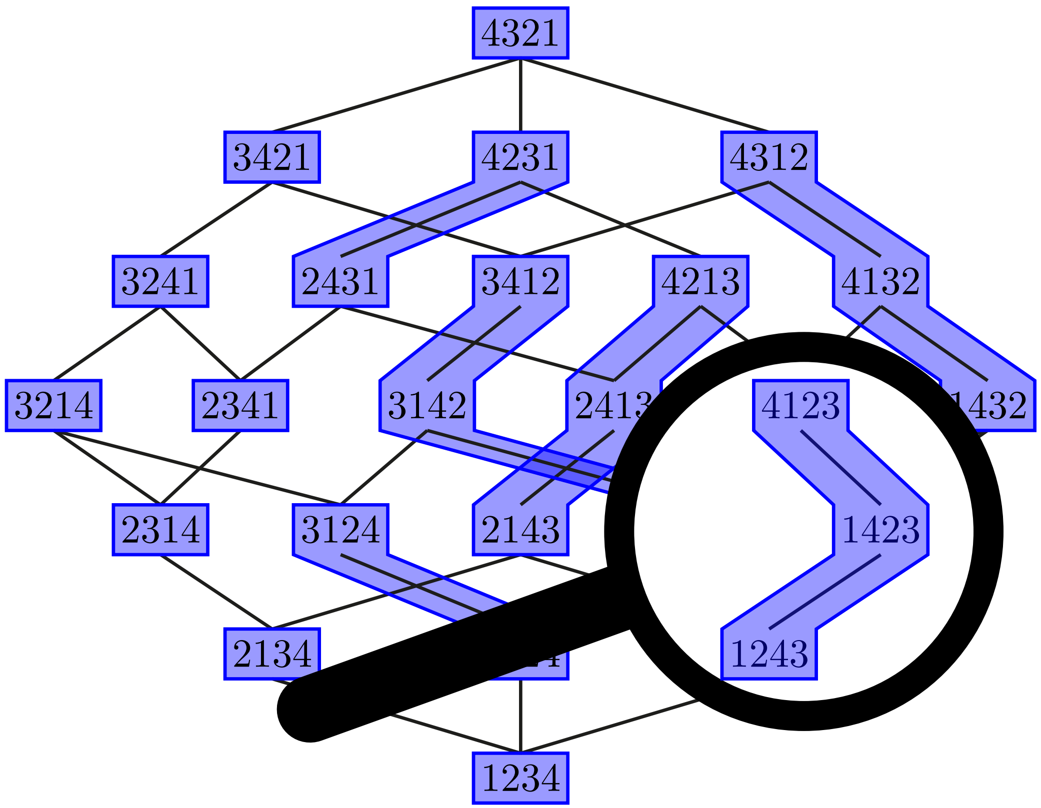

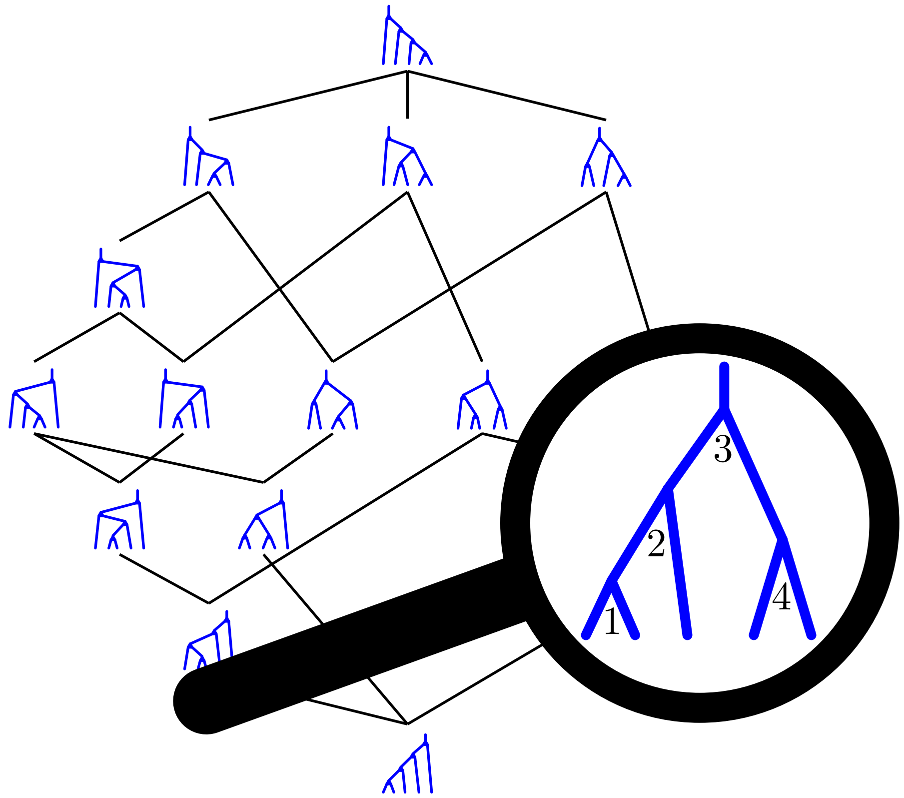

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POLYTOPES: PERMUTAHEDRON AND ASSOCIAHEDRON

fan = collection of polyhedral cones closed by faces and intersecting along faces

polytope = convex hull of a finite set = intersection of finitely many affine halfspaces

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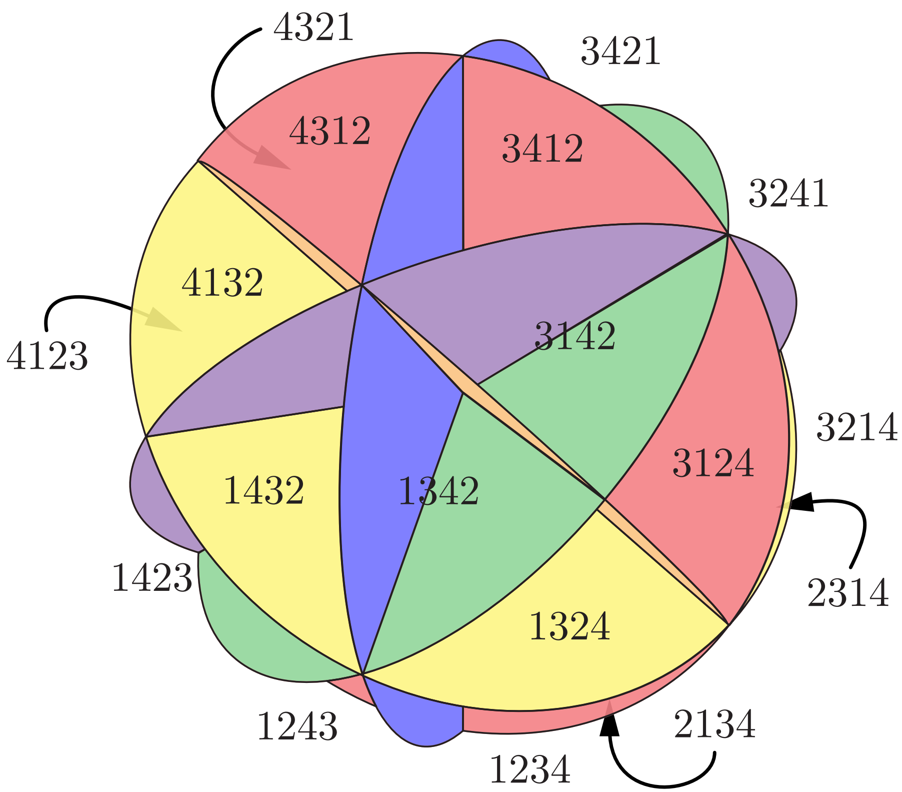
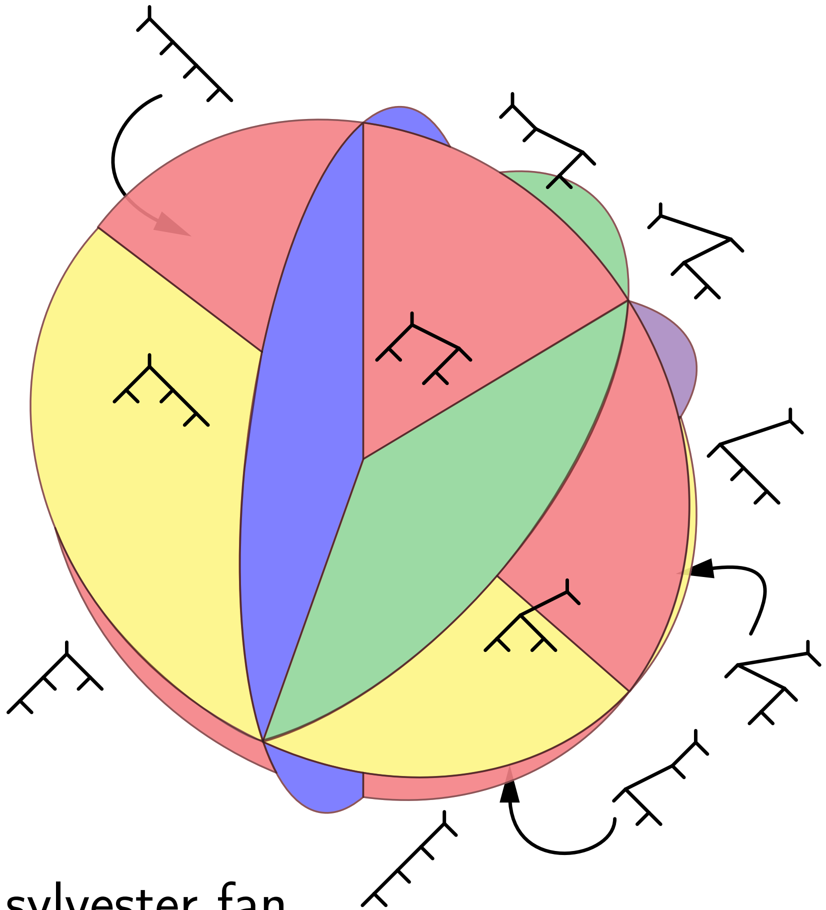


braid fan



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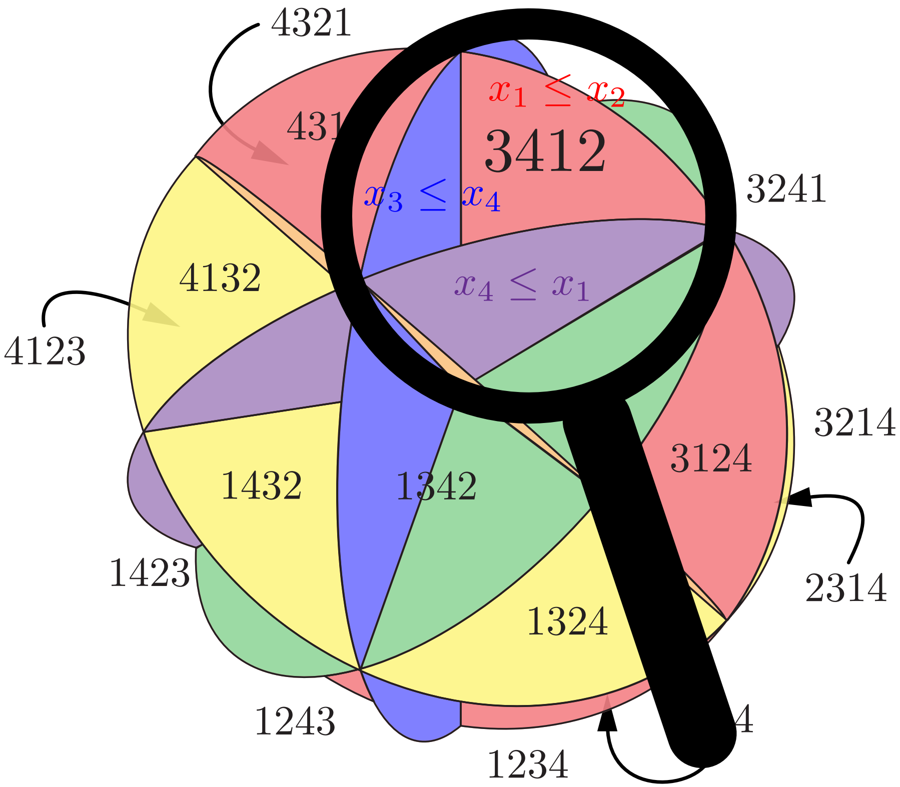

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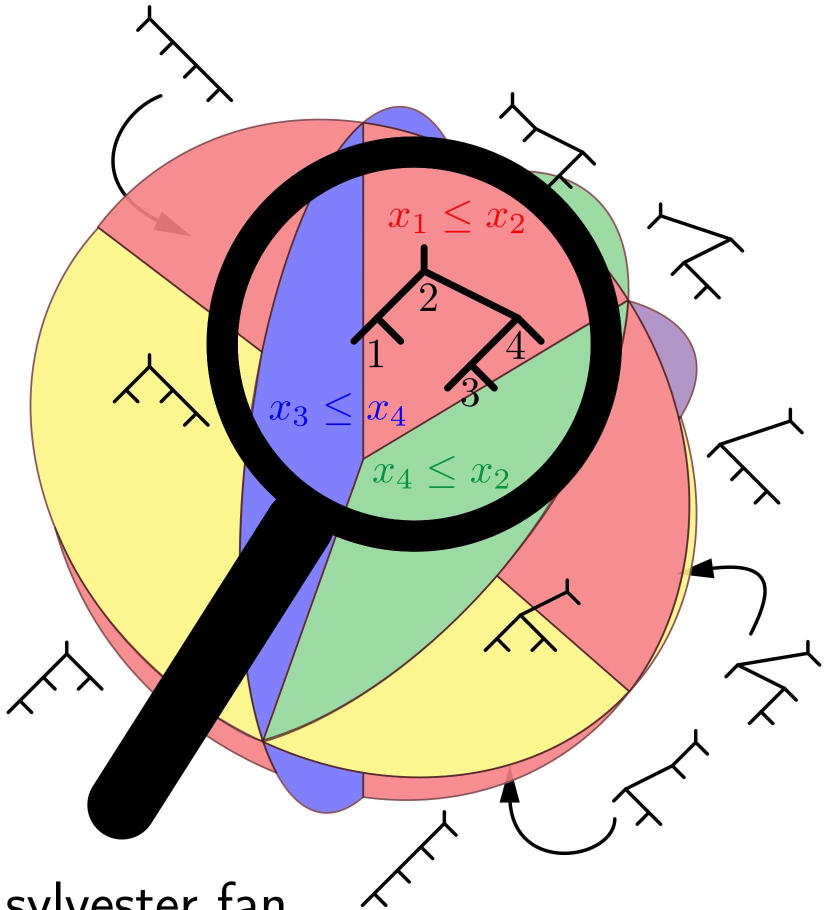

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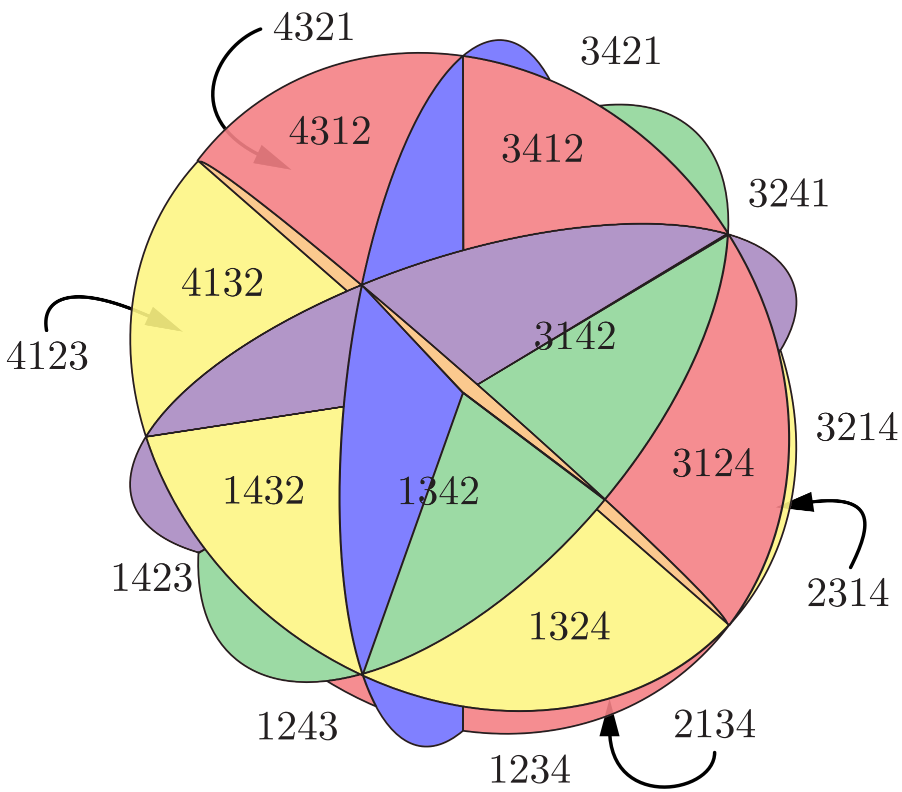

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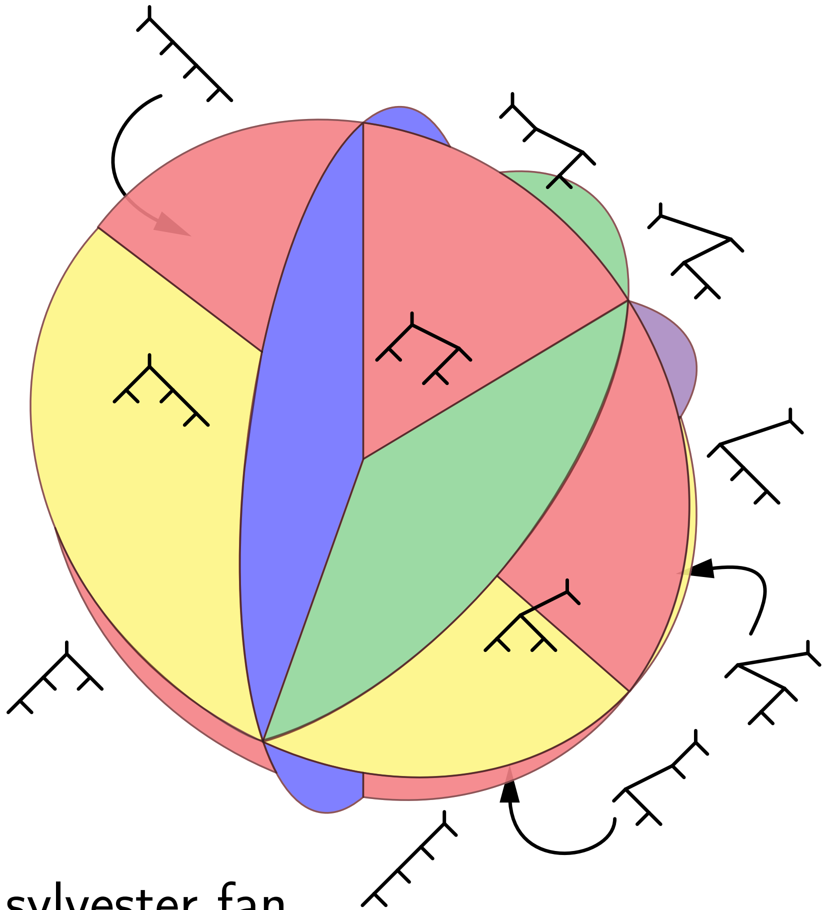

