Hypergraphs and hypergraphic polytopes

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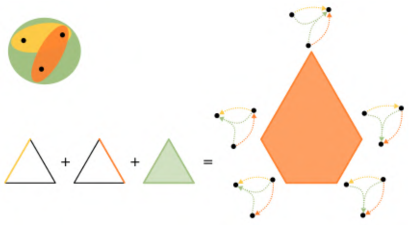

Motivation

Stanley (1973): For a graph g, $m \in \mathbb{Z}_{>0}$

$$\chi_g(m) := \#\text{proper } m\text{-colorings of } g$$

is a polynomial in m and

$$(-1)^d \chi_g(-m) = \#\text{pairs of compatible } m\text{-colorings and acyclic orientations.}$$

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Theorem (Aguiar, Ardila 2017; Billera, Jia, Reiner 2009)
For generalized permutahedra $\mathcal{P} \subset \mathbb{R}^d$, $m \in \mathbb{Z}_{>0}$

$$\chi_d(\mathcal{P})(m) := \# \left(\mathcal{P}\text{-generic directions } y \in (\mathbb{R}^d)^* \text{ with } y \in [m]^d\right)$$

agrees with a polynomial in m of degree d. Moreover,

$$(-1)^d \chi_d(\mathcal{P})(-m) = \sum_{y \in [m]^d} \# \left(\text{vertices of } \mathcal{P}_y\right) .$$

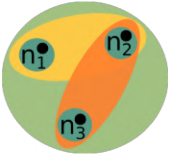
Outline

1. Hypergraphs, colorings, orientations
2. Hypergraphic polytopes
3. Generalized permutahedra
4. Reciprocity

Hypergraphs and colorings

A **hypergraph** $h = (I, E)$ is a pair of

- a finite set $I = \{n_1, \ldots, n_d\}$ of **nodes**
- a finite multiset E of non-empty subsets $e \subseteq I$ called **hyperedges**.



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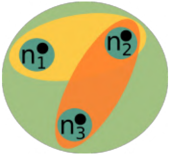
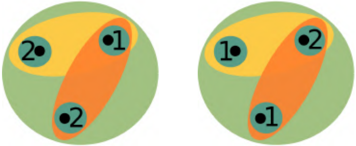


A m-**coloring** of $h = (I, E)$ is a map $c\colon I \to [m]$ assigning a **color** $c(n_i) \in [m]$ to every node $n_i \in I$.

A coloring $c : I \to [m]$ is **proper** if every hyperedge $e \in E$ contains a unique node $n_i \in e$ with maximal color.



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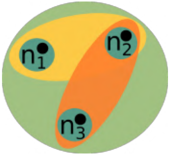

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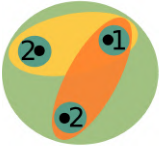
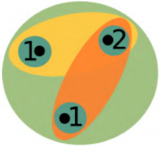


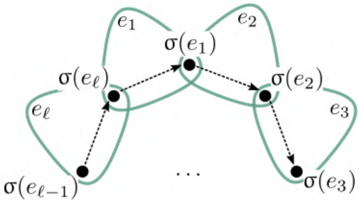


other notions of proper: not monochromatic, all nodes with different colors within a hyperedge [BDK12, EH66, BTV15, AH05]

Headings

A **heading** σ of $h = (I, E)$ is a map $\sigma\colon E \to I$ such that for $e \in E$ we have $\sigma(e) \in e$.

An **oriented cycle** in a heading σ of h is a sequence $e_1, \ldots, e_l$ s.t.



$$\sigma(e_1) \in e_2 \setminus \sigma(e_2)$$
$$\sigma(e_2) \in e_3 \setminus \sigma(e_3)$$
$$\vdots$$
$$\sigma(e_l) \in e_1 \setminus \sigma(e_1).$$

A heading σ of a hypergraph h is called **acyclic** if it does not contain an oriented cycle.

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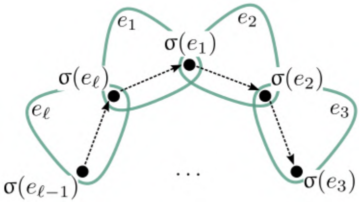

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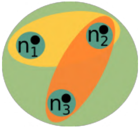
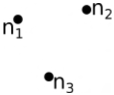
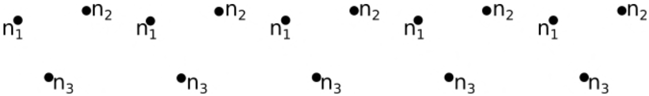
Different notions of orientations: [BBM19, Rus13]

Acyclic headings and compatible colorings

Example:







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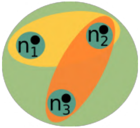

