Hypergraphs and hypergraphic polytopes

Sophie Rehberg

Doctorants en Géométrie Combinatoire (DGeCo)

June 24, 2021
Motivation

Stanley (1973): For a graph $g$, $m \in \mathbb{Z}_{>0}$

$$\chi_g(m) := \#\text{proper } m\text{-colorings of } g$$

is a polynomial in $m$ and

$$(-1)^d \chi_g(-m) = \#\text{pairs of compatible } m\text{-colorings}$$

and acyclic orientations.
Motivation

Stanley (1973): For a graph $g$, $m \in \mathbb{Z}_{>0}$

$$\chi_g(m) := \#\text{proper } m\text{-colorings of } g$$

is a polynomial in $m$ and

$$(-1)^d \chi_g(-m) = \#\text{pairs of compatible } m\text{-colorings and acyclic orientations}.$$ 

**Theorem (Aguiar, Ardila 2017; Billera, Jia, Reiner 2009)**

For generalized permutahedra $P \subset \mathbb{R}^d$, $m \in \mathbb{Z}_{>0}$

$$\chi_d(P)(m) := \# \left( P\text{-generic directions } y \in (\mathbb{R}^d)^* \text{ with } y \in [m]^d \right)$$

agrees with a polynomial in $m$ of degree $d$. Moreover,

$$(-1)^d \chi_d(P)(-m) = \sum_{y \in [m]^d} \# \text{(vertices of } P_y \text{)}.$$
Outline

1. Hypergraphs, colorings, orientations
2. Hypergraphic polytopes
3. Generalized permutahedra
4. Reciprocity
Hypergraphs and colorings

A hypergraph \( h = (I, E) \) is a pair of

- a finite set \( I = \{n_1, \ldots, n_d\} \) of nodes
- a finite multiset \( E \) of non-empty subsets \( e \subseteq I \) called hyperedges.
Hypergraphs and colorings

A **hypergraph** \( h = (I, E) \) is a pair of
- a finite set \( I = \{n_1, \ldots, n_d\} \) of **nodes**
- a finite multiset \( E \) of non-empty subsets \( e \subseteq I \) called **hyperedges**.

A **\( m \)-coloring** of \( h = (I, E) \) is a map \( c : I \rightarrow [m] \) assigning a **color** \( c(n_i) \in [m] \) to every node \( n_i \in I \).

A coloring \( c : I \rightarrow [m] \) is **proper** if every hyperedge \( e \in E \) contains a unique node \( n_i \in e \) with maximal color.
A hypergraph $h = (I, E)$ is a pair of
- a finite set $I = \{n_1, \ldots, n_d\}$ of nodes
- a finite multiset $E$ of non-empty subsets $e \subseteq I$ called hyperedges.

A $m$-coloring of $h = (I, E)$ is a map $c : I \to [m]$ assigning a color $c(n_i) \in [m]$ to every node $n_i \in I$.

A coloring $c : I \to [m]$ is proper if every hyperedge $e \in E$ contains a unique node $n_i \in e$ with maximal color.

Other notions of proper: not monochromatic, all nodes with different colors within a hyperedge [BDK12, EH66, BTV15, AH05]
A **heading** $\sigma$ of $h = (I, E)$ is a map $\sigma : E \rightarrow I$ such that for $e \in E$ we have $\sigma(e) \in e$.

An **oriented cycle** in a heading $\sigma$ of $h$ is a sequence $e_1, \ldots, e_l$ s.t.

\[
\begin{align*}
\sigma(e_1) &\in e_2 \setminus \sigma(e_2) \\
\sigma(e_2) &\in e_3 \setminus \sigma(e_3) \\
& \vdots \\
\sigma(e_l) &\in e_1 \setminus \sigma(e_1).
\end{align*}
\]

A heading $\sigma$ of a hypergraph $h$ is called **acyclic** if it does not contain an oriented cycle.
A **heading** $\sigma$ of $h = (I, E)$ is a map $\sigma : E \to I$ such that for $e \in E$ we have $\sigma(e) \in e$.

An **oriented cycle** in a heading $\sigma$ of $h$ is a sequence $e_1, \ldots, e_l$ s.t.

\[
\begin{align*}
\sigma(e_1) &\in e_2 \setminus \sigma(e_2) \\
\sigma(e_2) &\in e_3 \setminus \sigma(e_3) \\
&\quad \vdots \\
\sigma(e_l) &\in e_1 \setminus \sigma(e_1).
\end{align*}
\]

A heading $\sigma$ of a hypergraph $h$ is called **acyclic** if it does not contain an oriented cycle.