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quotient fan: obtained by glueing all cones of the same congruence class

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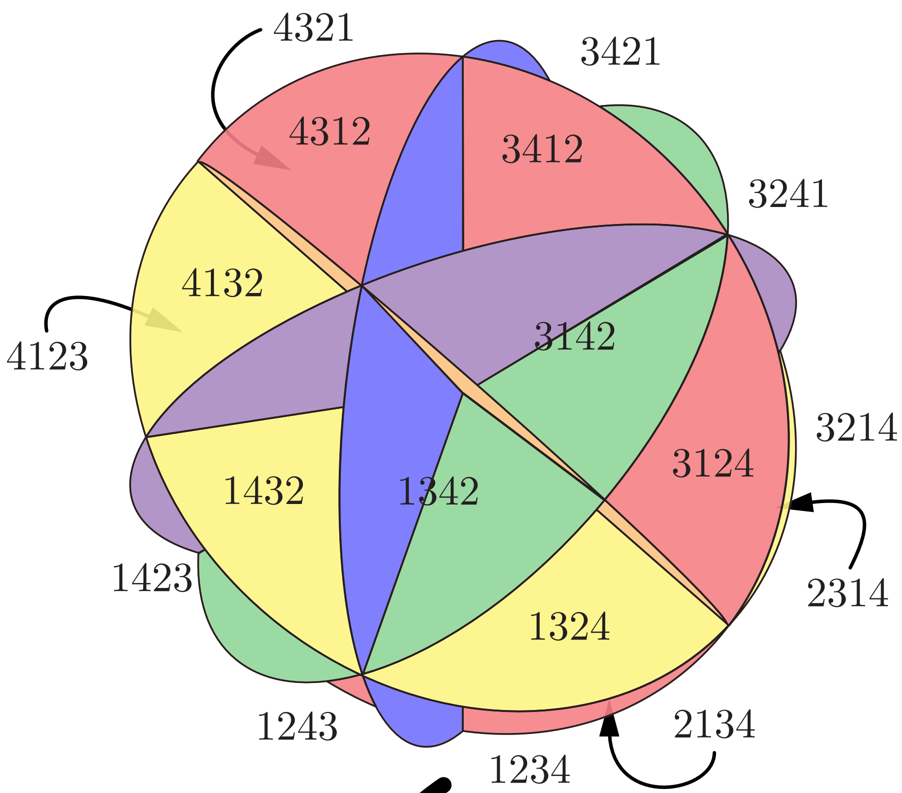
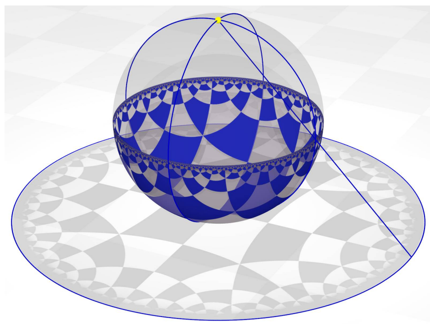
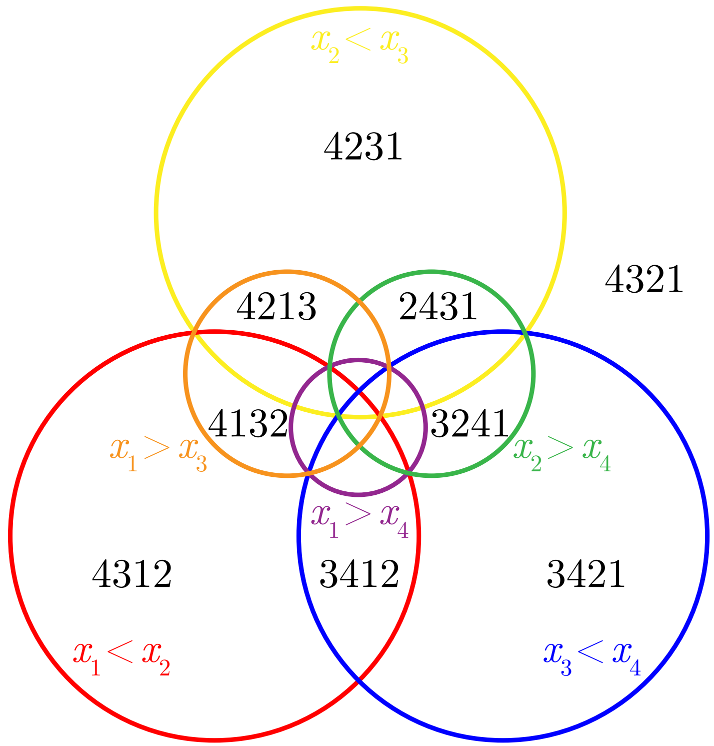


stereographic projection =





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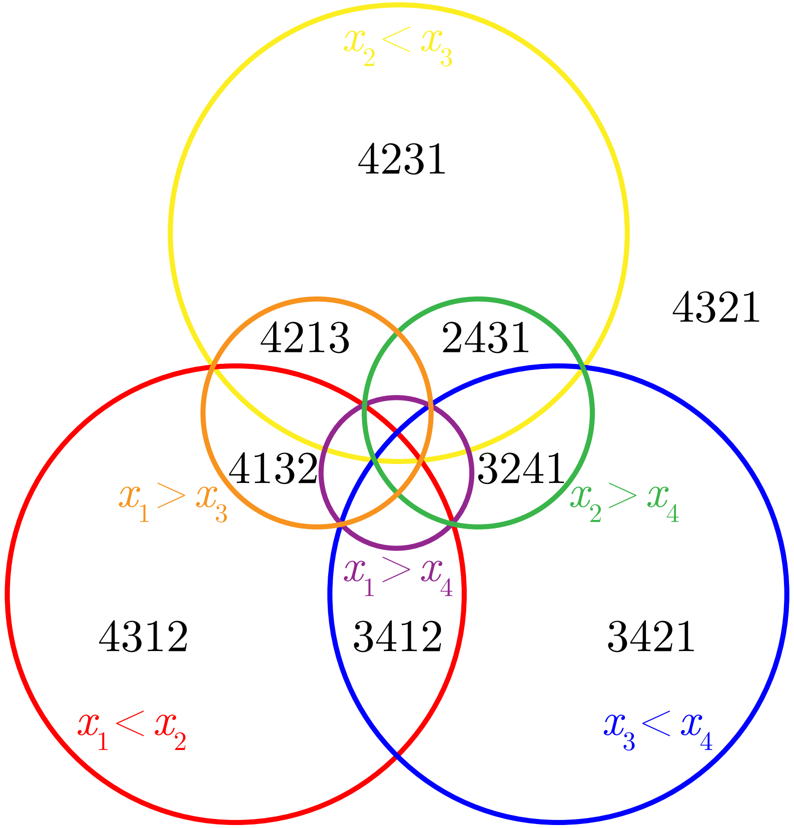

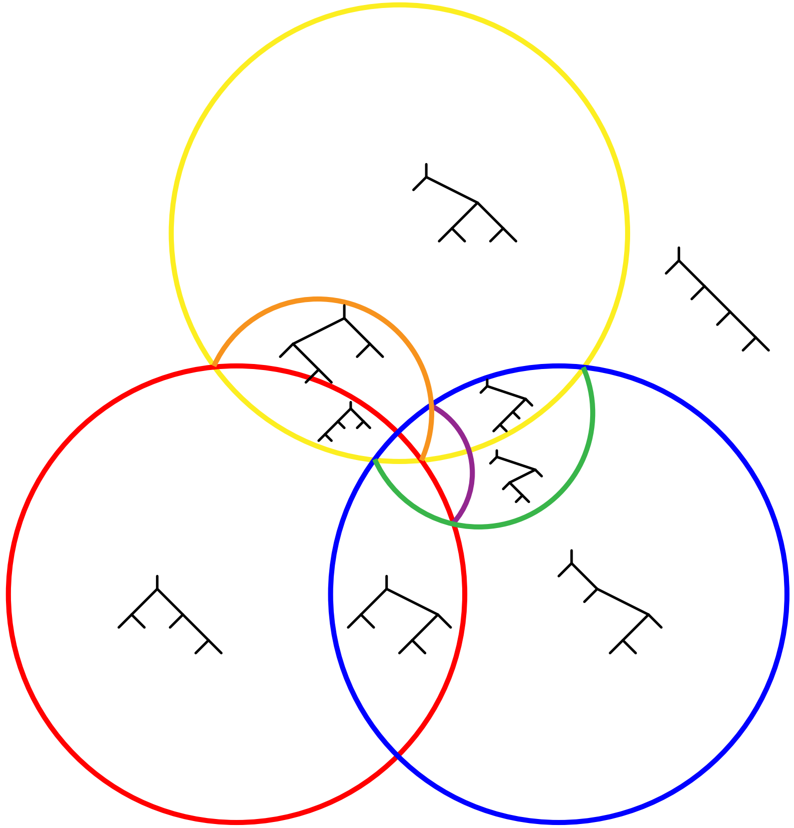

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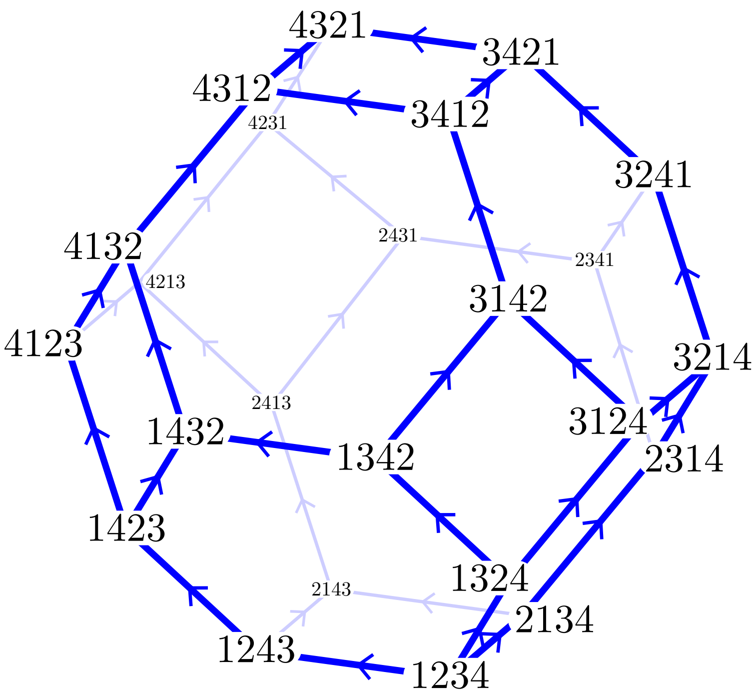
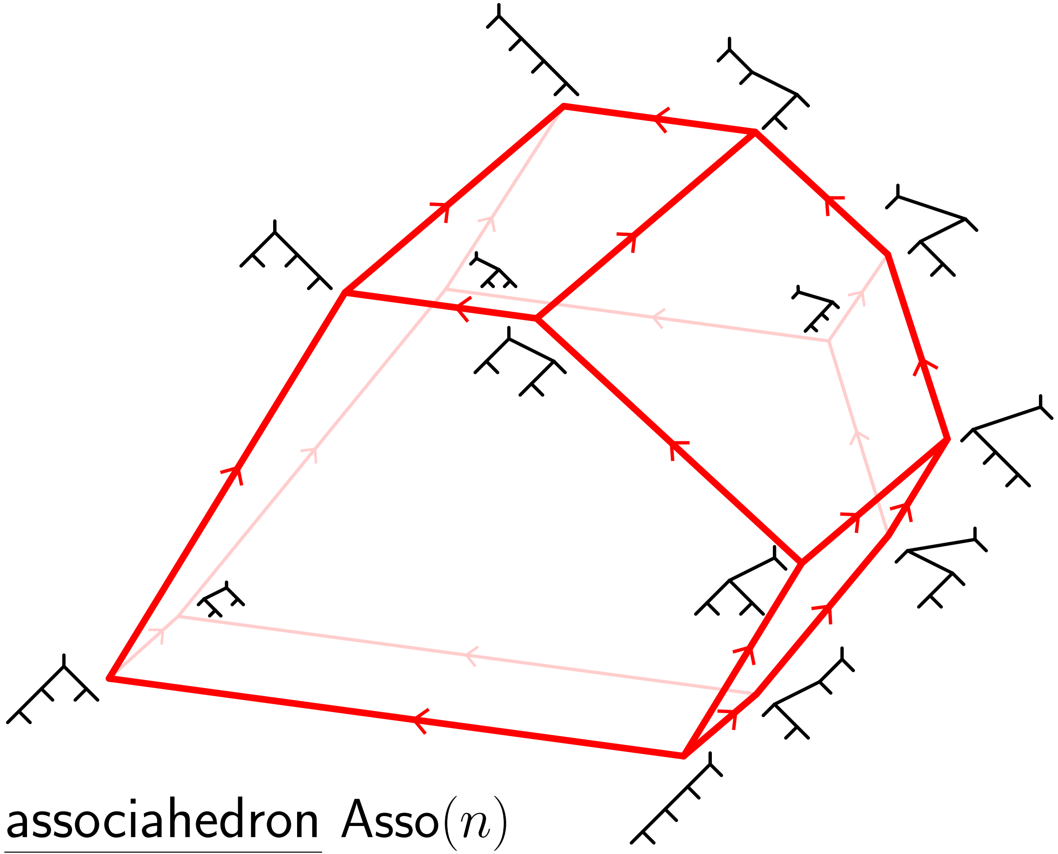


<u>permutahedron</u> Perm(n)



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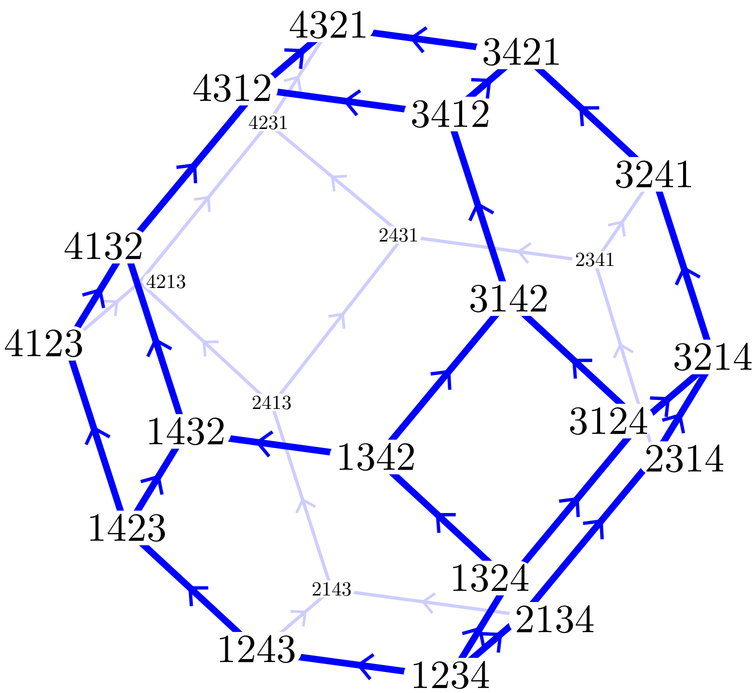
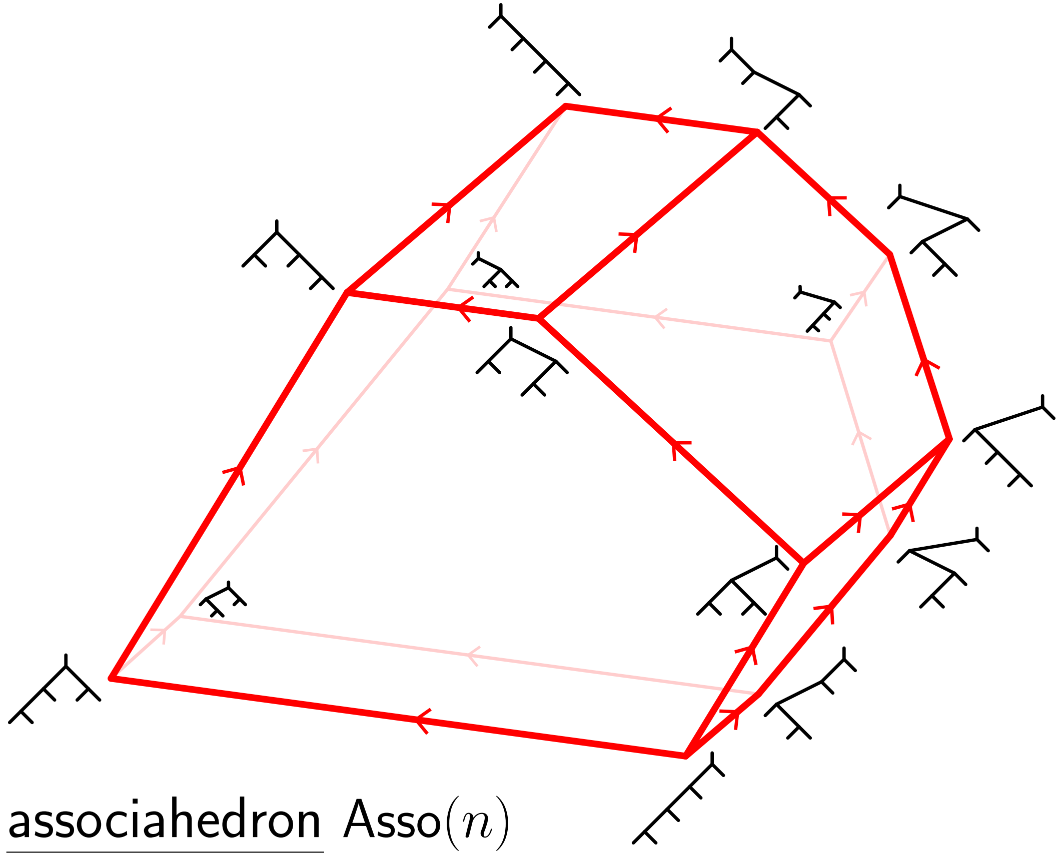


permutahedron Perm(n)

$\Longrightarrow$ weak order on permutations



associahedron Asso(n)

