Piecewise-Linear Rowmotion in Maximal Rectangles of Moon Polyominoes

Joe Johnson* and Ricky Liu

North Carolina State University

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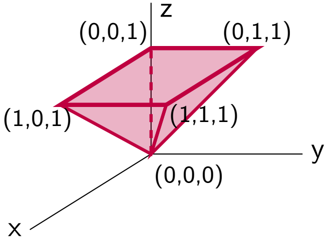
Words to Google

- Lattice Polytopes and Volumes
- Order Polytope
- Chain Polytope
- Moon Polyominoes
- Standard Young Tableaux
- $QSTAB(G)$
- Toggles in the Order Polytope
- Rowmotion
- RSK
- Promotion

Lattice Polytopes and Normalized Volumes

Definition

A *polytope* is the bounded intersection of finitely many half spaces. A polytope $Q \subset \mathbb{R}^n$ is a *lattice polytope* if each vertex of Q lies in $\mathbb{Z}^n$.



Theorem

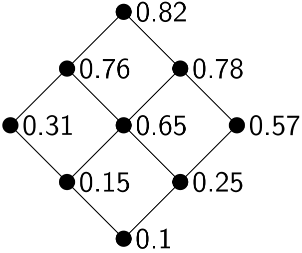
Let $Q \subset \mathbb{R}^n$ be a lattice polytope. Then the normalized volume $n!vol(Q)$ is an integer.

The Order Polytope

Definition

Let P be a finite poset. The *order polytope* of P, denoted $\mathcal{O}(P) \subseteq \mathbb{R}^{|P|}$, is the polytope satisfying the inequalities

- $0 \leq x_p \leq 1$ for all $p \in P$ and
- $x_p \leq x_q$ for all $p, q \in P$ with $p \preceq q$.

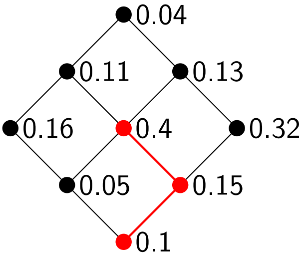


The Chain Polytope

Definition

Let P be a finite poset. A *chain* of P is a set of elements $\{x_1, \ldots, x_k\}$ satisfying $x_1 \prec x_2 \prec \cdots \prec x_k$. The *chain polytope* of P, denoted $\mathcal{C}(P) \subseteq \mathbb{R}^{|P|}$, is the polytope satisfying the inequalities

- $0 \leq x_p$ for all $p \in P$ and
- $\sum_{p \in C} x_p \leq 1$ for every (maximal) chain C in P.

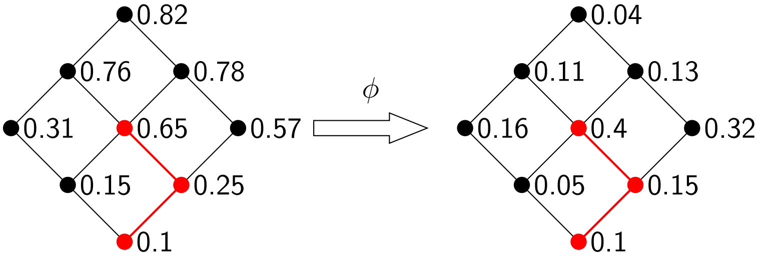


A Map Between The Two

Definition

The *transfer map* $\phi : k\,\mathcal{O}(P) \to k\,\mathcal{C}(P)$ is the map given by

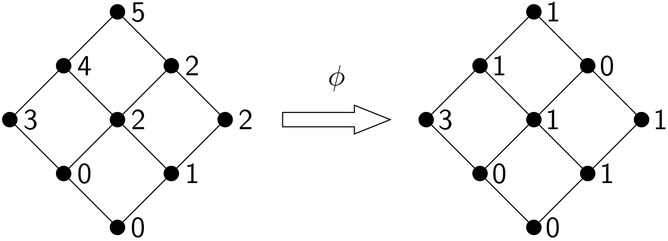
$\phi(x)_p = x_p - \max_{q \lessdot p} x_q$. Here we have $\phi^{-1}(x)_p = \max_{x_1 \lessdot x_2 \lessdot \cdots \lessdot x_j} \sum_{i=1}^{j} x_i$.



