On fluctuations of Birkhoff sums

Gaétan Leclerc

Abstract

We consider a smooth, transitive, area-preserving Axiom A diffeomorphism f on a surface M. We fix Ω a basic set, and we consider a smooth potential $\tau : \Omega \to \mathbb{R}$. In these notes, we explain how one can use the ideas found in M. Tsujii and Z. Zhang's paper on exponential mixing of mixing 3D anosov flows to check a non-concentration property for the Birkhoff sums of τ.

1 A bit of context

In a previous paper, we tried to study the Fourier decay properties of equilibrium states for transitive nonlinear Axiom A diffeomorphisms. Say that we fix an area-preserving diffeomorphism $f : M \to M$ (on some riemannian manifold of dimension 2 (M, g)). Suppose that f is Axiom A and smooth, fix Ω one of its basic sets, and then define, for $x \in \Omega$: $\tau_f(x) := \ln \partial_u f(x)$, where we denoted

$$\partial_u f(x) := \|(df_x)_{|E^u(x)}\|.$$

We can similarly define $\partial_s f : \Omega \to \mathbb{R}_+$ as

$$\partial_s f(x) := \|(df)_{|E^s(x)}\|.$$

The area-preserving hypothesis ensure the cohomology relation $\tau_f \sim -\ln \partial_s f$. Finally, fix some Holder potential $\varphi : \Omega \to \mathbb{R}$, and denote its associated equilibrium state $\mu \in \mathcal{P}(\Omega)$ (the set of borel probability measures supported on Ω). Using the "sum-product phenomenon", we (almost) proved the following criterion.

Theorem 1.1. *Suppose that there exists $C_0, \varepsilon_0, \gamma > 0$ such that:*

$$\forall n \geq 0,\ \forall a \in \mathbb{R},\ \mu\Big(x \in \Omega\ ,\ |S_n\tau_f(x) - a| \leq e^{-\varepsilon_0 n}\Big) \leq C_0 e^{-\gamma\varepsilon_0 n}, \qquad \text{(NC)}$$

where $S_n\tau_f(x) := \sum_{k=0}^{n-1} \tau_f(f^k(x))$ is a Birkhoff sum. Then, there exists $\rho > 0$ such that, for any small enough open set U, for any smooth bump function $\chi : M \to \mathbb{R}$ supported in U, for any local chart $\phi : U \to \mathbb{R}$, there exists C_1 such that:

$$\forall \xi \in \mathbb{R}^2 \setminus \{0\},\ \left|\int_\Omega e^{i\xi\cdot\phi(x)}\chi(x)d\mu(x)\right| \leq C_1|\xi|^{-\rho}.$$

In other words, $\phi_*(\chi d\mu)$ enjoy power decay of its Fourier transform.

The main difficulty is to check the "non-concentration hypothesis" (NC) found in the criterion. In these notes, we show how, in our 2-dimensional, area-preserving context, one can check theses kind of non-concentration estimates, adapting the ideas of M. Tsujii and Z. Zhang's paper [TZ20]. Our nonconcentration estimates will hold under a cohomology condition on a "longitudinal KAM cocyle" (defined later in the text: see remark 4.4).

Theorem 1.2. *Let $\tau : \Omega \to \mathbb{R}$ be any $C^{2+\alpha}$ potential. Denote by $\Phi_\tau : \Omega \to \mathbb{R}$ its associated "longitudinal KAM cocycle". If $\Phi_\tau \in C^\alpha$ is not cohomologous to zero (this is a $C^{2+\alpha}$-generic condition on τ), then there exists $C_0, \varepsilon_0, \gamma > 0$ such that:*

$$\forall n \geq 0,\ \forall a \in \mathbb{R},\ \mu\Big(x \in \Omega\ ,\ |S_n\tau(x) - a| \leq e^{-\varepsilon_0 n}\Big) \leq C_0 e^{-\gamma\varepsilon_0 n}.$$

Notice that, in this theorem, the potential has to be $C^{2+\alpha}$. In our case of interest, this is **not** true. So this isn't quite enough to tackle the case of the geometric potential τ_f, but the techniques that appear when dealing with this simpler context might have applications or be interesting enough to justify explaining them in detail here. Moreover, one might be able to adapt the ideas found in these notes in the case of τ_f by lifting the dynamics on the projective tangent bundle. One would have to replace τ_f by $(x, [v]) \mapsto \ln(\|(df)_x v\|/\|v\|)$ and replace the dynamics by $F(x, [v]) = (f(x), [(df)_x(v)])$. This idea was given to me by Zhiyuan Zhang (thank you!), whose help was very important for me to understand the ideas behind their paper and the possible generalizations of it. This will be explored in future work.
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2 A temporal distance function

Recall that we fixed a smooth Axiom A dffeomorphism on a surface M. We suppose that f is area-preserving on a basic set Ω. Suppose that (f, Ω) is transitive. We fix some smooth ($C^{2+\alpha}$ is enough) potential $\tau : \Omega \to \mathbb{R}$. Suppose also that Ω is not a periodic orbit: in particular, by transitivity of the dynamics, it is a perfect set. (Typically Ω is Cantor dust. One could also work in the case where $\Omega = M$ with M a closed surface.) Even better, using the local product structure, for each $x \in \Omega$, $\Omega \cap W^u_{loc}(x)$ and $\Omega \cap W^s_{loc}(x)$ are perfect sets. Recall that one can modify the riemannian structure on the surface M so that:

$$\forall x \in \Omega,\ |\lambda_x| := \partial_u f(x) \in (1, \infty) \quad , \quad |\mu_x| := \partial_s f(x) \in (0, 1).$$

In general, we will denote by ∂_u and ∂_s the vector fields defined on Ω of norm 1 that points in the stable/unstable direction (defined up to some sign... Hence the modulus in the definition of $\partial_u f$). By continuity of $x \in \Omega \mapsto (\lambda_x, \mu_x)$ and by compacity of Ω, there exists $0 < \mu_+ < \mu_- < 1 < \lambda_- < \lambda_+$ such that:

$$\forall x \in \Omega,\ \mu_+ \leq |\mu_x| \leq \mu_- < 1 < \lambda_- \leq |\lambda_x| \leq \lambda_+.$$

We can choose our constants so that $\mu_\pm \lambda_\pm = 1$. Notice that our area-preserving hypothesis implies the cohomology relation $\ln(|\mu_x| \cdot |\lambda_x|) \sim 0$. In particular, we can write, uniformly in $n \geq 0$ and $x \in \Omega$:

$$|\mu_x| \dots |\mu_{f^n(x)}| = |\lambda_x|^{-1} \dots |\lambda_{f^n(x)}|^{-1} e^{O(1)}.$$

If x is a n-periodic point, then $|\lambda_x| \dots |\lambda_{f^n(x)}||\mu_x| \dots |\mu_{f^n(x)}| = 1$.

Denote by $\widetilde{\text{Diag}} \subset \Omega^2$ a small enough neighborhood of the diagonal, so that for any $(p, q) \in \widetilde{\text{Diag}}$, $\{[p, q]\} := W^s_{loc}(p) \cap W^u_{loc}(q)$ and $\{[q, p]\} := W^s_{loc}(q) \cap W^u_{loc}(p)$ are well defined.

Definition 2.1. For $(p, q) \in \widetilde{\text{Diag}}$, and for any $n \in \mathbb{Z}$, define:

$$T_n(p, q) := \tau(f^n(p)) - \tau(f^n([p, q])) - \tau(f^n([q, p])) + \tau(f^n(q)).$$

Define also $\Delta, \Delta^+, \Delta^- : \widetilde{\text{Diag}} \longrightarrow \mathbb{R}$ by the formulas:

$$\Delta(p, q) := \sum_{n \in \mathbb{Z}} T_n(p, q) \quad , \Delta^+(p, q) := \sum_{n \geq 0} T_n(p, q) \quad , \Delta^-(p, q) := \sum_{n \geq 0} T_{-n}(p, q).$$

These functions are all well defined and continuous, since $|T_n(p, q)| \leq C_\tau \mu_-^{|n|} d(p, q)$. The point is that nonconcentration of either Δ, Δ^+ or Δ^- is enough to ensure the non-concentration estimates that we want. Notice also that $\Delta(p, q)$ is a temporal distance function associated to the suspension flow of f with roof function τ. First, recall a fact about rectangles. (In the following of theses notes, we will denote by d^s and d^u the arclenght distance induced along stable/unstable manifolds.)