Different notions of orientations: [BBM19, Rus13]
Acyclic headings and compatible colorings

Example:
Acyclic headings and compatible colorings

Example:
A coloring $c$ and a heading $\sigma$ are **compatible** if

$$c(\sigma(e)) = \max_{j \in e} c(j).$$
Reciprocity for hypergraphs

Theorem (Aval, Karaboghossian, Tanasa 2020)
For a hypergraph $h = (I, E)$ with $I = \{n_1, \ldots, n_d\}$, $m \in \mathbb{Z}_{>0}$

$$
\chi_d(h)(m) := \#(\text{proper colorings of } h \text{ with } m \text{ colors})
$$
agrees with a polynomial in $m$ of degree $d$. Moreover,

$$
(-1)^d \chi_d(h)(-m) = \#(\text{compatible pairs of } m\text{-colorings and acyclic headings of } h).
$$

In particular, $(-1)^d \chi_d(h)(-1) = \#(\text{acyclic headings of } h)$. 
Outline

1. Hypergraphs, colorings, orientations
2. Hypergraphic polytopes
3. Generalized permutahedra
4. Reciprocity
Hypergraphic polytope

$|I| = d$, $\mathbb{R}^d \cong \mathbb{R}^I$, for every node $n_i$ pick a standard basis vector $b_i$.

For $h = (I, E)$ define the **hypergraphic polytope** $\mathcal{P}(h) \subset \mathbb{R}^d$ as:

$$\mathcal{P}(h) = \sum_{e \in E} \Delta_e \subset \mathbb{R}^d$$

where $\Delta_e = \text{conv}\{b_i : n_i \in e\}$ for hyperedges $e \subseteq I$. 

![Diagram of a hypergraphic polytope](image)
Vertex description of hypergraphic polytopes

Proposition (?)
Vertex description of hypergraphic polytopes

Proposition (?)
Vertex description of hypergraphic polytopes

Proof:

\[ \mathcal{P}(h) = \sum_{e \in E} \text{conv}\{b_i : n_i \in e\} = \text{conv}\left\{ \sum_{e \in E} \{b_i : n_i \in e\} \right\} \]

So, \( \mathcal{P}(h) \) is the convex hull of indegree vectors of headings. Left to show: indegree vector is a vertex iff heading is acyclic.
Vertex description of hypergraphic polytopes

Proof:

\[ \mathcal{P}(h) = \sum_{e \in E} \text{conv}\{b_i : n_i \in e\} = \text{conv}\left\{ \sum_{e \in E} \{b_i : n_i \in e\} \right\} \]

So, \( \mathcal{P}(h) \) is convex hull of indegree vectors of headings.
Vertex description of hypergraphic polytopes

**Proof:**

\[ \mathcal{P}(h) = \sum_{e \in E} \text{conv}\{b_i : n_i \in e\} = \text{conv} \left\{ \sum_{e \in E} \{b_i : n_i \in e\} \right\} \]

So, \( \mathcal{P}(h) \) is convex hull of indegree vectors of headings. Left to show: indegree vector is vertex iff heading is acyclic.
Proof:
Claim 1: indegree vectors of cyclic headings can be written as convex combination.
Vertex description of hypergraphic polytopes

**Proof:**
Claim 2: indegree vectors of acyclic headings cannot be written as convex combination.
Subclasses

[AA17, AKT20]

- simplicial complexes $\leftrightarrow$ simplicial complex polytopes
- graphs $\leftrightarrow$ graphical zonotopes
- building sets $\leftrightarrow$ nestohedra
- ...

...
Outline

1. Hypergraphs, colorings, orientations
2. Hypergraphic polytopes
3. Generalized permutahedra
4. Reciprocity
Normal fan

For **directions** $y \in (\mathbb{R}^d)^*$ and a polytope $\mathcal{P} \subset \mathbb{R}^d$ call

$$\mathcal{P}_y := \{x \in \mathcal{P} : y(x) \geq y(x') \text{ for all } x' \in \mathcal{P}\}$$

the $y$-**maximal face**. For a face $F$ of $\mathcal{P}$ define the **normal cone**

$$N_{\mathcal{P}}(F) := \{y \in (\mathbb{R}^d)^* : \mathcal{P}_y \supseteq F\}.$$

**Facts:**

- $\dim N_{\mathcal{P}}(F) = d - \dim F = \text{codim } F$
- $F$ a face of $G \iff N_{\mathcal{P}}(G)$ a face of $N_{\mathcal{P}}(F)$

Define the **normal fan**:

$$\mathcal{N}(\mathcal{P}) := \{N_{\mathcal{P}}(F) : F \text{ a face of } \mathcal{P}\}.$$ 