Theorem (Stanley)

The transfer map is piecewise-linear, volume-preserving, and continuous.

A Map Between The Two



Theorem (Stanley)

Let P be a finite poset. Then the transfer map restricts to a bijection between $k\mathcal{O}(P) \cap \mathbb{Z}^d$ and $k\mathcal{C}(P) \cap \mathbb{Z}^d$.

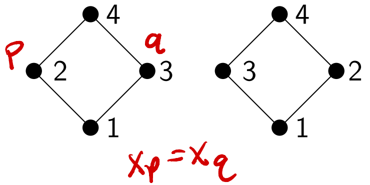
Corollary

Let $P = [0, r] \times [0, s]$. Then the transfer map restricts to a bijection between $r + 1 \times s + 1$ nonnegative integer matrices and $r + 1 \times s + 1$ nonnegative integer matrices with weakly increasing rows and columns.

Volume of $\mathcal{O}(P)$ and $\mathcal{C}(P)$

Definition

Let P be a finite poset with n elements. A *linear extension* of P is bijection $\phi : P \to [n]$ such that for all $p, q \in P$, if $p \prec q$, then $\phi(p) \prec \phi(q)$.



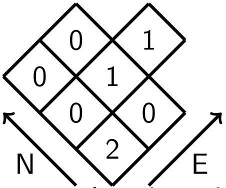
Theorem (Stanley)

Let P be a poset. The normalized volume of both $\mathcal{O}(P)$ and $\mathcal{C}(P)$ is the number of linear extensions of P.

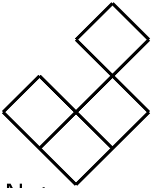
Moon Polyominoes

Definition

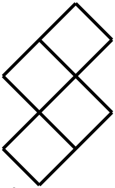
A *moon polyomino* is a convex diagram such that for any two rows R_1 and R_2, either $R_1 \subseteq R_2$ or $R_2 \subseteq R_1$.



A moon polyomino with a NE chain of weight 3



Not a convex diagram



Not a moon polyomino

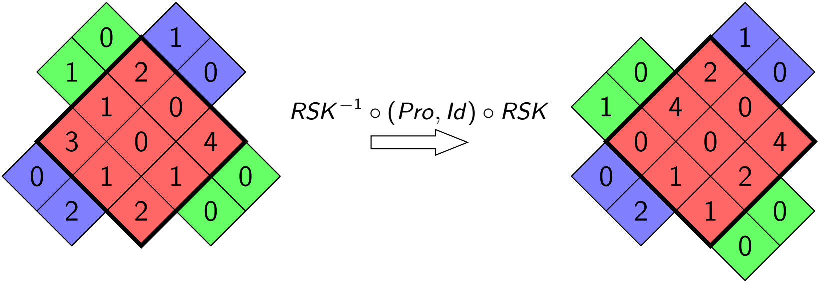
Definition

A (weak) k-NE chain (or a NE chain of weight k) of a filling of a moon polyomino is a NE path that lies in a rectangle and has weight sum k.

Rubey's Bijection

Theorem (Rubey)

Let $\mathcal{M}$ and $\mathcal{N}$ be moon polyominoes such that $\mathcal{N}$ can be obtained from $\mathcal{M}$ by permuting rows and columns. Then the number of nonnegative integer fillings of $\mathcal{M}$ with maximum weight among all NE chains at most k equals the number of nonnegative integer fillings of $\mathcal{N}$ with maximum weight among all NE chains at most k.



Main Questions

Question

Is there a piecewise-linear proof of Rubey's result?

Combinatorial Level	Polyhedral Level
Bijection	PL volume-preserving continuous map that preserves the lattice
fillings with maximum weight NE chains of weight $\leq k$	lattice points in the kth dilate of a polytope

Question

Are there additional statistics that are preserved by $RSK^{-1} \circ (Pro, Id) \circ RSK$*?*

Why do we care? Piecewise-linear proofs require different proof techniques and establish different relationships between objects than the combinatorial proofs.