Lemma 2.2. *Let any $\beta_Z > 1$ (a "zooming" parameter). There exists $c \in (0, 1)$ such that, for any $\sigma > 0$ small enough, there exists a finite partition (up to a zero-measure set) of Ω with rectangles $(R_i^{(\sigma)})_{i \in I}$ such that each $R_i^{(\sigma)}$ can be written $R_i^{(\sigma)} = [U_i(\sigma), S_i(\sigma^{\beta_Z})]$, with $U_i(\sigma) \subset \Omega \cap W^u_{loc}(p_i)$ some unstable curve of unstable diameter $\mathrm{diam}^u(U_i(\sigma)) \in [c\sigma, \sigma]$, and $S_i(\sigma^{\beta_Z}) \subset \Omega \cap W^s_{loc}(p_i)$ some stable curve of stable diameter $\mathrm{diam}^s(S_i(\sigma^{\beta_Z})) \in [c\sigma^{\beta_Z}, \sigma^{\beta_Z}]$.*

Proof. Fix some small Markov partition $(R_j)_{j\in J}$ of Ω. Recall that the boundary of those rectangles have zero measure. For each j, write $R_j = [U_j, S_j]$. Now, for each $n \geq 0$, define $(U_{\mathbf{a}}^{(n)})_{\mathbf{a}\in J^n}$ as the partition:

$$U_{\mathbf{a}}^{(n)} := U_{a_1} \cap f^{-1}(R_{a_2}) \cap \cdots \cap f^{-n}(R_{a_n}).$$

For almost every $x \in U_a$, for each $n \geq 0$, there exists a unique $\mathbf{a} \in J^n$ such that $x \in U_{\mathbf{a}^{(n)}}^{(n)}$. As n grows, the diameter of those goes to zero exponentially quickly. Let $n(x)$ be the smallest integer $n \geq 0$ such that $x \in U_{\mathbf{a}^{(n(x))}}^{(n(x))}$ and $\operatorname{diam}^u(U_{\mathbf{a}^{(n(x))}}^{(n(x))}) \leq \sigma$. The unstable part of the partition is then given by $(U(x))_{x\in\cup_j U_j}$, where $U(x) := U_{\mathbf{a}^{(n(x))}}^{(n(x))}$. This is a finite partition which satisfies the bounds that we want. To conclude, apply the same construction for the stable part (by reversing the dynamics), and then define $R_i^{(\sigma)}$ as the rectangles obtained from the unstable and stable parts. $\square$

From now on, we fix a parameter $\beta_Z > 1$. It will be chosen large enough later in the text (in the end of section 4).

In the next lemma, we will denote by $\mathrm{Rect}_{\beta_Z}(\sigma)$ the set of nonempty rectangles $R^{(\sigma)} \subset \Omega$ of stable (resp. unstable) diameter $\simeq \sigma^{\beta_Z}$ (resp $\simeq \sigma$). In particular, $R^{(\sigma)}$ have nonempty interior for the topology of Ω.

Lemma 2.3. *Suppose that there exists $\gamma > 0$ and $N \in \mathbb{N}^*$ such that, for any small enough $\sigma > 0$, for all rectangle $R^{(\sigma)} \in Rect_{\beta_Z}(\sigma)$, for all $a \in \mathbb{R}$:*

$$(\mu \otimes \mu)((p,q) \in (R^{(\sigma)})^2,\ |\Delta^+(p,q)| \leq \sigma^N) \leq \mu(R^{(\sigma)})^2 \cdot \sigma^\gamma.$$

Then (NC) (non-concentration) holds.

Proof. Let $n \geq 0$ be large enough, and let $\sigma \sim e^{-\varepsilon_0 n}$, where ε_0 is a small enough constant (we will see how small we need it later in the proof). Let $a \in \mathbb{R}$. Fix $\mathrm{Part}(\sigma)$, a partition of Ω by rectangles of size $\sim \sigma^{1/N} \times \sigma^{\beta_Z/N}$. Then:

$$\mu(x \in \Omega, |S_n\tau(x) - a| \leq \sigma) = \sum_{R^{(\sigma)} \in \mathrm{Part}(\sigma)} \mu(x \in R^{(\sigma)},\ S_n\tau(x) \in [a-\sigma, a+\sigma])$$

$$= \sum_{R^{(\sigma)} \in \mathrm{Part}(\sigma)} \mu(R^{(\sigma)}) \left(\mu(R^{(\sigma)})^{-1} \mu(x \in R^{(\sigma)},\ S_n\tau(x) \in [a-\sigma, a+\sigma]) \right).$$

Now, notice the following. If we fix any (small enough) rectangle $R = [U,S]$ of nonzero measure (fixing U and S, some reference unstable and stable curves in R), then the local product structure of the equilibrium state μ (see [Cl20] for example) allows one to write, for some measures μ^u defined on U and some measure μ^s defined on S:

$$\mu(x \in R,\ h(x) \in I_\sigma) = \int_U \int_S \mathbb{1}_{I_\sigma}(h([z,y])) d\mu^s(y) d\mu^u(z),$$

where we defined $h(x) := S_n\tau(x)$, and $I_\sigma := [a-\sigma, a+\sigma]$. One then can write, using Cauchy-Scwhartz:

$$\mu(x \in R,\ S_n\tau(x) \in I_\sigma)^2 = \left(\iint_{S\times U} \mathbb{1}_{I_\sigma}(h([z,y])) d\mu^u(z) d\mu^s(y) \right)^2$$

$$\leq \mu^s(S) \int_S \iint_{U\times U} \mathbb{1}_{I_\sigma}(h([z,y])) \mathbb{1}_{I_\sigma}(h([\tilde{z},y])) (d\mu^u)^2(z,\tilde{z}) d\mu^s(y)$$

$$\leq \mu^s(S) \int_S \iint_{U\times U} \mathbb{1}_{[-2\sigma,2\sigma]}(h([z,y]) - h([\tilde{z},y])) (d\mu^u)^2(z,\tilde{z}) d\mu^s(y).$$

Then, using Fubini and Cauchy-Schwartz again yields:

$$\mu(x \in R,\ S_n\tau(x) \in I_\sigma)^4 \leq \mu^s(S)^2 \left(\iint_{U\times U} \int_S \mathbb{1}_{[-2\sigma,2\sigma]}(h([z,y]) - h([\tilde{z},y])) d\mu^s(y) (d\mu^u)^2(z,\tilde{z}) \right)^2$$

$$\leq \mu^s(S)^2 \mu^u(U)^2 \iint_{U\times U} \iint_{S\times S} \mathbb{1}_{[-2\sigma,2\sigma]}(h([z,y]) - h([\tilde{z},y])) \mathbb{1}_{[-2\sigma,2\sigma]}(h([z,\tilde{y}]) - h([\tilde{z},\tilde{y}])) (d\mu^s)^2(y,\tilde{y}) (d\mu^u)^2(z,\tilde{z})$$

$$\leq \mu(R)^2 \iiiint_{U\times U\times S\times S} \mathbb{1}_{[\pm 4\sigma]}(h([z,y]) - h([\tilde z,y]) - h([z,\tilde y]) + h([\tilde z,\tilde y])(d\mu^u)^2(z,\tilde z)(d\mu^s)^2(y,\tilde y)$$

$$= \mu(R)^2 \iint_{R^2} \mathbb{1}_{[\pm 4\sigma]}(h(p) - h([p,q]) - h([q,p]) + h(q))d\mu(p)d\mu(q).$$

In other words, denoting by $H(p,q) := h(p) - h([p,q]) - h([q,p]) + h(q)$ (defined only on a neihborhood of the diagonal !), we have:

$$\frac{1}{\mu(R)}\mu(x\in R, h(x)\in I_\sigma) \leq \left(\frac{1}{\mu^2(R^2)}\mu^2((p,q)\in R^2,\ H(p,q)\in[-4\sigma,4\sigma])\right)^{1/4}.$$

In our particular case, $h(x) = S_n\tau(x)$, so we get the expression

$$H(p,q) = T_n(p,q) = \sum_{k=0}^{n-1}\big(\tau(f^k p) - \tau(f^k([p,q])) - \tau(f^k([q,p])) + \tau(f^k(q))\big).$$