A direction $y \in (\mathbb{R}^d)^*$ is called $\mathcal{P}$-**generic** if $\mathcal{P}_y$ is a vertex.
The **standard permutahedron** $\pi_d$ is the convex hull of $d!$ vertices, namely, all the permutations of the point $(1, \ldots, d)$. 
Braid fan and standard permutahedron

The **braid arrangement** $\mathcal{B}_d$ is the set of hyperplanes $H_{i,j} := \{x \in (\mathbb{R}^d)^*: x_i = x_j\}$. The **braid fan** is the fan induced by $\mathcal{B}_d$. 
A polytope $\mathcal{P} \subset \mathbb{R}^d$ is a **generalized permutahedron** if its normal fan $\mathcal{N}(\mathcal{P})$ is a coarsening of the braid fan.

Graphically:
all deformations of standard permutahedron by translating facets.
Submodular functions

A function $z : 2^I \rightarrow \mathbb{R}$ is **submodular** if

$$z(S) + z(T) \geq z(s \cap T) + z(s \cup T) \quad \text{for all } S, T \subseteq I.$$

**Theorem**

A polytope $\mathcal{P}$ is a generalized permutahedron if and only if there exists a unique submodular function $z : 2^I \rightarrow \mathbb{R}$ with $z(\emptyset) = 0$ such that

$$\mathcal{P} = \left\{ x \in \mathbb{R}^I : \sum_{i \in I} x_i = z(I) \quad \text{and} \quad \sum_{i \in T} x_i \leq z(T) \quad \text{for } T \subseteq I \right\}.$$
Submodular functions

A function \( z : 2^I \to \mathbb{R} \) is **submodular** if

\[
z(S) + z(T) \geq z(S \cap T) + z(S \cup T) \quad \text{for all} \ S, T \subseteq I.
\]

**Theorem**

A polytope \( \mathcal{P} \) is a generalized permutahedron if and only if there exists a unique submodular function \( z : 2^I \to \mathbb{R} \) with \( z(\emptyset) = 0 \) such that

\[
\mathcal{P} = \left\{ x \in \mathbb{R}^I : \sum_{i \in I} x_i = z(I) \quad \text{and} \quad \sum_{i \in T} x_i \leq z(T) \quad \text{for} \ T \subseteq I \right\}.
\]

**Corollary**: Hypergraphic polytopes are generalized permutahedra.

**Proof:**

\[
z(A) := |\{ e \in E : e \cap A \neq \emptyset \}|.
\]
Relational submodular functions

A relation $R \subseteq I \times J$ defines a function $z_R : 2^I \to \mathbb{Z}_{\geq 0}$ by

$$z_R(A) = |\{ b \in J : (a, b) \in R \}| \quad \text{for } A \subseteq I$$

called relational submodular function.
Relational submodular functions

A relation $R \subseteq I \times J$ defines a function $z_R : 2^I \to \mathbb{Z}_{\geq 0}$ by

$$z_R(A) = |\{ b \in J : (a, b) \in R \}| \quad \text{for } A \subseteq I$$

called **relational submodular function**.

**Theorem ([AA17, Prop.19.4])**

A submodular function $z$ is relational
iff the associated generalized permutahedron is a hypergrphical polytope

iff $z(\emptyset) = 0$ and for all $A \subseteq I$ we have $z(A) \in \mathbb{Z}$ and

$$\sum_{K \supseteq A} (-1)^{|A| - |K|} z(K) \leq 0.$$
Outline

1 Hypergraphs, colorings, orientations
2 Hypergraphic polytopes
3 Generalized permutahedra
4 Reciprocity
Reciprocity for generalized permutahedra

**Theorem (Aguiar, Ardila 2017; Billera, Jia, Reiner 2009)**

For generalized permutahedra $\mathcal{P} \subset \mathbb{R}^d$, $m \in \mathbb{Z}_{>0}$

$$\chi_d(\mathcal{P})(m) := \# \left( \mathcal{P}\text{-generic directions } y \in (\mathbb{R}^d)^* \text{ with } y \in [m]^d \right)$$

agrees with a polynomial in $m$ of degree $d$. 