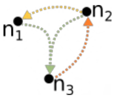

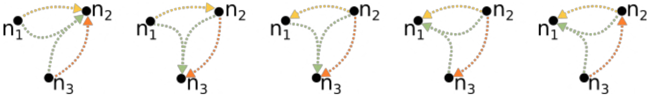

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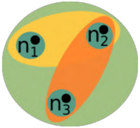
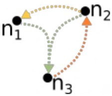
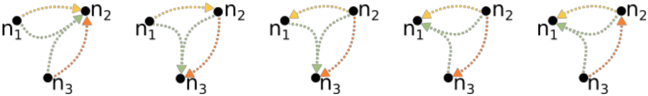
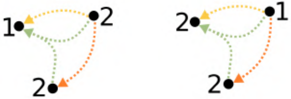






A coloring c and a heading σ are **compatible** if

$$c(\sigma(e)) = \max_{j \in e} c(j)\,.$$



Reciprocity for hypergraphs

Theorem (Aval, Karaboghossian, Tanasa 2020)
For a hypergraph $h = (I, E)$ with $I = \{n_1, \dots, n_d\}$, $m \in \mathbb{Z}_{>0}$

$$\chi_d(h)(m) := \#(\text{proper colorings of } h \text{ with m colors})$$

agrees with a polynomial in m of degree d. Moreover,

$$(-1)^d \chi_d(h)(-m) = \#(\text{compatible pairs of } m\text{-colorings and acyclic headings of } h).$$

In particular, $(-1)^d \chi_d(h)(-1) = \#(\text{acyclic headings of } h)$.

Outline

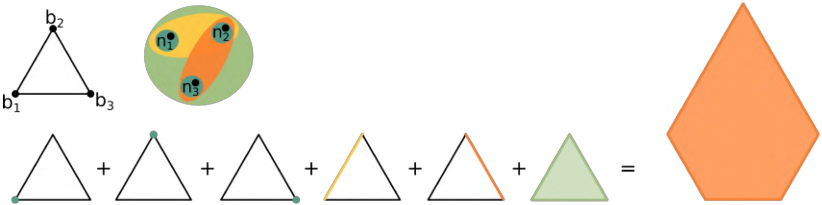
Hypergraphic polytope

$|I| = d$, $\mathbb{R}^d \cong \mathbb{R}^I$,
for every node n_i pick a standard basis vector b_i.

For $h = (I, E)$ define the **hypergraphic polytope** $\mathcal{P}(h) \subset \mathbb{R}^d$ as:

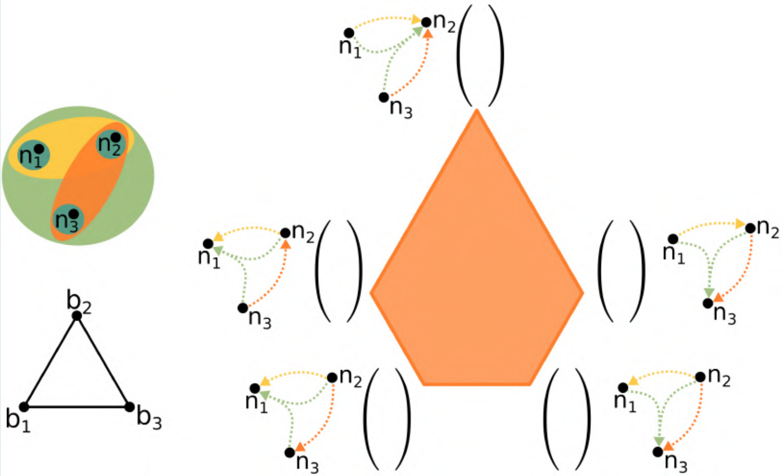
$$\mathcal{P}(h) = \sum_{e \in E} \Delta_e \subset \mathbb{R}^d$$

where $\Delta_e = \text{conv}\{b_i \ : \ n_i \in e\}$ for hyperedges $e \subseteq I$.



Vertex description of hypergraphic polytopes

Proposition (?)



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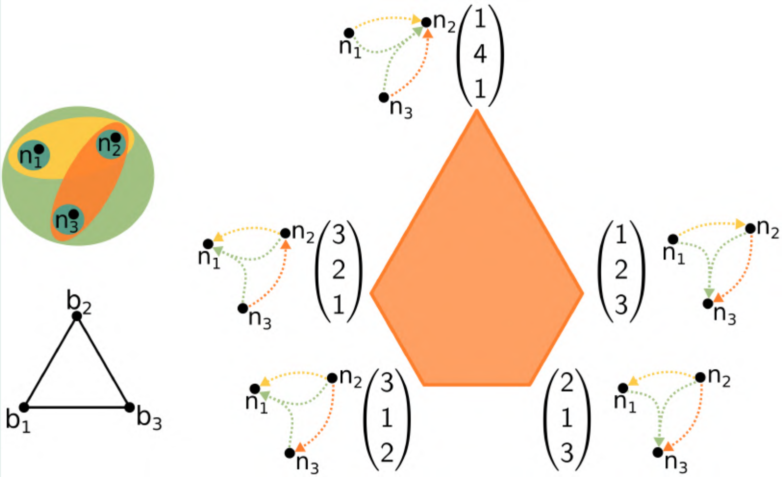
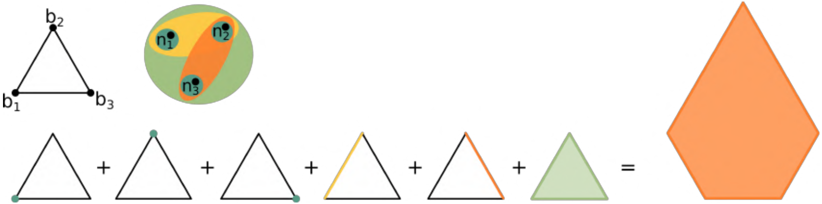