Injecting this estimate in our sum from the beginning of the proof yields:

$$\mu(x\in\Omega, |S_n\tau(x)-a|\leq\sigma) = \sum_{R^{(\sigma)}\in\mathrm{Part}(\sigma)} \mu(R^{(\sigma)})\left(\mu(R^{(\sigma)})^{-1}\mu(x\in R^{(\sigma)},\ S_n\tau(x)\in[a-\sigma,a+\sigma])\right)$$

$$\leq \sum_{R^{(\sigma)}\in\mathrm{Part}(\sigma)} \mu(R^{(\sigma)})\left(\frac{1}{\mu(R^{(\sigma)})^2}\mu^2\Big((p,q)\in(R^{(\sigma)})^2,\ \Big|\sum_{k=0}^{n-1}T_k(p,q)\Big|\leq 4\sigma\Big)\right)^{1/4}.$$

To conclude, notice that

$$\forall p,q\in R^{(\sigma)},\ \Big|\Delta^+(p,q) - \sum_{k=0}^{n-1}T_k(p,q)\Big| \lesssim \mu_-^n \leq \sigma$$

since ε_0 is small in front of $|\ln\mu_-|$, and $\sigma\sim e^{-\varepsilon_0 n}$. Hence:

$$\sum_{R^{(\sigma)}\in\mathrm{Part}(\sigma)} \mu(R^{(\sigma)})\left(\frac{1}{\mu(R^{(\sigma)})^2}\mu^2\Big((p,q)\in(R^{(\sigma)})^2,\ \Big|\sum_{k=0}^{n-1}T_k(p,q)\Big|\leq 4\sigma\Big)\right)^{1/4}$$

$$\leq \sum_{R^{(\sigma)}\in\mathrm{Part}(\sigma)} \mu(R^{(\sigma)})\left(\frac{1}{\mu(R^{(\sigma)})^2}\mu^2\Big((p,q)\in(R^{(\sigma)})^2,\ \Big|\Delta^+(p,q)\Big|\leq 5\sigma\Big)\right)^{1/4}$$

$$\leq \sum_{R^{(\sigma)}\in\mathrm{Part}(\sigma)} \mu(R^{(\sigma)})\cdot(5\sigma)^{\gamma/(N\cdot 4)} \leq 2\sigma^{\gamma/(4N)},$$

and we are done. □

Remark 2.4. Using the invariance of μ by f at the very beginning of the proof yields

$$\mu(x\in\Omega,|S_n\tau(x)-a|\leq\sigma) = \mu(x\in\Omega,|S_n\tau(f^{-n}(x))-a|\leq\sigma) = \mu(x\in\Omega,|S_n\tau(f^{-n/2}(x))-a|\leq\sigma).$$

Starting from the middle term or the last term and then following the previous proof shows that non-concentration holds also if one replaces Δ^+ by Δ^- or Δ.

Lemma 2.5. *Suppose that there exists $N\in\mathbb{N}^*$ and $\gamma>0$ such that, for any small enough $\sigma>0$, for all $p\in\Omega$, for all rectangle $R_p^{(\sigma)}\in Rect_{\beta_Z}(\sigma)$ containing a ball $[B_u(p,\sigma/10),B_s(p,\sigma^{\beta_Z}/10)]\cap\Omega$:*

$$\mu(q\in R_p^{(\sigma)},\ |\Delta^+(p,q)|\leq\sigma^N) \leq \mu(R_p^{(\sigma)})\sigma^\gamma.$$

Then (NC) holds.

Proof. We are going to check that the previous lemma applies. Let $R^{(\sigma)}$ be some rectangle of stable/unstable diameter $\sim\sigma$. Then:

$$\mu^2\Big((p,q)\in R^{(\sigma)}\ ,\ |\Delta(p,q)|\leq\sigma^N\Big)=\int_{R^{(\sigma)}}\mu(q\in R^{(\sigma)},\ |\Delta(p,q)|\leq\sigma^N)d\mu(p)$$

$$\leq\int_{R^{(\sigma)}}\mu(q\in R_p^{(10\sigma)},\ |\Delta(p,q)|\leq\sigma^N)d\mu(p),$$

where we denoted $R_p^{(10\sigma)}$ some rectangle containing $R^{(\sigma)}$ in its center, with unstable (resp stable) diameter $10^{\beta z}$ times larger (resp 10 larger). Now, we can use the hypothesis, since p is sufficiently close to the center of the square. We find:

$$\int_{R^{(\sigma)}}\mu(q\in R_p^{(10\sigma)},\ |\Delta(p,q)|\leq\sigma^N)d\mu(p)\leq\mu(R^\sigma)\mu(R_p^{(10\sigma)})\cdot 10^\gamma\sigma^\gamma.$$

The fact that the measure μ is **doubling** [Do98] gives us some constant $C>0$ such that $\mu(R^{(A\sigma)})\leq C\mu(R^\sigma)$, and this allows us to check the condition of the previous lemma. $\square$

Now we know that, to conclude, it suffices to understand the oscillations of $\Delta(p,q)$, for any fixed p, when q gets close to p. To do so, we will introduce some coordinate systems associated to the dynamics.

3 Construction of adapted coordinates.

In this section, we construct a family of adapted coordinates in which the dynamics is going to be (almost) linearized. We also define templates (linear form version, and vector field version).

Lemma 3.1. *There exists a family of uniformly smooth maps $(\Phi_x^s)_{x\in\Omega}$, such that for all $x\in\Omega$ $\Phi_x^s:\mathbb{R}\longrightarrow W^s(x)$ is a smooth parametrization of $W^s(x)$, $\Phi_x^s(0)=x$, $|(\Phi_x^u)'(0)|=1$, and:*

$$\forall x\in\Omega,\forall y\in\mathbb{R},\ f\left(\Phi_x^s(y)\right)=\Phi_{f(x)}^s(\mu_x y),$$

where $\mu_x:=\varepsilon_x^s|\mu_x|$, with $|\mu_x|:=\partial_s f(x)\in(\mu_+,\mu_x)\subset(0,1)$ and $\varepsilon_x^s\in\{-1,1\}$ is some sign that depends on x. The dependence in x of $(\Phi_x^s)_{x\in\Omega}$ is Hölder.

Proof. The proof is taken from [KK07]. The idea is to first define Φ_x^s on $W_{loc}^s(x)$ (on which $\partial_s f(x)$ makes sense and is smooth along $W_{loc}^s(x)$, even if $x\notin\Omega$), and then to extend our maps on $W^s(x)$ using the conjugacy relation that we want. Define, for any $x\in\Omega$, and for any $y\in W_{\mathrm{loc}}^s(x)$, the function $\rho_x(y)$ by the formula:

$$\rho_x(y):=\sum_{n=0}^{\infty}\left[(\ln\partial_s f)(f^n y)-(\ln\partial_s f)(f^n x)\right].$$

This is well defined and smooth along $W_{loc}^s(x)$ (and this, uniformly in $x\in\Omega$). We can then define, for $y\in W^s(x)$:

$$\Psi_x^s(y):=\int_{[x,y]\subset W_{loc}^s(x)}e^{\rho_x(y')}dy'$$

in the sense that we integrate from x to y, following the local stable manifold $W_{loc}^s(x)$ (w.r.t. the arclenght). This function is smooth along $W_{loc}^s(x)$, and is obviously invertible since its stable derivative is positive. We denote by $\Phi_x^s:(-\varepsilon,\varepsilon)\to W^u(x)$ its inverse. (One can choose a uniform ε for all these maps, but this is not very important.)

We check that the dynamics is linearized in these coordinates. Notice that, since $\Psi_x^s(x)=0$, the desired relation is equivalent to:

$$\forall y\in W_{loc}^s(x),\ \partial_s\Psi_{f(x)}^s(f(y))\partial_s f(y)=|\mu_x|\partial_s\Psi_x^s(y).$$

But this is obviously true, by construction of Ψ_x^s. It follows that, for all $y\in W_{loc}^s(x)$, $\Psi_{f(x)}^s(f(y))=\mu_x\Psi_x^s(y)$. In particular, notice that iterating this relation yields, for $y\in W_{loc}^s(x)$, $\Psi_{f^n(x)}^s(f^n(y))=$

$\mu_{f^{n-1}x} \dots \mu_x \Psi^s_x(y)$.

