Main result: type B

The type B Eulerian polynomial is:

$$B_d(z) = \sum_{k=0}^{d} h_k(\pi_d^B) z^k = \sum_{\tau \in \mathfrak{S}_d^{\pm}} z^{\mathrm{exc}(\tau) + \lfloor \frac{\mathrm{neg}(\tau)+1}{2} \rfloor}$$

$\tau : \pm[d] \longrightarrow \pm[d]$ such that $\tau(-i) = -\tau(i)$

$i \in [d-1]$ is an excedance of $\tau \in \mathfrak{S}_d^{\pm}$ if $\tau(i) > i$

$i \in [d]$ is a negation of $\tau \in \mathfrak{S}_d^{\pm}$ if $\tau(i) < 0$

in $\mathfrak{S}_2^{\pm}$

id
$(1\ \bar{1})$
$(2\ \bar{2})$
$(1\ \bar{1})(2\ \bar{2})$
$(1\ 2)(\bar{1}\ \bar{2})$
$(1\ \bar{2})(\bar{1}\ 2)$
$(1\ 2\ \bar{1}\ \bar{2})$
$(1\ \bar{2}\ \bar{1}\ 2)$

so $B_2(z) = 1 + 6z + z^2$

Main result: type B

The type B Eulerian polynomial is:

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Theorem (B- '20)

For any flat $\mathrm{X} \in \mathcal{L}[\mathcal{B}_d]$ *and* $r = 0, 1, \ldots, d$,

$$\eta_{\mathrm{X}}(\Pi_r(\pi_d^B)) = \#\Big\{\tau \in \mathfrak{S}_d^{\pm} \,:\, \mathrm{fix}(\tau) = \mathrm{X},\ \mathrm{exc}(\tau) + \lfloor \frac{\mathrm{neg}(\tau)+1}{2} \rfloor = r\Big\}.$$

Corollaries of the proofs

The generating functions for Eulerian polynomials are (Euler 1700s, Brenti '94):

$$A(z,x) = \sum_{d\geq 0} A_d(z)\frac{x^d}{d!} = \frac{z-1}{z-e^{x(z-1)}} \qquad B(z,x) = \sum_{d\geq 0} B_d(z)\frac{x^d}{2^d\, d!} = \frac{(1-z)e^{x(1-z)/2}}{1-ze^{x(1-z)}}$$

The proofs of the previous theorems show:

$$\sum_{d\geq 0}\Big(\sum_{\sigma\in\mathfrak{S}_d} t^{\dim(\mathrm{fix}(\sigma))} z^{\mathrm{exc}(\sigma)}\Big)\frac{x^d}{d!} = A(z,x)^t \qquad \text{(Brenti '00)}$$

$$\sum_{d\geq 0}\Big(\sum_{\tau\in\mathfrak{S}_d^{\pm}} t^{\dim(\mathrm{fix}(\tau))} z^{\mathrm{exc}_B(\tau)}\Big)\frac{x^d}{2^d\, d!} = B(z,x)\, A(z,x)^{\frac{t-1}{2}} \qquad \text{(B- '20)}$$

Question

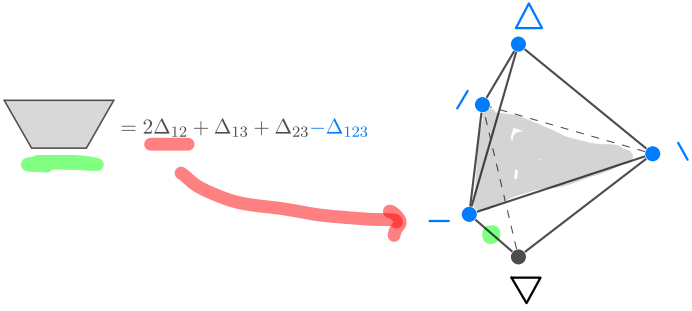
(Posed by Ardila-Castillo-Eur-Postnikov '20)

What is a *nice* type B analog of the following result?

Theorem (Postnikov '09, Ardila-Benedetti-Doker '10)

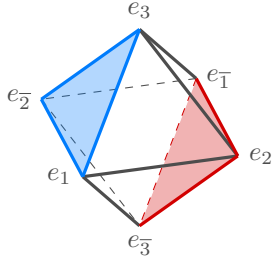
*Every generalized permutahedron in $\mathbb{R}^d$ can be written **uniquely** as a signed Minkowski sum of the faces of the standard simplex $\Delta_{[d]}$.*

Cone of generalized permutahedra (elements of the form $\log[\mathfrak{p}]$ in $\Pi_1(\pi_3)$):



Type B analog of Δ_d

The faces of the cross-polytope only generate a subspace of (roughly) half the required dimension.



$$\Delta_{\overline{1}2} = \Delta_{1\overline{2}}$$

$$\Delta_{\overline{1}2\overline{3}} = \Delta_{1\overline{2}} + \Delta_{13} + \Delta_{\overline{2}3} - \Delta_{1\overline{2}3}$$

(up to translation)

Second corollary

Corollary (B- '20)

Any family of generators (via signed Minkowski sums) for generalized permutahedra of type B contains at least 2^{d-1} full-dimensional polytopes.