Vertex description of hypergraphic polytopes

Proof:

$$\mathcal{P}(h) = \sum_{e \in E} \operatorname{conv}\{b_i : n_i \in e\} = \operatorname{conv}\left\{\sum_{e \in E}\{b_i : n_i \in e\}\right\}$$



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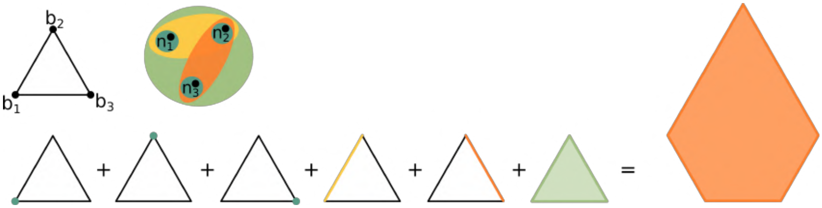

So, $\mathcal{P}(h)$ is convex hull of indegree vectors of headings.

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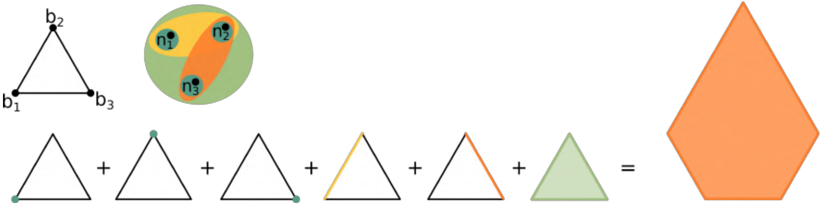
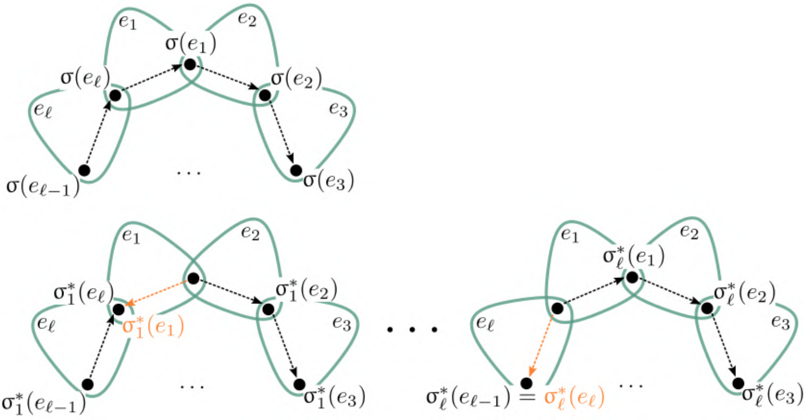


So, $\mathcal{P}(h)$ is convex hull of indegree vectors of headings.
Left to show: indegree vector is vertex iff heading is acyclic.

Vertex description of hypergraphic polytopes

Proof:

Claim 1: indegree vectors of cyclic headings can be written as convex combination.



Vertex description of hypergraphic polytopes

Proof:

Claim 2: indegree vectors of acyclic headings cannot be written as convex combination.

Subclasses

[AA17, AKT20]

- simplicial complexes $\leftrightarrow$ simplicial complex polytopes
- graphs $\leftrightarrow$ graphical zonotopes
- building sets $\leftrightarrow$ nestohedra
- ...

Outline

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3. Generalized permutahedra
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Normal fan

For **directions** $y \in (\mathbb{R}^d)^*$ and a polytope $\mathcal{P} \subset \mathbb{R}^d$ call

$$\mathcal{P}_y \coloneqq \{x \in \mathcal{P} \;:\; y(x) \geq y(x') \quad \text{for all } x' \in \mathcal{P}\}$$

the y-**maximal face**. For a face F of $\mathcal{P}$ define the **normal cone**

$$N_{\mathcal{P}}(F) \coloneqq \{y \in (\mathbb{R}^d)^* \;:\; \mathcal{P}_y \supseteq F\}.$$

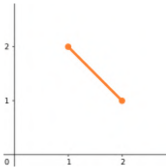
Facts:
- $\dim N_{\mathcal{P}}(F) = d - \dim F = \operatorname{codim} F$
- F a face of G $\Leftrightarrow$ $N_{\mathcal{P}}(G)$ a face of $N_{\mathcal{P}}(F)$