\[22\]
Reciprocity for generalized permutahedra

**Theorem (Aguiar, Ardila 2017; Billera, Jia, Reiner 2009)**
For generalized permutahedra $\mathcal{P} \subset \mathbb{R}^d$, $m \in \mathbb{Z}_{>0}$

$$\chi_d(\mathcal{P})(m) := \# \left( \mathcal{P} \text{-generic directions } y \in (\mathbb{R}^d)^* \text{ with } y \in [m]^d \right)$$

agrees with a polynomial in $m$ of degree $d$. Moreover,

$$(-1)^d \chi_d(\mathcal{P})(-m) = \sum_{y \in [m]^d} \# \text{(vertices of } \mathcal{P}_y) .$$
Theorem (Aval, Karaboghossian, Tanasa 2020)
For a hypergraph $h = (I, E)$ with $I = \{n_1, \ldots, n_d\}$, $m \in \mathbb{Z}_{>0}$

$$\chi_d(h)(m) := \#(\text{proper colorings of } h \text{ with } m \text{ colors})$$

agrees with a polynomial in $m$ of degree $d$. 
Reciprocity for hypergraphs I

**Theorem (Aval, Karaboghossian, Tanasa 2020)**
For a hypergraph $h = (I, E)$ with $I = \{n_1, \ldots, n_d\}$, $m \in \mathbb{Z}_{>0}$

$$\chi_d(h)(m) := \#(\text{proper colorings of } h \text{ with } m \text{ colors})$$

agrees with a polynomial in $m$ of degree $d$.

**Proof idea:**

$$\chi_d(h)(m) = \# \left( \mathcal{P}(h)\text{-generic directions } y \in (\mathbb{R}^d)^* \text{ with } y \in [m]^d \right)$$

$$= \chi_d(\mathcal{P}(h))(m)$$
Reciprocity for hypergraphs I

Claim:

\[ \# \text{(proper colorings of } h \text{ with } m \text{ colors)} = \# \left( \mathcal{P}(h) \text{-generic directions } y \in (\mathbb{R}^d)^* \text{ with } y \in [m]^d \right). \]
Reciprocity for hypergraphs II

**Theorem (Aval, Karaboghossian, Tanasa 2020)**
For a hypergraph $h = (I, E)$ with $I = \{n_1, \ldots, n_d\}$, $m \in \mathbb{Z}_{>0}$

$$(-1)^{d} \chi_d(h)(-m) = \#($$compatible pairs of $m$-colorings and acyclic headings of $h$).

In particular, $(-1)^{d} \chi_d(h)(-1) = \#($acyclic headings of $h$).
Theorem (Aval, Karaboghossian, Tanasa 2020)
For a hypergraph \( h = (I, E) \) with \( I = \{n_1, \ldots, n_d\} \), \( m \in \mathbb{Z}_{>0} \)

\[ (-1)^d \chi_d(h)(-m) = \#(\text{compatible pairs of } m\text{-colorings and acyclic headings of } h). \]

In particular, \( (-1)^d \chi_d(h)(-1) = \#(\text{acyclic headings of } h). \)

Recall:

\[ (-1)^d \chi_d(h)(-m) = (-1)^d \chi_d(\mathcal{P}(h))(-m) = \sum_{y \in [m]^d} \#(\text{vertices of } \mathcal{P}(h)_y). \]
Reciprocity for hypergraphs II

Claim:
\[ \sum_{y \in [m]^d} \# \left( \text{vertices of } \mathcal{P}(h)_y \right) = \sum_{m\text{-colorings}} \# \left( \text{compatible acyclic headings of } h \right) \]
Reciprocity for hypergraphs (summary)

Theorem (Aval, Karaboghossian, Tanasa 2020)
For a hypergraph $h = (I, E)$ with $I = \{n_1, \ldots, n_d\}$, $m \in \mathbb{Z}_{>0}$

$$\chi_d(h)(m) := \#(\text{proper colorings of } h \text{ with } m \text{ colors})$$

agrees with a polynomial in $m$ of degree $d$. Moreover,

$$(-1)^d \chi_d(h)(-m) = \#(\text{compatible pairs of } m\text{-colorings and acyclic headings of } h).$$
References I


References II


References III


Thank you!
Big Picture

Billera, Jia, Reiner (2009)
Hopf algebra on matroids and quasisymmetric functions

Aguiar, Ardila (2017)
Hopf monoid of generalized permutahedra

Aval, Karaboghossian, Tanasa (2020)
Hopf monoid of hypergraphs

reciprocity result for generalized permutahedra
Aguiar, Ardila (2017)
Billera, Jia, Reiner (2009)

reciprocity result for generalized permutahedra

new proof
(geometric)

pruned inside-out Ehrhart theory

inside-out polytopes

Ehrhart theory

Aval, Karaboghossian, Tanasa (2020)
reciprocity result for hypergraphs

Stanley (1970)
reciprocity result for graphs
Polyhedral fans

For a complete fan $\mathcal{N}$ in $\mathbb{R}^d$ define the \textbf{codimension-one fan} $\mathcal{N}^{\rm co\,1}$

$$\mathcal{N}^{\rm co\,1} := \{ N \in \mathcal{N} : \text{codim } N \geq 1 \} = \{ N \in \mathcal{N} : \text{dim } N \leq d - 1 \}.$$