Standard Young Tableaux

Definition

Let $\lambda \vdash n$. A *standard Young tableau* of shape λ is a filling of the boxes of λ with the numbers $1, .., n$ such that the columns and rows are both strictly increasing.

1	2	5
3	4	
6		

A SYT

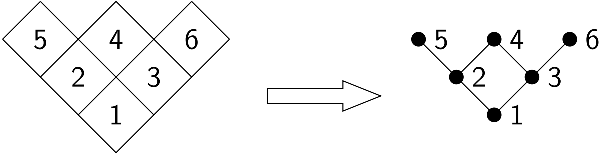
2	1	5
3	4	
6		

Not a SYT

An Associated Poset

Definition

Let P be a finite poset with n elements. A *linear extension* of P is bijection $\phi : P \to [n]$ such that for all $p, q \in P$, if $p \prec q$, then $\phi(p) \prec \phi(q)$.



Fact: The number of standard Young tableaux of shape λ is equal to the number of linear extensions of its corresponding poset.

Theorem (Stanley)

Let P be a poset. The normalized volume of both $\mathcal{O}(P)$ and $\mathcal{C}(P)$ is the number of linear extensions of P.

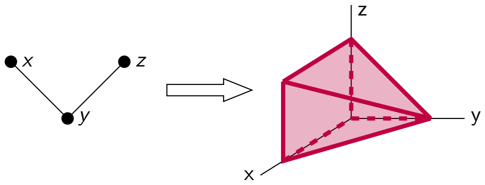
Clique-Constraint Polytope

Definition

Let G be a graph. A *clique* C of G is a set of vertices such that for any $v, w \in C$, $\{v, w\}$ is an edge of G.

Definition

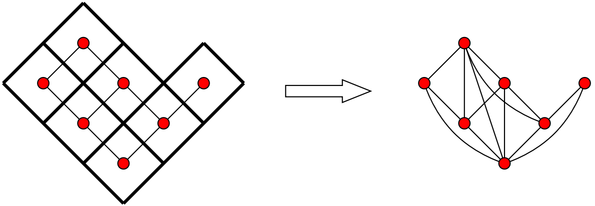
Let $G = (V, E)$ be a graph. The *clique-constraint stable set polytope* $QSTAB(G)$ is the polytope in $\mathbb{R}^{|V|}$ given by half spaces $x_v \geq 0$ for all $v \in V$ and $\sum_{v \in C} x_v \leq 1$ for all (maximal) cliques C of G.



Generalizing the Order and Chain Polytopes

Definition

Let G be a graph. We say G is a *comparability graph* if there exists a poset P on the vertices of G such that $\{v, w\}$ is an edge of G if and only if v and w are comparable in P.



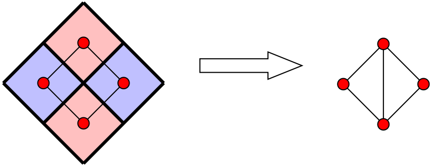
Let λ be a partition shape, let P_λ be the poset corresponding to λ, and let G_λ be the comparability graph of P_λ. Then

$$\mathcal{C}(P_\lambda) = QSTAB(G_\lambda)$$

Generalizing the Order and Chain Polytopes

We draw an edge between two nodes if

- they lie in the same row,
- they lie in the same column,
- or if they follow the following diagonal rule:

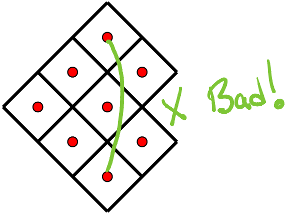


If the red boxes are comparable and the blue boxes exist (where the corresponding rows and columns intersect), then we draw an edge between the red boxes.