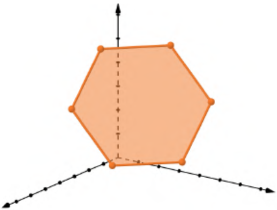
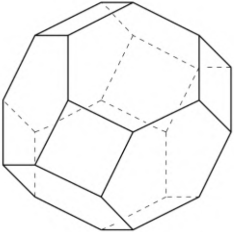
Define the **normal fan**:

$$\mathcal{N}(\mathcal{P}) \coloneqq \{N_{\mathcal{P}}(F) \;:\; F \text{ a face of } \mathcal{P}\}.$$

A direction $y \in (\mathbb{R}^d)^*$ is called $\mathcal{P}$-**generic** if $\mathcal{P}_y$ is a vertex.

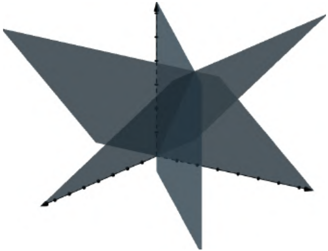
Standard permutahedra



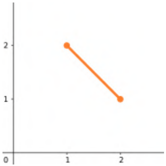



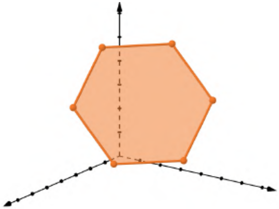
The **standard permutahedron** π_d is the convex hull of $d!$ vertices, namely, all the permutations of the point $(1, \ldots, d)$.

Braid fan and standard permutahedron



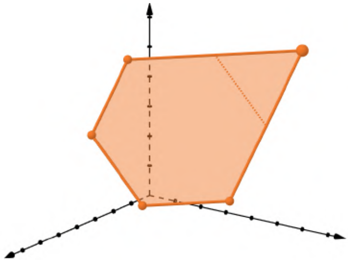
The **braid arrangement** $\mathcal{B}_d$ is the set of hyperplanes $H_{i,j} := \{x \in (\mathbb{R}^d)^* \colon x_i = x_j\}$. The **braid fan** is the fan induced by $\mathcal{B}_d$.

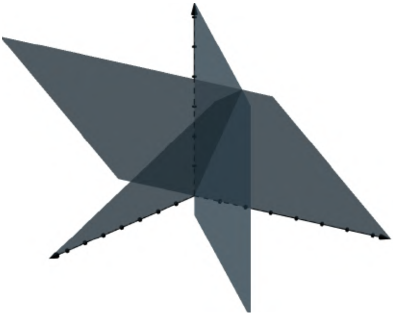




Generalized Permutahedra

A polytope $\mathcal{P} \subset \mathbb{R}^d$ is a **generalized permutahedron** if its normal fan $\mathcal{N}(\mathcal{P})$ is a coarsening of the braid fan.





Graphically:
all deformations of standard permutahedron by translating facets.

Submodular functions

A function $z\colon 2^I \to \mathbb{R}$ is **submodular** if

$$z(S) + z(T) \geq z(s \cap T) + z(s \cup T) \quad \text{for all} S, T \subseteq I\,.$$

Theorem
A polytope $\mathcal{P}$ is a generalized permutahedron if and only if there exists a unique submodular function $z\colon 2^I \to \mathbb{R}$ with $z(\emptyset) = 0$ such that

$$\mathcal{P} = \left\{ x \in \mathbb{R}I : \sum_{i \in I} x_i = z(I) \quad \text{and} \quad \sum_{i \in T} x_i \leq z(T) \quad \text{for } T \subseteq I \right\}.$$

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Corollary: Hypergraphic polytopes are generalized permutahedra.

Proof:

$$z(A) := |\{e \in E\colon e \cap A \neq \emptyset\}|\,.$$

Relational submodular functions

A relation $R \subseteq I \times J$ defines a funtion $z_R \colon 2^I \to \mathbb{Z}_{\geq 0}$ by

$$z_R(A) = |\{b \in J \colon (a, b) \in R\}| \quad \text{for } A \subset I$$

called **relational submodular function**.

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Theorem ([AA17, Prop.19.4])
A submodular function z is relational
iff the associated generalized permutahedron is a hypergrphical polytope
iff $z(\emptyset) = 0$ and for all $A \subseteq I$ we have $z(A) \in \mathbb{Z}$ and $\sum_{K \supseteq A} (-1)^{|A-K|} z(K) \leq 0$.

