① Tropical fans

$N \simeq \mathbb{Z}^r$ lattice
$N_{\mathbb{R}} \simeq \mathbb{R}^r$ $\mathbb{R}$-vector space

Def (fan)

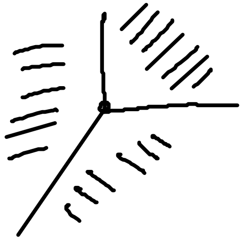
Σ finite collection of cones

(i) $\sigma \in \Sigma$, $\underset{\text{face}}{\tau \leq \sigma} \Rightarrow \tau \in \Sigma$

(ii) $\sigma, \eta \in \Sigma$, $\sigma \cap \eta$ common face

fans are unimodular.

Ex.



Ex. Normal fans of Polytopes

Ex. $\mathcal{P} \subseteq 2^E$ family of subsets of a finite set E

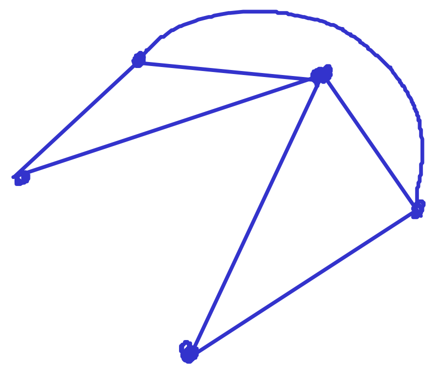
$F \in \mathcal{P} \rightsquigarrow \mathbb{1}_F = (0,0,\ldots,1,0,\ldots,0,1,\ldots,0) \in \mathbb{R}^E$

$\underbrace{\emptyset \neq F_1 \subsetneq \cdots \subsetneq F_\ell \neq E}_{\mathcal{F} \text{ flag}}$ $\quad \sigma_{\mathcal{F}} :=$ cone generated by $\mathbb{1}_{F_1}, \ldots, \mathbb{1}_{F_\ell}$

the collection $\Sigma_{\mathcal{P}}$ is a fan in $\mathbb{R}^E / \mathbb{R}\mathbb{1}_E$

Ex. $G = (V, E)$ graph

$\mathcal{P} \subseteq 2^E$



Def (Chow ring)

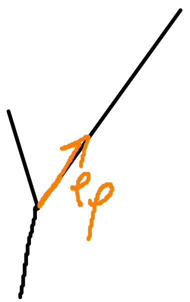
$\Sigma \subseteq N_{\mathbb{R}}$ unimodular fan

For each ray $\rho \in \Sigma$, x_ρ

$$A(\Sigma) := \mathbb{Z}[x_\rho]_{\rho \text{ ray}} \big/ I+J$$

I: $x_{\rho_1} \cdots x_{\rho_k} = 0$ if $\rho_1, \dots, \rho_k$ do not form a cone

J: for each linear form ℓ $\quad \sum \ell(e_\rho) x_\rho = 0$



$\ell : N \to \mathbb{Z}$

<u>Example</u> $\mathcal{P} \subseteq 2^E$ $\qquad$ $\Sigma_{\mathcal{P}} \subseteq \mathbb{R}^E / \mathbb{R} 1_E$

$F \subsetneq E$ $\qquad$ X_F $\qquad$ $\ell : \mathbb{Z}^E \to \mathbb{Z}$

$\ell(1_E) = 0$

① $X_{F_1} \cdots X_{F_\ell} = 0$ if F_j s are not comparable.

② $\sum_{F \ni i} X_F = \sum_{F \ni j} X_F$ $\quad \forall\ i, j \in E.$

$\rightsquigarrow$ $\boxed{\alpha := \sum_{F \ni i} X_F}$ $\qquad$ $\boxed{\beta := \sum_{F \not\ni i} X_F}$

<u>Def</u> G conn. graph Σ_G corresponding fan
n vertices
(matroid)

$$A(G) = A^0 \oplus A^1 \oplus \cdots \oplus A^{n-2}$$

<u>Thm</u> (AHK) $r = n-2$

1. $A^0 \simeq A \simeq \mathbb{Z}$
2. $A^j \times A^{r-j} \longrightarrow A^r \simeq \mathbb{Z}$ Poincaré duality
3. $A^j \xrightarrow[\omega^{r-2j}]{\simeq} A^{r-j}$ Lefschetz
4. $A^1 \times A^1 \longrightarrow \mathbb{Z}$ Hodge-index
$(X, Y) \longrightarrow \omega^{r-2} XY$ one positive eigenvalue

$$f_G(x) = x\sum (-1)^i a_i x^{r-i}$$

Proof of the log-concavity

$$\alpha = \sum_{F \ni i} x_F \qquad \beta = \sum_{F \ni 0} x_F$$

$$f_G(x)/_{1-x} = x \sum (-1)^i b_i x^{r-i}$$

Prop $b_i = \alpha^i \beta^{r-i} \in A^r = \mathbb{Z}$

$\rightsquigarrow$ Log-concavity follows from Hodge-index.

Def (Tropical fan)

$\Sigma \subseteq N_{\mathbb{R}}$ with Balancing condition

Thm (A.–Piquerez 21)

Smooth convex tropical fans verify

Poincaré duality, Lefschetz, Hodge-index.