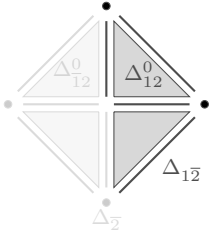
- From the previous theorem, $\eta_{\{\mathbf{0}\}}(\Pi_1(\pi_d^B)) = 2^{d-1}$.
- We employ Eulerian idempotents of $\mathbb{R}\Sigma[\mathcal{B}_d]$ (Saliola '06, Aguiar-Mahajan '17) to deduce that, if $\mathfrak{q}$ is not full-dimensional, the projection of $\log[\mathfrak{q}]$ to certain $\eta_{\{\mathbf{0}\}}(\Pi_1(\pi_d^B))$-dimensional subspace is zero.

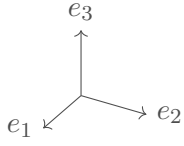
14 full-dimensional *type B shard polytopes* in $\mathbb{R}^3$ (Padrol-Pilaud-Ritter '20)

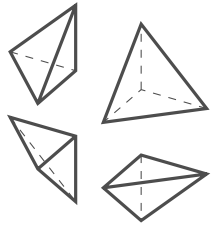
Generators for type B generalized permutahedra

For nonempty $S \subseteq [\pm d]$ with $S \cap \overline{S} = \emptyset$ define

$$\Delta_S = \mathrm{Conv}\{e_j \mid j \in S\} \quad \text{and} \quad \Delta_S^0 = \mathrm{Conv}(\Delta_S \cup \{\mathbf{0}\}) \qquad (e_{\bar{i}} = -e_i)$$







Theorem (B- '20)

Any type B generalized permutahedron can be written uniquely as a signed Minkowski sum of the simplices

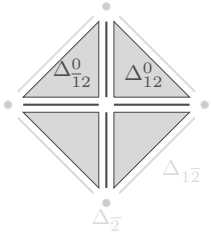
$$\left\{\Delta_S, \Delta_S^0 \,:\, \min\{|j| \,:\, j \in S\} \in S\right\}.$$

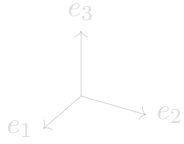
The proof uses a *valuation* $\Pi_1(\pi_d^B) \to \mathbb{R}\Sigma[\mathcal{B}_d]$.

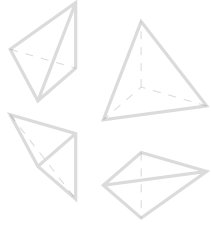
Generators for type B generalized permutahedra

For nonempty $S \subseteq [\pm d]$ with $S \cap \overline{S} = \emptyset$ define

$$\Delta_S = \mathrm{Conv}\{e_j \mid j \in S\} \quad \text{and} \quad \Delta_S^0 = \mathrm{Conv}(\Delta_S \cup \{\mathbf{0}\}) \qquad (e_{\bar{i}} = -e_i)$$





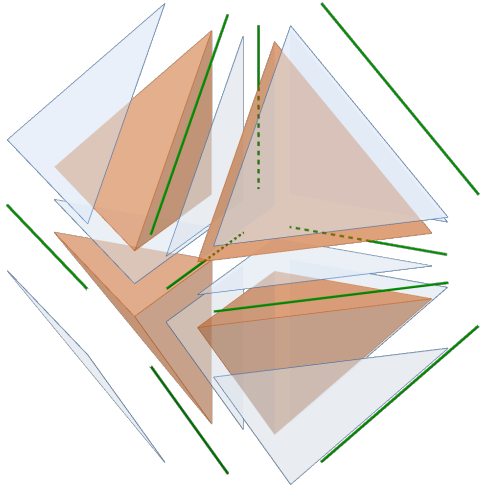


Theorem (B- '20)

Any type B generalized permutahedron can be written uniquely as a signed Minkowski sum of the simplices Δ_S^0.

"Change of basis" bisubmodular functions $\longleftrightarrow$ signed Mink. sum of Δ_S°

THANK YOU!



arXiv:2009.05876