To conclude the proof, we need to extend Ψ^s_x on the whole stable manifold of $x \in \Omega$. We proceed as follow. Let $y \in W^s(x)$. If n is large enough (depending on y), one sees that $f^n y \in W^s_{loc}(f^n x)$. Hence, it makes sense to define:

$$\Psi^s_x(y) := \mu_x^{-1} \dots \mu_{f^{n-1}x}^{-1} \Psi^s_{f^n x}(f^n(y)).$$

The previous discussion ensure that this is well defined. Moreover, it is easy to check that the map $\Psi^s_x : W^s(x) \to \mathbb{R}$ is a smooth diffeomorphism (when we see $W^s(x)$ as a manifold equipped with the arclenght.) The inverse of Ψ^s_x is defined to be $\Phi^s_x : \mathbb{R} \to W^s(x)$. The commutation relation is then easy to check. The Holder regularity in x is tedious to detail but shouldn't be surprising. $\square$

Lemma 3.2. *There exists a family of uniformly smooth maps $(\Phi^u_x)_{x\in\Omega}$, such that for all $x \in \Omega$ $\Phi^u_x : \mathbb{R} \longrightarrow W^u(x)$ is a smooth parametrization of $W^u(x)$, $\Phi^u_x(0) = x$, $|(\Phi^u_x)'(0)| = 1$, and:*

$$\forall x \in \Omega, \forall z \in \mathbb{R},\ f\left(\Phi^u_x(z)\right) = \Phi^u_{f(x)}(\lambda_x z),$$

where $\lambda_x := \varepsilon^u_x|\lambda_x|$, with $|\lambda_x| = \partial_u f(x) \in (\lambda_-, \lambda_+) \subset (1, \infty)$ and $\varepsilon^u_x \in \{-1, 1\}$ is some sign that depends on x. The dependence in x of (Φ^u_x) is Hölder.

Definition 3.3. These parametrizations often goes outside Ω, but we are only interested by what's happening inside Ω. So let us define:

$$\Omega^u_x := (\Phi^u_x)^{-1}(\Omega) \subset \mathbb{R} \quad , \quad \Omega^s_x := (\Phi^s_x)^{-1}(\Omega) \subset \mathbb{R}.$$

Notice that, for all $x \in \Omega$, $0 \in \Omega^u_x$. Moreover:

$$\forall x \in \Omega, \forall z \in \Omega^u_x,\ \lambda_x z \in \Omega^u_{f(x)} \subset \mathbb{R}.$$

A similar statement hold for Ω^s_x.

Remark 3.4. Let us define some further notations. Define, for $n \in \mathbb{Z}$ and $x \in \Omega$:

$$|\lambda_x^{\langle n\rangle}| := \partial_u(f^n)(x) \quad ; \quad |\mu_x^{\langle n\rangle}| := \partial_s(f^n)(x).$$

Notice that $|\lambda_x^{\langle 0\rangle}| = |\mu_x^{\langle 0\rangle}| = 1$, $|\lambda_x^{\langle -n\rangle}| = |\lambda_{f^{-n}(x)}^{\langle n\rangle}|^{-1}$ and $|\mu_x^{\langle -n\rangle}| = |\mu_{f^{-n}(x)}^{\langle n\rangle}|^{-1}$. Moreover, we can write some relations involving (Φ^u_x) and (Φ^s_x). For all $n \in \mathbb{Z}$, $x \in \Omega$, $y \in \Omega^s_x$, $z \in \Omega^u_x$, we have:

$$f^n(\Phi^u_x(z)) = \Phi^u_{f^n(x)}(\lambda_x^{\langle n\rangle} z) \quad ; \quad f^n(\Phi^s_x(y)) = \Phi^s_{f^n(x)}(\mu_x^{\langle n\rangle} y),$$

where $\lambda_x^{\langle n\rangle}$ (resp. $\mu_x^{\langle n\rangle}$) is $|\lambda_x^{\langle n\rangle}|$ (resp. $|\mu_x^{\langle n\rangle}|$) multiplied by the obvious associated sign.

Lemma 3.5 (change of parametrizations)**.** *Let $x \in \Omega$ and let $\tilde{x} \in \Omega \cap W^u_{loc}(x)$. Then the real map $\mathit{aff}_{\tilde{x},x} := (\Phi^u_{\tilde{x}})^{-1} \circ \Phi^u_x : \mathbb{R} \longrightarrow \mathbb{R}$ is affine. Moreover, there exists $C \geq 1$ and $\alpha > 0$ such that $\ln|\mathit{aff}_{\tilde{x},x}{}'(0)| \leq C d(x,\tilde{x})^\alpha$.*

Proof. Notice that, for all $z \in \mathbb{R}$, and for all $n \geq 0$:

$$(\Phi^u_{\tilde{x}})^{-1}\left(\Phi^u_x(z)\right) = \lambda_{\tilde{x}}^{\langle n\rangle} \left(\Phi^u_{f^{-n}(\tilde{x})}\right)^{-1} \left(\Phi^u_{f^{-n}(x)}(\lambda_x^{\langle -n\rangle} z)\right).$$

In particular, without loss of generality, we see that we can reduce our problem to show that $\mathrm{aff}_{x,\tilde{x}}$ is affine on a neighborhood of zero, and this property should spread. In this case, we can compute the log of the absolute value of the differential of $(\Phi^u_{\tilde{x}})^{-1} \circ \Phi^u_x$, and we get:

$$\ln\left(\left|\left((\Phi^u_{\tilde{x}})^{-1} \circ \Phi^u_x\right)'(z)\right|\right)$$
$$= \rho_{\tilde{x}}(\Phi^u_x(z)) - \rho_x(\Phi^u_x(z)) = \rho_{\tilde{x}}(x),$$

which is constant in z. The proof is done: the bound on $\mathrm{aff}_{x,\tilde{x}}{}'(0)$ follows from an easy bound on $\rho_{\tilde{x}}(x)$. $\square$

These coordinates are interesting but only linearize the dynamics along the stable or unstable direction. Of course, we can't expect to fully linearize the dynamics in smooth coordinates, but we can still try to introduce coordinates that will linearize the dynamics in a weaker sense, in some particular places. This construction is directly taken from [TZ20], appendix B.

Lemma 3.6 (Nonstationary normal coordinates)**.** *The exists two small constants $\rho_1 < \rho_0 < 1$, and a family of uniformly smooth coordinates charts $\{\iota_x : (-\rho_0, \rho_0)^2 \to M\}_{x\in\Omega}$ such that:*

- *For every $x \in \Omega$, we have*
$$\iota_x(0,0) = x, \quad \iota_x(z,0) = \Phi_x^u(z), \quad \iota_x(0,y) = \Phi_x^s(y),$$
- *the map $f_x := \iota_{f(x)}^{-1} \circ f \circ \iota_x : (-\rho_1, \rho_1)^2 \longrightarrow (-\rho_0, \rho_0)^2$ is smooth (uniformly in x) and satisfies*
$$\pi_y(\partial_y f_x(z,0)) = \mu_x, \quad \pi_z(\partial_z f_x(0,y)) = \lambda_x,$$
where π_z (resp. π_y) is the projection on the first (resp. second) coordinate.

Furthermore, one can assume the dependence in x of $(\iota_x)_{x\in\Omega}$ to be Hölder regular.

