Reconstructing Polytopes

Joseph Doolittle

Technische Universität Graz

nerdyjoe@gmail.com

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Outline

- Background
- Reconstruction
- Prior Results
- Non-simple polytopes
- Spheres

Background

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- The *dual* of a simple polytope is a simplicial polytope.

Background

- A *simplicial complex* Δ with ground set V is a subset of 2^V such that if $\sigma \in \Delta$ and $\tau \subset \sigma$, then $\tau \in \Delta$.

- The *faces* of a simplicial complex are its elements, and the facets are the maximal faces.

- The dimension of a face is its cardinality minus 1. A simplicial complex is *pure* if all its facets are the same dimension.

- A simplicial sphere is a simplicial complex whose geometric realization is homeomorphic to a sphere.

Reconstruction

What is reconstruction?
Given $f : D \to C$ where f is injective, we say that D is *reconstructable* from f and C.

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Example

Let D be the set of polytope face lattices, C be the set of height one posets. Then f takes the face lattice to the rank-restriction of the poset to facets and vertices. That this map is injective is equivalant to the standard result that writing facets of a polytope as a list of vertices fully determines the face lattice.

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We often omit f or C when they are clear from context.

Reconstruction of Simple Polytopes

Theorem (Blind, Mani 1987, "well known")

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Theorem (Blind, Mani 1987)

An isomorphism between the graphs of two simple polytope extends to an isomorphism of the 2-faces of those polytopes.

Together, this is the first proof that simple polytopes are reconstructable from their graph.

Reconstruction of Simple Polytopes

For an orientation O of a graph $G = (V, E)$, Kalai defines the parameter

$$F(O) = \sum_{v \in V} 2^{indeg_O(v)}$$

In any acyclic orientation of a polytope, for each face, there is at least one sink vertex, and this face is counted by that sink vertex and collection of incident edges.

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Lemma (Kalai 1988)

The minimum of $F(O)$ for an acyclic orientation of the graph of a simple polytope is the number of faces of that polytope.

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Theorem (D., Nevo, Pineda-Villavicencio, Ugon, Yost 2019)

For a polytope P with a single non-simple vertex v, every 2-face not containing v appears as an initial cycle of some acyclic orientation minimizing $F(O)$ among all orientations with v as a global sink. Therefore, polytopes with one nonsimple vertex are reconstructable from their graphs.

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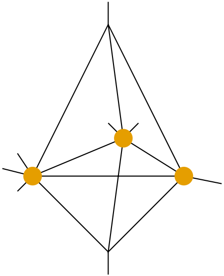
Theorem (D., Nevo, Pineda-Villavicencio, Ugon, Yost 2019)

Polytopes with two nonsimple vertices are reconstructable from their graphs.

More nonsimple vertices?

If a polytope has too many nonsimple vertices, the graph may no longer be unique.



Conjecture (D., Nevo, Pineda-Villavicencio, Ugon, Yost 2019)

If a d-polytope has at most $d - 2$ nonsimple vertices, it is reconstructable from its graph.

Perles' Conjecture

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The facets of a simple d-polytope are the $(d-1)$-connected, $(d-1)$-regular, induced, non-separating subgraphs of that polytope's graph.

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It is helpful to think in the dual setting. Counterexamples are then simplicial complexes satisfying several properties.

The first counterexample

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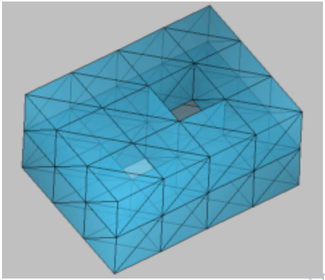

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(Dual) counterexamples must satisfy:

- Not a cone (counterexample)
- No top dimensional homology (non-separating)
- Every facet contains exactly one free ridge ($(d-1)$-regular)
- Strongly connected ($(d-1)$-connected)

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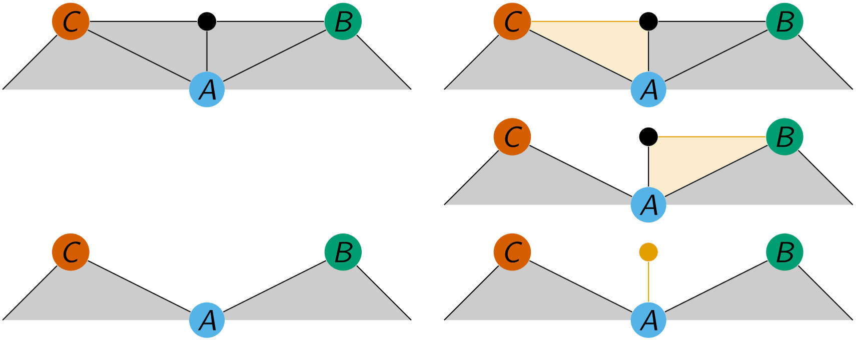

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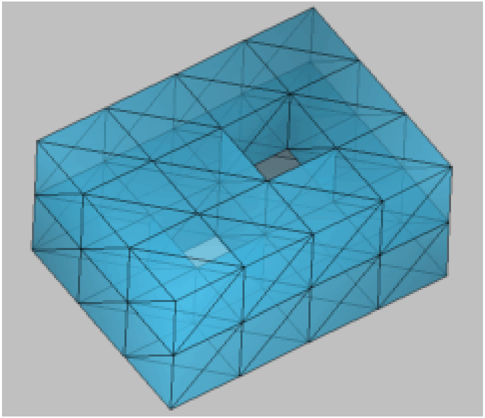
The core has no:

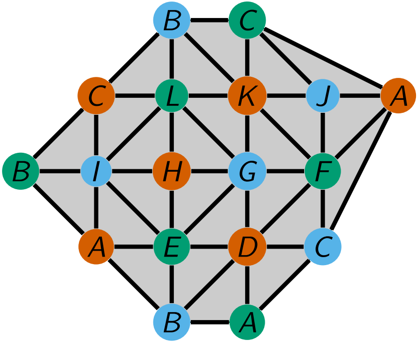
- top dimensional homology
- $(d-1)$-faces
- free faces

Bing's House with two rooms



A core

The following is an example of a core.

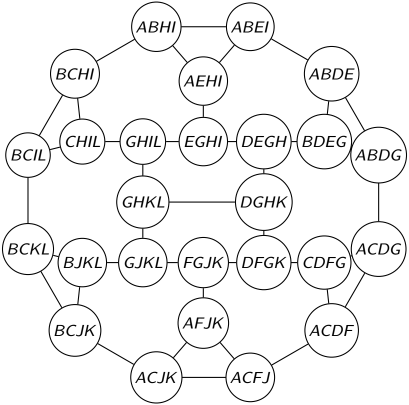


$f(\Delta) = (12, 36, 25)$
$\tilde{H}_1(\Delta) = 0$
$\tilde{H}_2(\Delta) = 0$
Related to the dunce hat

A core

This core gives a counterexample with f-vector $(12, 50, 76, 38)$.

A = (27, -95, 120, 0)
B = (-50, -45, 101, -94)
C = (-9, -67, 126, -35)
D = (195, -145, 11, 125)
E = (-40, -10, 8, -65)
F = (232, -102, -21, 198)
G = (-63, -25, 94, -139)
H = (-80, 45, -49, -65)
I = (-72, 4, 24, -90)
J = (-30, 167, -154, 92)
K = (-43, 190, -199, 100)
L = (-67, 80, -61, -26)



Theorem (D, 2018)

This 3-regular, 3-connected, induced, nonseparating planar subgraph of the dual graph is not the dual graph of the facets containing a vertex.

A conjecture of Kalai

Conjecture (Kalai 2009)

Simplicial spheres are reconstructable from their facet and ridge incidences.

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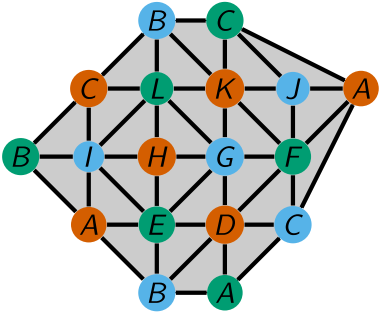
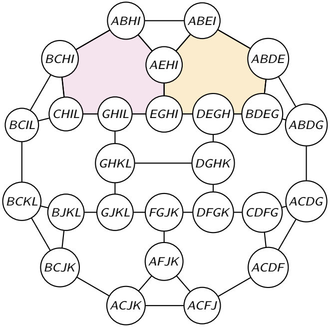
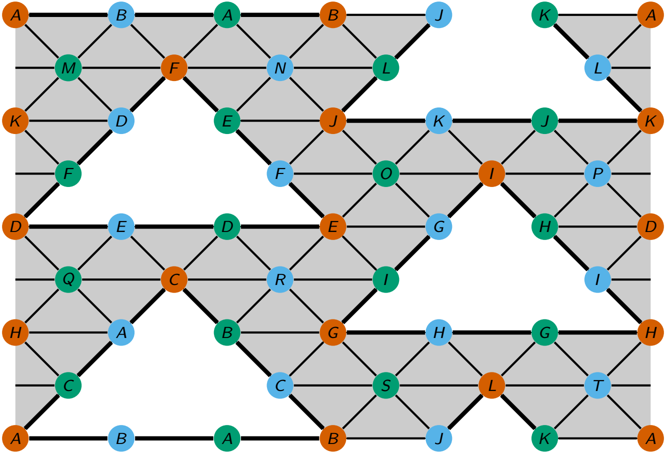
Counterexamples to Kalai's conjecture require many counterexamples to Perles' conjecture.

These many counterexamples need to be compatable with eachother and complete.

The pink shaded apparent 2-face is the dual of *HI*
The orange shaded apparent 2-face is not a 2-face at all!






A B A B J K A
M F N L L
K D E J K J K
F F O I P
D E D E G H D
Q C R I I
H A B G H G H
C C S L T
A B A B J K A

Back to the Beginning

Given two simplicial complexes Δ, Γ and an isomorphism f between their facets and ridges, define g a map from the faces of Δ to the faces of Γ.

$$g(\sigma) = \bigcap_{\sigma \subset F} f(F)$$

Theorem (Blind, Mani 1987)

If g takes peaks to peaks, then g is a simplicial isomorphism.

What could the image of a peak be under g?

Drawing!

Conclusion

Reconstruction of simplicial complexes from the top three layers is solved.

Reconstruction from the top two layers requires some additional structure.

Conjecture (Kalai 2009)

Simplicial spheres are reconstructable from their facet and ridge incidences.

Thank you, and stay healthy!
Merci, et restez en bonne santé!

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