A Polytope for Moon Polyominoes

Definition

A (weak) k-NE chain (or a NE chain of weight k) of a filling of a moon polyomino is a NE path that lies in a rectangle and has weight sum k.



$$\text{NE Chain } C \longleftrightarrow \text{Clique } C \longleftrightarrow \text{Facet } \sum_{v \in C} x_v \leq 1$$

$$\text{Max weight sum} \leq k \longleftrightarrow \sum_{v \in C} x_v \leq k \longleftrightarrow \text{lattice point in our polytope!}$$

Our Main Theorem

Notation: For a moon polyomino $\mathcal{M}$, we let $QSTAB(\mathcal{M})$ denote the associated clique-constraint stable set polytope.

Theorem (J., Liu)

Let $\mathcal{M}$, $\mathcal{N}$ be moon polyominoes and suppose $\mathcal{N}$ can be obtained from $\mathcal{M}$ by permuting rows and columns. There exists a piecewise-linear, volume-preserving, continuous map $\psi : \mathbb{R}_{\geq 0}^{|\mathcal{M}|} \to \mathbb{R}_{\geq 0}^{|\mathcal{N}|}$ that restricts to a bijection between lattice points and for all $k \in \mathbb{Z}_{\geq 0}$ satisfies

$$\psi(kQSTAB(\mathcal{M})) = kQSTAB(\mathcal{N}).$$

Corollary

Consequently $kQSTAB(\mathcal{M})$ and $kQSTAB(\mathcal{N})$ have the same volume. The normalized volume of $QSTAB(\mathcal{M})$ is equal to the number of standard Young tableaux of the unique Ferrers shape obtained by permuting rows and columns.

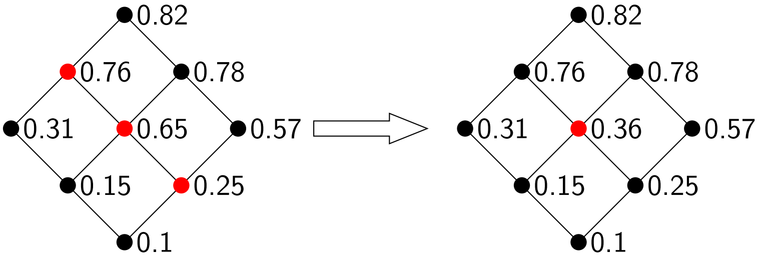
Toggles in the Order Polytope

Definition

Let P be a finite poset and let $a \in P$. The *toggle* $t_a : \mathcal{O}(P) \to \mathcal{O}(P)$ is

$$t_a(x)_b = \begin{cases} x_b & \text{if } a \neq b \\ \min\limits_{a \lessdot c} x_c + \max\limits_{c \lessdot a} x_c - x_a & \text{if } a = b \end{cases}$$

for all $x \in \mathcal{O}(P)$.



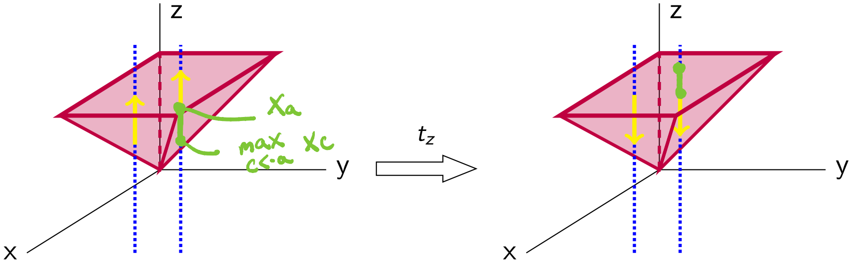
$$\min_{a \lessdot c} p_c + \max_{c \lessdot a} p_c - p_a = 0.76 + 0.25 - 0.65 = 0.36$$

Toggles are Fiber Reversing

$$t_a(x)_a = \min_{a \lessdot c} x_c + \max_{c \lessdot a} x_c - x_a$$

$$t_a(x)_a = \min_{a \lessdot c} x_c - \left(-\max_{c \lessdot a} x_c + x_a\right)$$

Let t_z be the toggle on the z coordinate.



In particular, toggles are involutions.