Proof. Since the stable/unstable manifolds are smooth, and since they intersect uniformly transversely, we know that we can construct a system of smooth coordinate charts $(\check{\iota}_x)_{x\in\Omega}$ such that, for all $x \in \Omega$,
$$\check{\iota}_x(0,0) = x, \quad \check{\iota}_x(z,0) = \Phi_x^u(z), \quad \check{\iota}_x(0,y) = \Phi_x^s(y).$$
One can also assume the dependence in x of these to be Hölder regular, since the stable/unstable laminations are Hölder (in our context, they are even $C^{1+\alpha}$). Define $\check{f}_x := \check{\iota}_{f(x)}^{-1} \circ f \circ \check{\iota}_x$. This is a smooth map defined on a neighborhood of zero, with a (hyperbolic) fixed point at zero. Notice also that $(d\check{f}_x)_0$ is a diagonal map with coefficients (λ_x, μ_x). Those coordinates won't do, but we can straighten them into doing what we want. Define:
$$\check{\rho}_x^u(z) := \sum_{n=1}^{\infty} \left(\ln |\pi_y \partial_y \check{f}_{f^{-n}(x)}(\lambda_x^{\langle -n \rangle} z, 0)| - \ln |\mu_{f^{-n}(x)}| \right)$$
and
$$\check{\rho}_x^s(y) := \sum_{n=0}^{\infty} \left(\ln |\pi_z \partial_z \check{f}_{f^n(x)}(0, \mu_x^{\langle n \rangle} y)| - \ln |\lambda_{f^n(x)}| \right).$$
Finally, set $\check{\mathcal{D}}_x^u(z,y) := (z, y e^{\check{\rho}_x^u(z)})$, $\check{\mathcal{D}}_x^s(z,y) := (z e^{-\check{\rho}_x^s(y)}, y)$, $\check{\mathcal{D}}_x := \check{\mathcal{D}}_x^u \circ \check{\mathcal{D}}_x^s$ and $\iota_x := \check{\iota}_x \circ \check{\mathcal{D}}_x$. Let us check that $f_x := \iota_{f(x)}^{-1} \circ f \circ \iota_x$ satisfies the desired relations. First of all, notice that $\check{\rho}_x^u$ and $\check{\rho}_x^s$ are smooth and satisfy $\check{\rho}_x^u(0) = \check{\rho}_x^s(0) = 0$. In particular, $\check{\mathcal{D}}_x$, $\check{\mathcal{D}}_x^u$ and $\check{\mathcal{D}}_x^s$ are smooth, and coincide with the identity on $\{(z,y)\ ,\ z = 0 \text{ or } y = 0\}$. Moreover,
$$\check{\rho}_{f(x)}^u(\lambda_x z) = \ln |\pi_y \partial_y \check{f}_x(z,0)| - \ln |\mu_x| + \check{\rho}_x^u(z)$$
and
$$\check{\rho}_x^s(y) = \ln |\pi_z \partial_z \check{f}_x(0,y)| - \ln |\lambda_x| + \check{\rho}_{f(x)}^s(\mu_x y).$$
Now let us write f_x in terms of $\check{f}_x$: we have
$$f_x = \iota_{f(x)}^{-1} \circ f \circ \iota_x = (\check{\mathcal{D}}_{f(x)}^s)^{-1} \circ (\check{\mathcal{D}}_{f(x)}^u)^{-1} \circ \check{f}_x \circ \check{\mathcal{D}}_x^u \circ \check{\mathcal{D}}_x^s.$$
Hence:
$$(df_x)_{(z,0)} = d((\mathcal{D}_{f(x)}^s)^{-1})_{(\lambda_x z,0)} \circ d((\mathcal{D}_{f(x)}^u)^{-1})_{(\lambda_x z,0)} \circ (d\check{f}_x)_{(z,0)} \circ (d\mathcal{D}_x^u)_{(z,0)} \circ (d\mathcal{D}_x^s)_{(z,0)}$$
Abusing a bit notations, we can write in matrix form:
$$(df_x)_{(z,0)} = \begin{pmatrix} 1 & (*) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & e^{-\check{\rho}_{f(x)}(\lambda_x z)} \end{pmatrix} \begin{pmatrix} \lambda_x & \pi_z \partial_y \check{f}_x(z,0) \\ 0 & \pi_y \partial_y \check{f}_x(z,0) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & e^{\check{\rho}_x^u(z)} \end{pmatrix} \begin{pmatrix} 1 & (*) \\ 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} \lambda_x & (*) \\ 0 & e^{-\check{\rho}_{f(x)}^u(\lambda_x z) + \check{\rho}_x^u(z)} \pi_y \partial_y \check{f}_x(z,0) \end{pmatrix} = \begin{pmatrix} \lambda_x & (*) \\ 0 & \mu_x \end{pmatrix},$$
which implies in particular that $\pi_y \partial_y f_x(z,0) = \mu_x$. A similar computation shows that
$$(df_x)_{(0,y)} = \begin{pmatrix} \lambda_x & 0 \\ (*) & \mu_x \end{pmatrix}.$$
In particular, $\pi_z \partial_z f_x(0,y) = \lambda_x$. $\square$

Remark 3.7. Notice the following pretty convenient property. As soon as the quantity writen down makes sense, we have the identities:

$$(df_x^{\langle n\rangle})_{(z,0)} = \begin{pmatrix} \lambda_x^{\langle n\rangle} & (*) \\ 0 & \mu_x^{\langle n\rangle} \end{pmatrix} \quad ; \quad (df_x^{\langle n\rangle})_{(0,y)} = \begin{pmatrix} \lambda_x^{\langle n\rangle} & 0 \\ (*) & \mu_x^{\langle n\rangle} \end{pmatrix}$$

where $f_x^{\langle n\rangle} := \iota_{f^n(x)} \circ f^n \circ \iota_x : (-(\rho_1/\rho_0)^n\rho_0, (\rho_1/\rho_0)^n\rho_0)^2 \longrightarrow (-\rho_0, \rho_0)^2$.

This coordinate system is not "canonically attached" to the dynamics, since the behavior of ι_x outside the "cross" $\{(z,y)\ ,\ zy = 0\}$ might be completely arbitrary. But the behavior of those coordinates near the cross seems to give rise to less arbitrary objects. Those objects will be called "templates" in these notes. They are inspired from the "templates" appearing in [TZ20].

Definition 3.8 (Templates, dual version)**.** Let $x \in \Omega$. A template based at x is a continuous 1-form $\xi_x : W^u_{loc}(x) \to \Omega^1(M)$ such that:

$$\forall z \in W^u_{loc}(x),\ \mathrm{Ker}(\xi_x)_z \supset E^u(z).$$

We will denote by $\Xi(x)$ the space of templates based at x.

Remark 3.9. Notice that, since $E^u(z)$ moves smoothly along the unstable local manifold $W^u_{loc}(x)$, it makes sense to consider **smooth** templates. Notice further that, since $(df)_z E^u(z) = E^u(f(z))$, the diffeomorphism f acts naturally on templates by taking the pushforward. This yields a map $f_* : \Xi(x) \to \Xi(f(x))$.

Lemma 3.10 (Some interesting templates.)**.** *There exists a family $(\xi^s_x)_{x\in\Omega}$ of smooth templates, where $\xi^s_x \in \Xi(x)$, that satifies the following invariance relation:*

$$\forall z \in W^u_{loc}(x),\ (f_*(\xi^s_x))_{f(z)} = \mu_x^{-1}(\xi^s_x)_{f(z)}.$$

(Moreover, the dependence in x of (ξ^s_x) is Hölder.)

Proof. For all $x \in \Omega$, define $\xi^s_x := (\iota_x)_*(dy)$. It is clear that this defines a smooth template at x. Furthermore, using remark 3.7 for $n = -1$:

$$f_*\xi^s_x = f_*((\iota_x)_*(dy)) = (\iota_{f(x)})_*((f_x)_*(dy)) = (\iota_{f(x)})_*(d(\pi_y f_x^{-1})) = (\iota_{f(x)})_*(\mu_x^{-1}dy) = \mu_x^{-1}\xi^s_{f(x)}.$$

The dependence is Hölder because (ι_x) depends on x in a Hölder manner. □

It is natural to try to find a "vector field" version of those templates. We suggest a way to proceed in the following.

Definition 3.11 (Templates, vector version)**.** Let $x \in \Omega$. A (vector) template based at x is a continuous section of the line bundle TM/E^u along $W^u_{loc}(x)$. We will denote by $\Gamma(x)$ the space of (vector) templates at x.

Remark 3.12. If X is some continuous vector field defined along $W^u_{loc}(x)$, we can take its class modulo E^u to get a (vector) template $[X]$. Notice also that it makes sense to talk about smooth (vector) templates. Notice further that f acts naturally on (vector) templates, since $(df)_z E^u(z) = E^u(f(z))$. This define a map $f_* : \Gamma(x) \to \Gamma(f(x))$.