$\Longrightarrow$ Tamari lattice on binary trees

QUOTIENT FANS AND QUOTIENTOPES

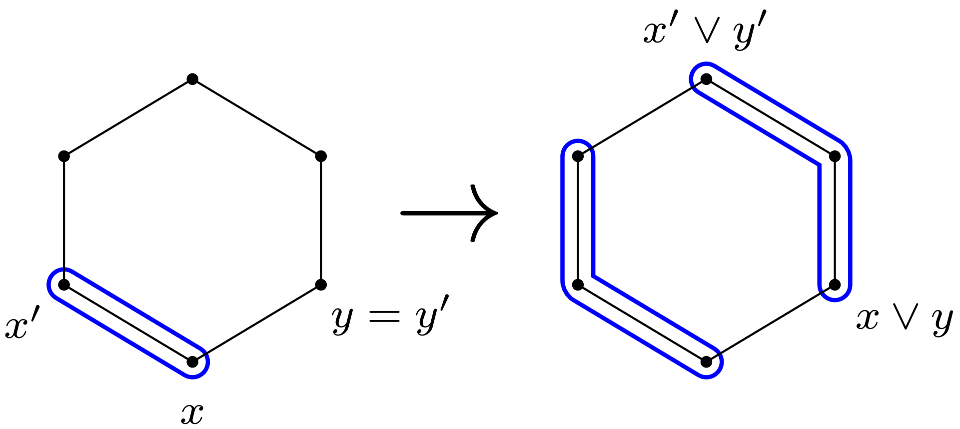
QUOTIENT FAN

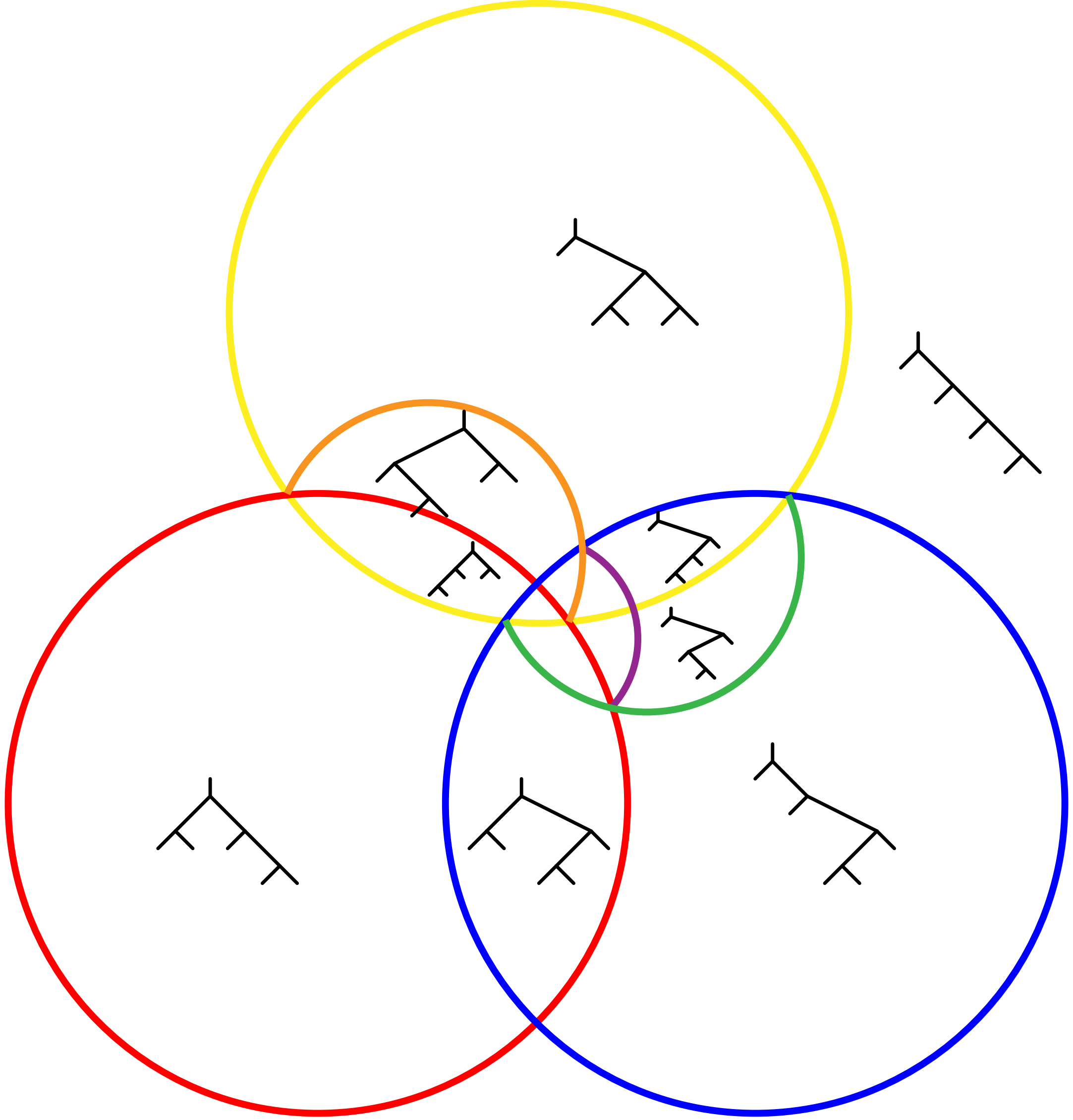
quotient fan: obtained by glueing all cones of the same congruence class

Reading ('05)

$\boldsymbol{W}_{\equiv}$ = walls of the quotient fan $\mathcal{F}_{\equiv}$

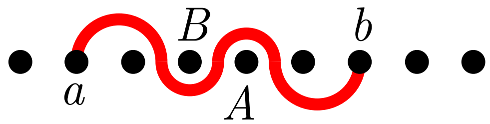
Describe the possible sets of walls $\boldsymbol{W}_{\equiv}$



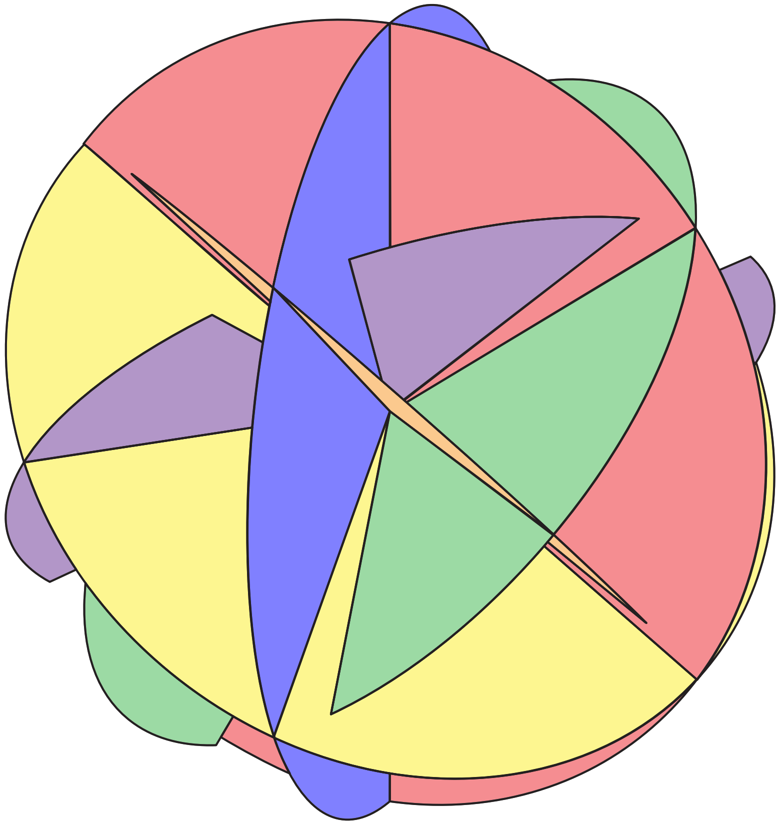


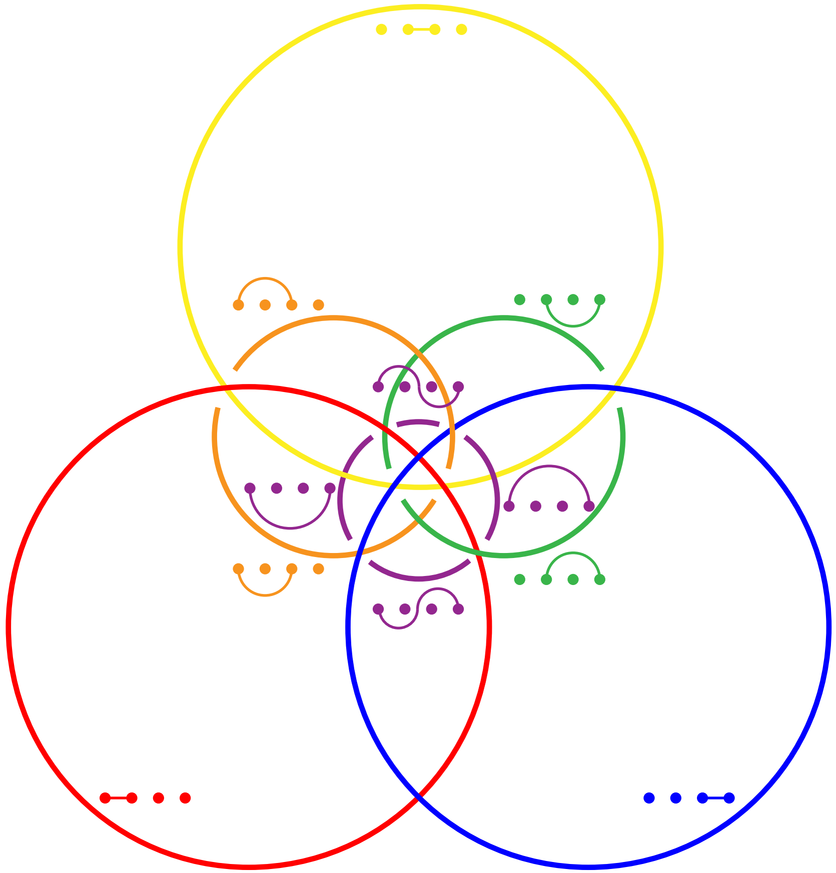
ARCS AND SHARDS

arc (a, b, A, B) with $1 \leq a < b \leq n$ and $A \sqcup B = \,]a, b[$



shard $\Sigma(a, b, A, B) = \left\{ \boldsymbol{x} \in \mathbb{R}^n \;\middle|\; x_{a'} \leq x_a = x_b \leq x_{b'} \text{ for all } a' \in A \text{ and } b' \in B \right\}$





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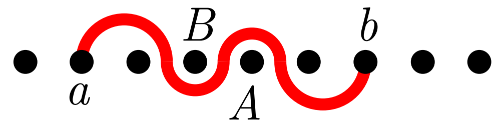

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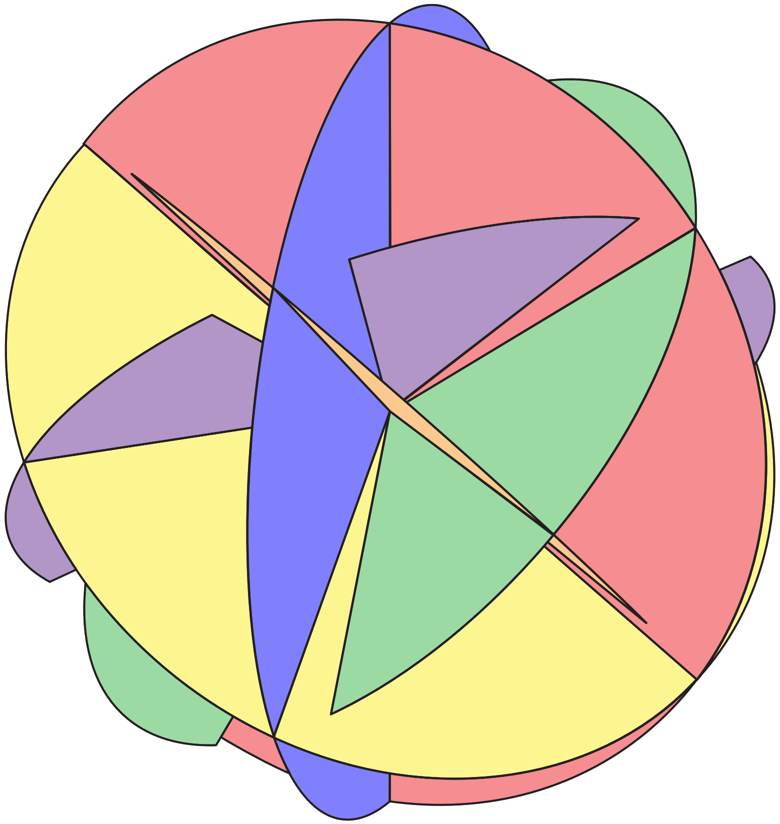

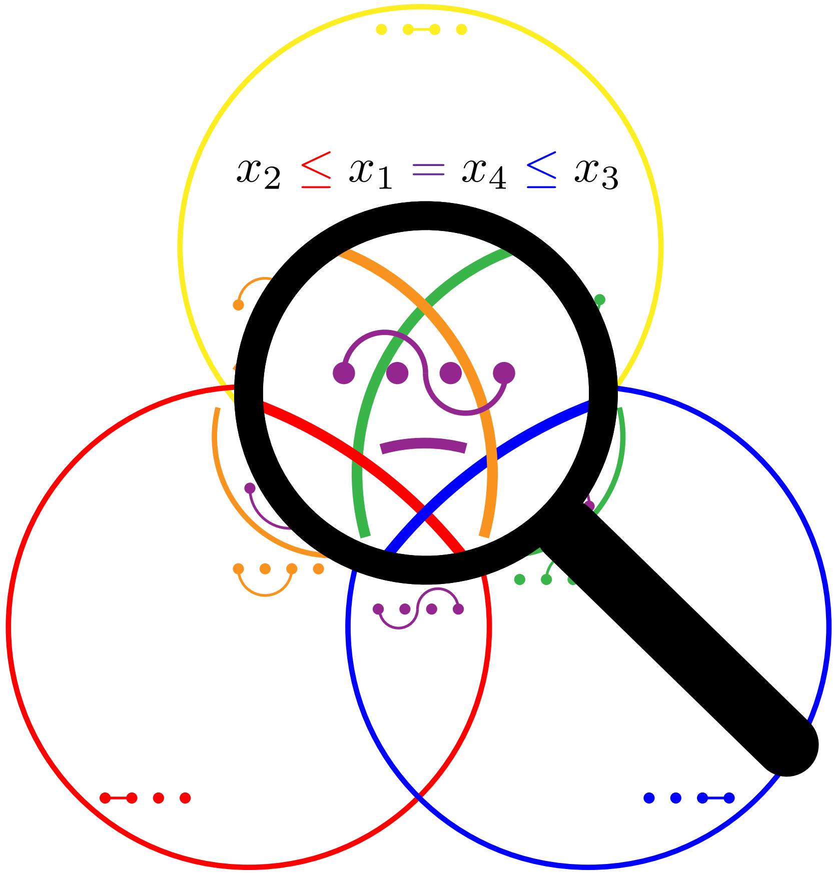

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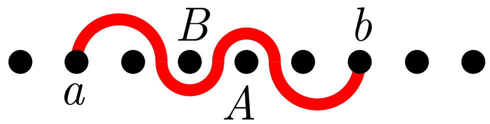

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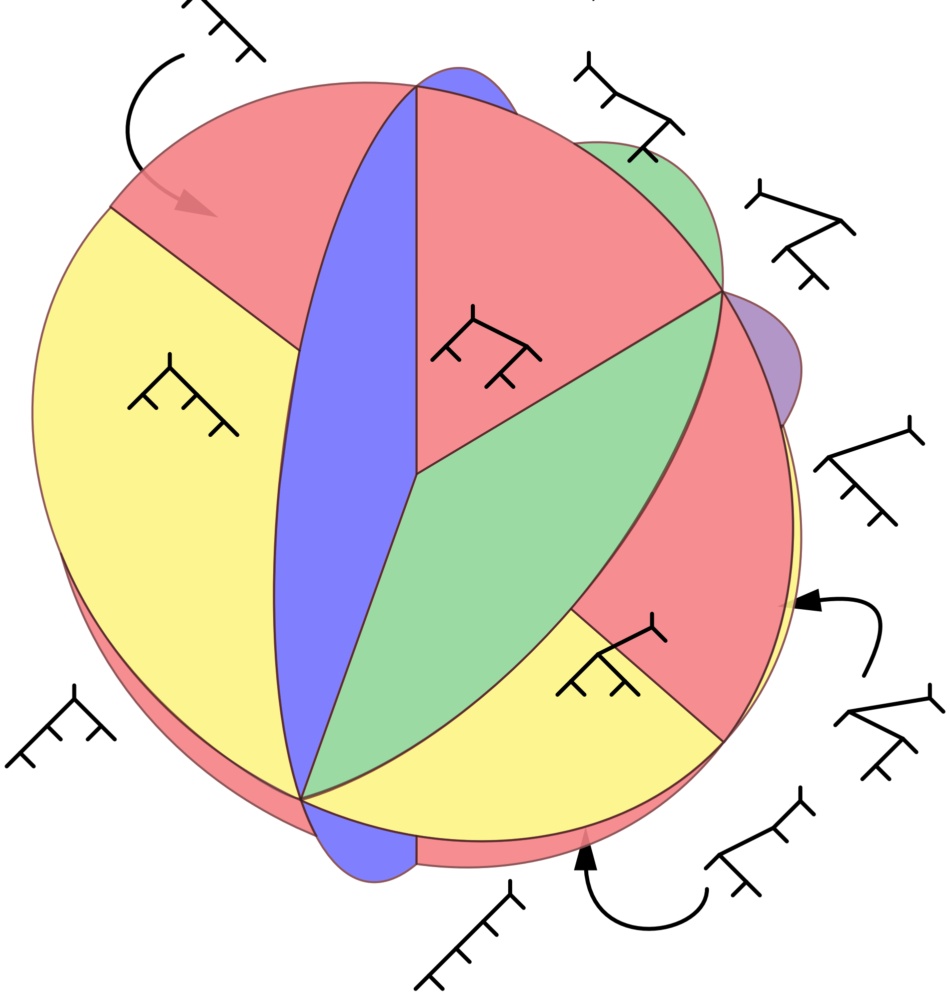
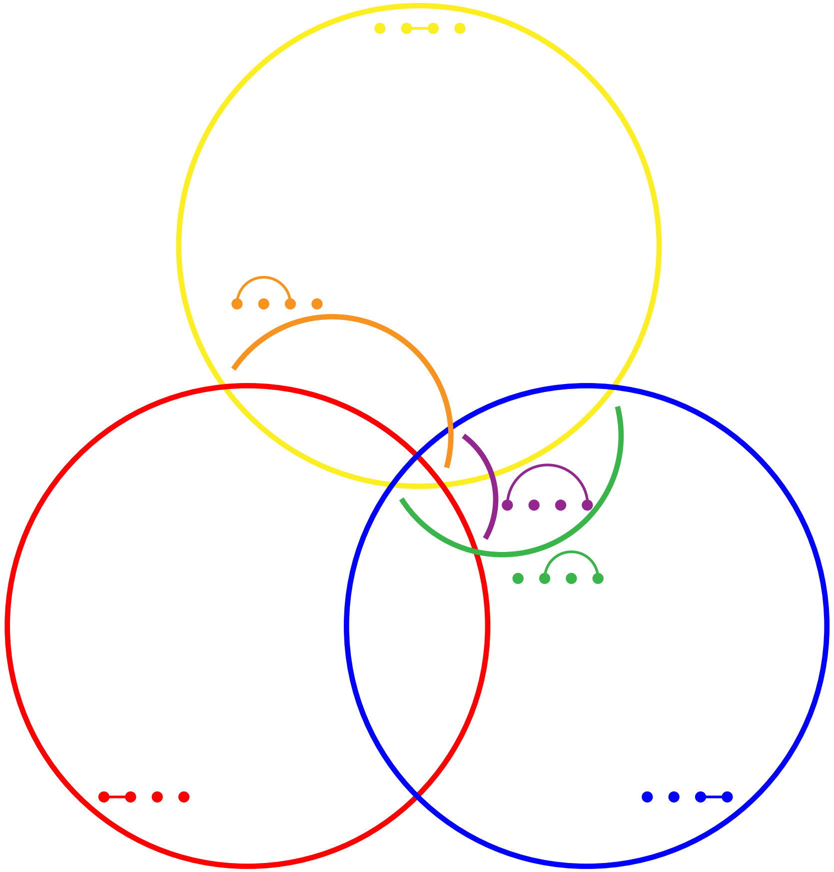
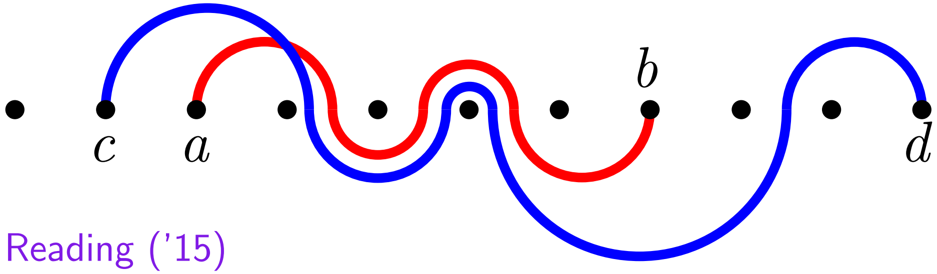