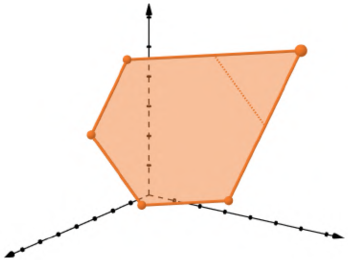
Outline

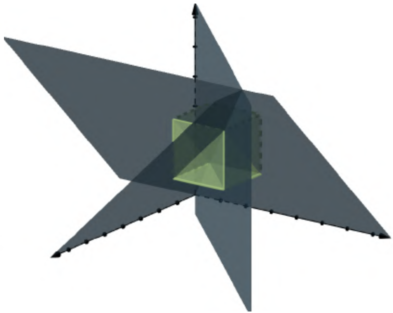
Reciprocity for generalized permutahedra

Theorem (Aguiar, Ardila 2017; Billera, Jia, Reiner 2009)
For generalized permutahedra $\mathcal{P} \subset \mathbb{R}^d$, $m \in \mathbb{Z}_{>0}$

$$\chi_d(\mathcal{P})(m) := \# \left(\mathcal{P}\text{-generic directions } y \in (\mathbb{R}^d)^* \text{ with } y \in [m]^d\right)$$

agrees with a polynomial in m of degree d.





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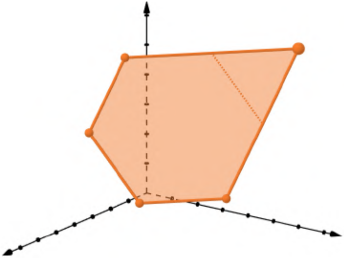
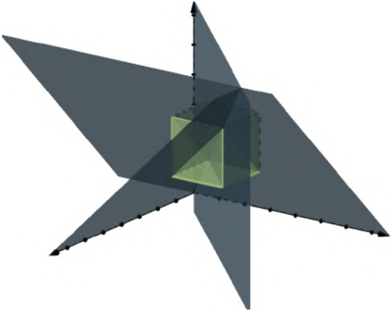
$$(-1)^d \chi_d(\mathcal{P})(-m) = \sum_{y \in [m]^d} \#\,(\text{vertices of } \mathcal{P}_y)\ .$$





Reciprocity for hypergraphs I

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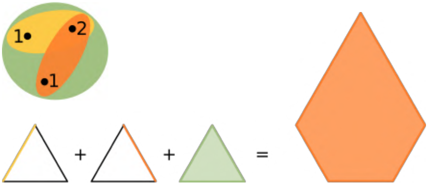

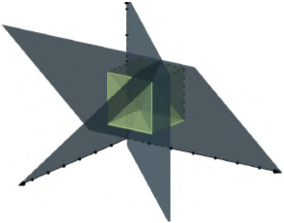

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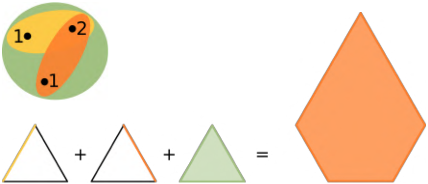
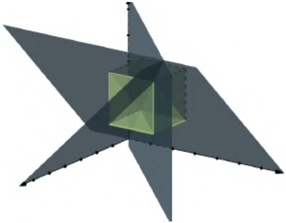
$$\chi_d(h)(m) := \#(\text{proper colorings of } h \text{ with m colors})$$

agrees with a polynomial in m of degree d.

Proof idea:

$$\chi_d(h)(m) = \# \left(\mathcal{P}(h)\text{-generic directions } y \in (\mathbb{R}^d)^* \text{ with } y \in [m]^d\right)$$
$$= \chi_d\left(\mathcal{P}(h)\right)(m)$$





Reciprocity for hypergraphs I

Claim:

#(proper colorings of h with m colors)

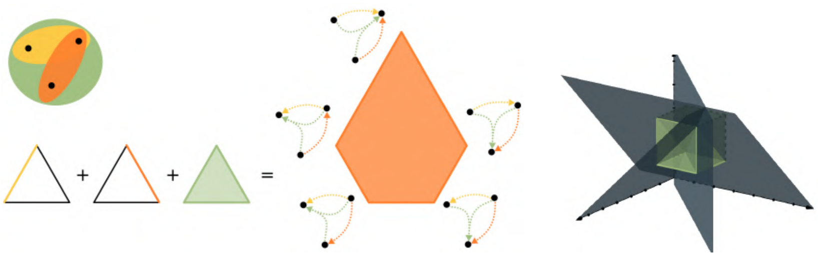
$$= \# \left(\mathcal{P}(h)\text{-generic directions } y \in (\mathbb{R}^d)^* \text{ with } y \in [m]^d\right).$$

Reciprocity for hypergraphs II

Theorem (Aval, Karaboghossian, Tanasa 2020)
For a hypergraph $h = (I, E)$ with $I = \{n_1, \ldots, n_d\}$, $m \in \mathbb{Z}_{>0}$

$$(-1)^d \chi_d(h)(-m) = \#(\text{compatible pairs of } m\text{-colorings and acyclic headings of } h).$$

In particular, $(-1)^d \chi_d(h)(-1) = \#(\text{acyclic headings of } h)$.