For a normal fan $\mathcal{N}(\mathcal{P})$ we get

$$\mathcal{N}^{\rm co\,1}(\mathcal{P}) = \{ N_{\mathcal{P}}(F) : F \text{ a face of } \mathcal{P} \text{ with dim}(F) \geq 1 \}.$$
Pruned inside-out polytopes

For a polytope $Q \subset \mathbb{R}^d$ and a complete fan $\mathcal{N}$ in $\mathbb{R}^d$ we call

$$Q \setminus \left( \bigcup \mathcal{N}^{\text{co}1} \right) = \biguplus_{N \in \mathcal{N}, \ N \text{ full-dimensional}} (Q \cap N^\circ)$$

a pruned inside-out polytope and we call the connected components in $Q \setminus \left( \bigcup \mathcal{N}^{\text{co}1} \right)$ regions.
Pruned inside-out counting

For $t \in \mathbb{Z}_{>0}$ define the **inner pruned Ehrhart function** as

$$\text{in}_{Q \setminus \mathcal{N}^{\text{co1}}}(t) := \# \left( t \cdot (Q \setminus \bigcup \mathcal{N}^{\text{co1}}) \cap \mathbb{Z}^d \right).$$

**Note:**

$$\text{in}_{Q^{\circ} \setminus \mathcal{N}^{\text{co1}}}(t) = \sum_{i=1}^{k} ehr_{R_i^{\circ}}(t)$$

and it is a polynomial if regions $R_i$ are integral.
Define the **cumulative pruned Ehrhart function** for $t \in \mathbb{Z}_{>0}$ as

$$
c_{\mathcal{Q}, \mathcal{N}_{\text{co1}}}(t) := \sum_{y \in t \mathcal{Q} \cap \mathbb{Z}^d} \#(\mathcal{N} \in \mathcal{N}, \mathcal{N} \text{ full.dim.}, y \in \mathcal{N})
$$

**Note:**

$$
c_{\mathcal{Q}, \mathcal{N}_{\text{co1}}}(t) = \sum_{i=1}^{k} e_{\mathcal{Q}, \mathcal{N}_{\text{co1}}} R_i(t)
$$

and it is a polynomial if regions $R_i$ are integral.
Theorem (R.)
For a polytope $Q \subset \mathbb{R}^d$ and a complete fan $\mathcal{N}$ in $\mathbb{R}^d$ we have

$(-1)^{\dim Q} \text{in}_{Q^\circ,\mathcal{N}_{\text{co}1}}(-t) = \text{cu}_{Q,\mathcal{N}_{\text{co}1}}(t)$.

Proof.
Reciprocity for generalized permutahedra

**Theorem (Aguiar, Ardila 2017; Billera, Jia, Reiner 2009)**
For generalized permutahedra $\mathcal{P} \subset \mathbb{R}^d$, $m \in \mathbb{Z}_{>0}$

$$\chi_d(\mathcal{P})(m) := \# \left( \mathcal{P} \text{-generic directions } y \in (\mathbb{R}^d)^* \text{ with } y \in [m]^d \right)$$

agrees with a polynomial in $m$ of degree $d$. 
Reciprocity for generalized permutahedra

**Theorem (Aguiar, Ardila 2017; Billera, Jia, Reiner 2009)**

For generalized permutahedra $\mathcal{P} \subset \mathbb{R}^d$, $m \in \mathbb{Z}_{>0}$

$$\chi_d(\mathcal{P})(m) \:= \# \left( \mathcal{P}\text{-\text{generic directions}} \ y \in (\mathbb{R}^d)^* \text{ with } y \in [m]^d \right)$$

agrees with a polynomial in $m$ of degree $d$. Moreover,

$$(-1)^d \chi_d(\mathcal{P})(-m) = \sum_{y \in [m]^d} \# (\text{vertices of } \mathcal{P}_y).$$
**Thm:** polynomiality of

\[ \chi_d(\mathcal{P})(m) := \# (\mathcal{P}-\text{generic directions } y \in (\mathbb{R}^d)^* \text{ with } y \in [m]^d). \]
Reciprocity for generalized permutahedra

Thm: \((-1)^d \chi_d(\mathcal{P})(-m) = \sum_{y \in [m]^d} \# \text{(vertices of } \mathcal{P}_y \text{)} \).