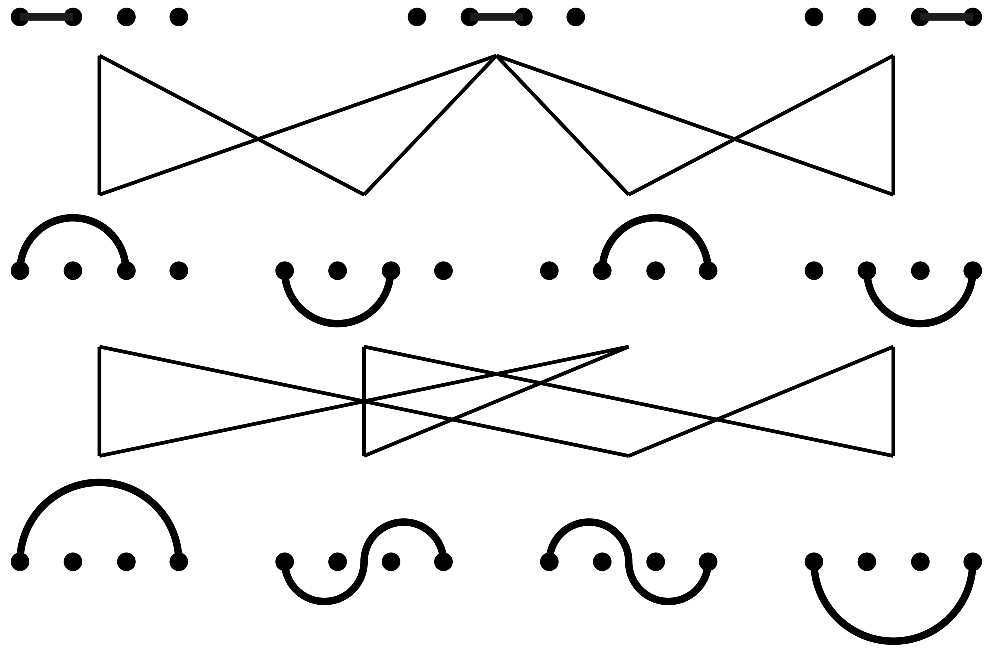


The set of walls $\boldsymbol{W}_{\equiv}$ of the quotient fan $\mathcal{F}_{\equiv}$ is a union of shards $\boldsymbol{\Sigma}_{\equiv}$ Reading ('05)

$\Sigma(a,b,A,B)$ forces $\Sigma(c,d,C,D) =$
$c \leq a < b \leq d$ and $A \subseteq C$ and $B \subseteq D$



Reading ('15)



FORCING

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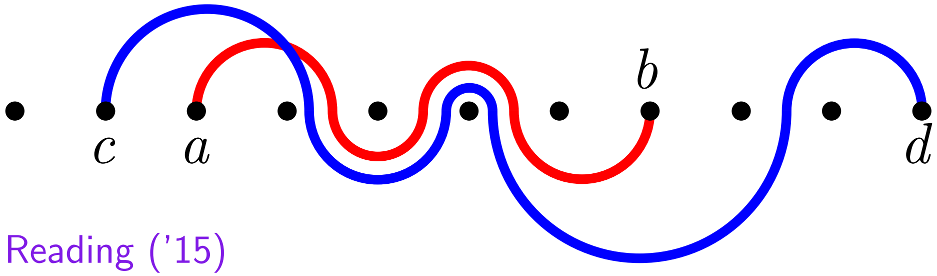

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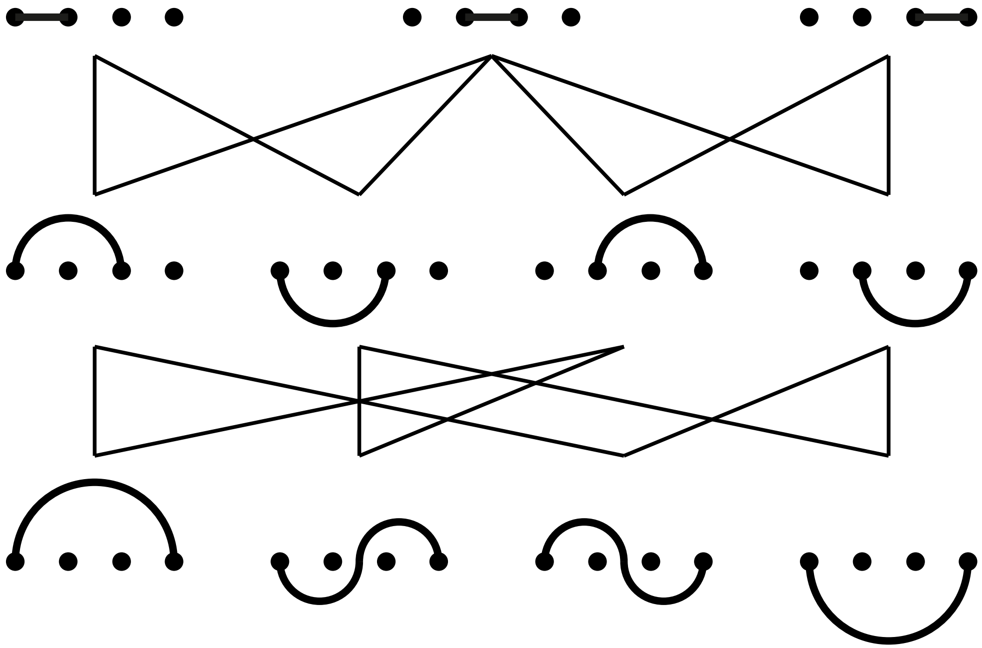
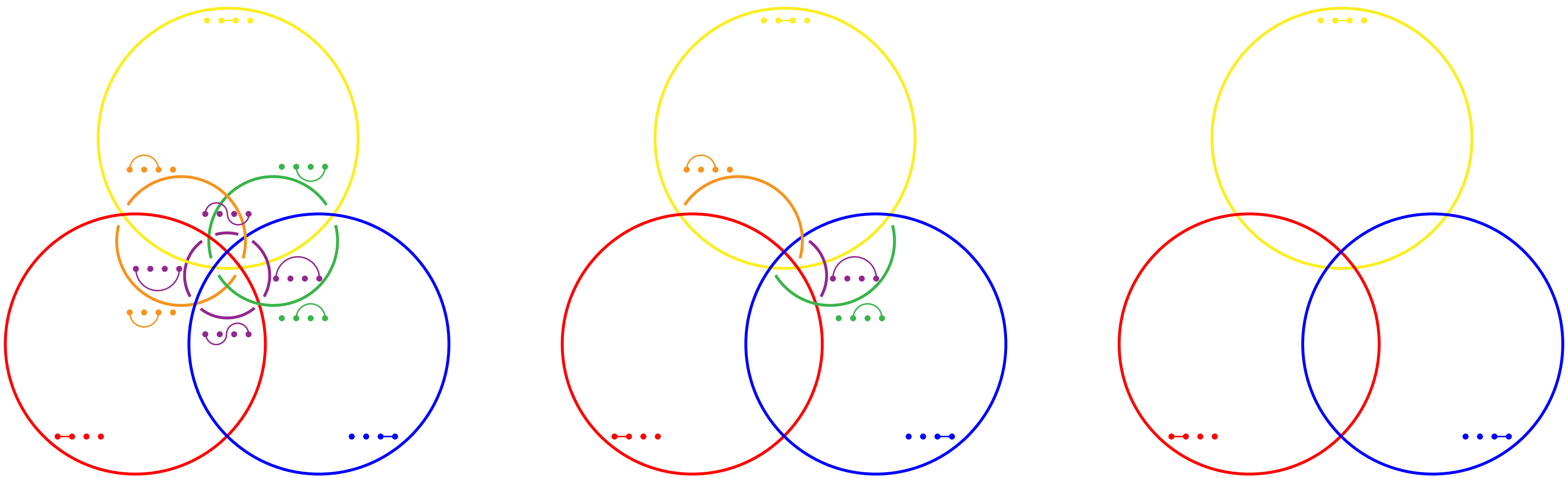


TFAE for a set of shards $\boldsymbol{\Sigma}$:

- there is a congruence $\equiv$ with $\boldsymbol{\Sigma} = \boldsymbol{\Sigma}_{\equiv}$
- $\boldsymbol{\Sigma}$ is an upper ideal in forcing order



SHARD IDEALS

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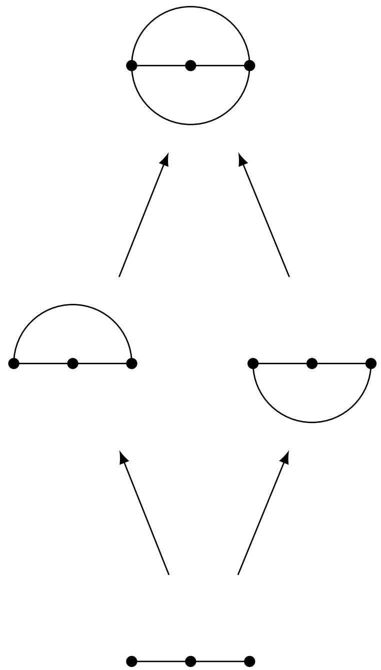
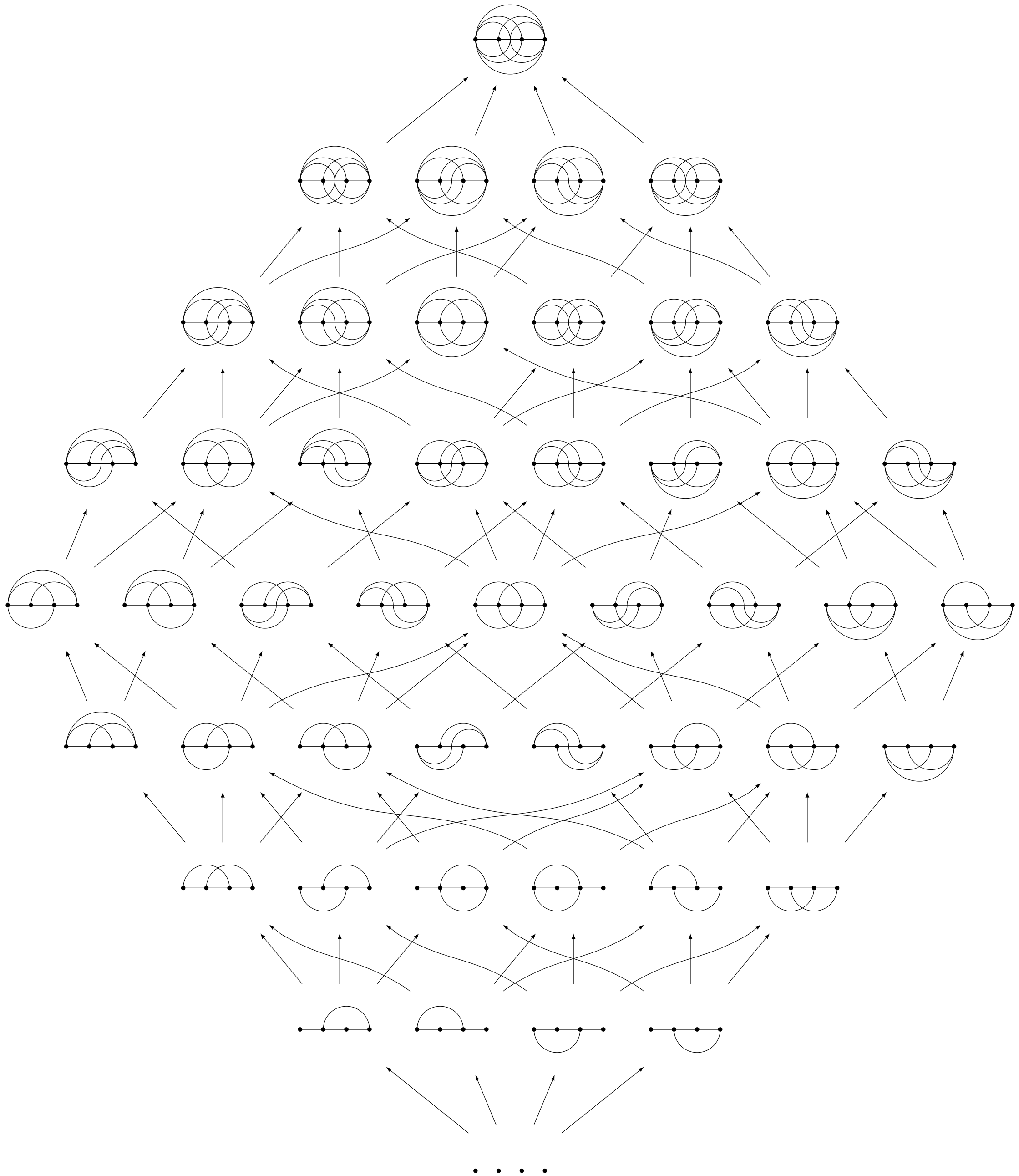


essential congruences:
1, 1, 4, 47, 3322, ...
OEIS A330039

all congruences
1, 2, 7, 60, 3444, ...
OEIS A091687



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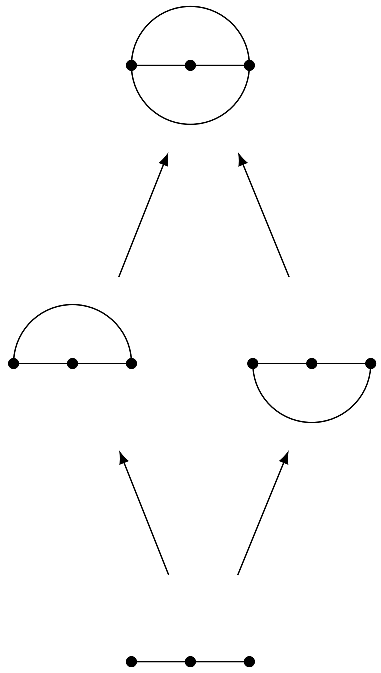