Piecewise-Linear Rowmotion

Lemma

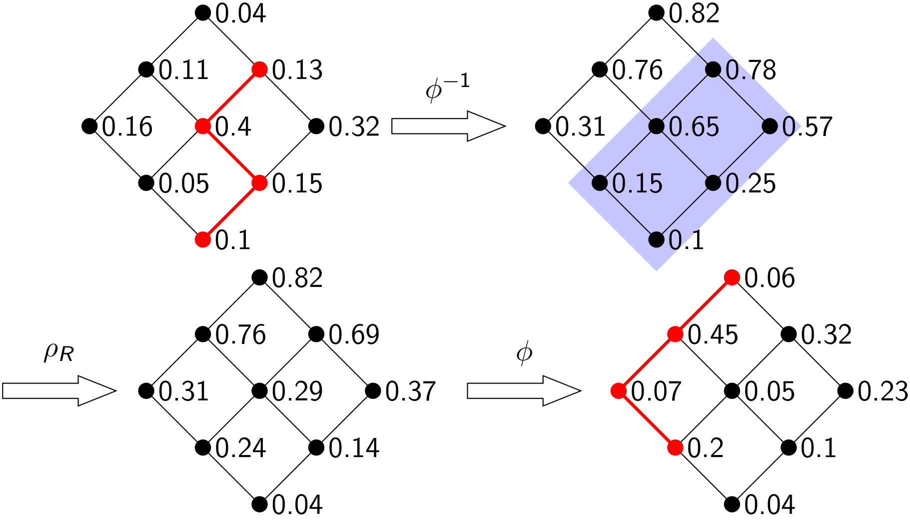
For any poset P and $a, b \in P$, the toggles t_a and t_b commute with each other if and only if neither $a \lessdot b$ nor $b \lessdot a$.

Definition

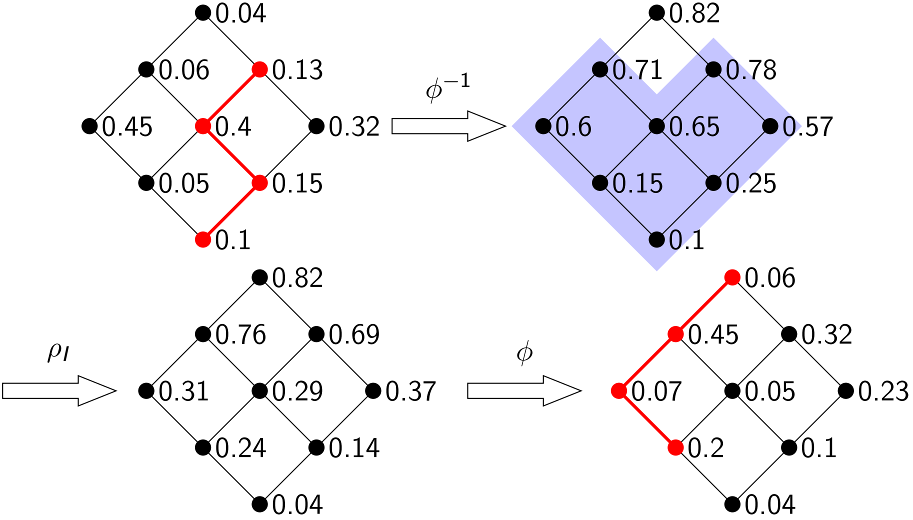
Let P be a finite poset. Then *rowmotion* is the map given by applying toggles along a linear extension from top to bottom.

For any order ideal I, we will let ρ_I be the composition of a sequence of toggles corresponding to elements in I applied in the order of a linear extension from top to bottom. We will also refer to ρ_I as a rowmotion.