Lemma 3.13 (Some interesting templates.)**.** *There exists a family $([\partial^x_s])_{x\in\Omega}$ of smooth (vector) templates, where $[\partial^x_s] \in \Gamma(x)$, such that:*

$$\forall z \in W^u_{loc}(x),\ (f_*[\partial^x_s])_{f(z)} = \mu_x[\partial^{f(x)}_s]_{f(z)}.$$

(Moreover, the dependence in x of $([\partial^x_s])$ is Hölder.)

Proof. For all $x \in \Omega$, define $\partial^x_s := (\iota_x)_*(\partial/\partial y)$ along $W^u_{loc}(x)$. This is smooth. Moreover, using remark 3.7, we find:

$$f_*\partial^s_x = f_*(\iota_x)_*(\partial_y) = (\iota_{f(x)})_*((f_x)_*\partial_y) = (\iota_{f(x)})_*(\mu_x\partial_y + (*)\partial_z) = \mu_x\partial^{f(x)}_s + (*)\partial_u.$$

Hence, taking the class modulo E^u, we find

$$f_*[\partial^x_s] = \mu_x[\partial^{f(x)}_s]$$

which is what we wanted. The dependence in x is then Hölder because of the properties of ι_x. □

Remark 3.14 (A quick duality remark)**.** We can define a sort of "duality bracket" $\Xi(x) \times \Gamma(x) \to C^0(W^u_{loc}(x), \mathbb{R})$ by the following formula:

$$\langle \xi, [X] \rangle := \xi(X).$$

Our special templates $(\xi^s_x)_{x\in\Omega}$ and (∂^x_s) can be chosen normalised so that $\langle \xi^s_x, \partial^x_s \rangle = 1$. This will not be usefull, but this is an indication that $\Xi(x)$ and $\Gamma(x)$ could countain the same informations.

Remark 3.15 (Templates acting on a space of functions)**.** It is natural to search for a space of functions on which (vector) templates could acts. A way to do it is as follow. For each $x \in \Omega$, define $\mathcal{F}(x)$ as the set of functions h defined on a neighborhood of $W^u_{loc}(x)$ that are C^1 along the stable direction and that vanish along $W^u_{loc}(x)$. In this case, for any point $z \in W^u_{loc}(x)$, we know that $\partial_s h_x(z)$ makes sense, and we know that $\partial_u h_x(z) = 0$ also makes sense. So one can make (vector) templates $[X]$ acts on h by setting:

$$\forall z \in W^u_{loc}(x), \ ([X] \cdot h_x)(z) := (X \cdot h_x)(z).$$

This is well defined. In the particular case where $[X] = [\partial^x_s]$, we get the formula:

$$\forall z \in (-1,1), \ ([\partial^x_s] \cdot h_x)(\Phi^u_x(z)) = \partial_y (h_x \circ \iota_x)(z,0)$$

Notice that f acts naturally on these space of functions, by taking a pullback $f^* : \mathcal{F}(f(x)) \to \mathcal{F}(x)$. If we fix $h_{f(x)} \in \mathcal{F}(f(x))$, and if we set $h_x := h_{f(x)} \circ f \in \mathcal{F}(x)$, notice finally that one can write

$$[\partial^x_s] \cdot h_x = [\partial^x_s] \cdot f^* h_{f(x)} = f_* [\partial^x_s] \cdot h_{f(x)} = \mu_x [\partial^{f(x)}_s] \cdot h_{f(x)}.$$

Lemma 3.16 (Changing basepoint)**.** *Let $x \in \Omega$. For $\tilde{x} \in \Omega \cap W^u_{loc}(x)$, let*

$$H(x,\tilde{x}) = \exp\Big(\sum_{n=0}^{\infty} \big(\ln \mu_{f^{-n}(x)} - \ln \mu_{f^{-n}(\tilde{x})} \big) \Big).$$

Then:

$$\forall z \in W^u_{loc}(x), \quad ([\partial^{\tilde{x}}_s])_z = H(x,\tilde{x}) ([\partial^x_s])_z.$$

Proof. Remember that TM/E^u is a line bundle, and that $[\partial^x_s]$ doesn't vanish. In particular, there exists a function $a_{x,\tilde{x}} : W^u_{loc}(x) \to \mathbb{R}$ such that:

$$\forall z \in W^u_{loc}(x), \quad ([\partial^{\tilde{x}}_s])_z = a_{x,\tilde{x}}(z) ([\partial^x_s])_z.$$

The main point is to show that $a_{x,\tilde{x}}$ is z-constant. Since the familly $([\partial^x_s])$ depends in x in a Hölder manner (and locally uniformly in z), we know that $a_{x,\tilde{x}}(z) = 1 + O(d(x,\tilde{x})^\alpha)$ for some α. The invariance properties of those (vector) templates yields an invariance property for $a_{x,\tilde{x}}(z)$:

$$\forall z \in W^u_{loc}(x), \ a_{x,\tilde{x}}(z) = \frac{\mu_x^{\langle -n \rangle}}{\mu_{\tilde{x}}^{\langle -n \rangle}} a_{f^{-n}(x), f^{-n}(\tilde{x})}(f^{-n}(z)).$$

Taking the limit as $n \to +\infty$ gives the result. □

4 Templates acting on Δ^+.

We return on our study of Δ^+. Recall that $\Delta^+ : \widetilde{\mathrm{Diag}} \to \mathbb{R}$ is defined as

$$\Delta^+(p,q) := \sum_{n=0}^{\infty} T_n(p,q),$$

where $T_n(p,q) := \tau(f^n(p)) - \tau(f^n([p,q])) - \tau(f^n([q,p])) + \tau(f^n(q))$. Let us fix some $p \in \Omega$, and set:

$$\Delta^+_p(q) := \Delta^+(p,q) \quad , \quad T_{p,n}(q) := T_n(p,q).$$

For each p and n, $T_{p,n}$ is $C^{1+\alpha}$, and moreover taking the derivative along the local stable lamination yields: $\partial_s T_{p,n}(q) = -(\partial_s \tau)(f^n([p,q])) |\mu^{\langle n \rangle}_{[p,q]}| \partial_s \pi_p(q) + \partial_s \tau(f^n(q))) |\mu^{\langle n \rangle}_q|$, where $\pi_p(q) := [p,q]$. It follows that Δ_p^+ is C^1 along the local stable lamination. Moreover, $T_{p,n}$ vanish on $W^u_{loc}(p)$, and so does Δ_p^+. It follows that

$$\Delta_p^+ \in \mathcal{F}(p) \quad , \quad T_{p,n} \in \mathcal{F}(p),$$

where $\mathcal{F}(p)$ denotes the space of function defined in remark 3.15. This ensure that the next definition makes sense.

Definition 4.1. For each $x \in \Omega$, for each $z \in \Omega_x^u \subset \mathbb{R}$, define $X_x \in C^\alpha((-\rho_0, \rho_0), \mathbb{R})$ by:

$$X_x(z) := ([\partial_s^x] \cdot \Delta_x^+)(\Phi_x^u(z)).$$

The family $(X_x)_{x\in\Omega}$ depends on x in a Hölder manner.

Lemma 4.2 (autosimilarity)**.** *We have $X_x(0) = 0$. Moreover, the family $(X_x)_{x\in\Omega}$ satisfies the following autosimilarity relation:*

$$\forall z \in \Omega_x^u, \ X_x(z) = \widehat{\tau}_x(z) + \mu_x X_{f(x)}(\lambda_x z),$$

where $\widehat{\tau}_x(z) := ([\partial_x^s] \cdot T_{x,0})(\Phi_x^u(z)) \in C^\alpha$.