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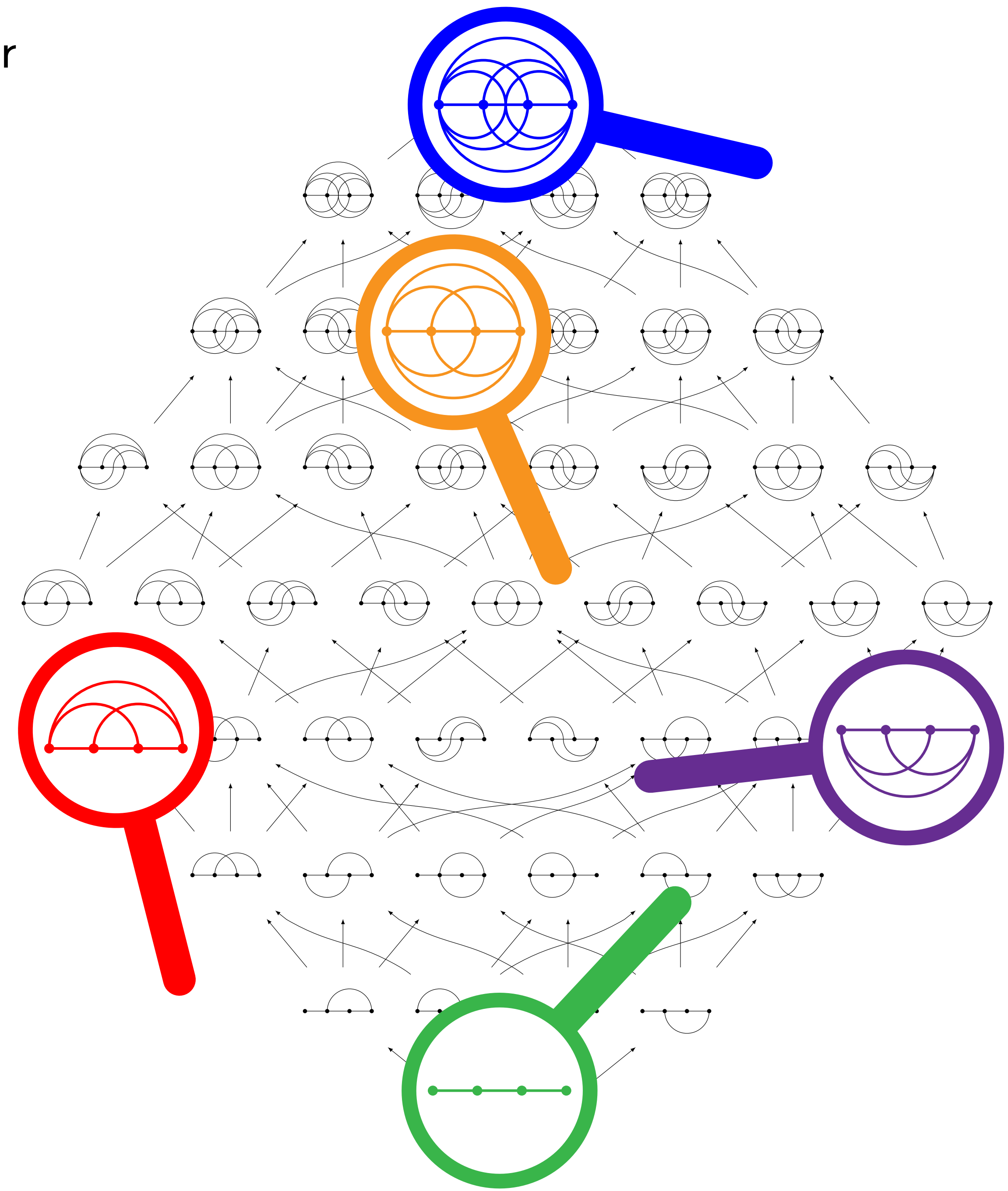
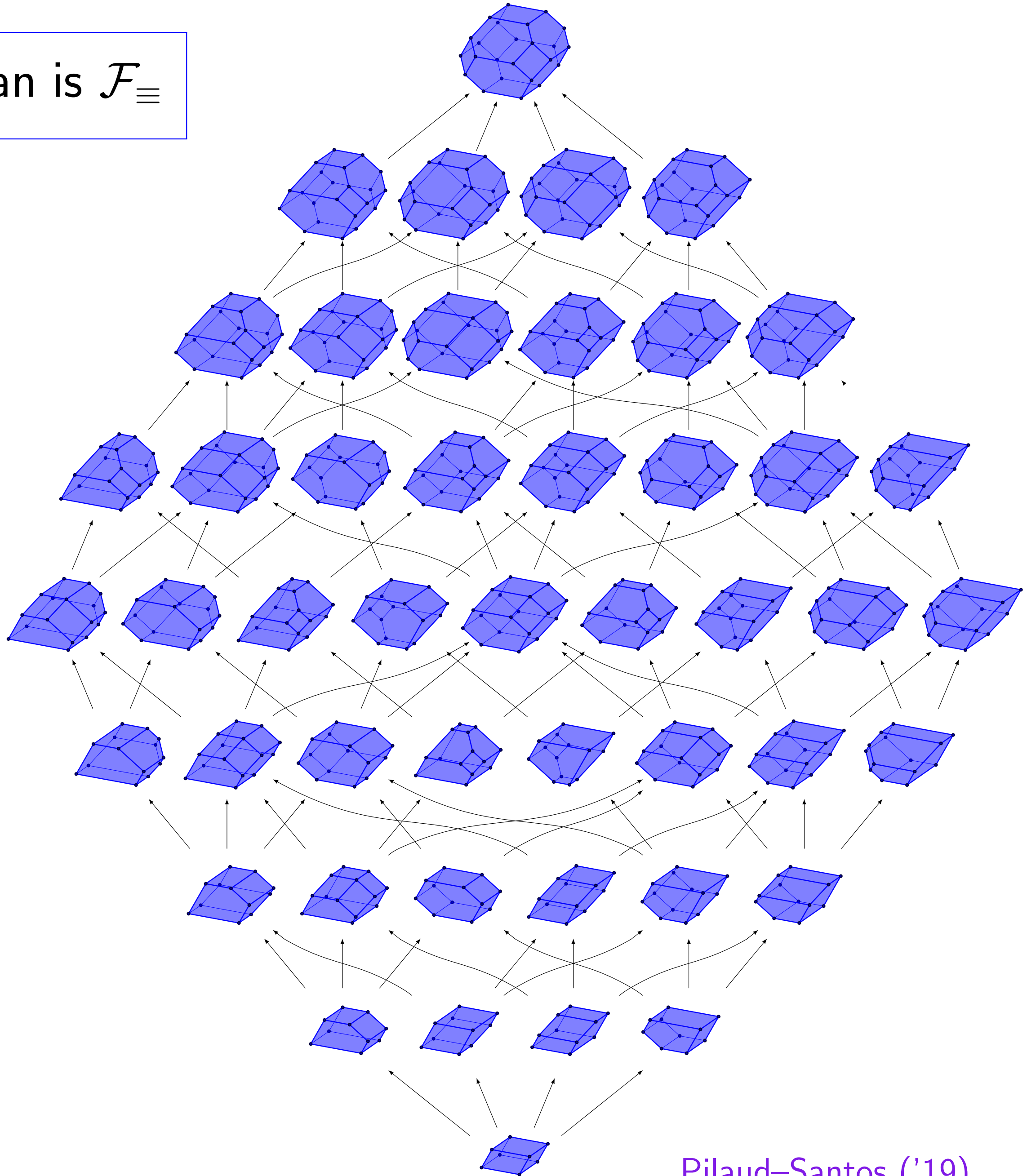


QUOTIENTOPES

quotientope $=$ polytope whose normal fan is $\mathcal{F}_{\equiv}$



Pilaud–Santos ('19)

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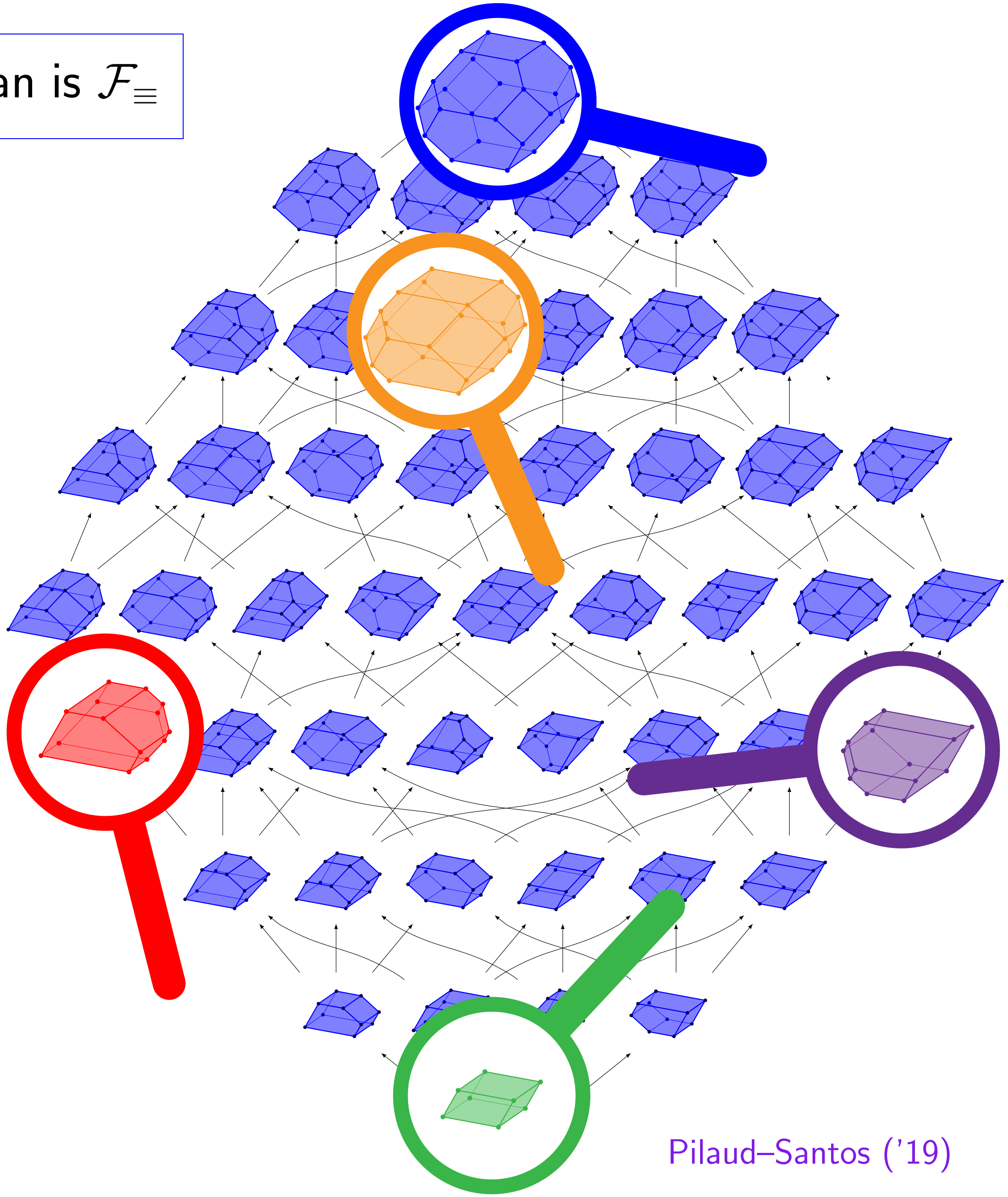

Pilaud–Santos ('19)

MINKOWSKI SUMS OF ASSOCIAHEDRA

INTERSECTIONS OF CONGRUENCES

If the congruence $\equiv$ is the intersection of the congruences $\equiv_1, \dots, \equiv_k$,
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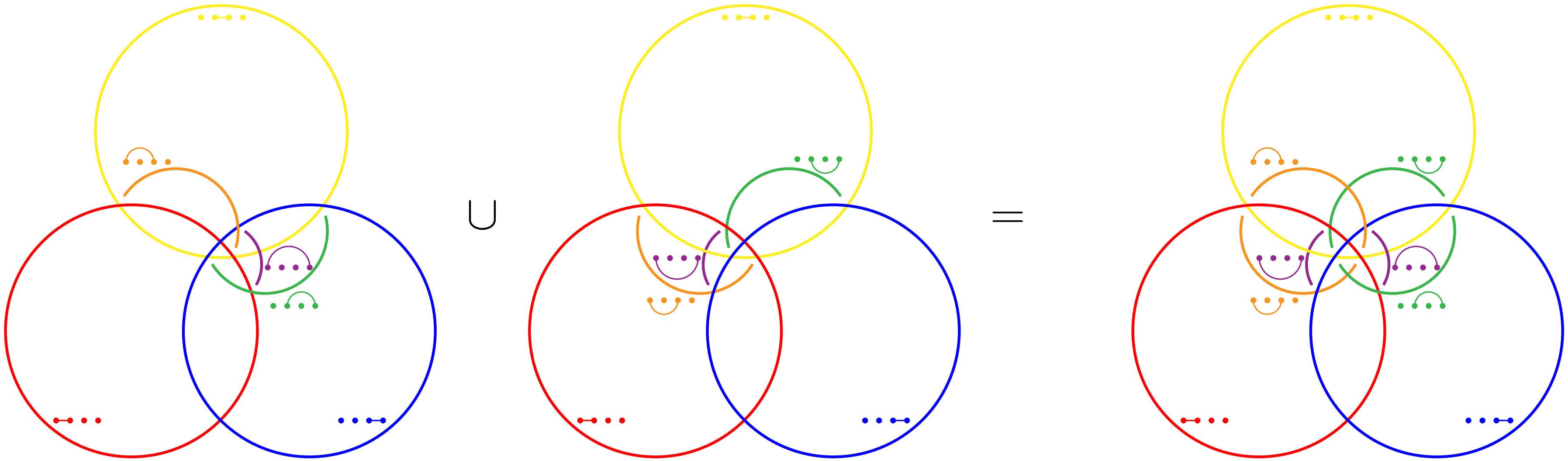

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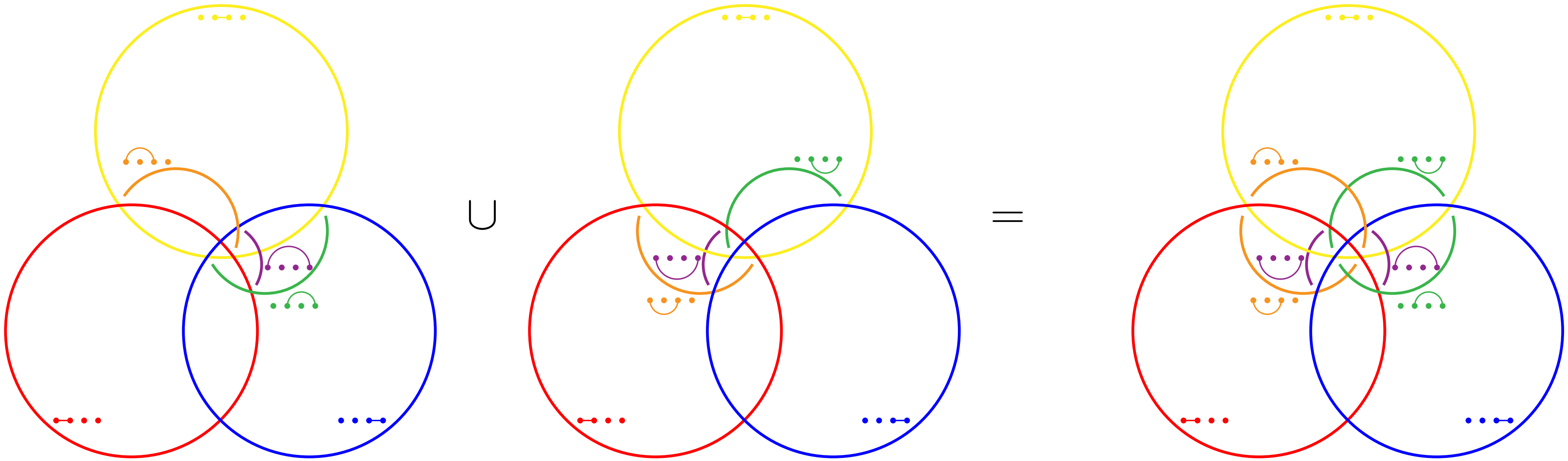
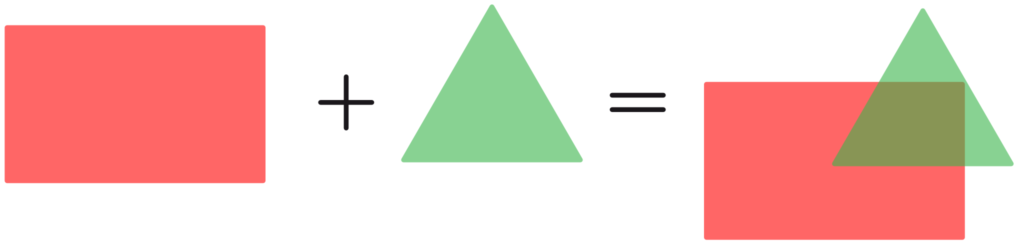


Minkowski sum $\mathsf{P} + \mathsf{Q} = \{\boldsymbol{p} + \boldsymbol{q} \mid \boldsymbol{p} \in \mathsf{P},\ \boldsymbol{q} \in \mathsf{Q}\}$



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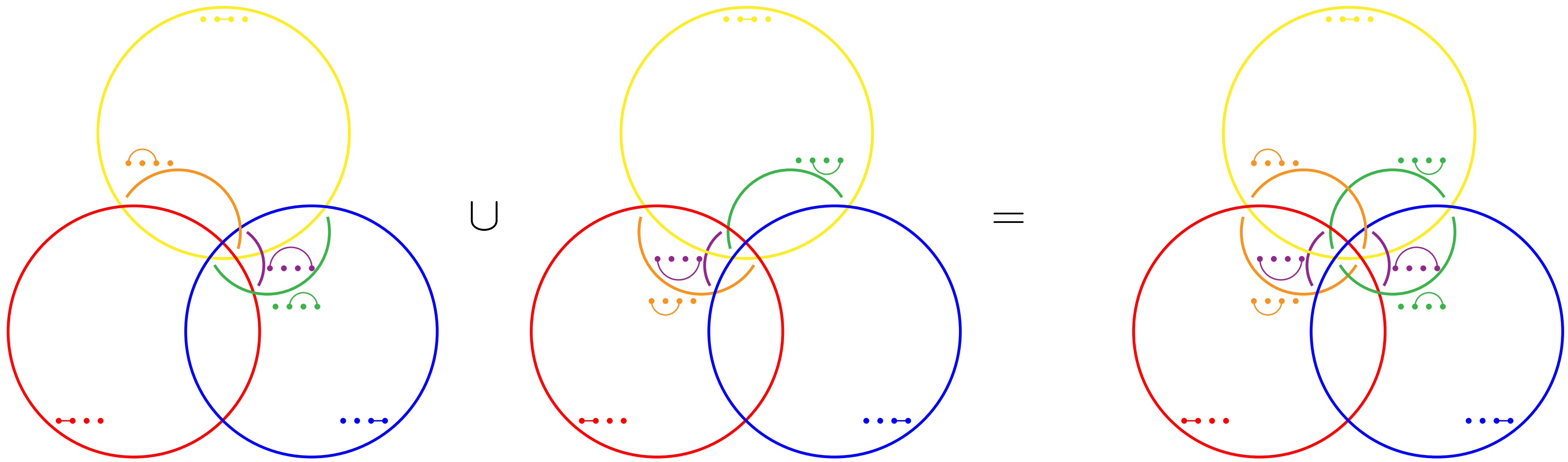

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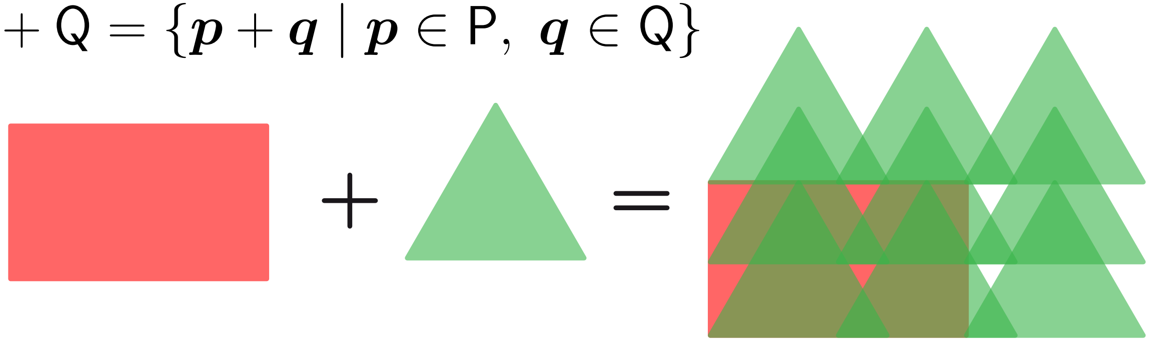

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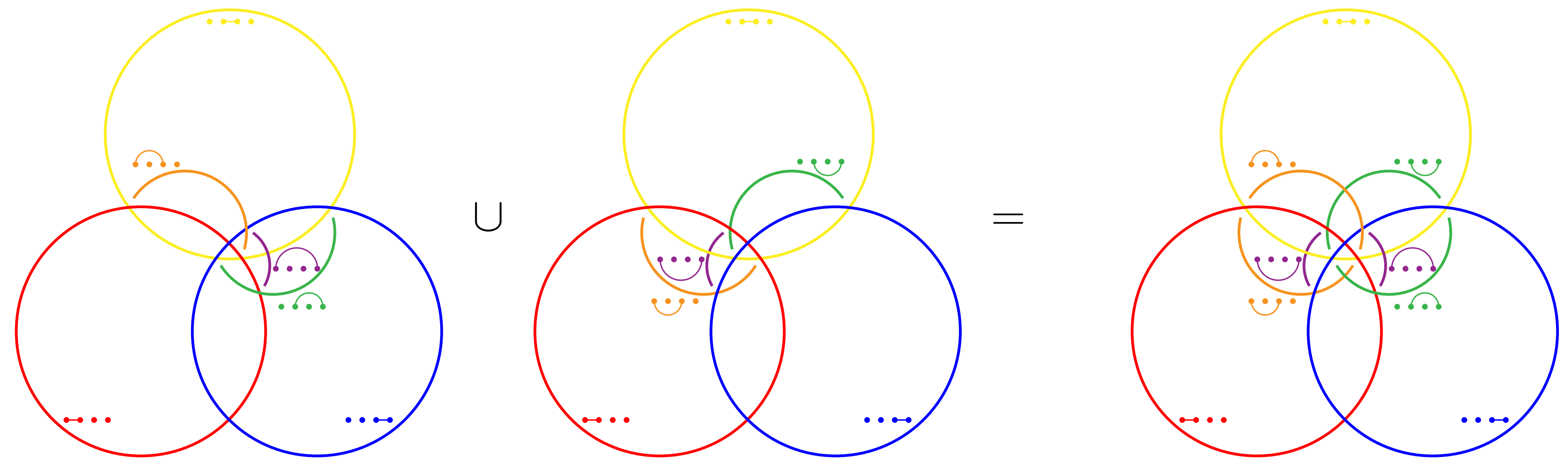