A Chain Shifting Lemma for Piecewise-Linear Rowmotion

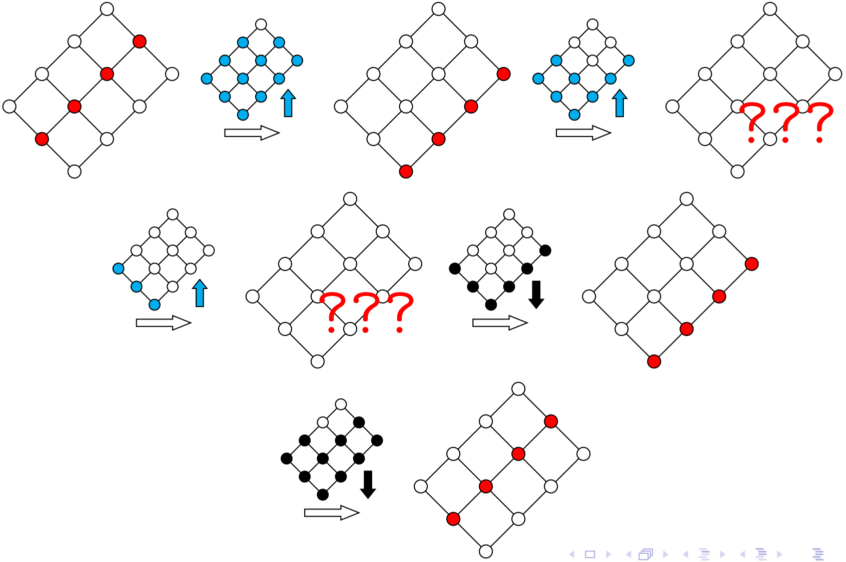


Other Order Ideals



A Composition of Rowmotions

Conjugate the given toggle sequence by ϕ:



A Composition of Rowmotions

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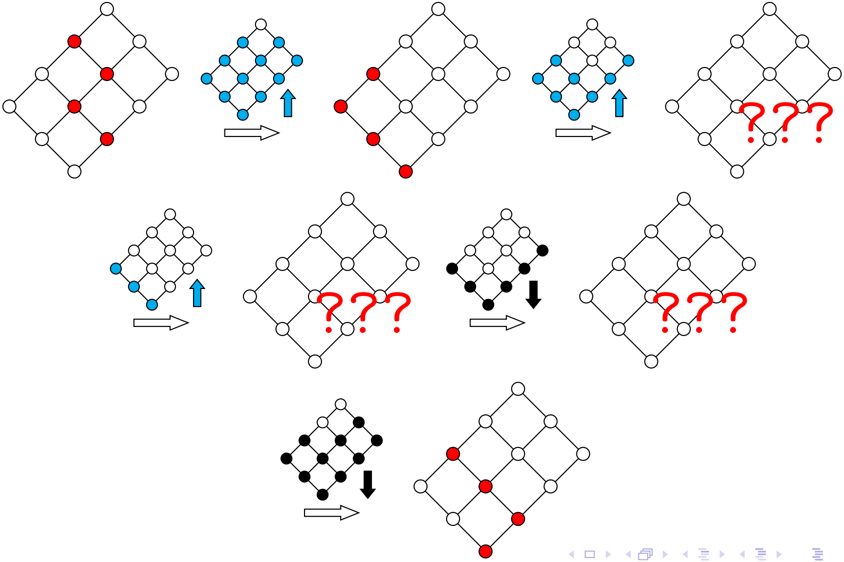
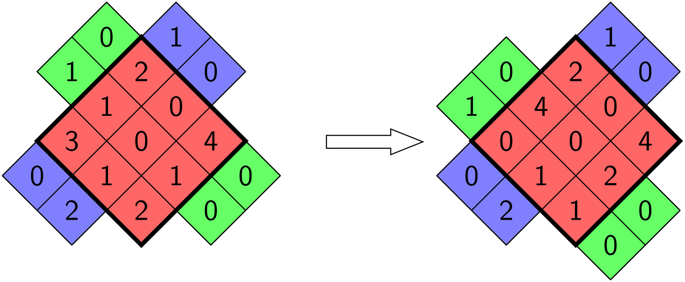


The Map Between Moon Polyominoes

It suffices to find a volume-preserving, piecewise linear, continuous map that is a bijection on lattice points for moon polyominoes obtained by a cyclic shifting of the columns.

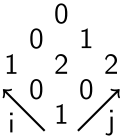


Blue Boxes - Stay the same
Green Boxes - Shift
Red Boxes - ???