Proof. Notice that $T_{p,n+1}(q) = T_{f(p),n}(f(q))$. It follows that:

$$\Delta_p^+(q) = T_{p,0}(q) + \sum_{n=0}^{\infty} T_{f(p),n}(f(q)) = T_{p,0}(q) + \Delta_{f(p)}^+(f(q)).$$

Making the vector template $[\partial_s^p]$ acts on this along $W^u_{loc}(p)$ yields (using the invariance properties of the family $([\partial_s^x])$):

$$([\partial_s^p] \cdot \Delta_p^+) = ([\partial_s^p] \cdot T_{p,0}) + ([\partial_s^p] \cdot (\Delta_{f(p)}^+ \circ f)) = ([\partial_s^p] \cdot T_{p,0}) + \mu_p([\partial_s^{f(p)}] \cdot \Delta_{f(p)}^+) \circ f.$$

Testing this equality on $\Phi_p^u(z)$ gives the desired equality, since $f \circ \Phi_p^u = \Phi_{f(p)}^u \circ (\lambda_p I_d)$. $\square$

Lemma 4.3 (regularity of $\widehat{\tau}_x$)**.** *The function $\widehat{\tau}_x$ is smoother than expected: it is $C^{1+\alpha}((-\rho_0, \rho_0), \mathbb{R})$. It vanish at $z = 0$, and its derivative at zero is:*

$$(\widehat{\tau}_x)'(0) = \partial_z \partial_y (\tau \circ \iota_x)(0, 0) + n_x'(0) \partial_z (\tau \circ \iota_x)(0, 0),$$

where $n_x(z) \in C^{1+\alpha}$ is defined such that $\partial_y + n_x(z)\partial_z \in \iota_x^{-1}(E^s)$ points in the stable direction at coordinates $(z, 0)$.

Proof. Let us do an explicit computation of $\widehat{\tau}_x$. By definition of $[\partial_s^x]$:

$$\widehat{\tau}_x(z) = ([\partial_s^x] \cdot T_{x,0})(\Phi_x^u(z)) = \partial_y (T_{x,0} \circ \iota_x)(z, 0).$$

Recall that $T_{x,0}(\iota_x(z, y)) = \tau(\iota_x(z, y)) - \tau([x, \iota_x(z, y)]) - \tau([\iota_x(z, y), x]) + \tau(x) \in C^{1+\alpha}$. Define

$$\pi_x^s(z, y) := \iota_x^{-1}([x, \iota_x(z, y)]) \in \{(0, y'), y' \in (-\rho_0, \rho_0)\}$$

and

$$\pi_x^u(z, y) := \iota_x^{-1}([\iota_x(z, y), x]) \in \{(z', 0), z' \in (-\rho_0, \rho_0)\}.$$

Define also $\tau_x := \tau \circ \iota_x$. Then:

$$T_{x,0}(\iota_x(z, y)) = \tau_x(z, y) - \tau_x(\pi_x^u(z, y)) - \tau_x(\pi_x^s(z, y)) + \tau_x(0).$$

For each point $z \in (-\rho_0, \rho_0)$, let $\vec{N}_x(z)$ be a vector pointing along the direction $\iota_x^{-1}(E^s)$, and normalize it so that $\vec{N}_x(z) = \partial_y + n_x(z)\partial_z$. By regularity of E^s, we can choose $\vec{N}_x(z)$ to be $C^{1+\alpha}$ in z. We can then, for each x, z, find a (smooth) path $t \mapsto \gamma_x(z, t)$ such that $\pi_x^u \circ \gamma_x(z, t) = (z, 0)$ and such that $\partial_t \gamma_x(z, 0) = \vec{N}_x(z)$. (We just follow the stable lamination in coordinates.) Using this path, we can compute the derivative of $T_{x,0} \circ \iota_x$ as follow:

$$\partial_y (T_{x,0} \circ \iota_x)(z, 0) = \Big((\partial_y + n_x(z)\partial_z) \cdot (T_{x,0} \circ \iota_x) \Big)(z, 0) = \frac{d}{dt} T_{x,0}(\iota_x(\gamma_x(z, t)))_{|t=0}$$

$$
\begin{aligned}
&= \frac{d}{dt}_{|t=0}\Big(\tau_x(\gamma_x(z,t)) - \tau_x(\pi^u_x(\gamma_x(z,t))) - \tau_x(\pi^s_x(\gamma_x(z,t))) + \tau_x(0)\Big)\\
&= \partial_y\tau_x(z,0) + n_x(z)\partial_z\tau_x(z,0) - (d\tau_x)_{(0,0)} \circ (d\pi^s_x)_{(z,0)}(\vec{N}_x(z))\\
&= \partial_y\tau_x(z,0) + n_x(z)\partial_z\tau_x(z,0) - \partial_y\tau_x(0,0)\pi_y(\partial_y\pi^s_x(z,0)),
\end{aligned}
$$

since $(d\pi^s_x)_{(z,0)}(\partial_z) = 0$, as $\pi^s_x(z,0) = (0,0)$, and since $\pi_z \circ \pi^s_x = 0$. In this expression, everything is $C^{1+\alpha}$; except eventually the last term $\eta_x(z) := \pi_y\partial_y\pi^s_x(z,0)$. Let us prove that $\eta_x(z)$ is, in fact, constant and equal to one. First of all, the maps (η_x) are at least continuous (and this, uniformly in x). Moreover, we have $\pi_y\pi^s_x(0,y) = y$, and hence $\eta_x(0) = 1$. To conclude, let us use the fact that the stable lamination is f invariant. This remark, written in coordinates, yields (as soon as the relation makes sense):

$$\pi^s_x = f^{\langle n\rangle}_{f^{-n}(x)} \circ \pi^s_{f^{-n}(x)} \circ f^{\langle -n\rangle}_x.$$

Taking the differential at $(z,0)$ yields, using remark 3.7:

$$\forall n \geq 0, \forall z \in (-\rho_0,\rho_0),\ (d\pi^s_x)_{(z,0)} = (df^{\langle n\rangle}_{f^{-n}(x)})_{(0,0)} \circ (d\pi^s_{f^{-n}(x)})_{(\lambda^{\langle -n\rangle}_x z,0)} \circ (df^{\langle -n\rangle}_x)_{(z,0)}$$

$$= \begin{pmatrix} \lambda^{\langle n\rangle}_{f^{-n}(x)} & 0 \\ 0 & \mu^{\langle n\rangle}_{f^{-n}(x)} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & \eta_{f^{-n}(x)}(\lambda^{\langle -n\rangle}_x z) \end{pmatrix} \begin{pmatrix} \lambda^{\langle -n\rangle}_x & (*) \\ 0 & \mu^{\langle -n\rangle}_x \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \eta_{f^{-n}(x)}(\lambda^{\langle -n\rangle}_x z) \end{pmatrix}$$

It follows that:

$$\forall z, \forall n \geq 0,\ \eta_x(z) = \eta_{f^{-n}(x)}(\lambda^{\langle -n\rangle}_x z) \underset{n\to\infty}{\longrightarrow} 1.$$

In conclusion, we get the following expression for $\widehat{\tau}_x$:

$$\widehat{\tau}_x(z) = \partial_y\tau_x(z,0) + n_x(z)\partial_z\tau_x(z,0) - \partial_y\tau_x(0,0),$$

which is a $C^{1+\alpha}$ function that vanish at zero. Let us compute its derivative at zero: we have

$$(\widehat{\tau}_x)'(0) = \partial_z\partial_y\tau_x(0,0) + n'_x(0)\partial_z\tau_x(0,0) + n_x(0)\partial^2_z\tau_x(0,0).$$

The fact that $n_x(0) = 0$ gives us the desired formula. □

Remark 4.4. Recall that, in our area-preserving context, there exists a Hölder map $h : \Omega \to \mathbb{R}$ such that $\lambda_x\mu_x = \exp(h(f(x))-h(x))$. Fix one such h for the rest of the paper. For each $\tau \in C^{2+\alpha}(\Omega,\mathbb{R})$, let us denote by $\Phi_\tau \in C^\alpha(\Omega,\mathbb{R})$ the map defined by

$$\Phi_\tau : x \in \Omega \longmapsto (\widehat{\tau}_x)'(0)e^{h(x)} \in \mathbb{R}.$$

The linear map $\tau \in C^{2+\alpha} \mapsto \Phi_\tau \in C^\alpha$ is obviously continuous. Moreover, it is easy to see that, genericaly in $\tau \in C^{2+\alpha}$, Φ_τ is not cohomologous to zero. Indeed, one can take a fixed point p_0 (or a periodic ordit) and look at the value of $\Phi_\tau(p_0)$: if it is zero, then it is easy to $C^{2+\alpha}-$perturb τ on a neighborhood of p_0 so that $\Phi_\tau(p_0)$ becomes non-vanishing. In the following of these notes, the potential $\Phi_\tau : \Omega \to \mathbb{R}$ will have the same kind of role for us as a "longitudinal KAM cocycle" would. (See [FH03] for details on this notion.)