Reciprocity for hypergraphs II

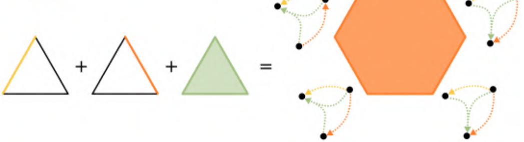
Theorem (Aval, Karaboghossian, Tanasa 2020)
For a hypergraph $h = (I, E)$ with $I = \{n_1, \dots, n_d\}$, $m \in \mathbb{Z}_{>0}$

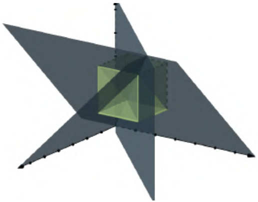
$$(-1)^d \chi_d(h)(-m) = \#(\text{compatible pairs of } m\text{-colorings and acyclic headings of } h).$$

In particular, $(-1)^d \chi_d(h)(-1) = \#(\text{acyclic headings of } h)$.

Recall:

$$(-1)^d \chi_d(h)(-m) = (-1)^d \chi_d(\mathcal{P}(h))(-m) = \sum_{y \in [m]^d} \#\left(\text{vertices of } \mathcal{P}(h)_y\right).$$

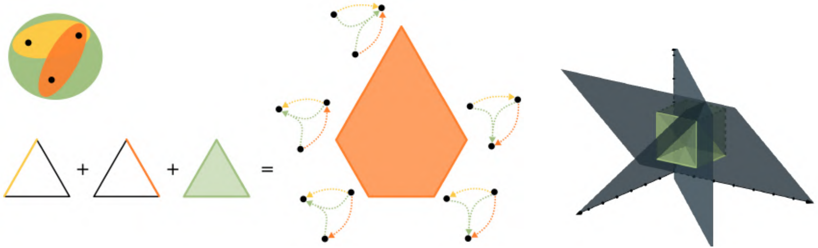




Reciprocity for hypergraphs II

Claim:

$$\sum_{y \in [m]^d} \#\,(\text{vertices of } \mathcal{P}(h)_y) = \sum_{m\text{-colorings}} \#\,(\text{compatible acyclic headings of } h)$$



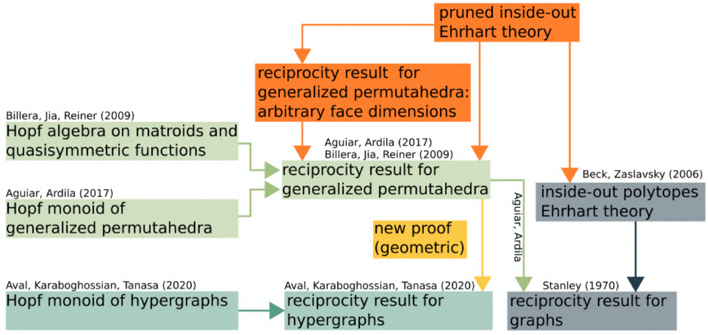
Reciprocity for hypergraphs (summary)

Theorem (Aval, Karaboghossian, Tanasa 2020)
For a hypergraph $h = (I, E)$ with $I = \{n_1, \ldots, n_d\}$, $m \in \mathbb{Z}_{>0}$

$$\chi_d(h)(m) := \#(\text{proper colorings of } h \text{ with m colors})$$

agrees with a polynomial in m of degree d. Moreover,

$$(-1)^d \chi_d(h)(-m) = \#(\text{compatible pairs of } m\text{-colorings and acyclic headings of } h).$$



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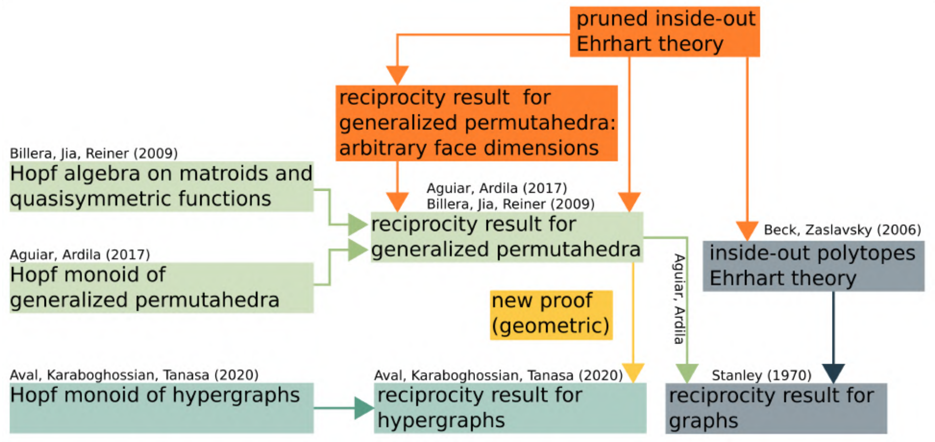
Thank you!