RSK on Matrices

RSK is a bijection between generalized permutations and pairs of semistandard Young tableaux of the same shape.

$$M \in \mathbb{Z}_{\geq 0}^{r+1 \times s+1} \mapsto w \mapsto (P, Q)$$



$$\longmapsto \begin{pmatrix} 1 & 1 & 1 & 2 & 2 & 3 & 3 \\ 1 & 3 & 3 & 2 & 2 & 1 & 2 \end{pmatrix}$$

The column (i, j) appears a_{ij} times.

$$\longmapsto P = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 2 & 2 \\ \hline 2 & 3 \\ \cline{1-2} 3 \\ \cline{1-1} \end{array} \qquad Q = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 3 \\ \hline 2 & 2 \\ \cline{1-2} 3 \\ \cline{1-1} \end{array}$$

RSK on Matrices

RSK is a bijection between generalized permutations and pairs of semistandard Young tableaux of the **same shape.**

$$M \in \mathbb{Z}_{\geq 0}^{r+1 \times s+1} \mapsto w \mapsto (P, Q) \mapsto (GT(P), GT(Q)) \mapsto RSK(M)$$

$$P = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 2 & 2 \\ \hline 2 & 3 \\ \cline{1-2} 3 \\ \cline{1-1} \end{array} \qquad Q = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 3 \\ \hline 2 & 2 \\ \cline{1-2} 3 \\ \cline{1-1} \end{array}$$

$$\longmapsto \; GT(P) = \begin{array}{ccccc} 4 & & 2 & & 1 \\ & 4 & & 1 & \\ & & 2 & & \end{array} \qquad GT(Q) = \begin{array}{ccccc} 4 & & 2 & & 1 \\ & 3 & & 2 & \\ & & 3 & & \end{array}$$

$$\longmapsto \quad \begin{array}{ccccc} & & 4 & & \\ & 4 & & 3 & \\ 2 & & 2 & & 3 \\ & 1 & & 2 & \\ & & 1 & & \end{array}$$

RSK the Slow Way

Theorem (O'Connell, Seppäläinen, Zygouras)

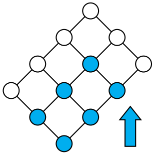
Let $\rho_{i,j}^{-1}$ denote the map given by applying toggles at elements in the order ideal of (i,j) applied in the order of a linear extension from bottom to top. Then RSK is given by

$$\rho^{-1}_{r-\min(r,s),s-\min(r,s)} \circ \cdots \rho^{-1}_{r-2,s-2} \circ \rho^{-1}_{r-1,s-1} \circ \phi^{-1}$$

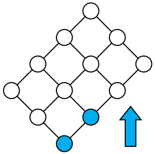
First

ϕ^{-1}

Second



Third

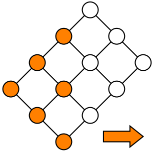


Promotion

Definition

Given $M \in \mathbb{Z}_{\geq 0}^{(r+1)\times(s+1)}$ with weakly increasing rows and columns, the *promotion-identity map*, denoted (Pro, Id), is given by toggling the files from left to right, starting with the leftmost and ending with the file just before the maximum element.

(Pro, Id)



Theorem (J., Liu)

The map $\psi : QSTAB(\mathcal{M}) \to QSTAB(\mathcal{N})$ restricts to $RSK^{-1} \circ (Pro, Id) \circ RSK$ on the appropriate rectangle.

Summary

1. We gave a rowmotion perspective to Rubey's map involving promotion. In dynamical algebraic combinatorics, objects with a notion of promotion frequently correspond to objects with a notion of rowmotion.

2. A collection of nonlattice polytopes have integer normalized volume.

3. RSK and Promotion have piecewise-linear analogues.

If G has no C_{2n+1} for $n \geq 2$ or $\overline{C_{2n+1}}$

$\Rightarrow$ vertices = independent set

Questions?

(1,0,0) (1,0,1)

(0,0,0)

(0,1,0)

(0,0,1)