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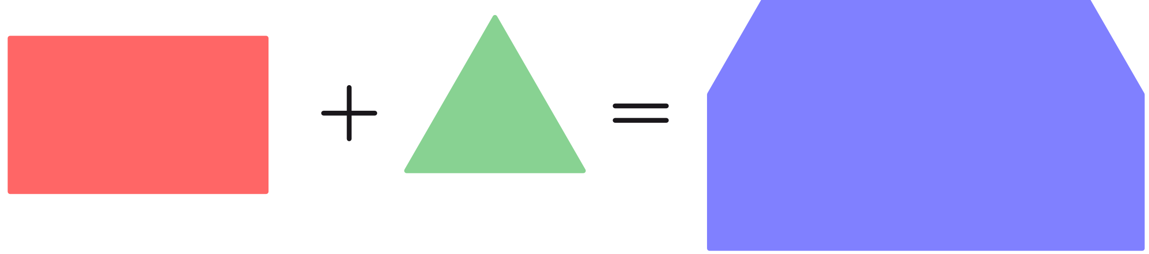

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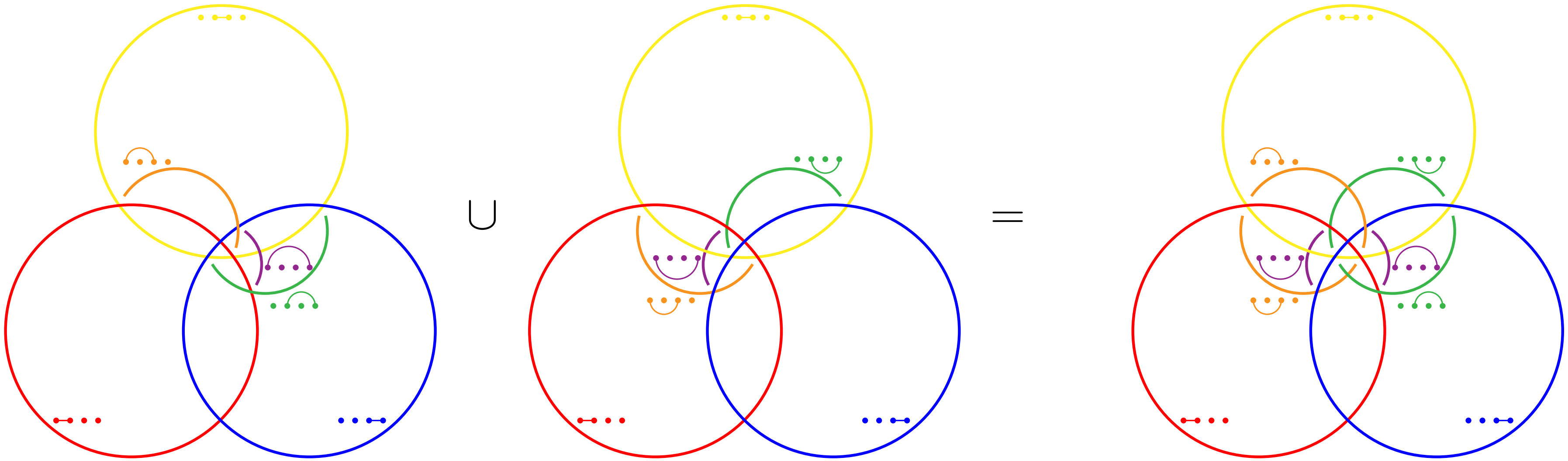

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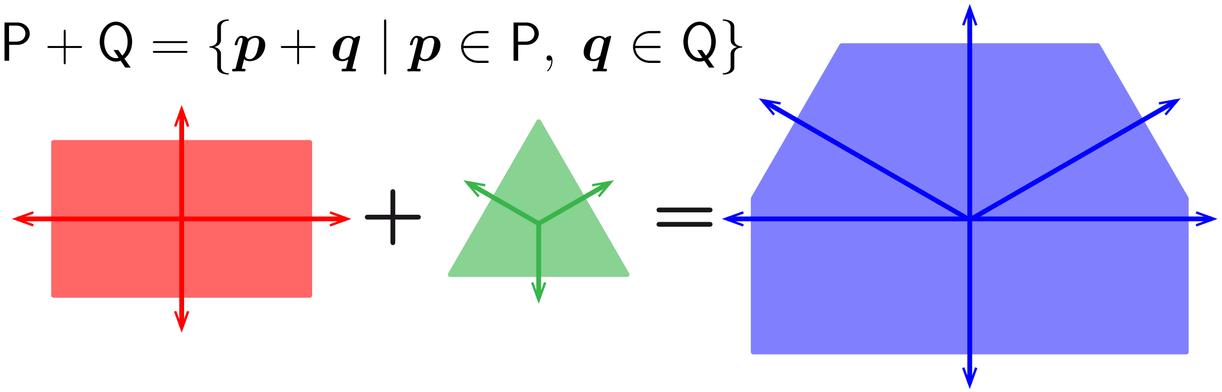

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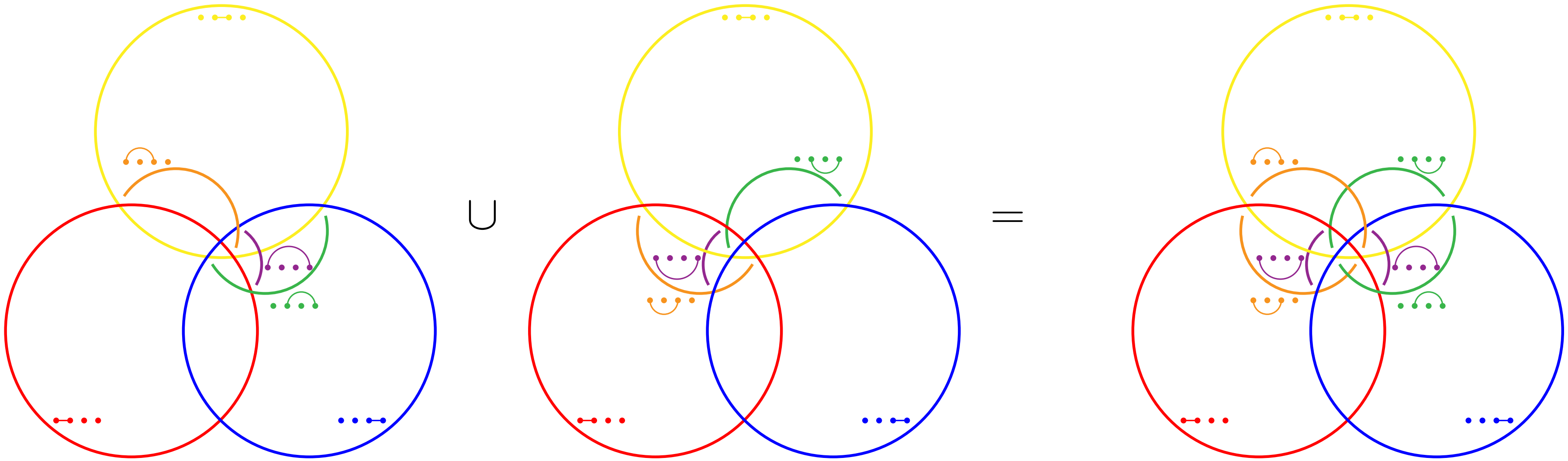
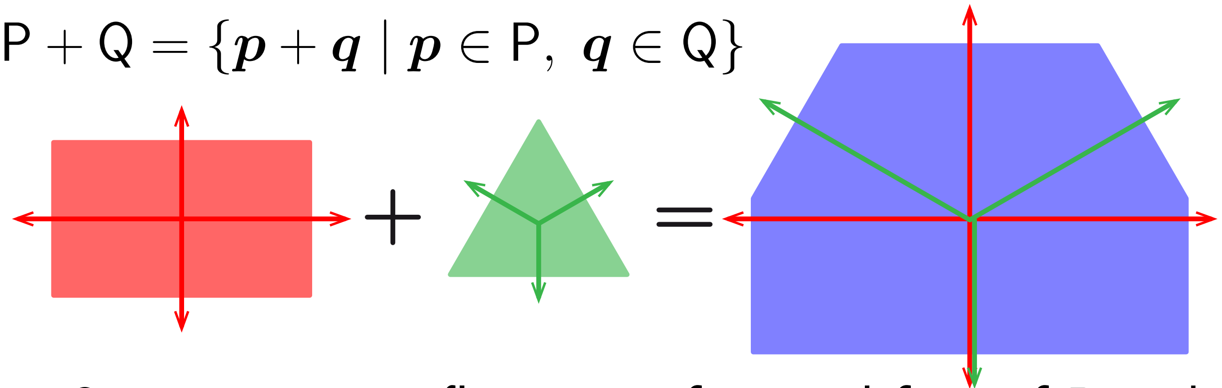


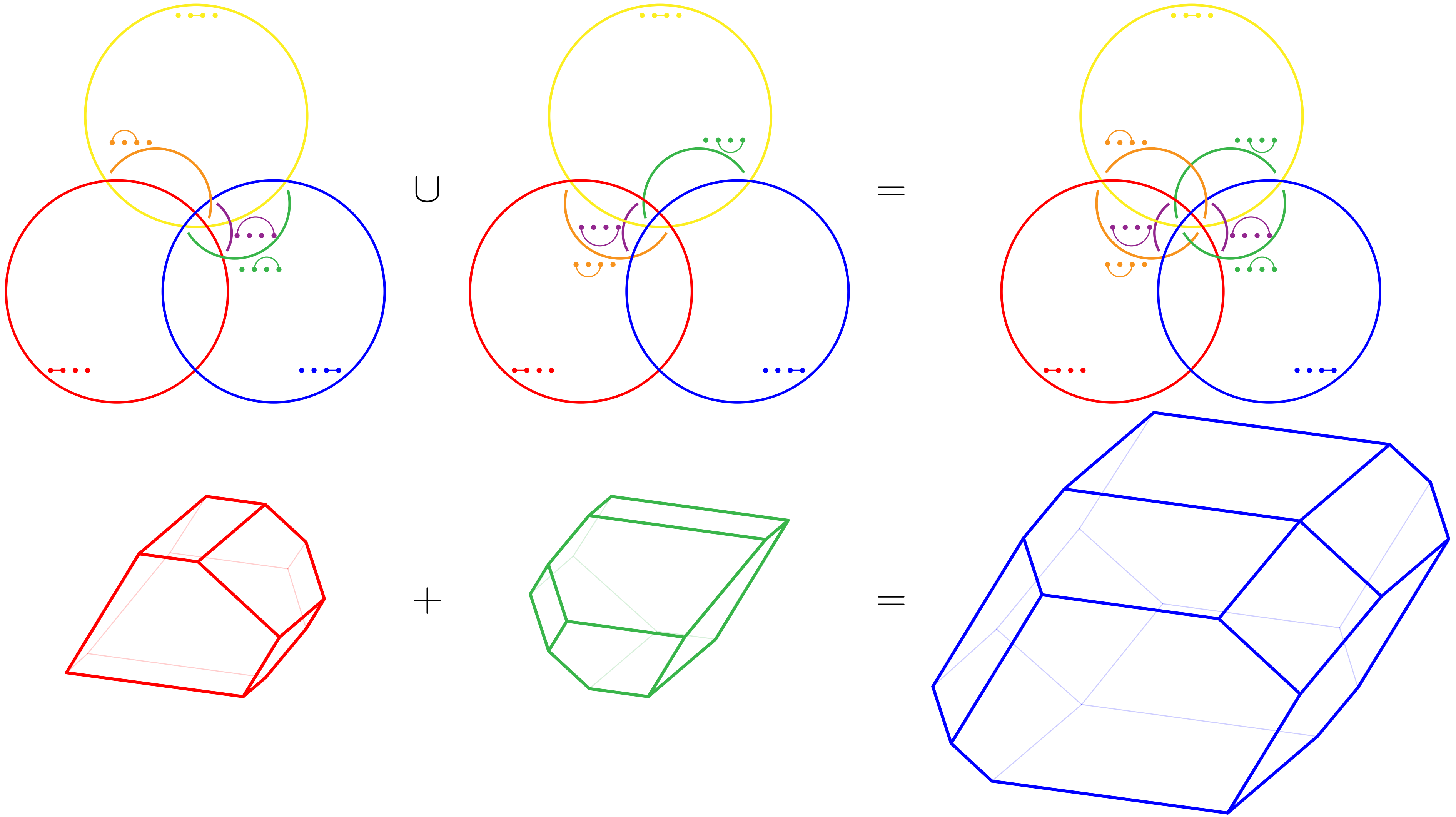
Minkowski sum $\mathsf{P} + \mathsf{Q} = \{\boldsymbol{p} + \boldsymbol{q} \mid \boldsymbol{p} \in \mathsf{P},\ \boldsymbol{q} \in \mathsf{Q}\}$



Normal fan of $\mathsf{P} + \mathsf{Q}$ = common refinement of normal fans of P and Q

MINKOWSKI SUMS OF QUOTIENTOPES

If the congruence $\equiv$ is the intersection of the congruences $\equiv_1, \dots, \equiv_k$,
then the quotient fan $\mathcal{F}_\equiv$ is the common refinement of the quotient fans $\mathcal{F}_{\equiv_1}, \dots, \mathcal{F}_{\equiv_k}$,
and a Minkowski sum of quotientopes for $\mathcal{F}_{\equiv_1}, \dots, \mathcal{F}_{\equiv_k}$ is a quotientope for $\mathcal{F}_\equiv$



MINKOWSKI SUMS OF ASSOCIAHEDRA

If the congruence $\equiv$ is the intersection of the congruences $\equiv_1, \dots, \equiv_k$,
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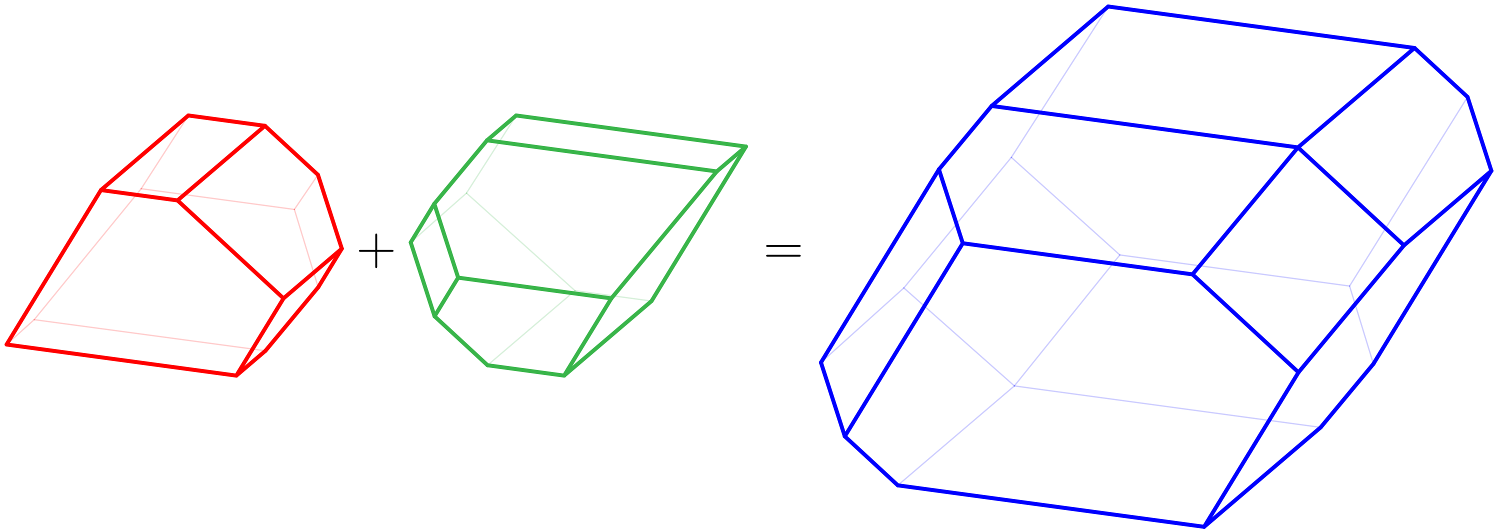
Any quotient fan is realized by a Minkowski sum of (low dim.) associahedra



Padrol-Pilaud-R. ('20$^+$)

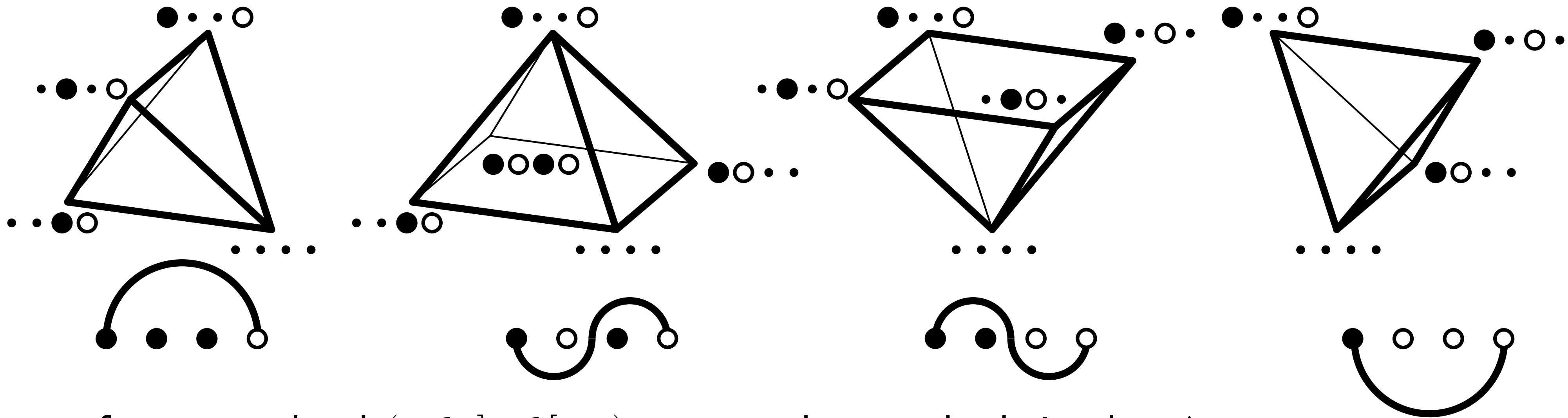
SHARD POLYTOPES

SHARD POLYTOPES

for a shard $\Sigma = \Sigma(a, b, A, B)$, define

- Σ-matching $=$ sequence $a \le a_1 < b_1 < \cdots < a_k < b_k \le b$ where $\begin{cases} a_i \in \{a\} \cup A \\ b_i \in B \cup \{b\} \end{cases}$
- characteristic vector $\chi(M) = \sum_{i \in [k]} \boldsymbol{e}_{a_i} - \boldsymbol{e}_{b_i}$

shard polytope $\mathrm{SP}(\Sigma) \; = \operatorname{conv}\left\{ \chi(M) \;\middle|\; M \; \Sigma\text{-matching} \right\}$