Big Picture



pruned inside-out Ehrhart theory
reciprocity result for generalized permutahedra: arbitrary face dimensions
Billera, Jia, Reiner (2009)
Hopf algebra on matroids and quasisymmetric functions
Aguiar, Ardila (2017)
Billera, Jia, Reiner (2009)
reciprocity result for generalized permutahedra
Beck, Zaslavsky (2006)
inside-out polytopes Ehrhart theory
Aguiar, Ardila (2017)
Hopf monoid of generalized permutahedra
new proof (geometric)
Aguiar, Ardila
Aval, Karaboghossian, Tanasa (2020)
Hopf monoid of hypergraphs
Aval, Karaboghossian, Tanasa (2020)
reciprocity result for hypergraphs
Stanley (1970)
reciprocity result for graphs

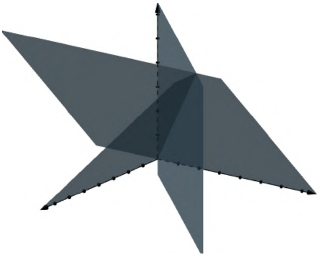
Polyhedral fans

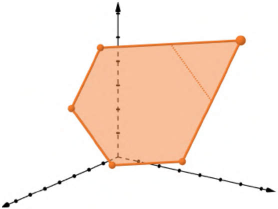
For a complete fan $\mathcal{N}$ in $\mathbb{R}^d$ define the **codimension-one fan** $\mathcal{N}^{\text{co}\,1}$

$$\begin{aligned}\mathcal{N}^{\text{co}\,1} &:= \{N \in \mathcal{N} : \ \text{codim}\, N \geq 1\} \\ &= \{N \in \mathcal{N} : \ \dim N \leq d-1\}\,.\end{aligned}$$

For a normal fan $\mathcal{N}(\mathcal{P})$ we get

$$\mathcal{N}^{\text{co}\,1}(\mathcal{P}) = \{N_{\mathcal{P}}(F) \ : \ F \text{ a face of } \mathcal{P} \text{ with } \dim(F) \geq 1\}.$$



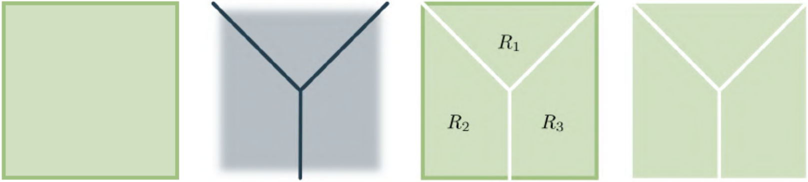


Pruned inside-out polytopes

For a polytope $\mathcal{Q} \subset \mathbb{R}^d$ and a complete fan $\mathcal{N}$ in $\mathbb{R}^d$ we call

$$\mathcal{Q} \setminus \left(\bigcup \mathcal{N}^{\mathrm{co}\,1} \right) = \biguplus_{\substack{N \in \mathcal{N}, \\ N \text{ full-dimensional}}} (\mathcal{Q} \cap N^\circ)$$

a **pruned inside-out polytope** and we call the connected components in $\mathcal{Q} \setminus \left(\bigcup \mathcal{N}^{\mathrm{co}\,1} \right)$ **regions**.

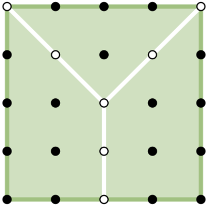


$\mathcal{Q}$ | $\mathcal{N}$ and $\mathcal{N}^{\mathrm{co}\,1}$ | $\mathcal{Q} \setminus \bigcup \mathcal{N}^{\mathrm{co}\,1}$ | $\mathcal{Q}^\circ \setminus \bigcup \mathcal{N}^{\mathrm{co}\,1}$

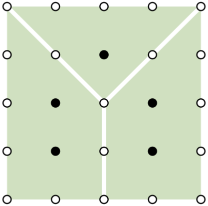
Pruned inside-out counting

For $t \in \mathbb{Z}_{>0}$ define the **inner pruned Ehrhart function** as

$$\mathrm{in}_{\mathcal{Q},\mathcal{N}^{\mathrm{co}\,1}}(t) := \# \left(t \cdot \left(\mathcal{Q} \setminus \bigcup \mathcal{N}^{\mathrm{co}\,1} \right) \cap \mathbb{Z}^d \right) .$$



$\mathrm{in}_{\mathcal{Q},\mathcal{N}^{\mathrm{co}\,1}}$



$\mathrm{in}_{\mathcal{Q}^\circ,\mathcal{N}^{\mathrm{co}\,1}}$

Note:

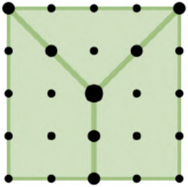
$$\mathrm{in}_{\mathcal{Q}^\circ,\mathcal{N}^{\mathrm{co}\,1}}(t) = \sum_{i=1}^{k} \mathrm{ehr}_{R_i^\circ}(t)$$

and it is a polynomial if regions R_i are integral.