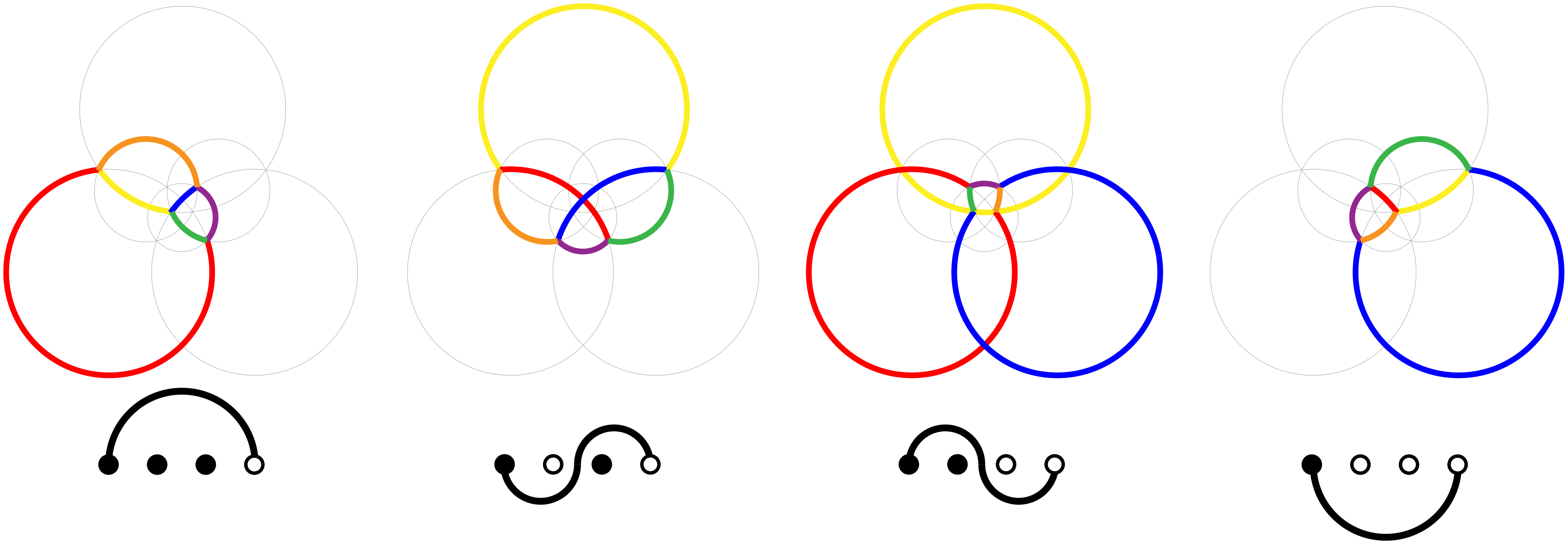
exm: for an up shard $(a, b,]a, b[, \varnothing)$, we get the standard simplex $\triangle_{[a,b]} - \boldsymbol{e}_b$

SHARD POLYTOPES

<u>shard polytope</u> $\mathsf{SP}(\Sigma) = \operatorname{conv}\left\{\chi(M) \;\middle|\; M\ \Sigma\text{-matching}\right\}$

The normal fan of the shard polytope $\mathsf{SP}(\Sigma)$

- contains the shard Σ,
- is contained in the union of the shards forcing Σ



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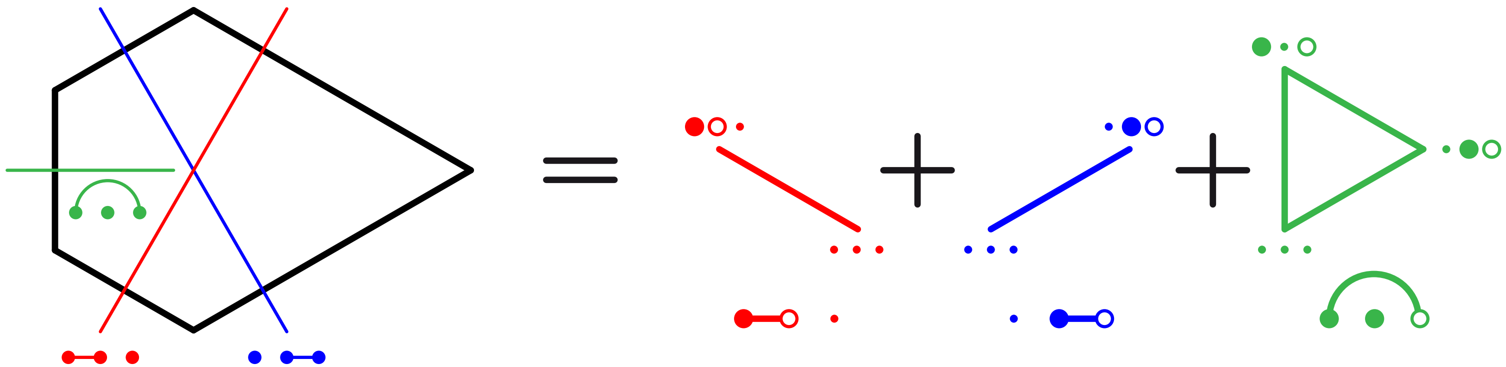
For any lattice congruence $\equiv$, the quotient fan $\mathcal{F}_\equiv$ is the normal fan of the Minkowski sum of the shard polytopes $\mathsf{SP}(\Sigma)$ for $\Sigma \in \boldsymbol{\Sigma}_\equiv$ Padrol-Pilaud-R. (20^+)



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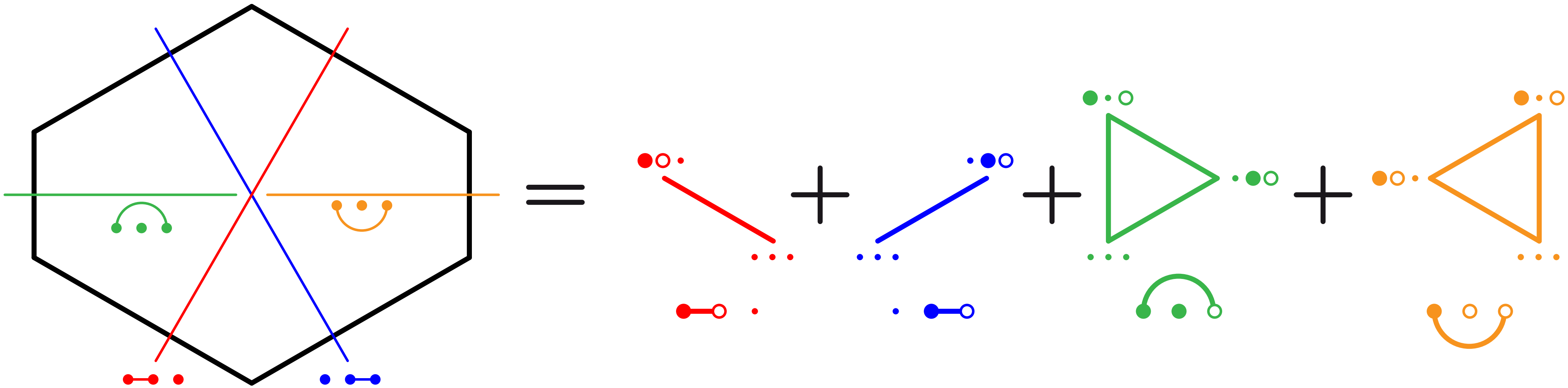


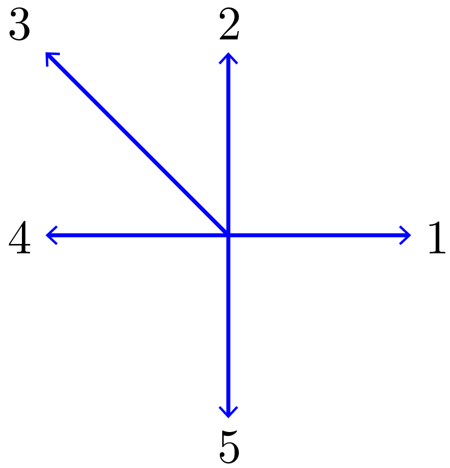
SHARD POLYTOPES AND TYPE CONES

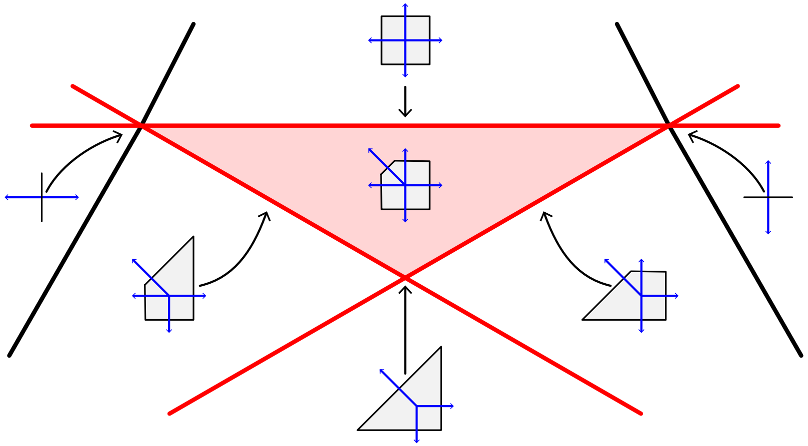
TYPE CONE

$\mathcal{F}$ = complete simplicial fan in $\mathbb{R}^n$ with N rays

type cone $\mathsf{TC}(\mathcal{F})$ = realization space of $\mathcal{F}$ — McMullen ('73)

$= \{ \boldsymbol{h} \in \mathbb{R}^N \mid \mathcal{F}$ is the normal fan of $\mathsf{P}_{\boldsymbol{h}} \}$

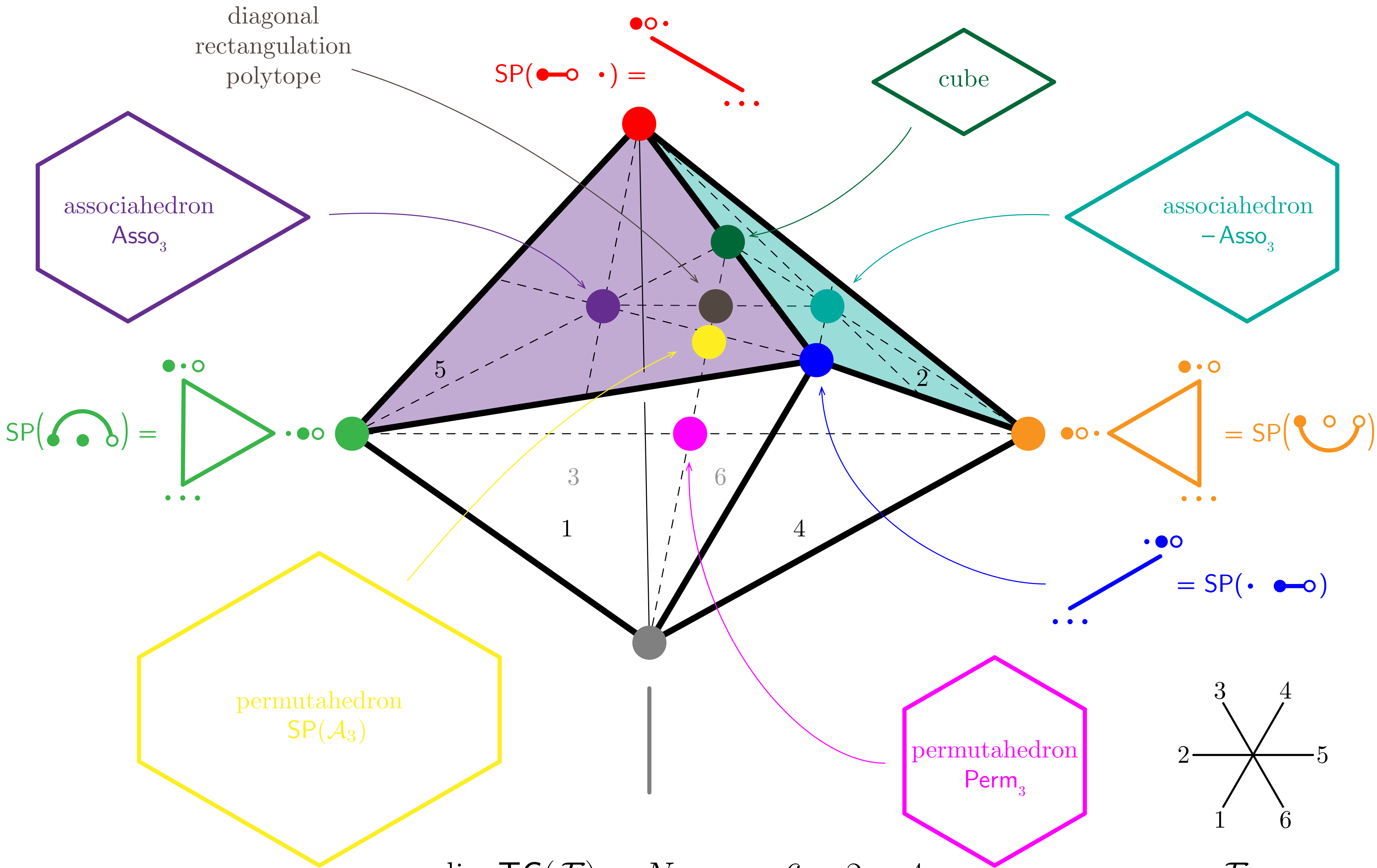




some properties of $\mathsf{TC}(\mathcal{F})$:

- closure of $\mathsf{TC}(\mathcal{F})$ = polytopes whose normal fan coarsens $\mathcal{F}$ = deformation cone
- Minkowski sums $\longleftrightarrow$ positive linear combinations

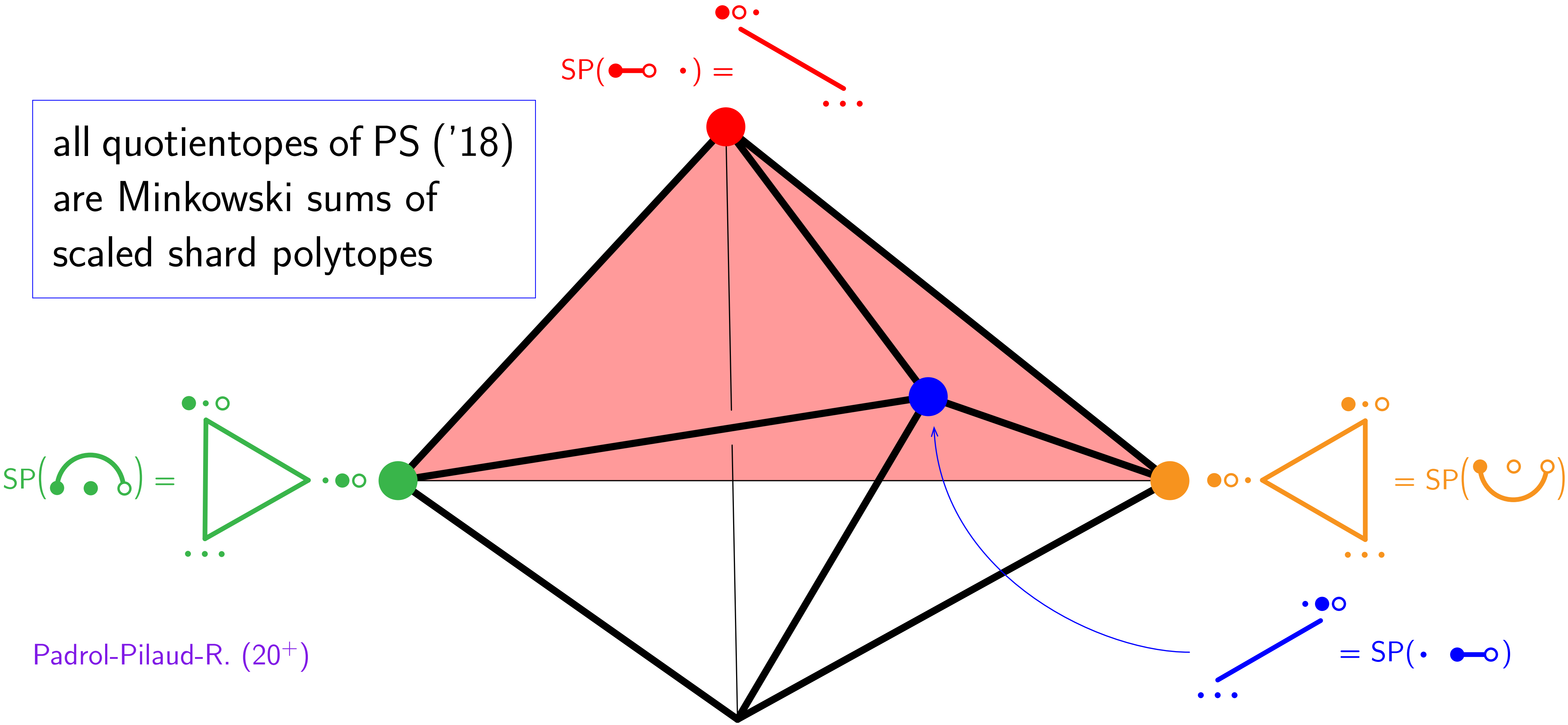
TYPE POLYTOPE



$$\dim \mathsf{TC}(\mathcal{F}) = N - n = 6 - 2 = 4$$

TYPE POLYTOPE

all quotientopes of PS ('18)
are Minkowski sums of
scaled shard polytopes



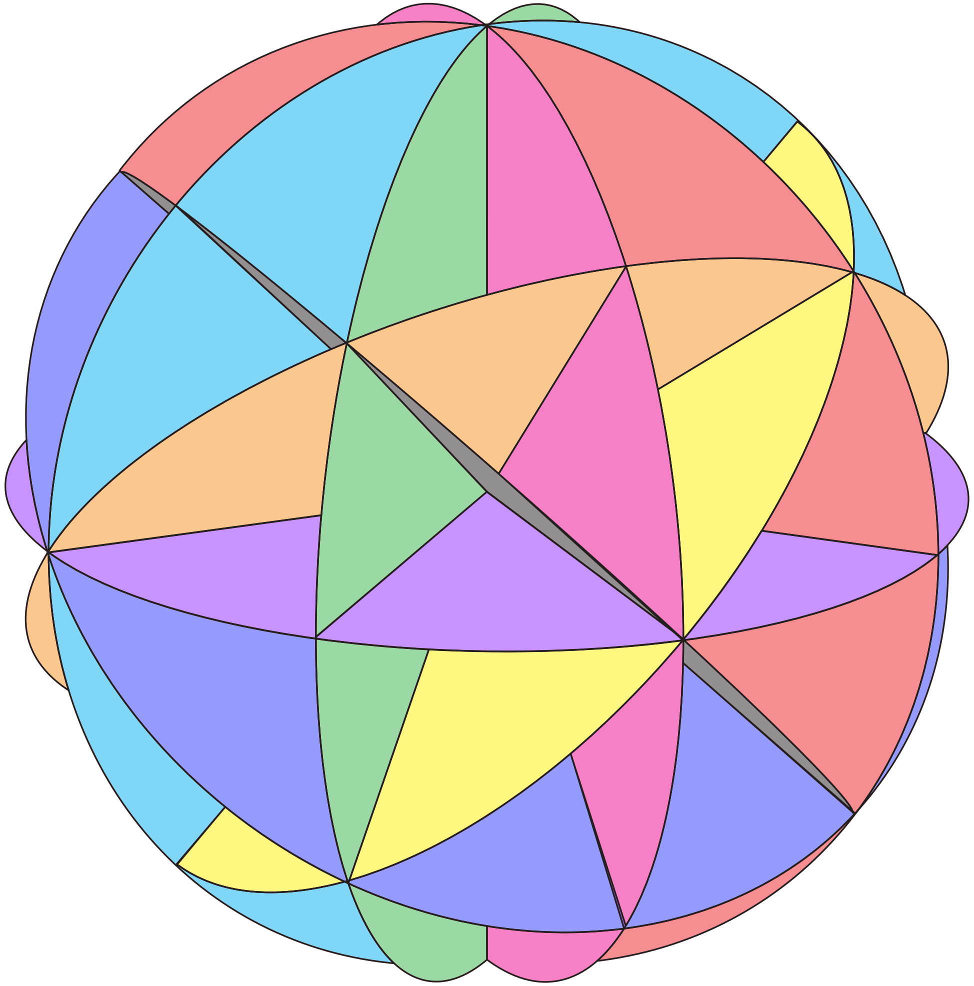
Padrol-Pilaud-R. (20^+)