Pruned inside-out counting

Define the **cumulative pruned Ehrhart function** for $t \in \mathbb{Z}_{>0}$ as

$$\mathrm{cu}_{\mathcal{Q},\mathcal{N}^{\mathrm{co}\,1}}(t) := \sum_{y \in t\mathcal{Q} \cap \mathbb{Z}^d} \#\left(N \in \mathcal{N},\ N \text{ full.dim.},\ y \in N\right) .$$



$\mathrm{cu}_{\mathcal{Q},\mathcal{N}^{\mathrm{co}\,1}}$

Note:

$$\mathrm{cu}_{\mathcal{Q},\mathcal{N}^{\mathrm{co}\,1}}(t) = \sum_{i=1}^{k} \mathrm{ehr}_{\overline{R_i}}(t)$$

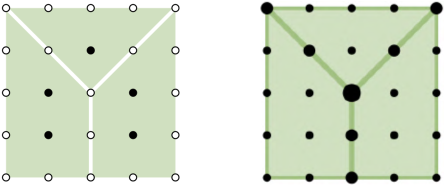
and it is a polynomial if regions R_i are is integral.

Pruned inside-out reciprocity

Theorem (R.)
For a polytope $\mathcal{Q} \subset \mathbb{R}^d$ and a complete fan $\mathcal{N}$ in $\mathbb{R}^d$ we have

$$(-1)^{\dim \mathcal{Q}} \operatorname{in}_{\mathcal{Q}^\circ, \mathcal{N}^{\mathrm{co}\,1}}(-t) = \operatorname{cu}_{\mathcal{Q}, \mathcal{N}^{\mathrm{co}\,1}}(t) \,.$$

Proof.

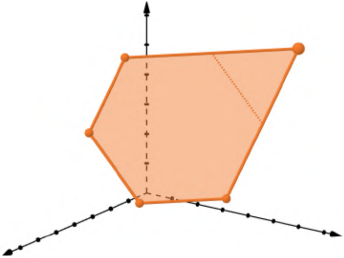


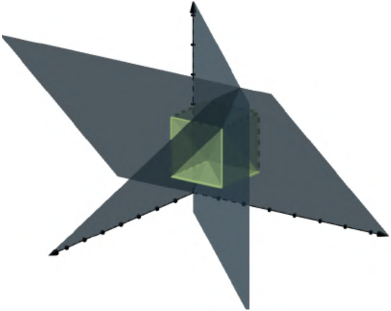
Reciprocity for generalized permutahedra

Theorem (Aguiar, Ardila 2017; Billera, Jia, Reiner 2009)
For generalized permutahedra $\mathcal{P} \subset \mathbb{R}^d$, $m \in \mathbb{Z}_{>0}$

$$\chi_d(\mathcal{P})(m) := \# \left(\mathcal{P}\text{-generic directions } y \in (\mathbb{R}^d)^* \text{ with } y \in [m]^d\right)$$

agrees with a polynomial in m of degree d.





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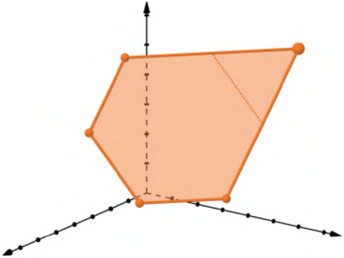
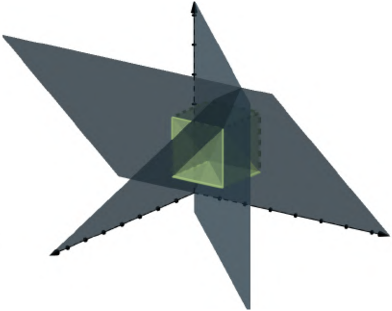
agrees with a polynomial in m of degree d. Moreover,

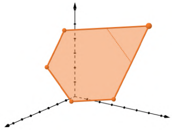
$$(-1)^d \chi_d(\mathcal{P})(-m) = \sum_{y \in [m]^d} \# \left(\text{vertices of } \mathcal{P}_y \right) .$$

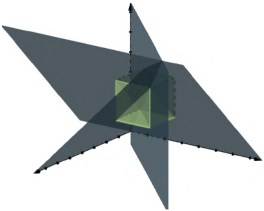


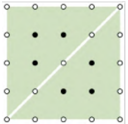


Reciprocity for generalized permutahedra

Thm: polynomiality of
$\chi_d(\mathcal{P})(m) := \#\ \big(\mathcal{P}\text{-generic directions } y \in (\mathbb{R}^d)^* \text{ with } y \in [m]^d\big)$.







Reciprocity for generalized permutahedra

Thm: $$(-1)^d \chi_d(\mathcal{P})(-m) = \sum_{y \in [m]^d} \#\left(\text{vertices of } \mathcal{P}_y\right) .$$