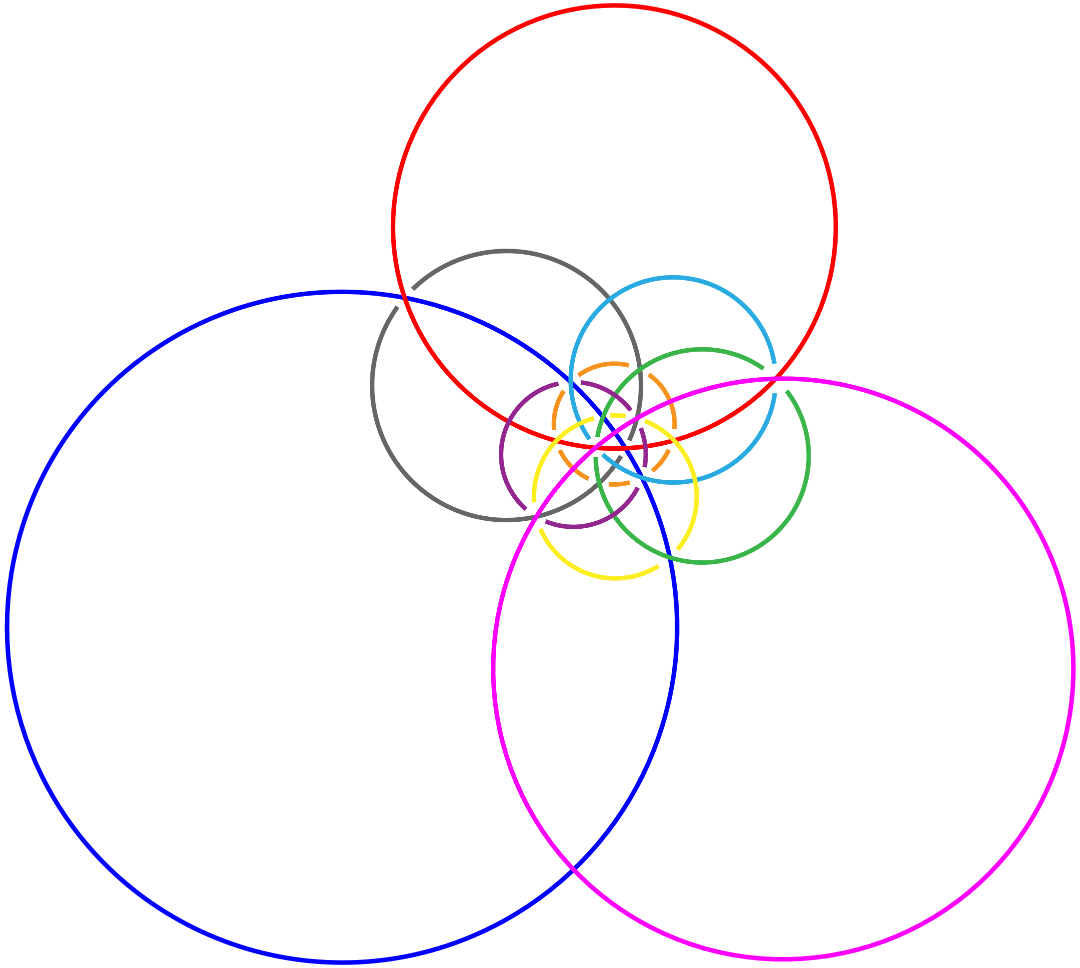
OPEN QUESTIONS

SHARDS FOR HYPERPLANE ARRANGEMENTS

shard = piece of hyperplane obtained after cutting all rank 2 subgroups

shard poset = (pre)poset of forcing relations among shards





Reading ('03)

SHARDS FOR HYPERPLANE ARRANGEMENTS

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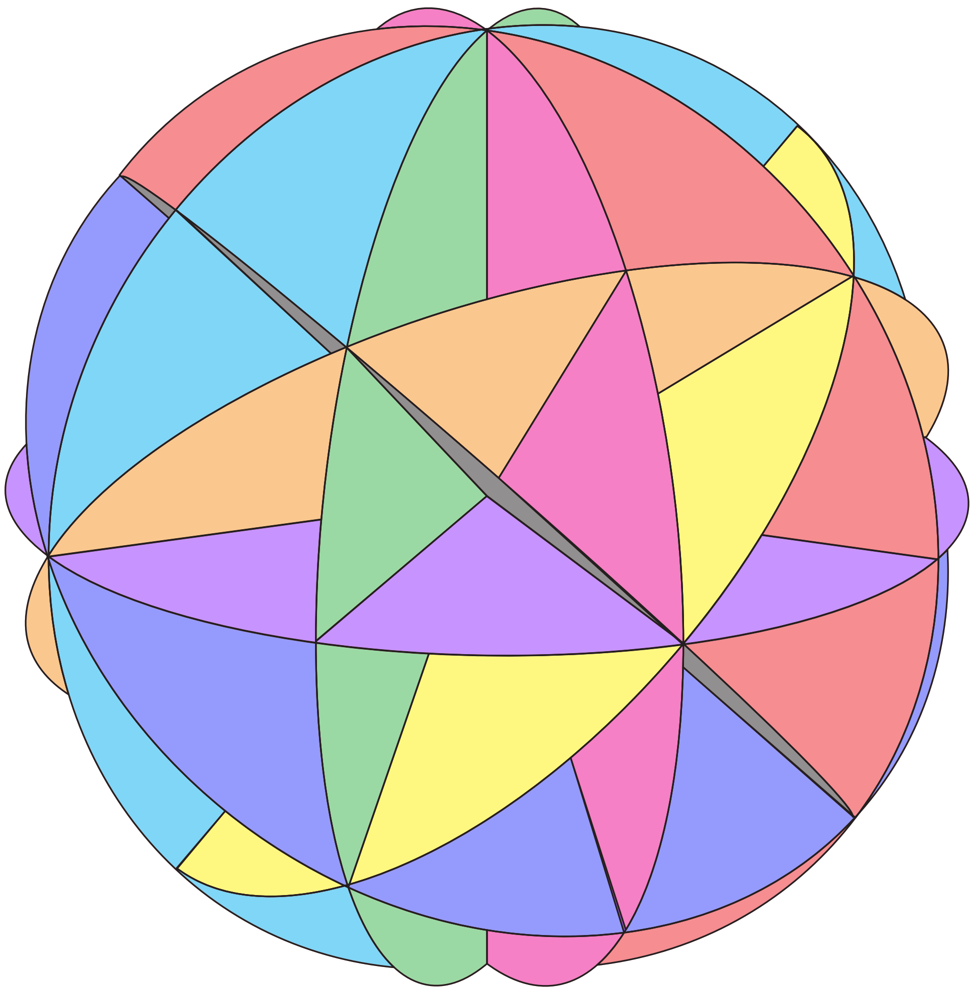

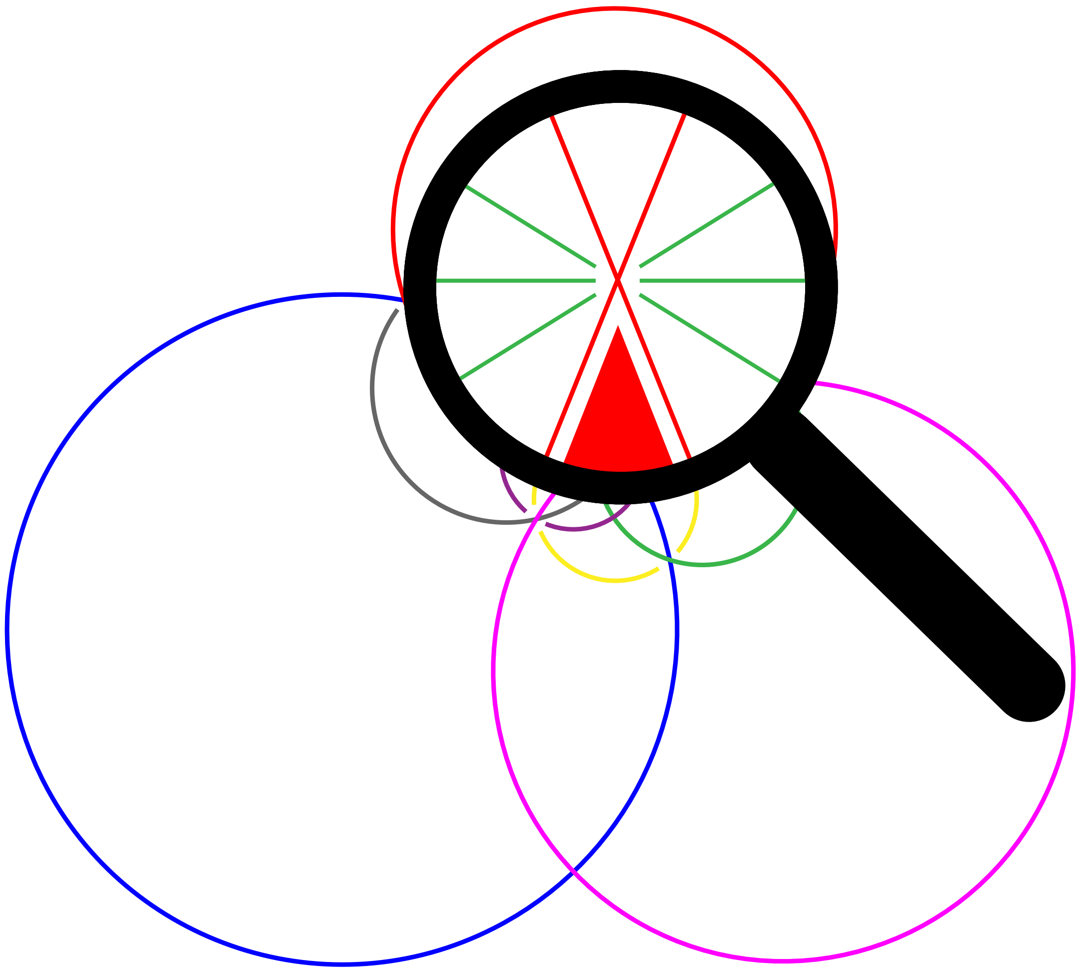

Reading ('03)

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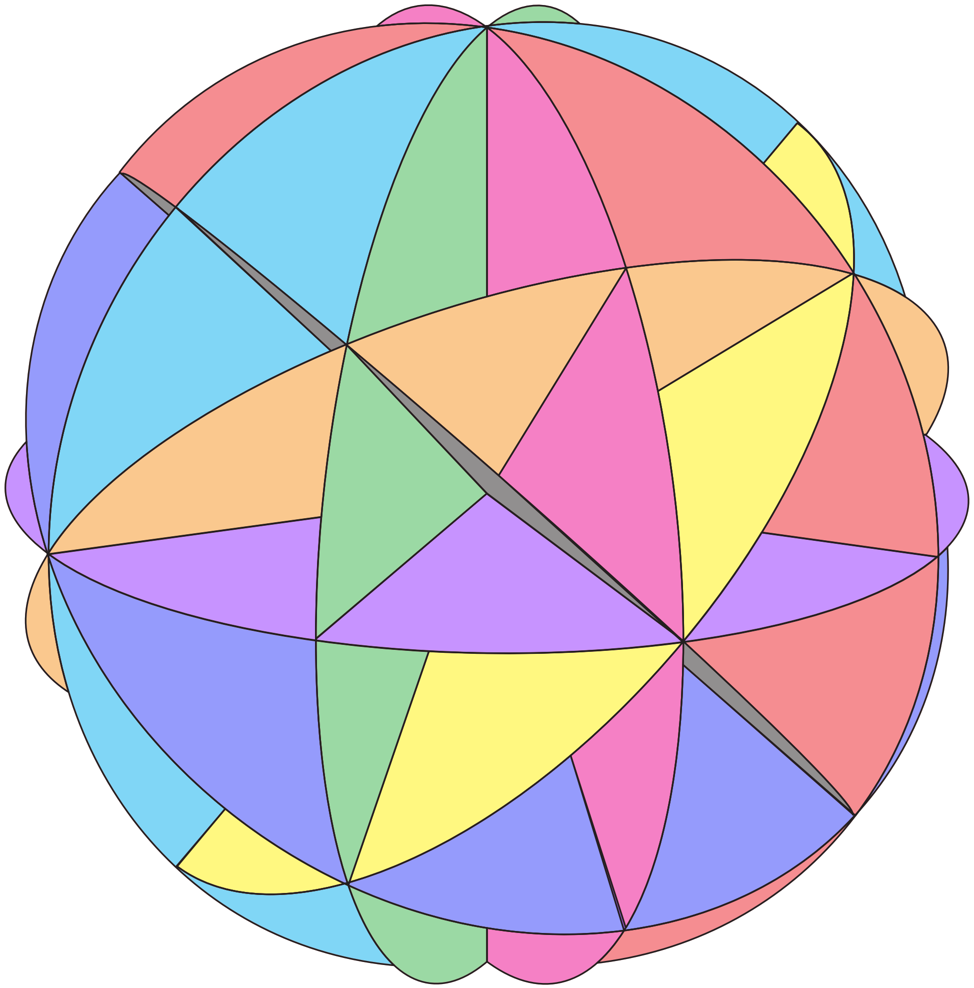

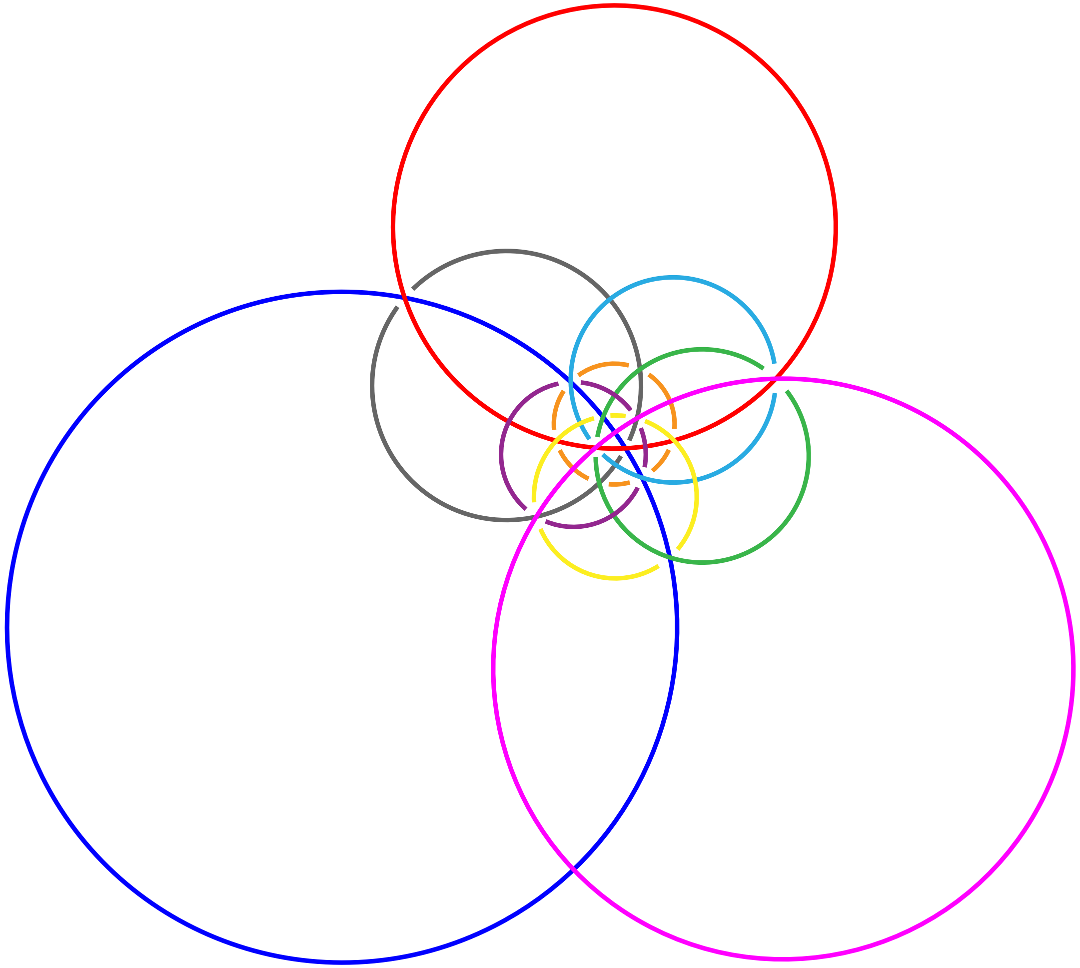

Reading ('03)

SHARD POLYTOPES FOR HYPERPLANE ARRANGEMENTS?

shard = piece of hyperplane obtained after cutting all rank 2 subgroups

shard poset = (pre)poset of forcing relations among shards

shard polytope for a shard Σ = polytope whose normal fan

- contains the shard Σ,
- is contained in the union of the shards forcing Σ

Find shard polytopes for arbitrary hyperplane arrangement with a lattice of regions

THANKS