Lemma 4.5 (Change of basepoint)**.** *Let $x \in \Omega$. Let $\tilde{x} \in W^u_{loc}(x)$ be close enough to x. Then, there exists $Aff_{x,\tilde{x}} : \mathbb{R} \to \mathbb{R}$, an (invertible) affine map, such that:*

$$Aff_{x,\tilde{x}}\Big(X_{\tilde{x}}(z)\Big) = X_x(aff_{x,\tilde{x}}(z)),$$

where $aff_{x,\tilde{x}} = (\Phi^u_x)^{-1} \circ \Phi^u_{\tilde{x}}$ is the affine change of charts defined in lemma 3.5. Moreover, there exists $C \geq 1$ and $\alpha > 0$ such that $\ln|{Aff_{x,\tilde{x}}}'(0)| \leq Cd(x,\tilde{x})^\alpha$.

Proof. Let $x \in \Omega$ and let $\tilde{x} \in W^u_{loc}(x)$ be close enough to x. We have, for p in a neighborhood of $\tilde{x}$:

$$\Delta^+(\tilde{x},q) = \Delta^+(\tilde{x},[x,q]) + \Delta^+(x,q).$$

We differentiate (w.r.t. q) with the vector template $[\partial^{\tilde{x}}_s]$ along $W^u_{loc}(\tilde{x})$ to find:

$$X_{\tilde{x}}(z) = [\partial^{\tilde{x}}_s] \cdot (\Delta^+_{\tilde{x}} \circ [x,\cdot])(\Phi^u_{\tilde{x}}(z)) + ([\partial^{\tilde{x}}_s] \cdot \Delta^+_x)(\Phi^u_{\tilde{x}}(z)).$$

The first thing to recall is that $\Phi^u_{\tilde{x}} = \Phi^u_x \circ (\Phi^u_x)^{-1} \circ \Phi^u_{\tilde{x}} = \Phi^u_x \circ \mathrm{aff}_{x,\tilde{x}}$, and moreover, by lemma 3.16, $[\partial^{\tilde{x}}_s] = H(x,\tilde{x})[\partial^x_s]$. From this, we see that the last term is $H(x,\tilde{x})X_x(\mathrm{aff}_{x,\tilde{x}}(z))$. To conclude, we only need to show that

$$[\partial^{\tilde{x}}_s] \cdot (\Delta^+_{\tilde{x}} \circ [x,\cdot]) = H(x,\tilde{x})[\partial^x_s] \cdot (\Delta^+_{\tilde{x}} \circ [x,\cdot])$$

is constant along $W^u_{loc}(x)$. In coordinates, we see that:

$$[\partial^x_s] \cdot (\Delta^+_{\tilde{x}} \circ [x,\cdot])(\Phi_x(z)) = \partial_y(\Delta^+_{\tilde{x}} \circ [x,\cdot] \circ \iota_x)(z,0) = \partial_y(\Delta^+_{\tilde{x}} \circ \iota_x \circ \pi^s_x)(z,0),$$

where π^s_x is defined in the proof of lemma 4.3. Recall from this proof that we have

$$(d\pi^s_x)_{(z,0)} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

is constant in z (and in x). It follows that: $[\partial^x_s] \cdot (\Delta^+_{\tilde{x}} \circ [\cdot,x])(\Phi_x(z)) = \partial_y(\Delta^+_{\tilde{x}} \circ \iota_x)(0,0)$, which is a constant expression in z. The proof is done. $\square$

We conclude this section by showing that one can reduce the study of the oscillations of Δ^+_x to the study of the oscillations of X_x. The proof is in two parts: we first establish a proper asymptotic expansion for Δ^+_p, and then we reduce (NC) to a statement about $(X_x)_{x\in\Omega}$.

Theorem 4.6. *Let $p,q \in \Omega$ be close enough. We will denote $\pi^S_p(q) := [p,q] =: s \in \Omega \cap W^s_{loc}(p)$, and $\pi^U_p(q) := [q,p] =: r \in \Omega \cap W^u_{loc}(p)$. We have $q = [r,s]$. These "coordinates" are $C^{1+\alpha}$. Suppose that $d^s(p,s) \le \sigma^{\beta_Z}$ and $d^u(p,r) \le \sigma$ for $\sigma > 0$ small enough. If $\beta_Z > 1$ is fixed large enough, then the following asymptotic expansion hold:*

$$\Delta^+_p([r,s]) = \pm \partial_s \Delta^+_p(r) d^s([r,s],r) + O(\sigma^{1+\beta_Z+\alpha}),$$

where ∂_s denotes the derivative in the stable direction.

Proof. Let us introduce some notations. Define, for any $p \in \Omega$, the $C^{2+\alpha}$ map $\nabla_p : W^s_{loc}(p) \to \mathbb{R}$ as

$$\nabla_p(s) := \sum_{n=0}^{\infty} \left(\tau(f^n(p)) - \tau(f^n(s))\right),$$

and notice that

$$\Delta^+_p(q) = \nabla_p(s) - \nabla_r(q).$$

A Taylor expansion yields:

$$\begin{aligned}\nabla_p(s) &= \nabla_p(p) \pm \partial_s \nabla_p(p) d^s(p,s) + \frac{1}{2}\partial_s^2 \nabla_p(p) d^s(p,s)^2 + O(d^s(p,s)^{2+\alpha}) \\ &= \pm\partial_s \nabla_p(p) d^s(p,s) + \frac{1}{2}\partial_s^2 \nabla_p(p) d^s(p,s)^2 + O(\sigma^{(2+\alpha)\beta_Z}) \\ &\pm\partial_s \nabla_p(p) d^s(p,s) + \frac{1}{2}\partial_s^2 \nabla_p(p) d^s(p,s)^2 + O(\sigma^{1+\alpha+\beta_Z})\end{aligned}$$

since $\beta_Z > 1$. Hence,

$$\begin{aligned}\Delta^+_p(q) &= \pm\left(\partial_s \nabla_p(p) d^s(p,s) - \partial_s \nabla_r(r) d^s(r,s)\right) \\ &+\frac{1}{2}\left(\partial_s^2 \nabla_p(p) d^s(p,s)^2 - \partial_s^2 \nabla_r(r) d^s(r,q)^2\right) + O(\sigma^{1+\alpha+\beta_Z}).\end{aligned}$$

Then, notice the following. The function

$$(r,s) \in \Omega^2 \cap (W^u_{loc}(p) \times W^s_{loc}(p)) \mapsto \frac{d^s([r,s],r)}{d^s(s,p)} \in \mathbb{R}$$

is $C^{1+\alpha}$. Indeed, in our 2 dimensionnal hyperbolic context, the bracket $[\cdot,\cdot]$ is (since the holonomies are $C^{1+\alpha}$), and the distance "arclenght" functions d^s have the same regularity than E^s, which is also $C^{1+\alpha}$. A Taylor expansion in the s variable around p yields:

$$\frac{d^s(q,r)}{d^s(s,p)} = \partial_s \pi^S_p(r) + O(d^s(s,p)^\alpha) = \partial_s \pi^S_p(r) + O(\sigma^{\alpha\beta_Z}).$$

If β_Z is chosen so large that $\alpha\beta_Z > 1+\alpha$, then we find the expansion:

$$\frac{d^s(q,r)}{d^s(s,p)} = \partial_s \pi_p^S(r) + O(\sigma^{1+\alpha}).$$

Next, notice that $r \in W^u_{loc}(p) \mapsto \partial_s \pi_p^S(r)$ is actually $C^{1+\alpha}$. This can be seen as follow. We at least know that this function is Hölder, and moreover, that $\ln \partial_s \pi_p^S(q) = O(d^u(p,q)^\alpha)$. Invariance by the dynamics yields, if q is close enough to $W^u_{loc}(p)$ depending on n :

$$\pi_p^S(q) = f^n \pi_{f^{-n}(p)}(f^{-n}(q)).$$

Taking the derivative in the stable direction, taking the limit as $n \to \infty$ and taking the log yields the formula:

$$\ln \partial_s \pi_p(r) = \sum_{n=0}^{\infty} \left(\ln(\partial_s f)(f^{-n}p) - \ln(\partial_s f)(f^{-n}r) \right).$$

This expression is clearly $C^{1+\alpha}$ on $r \in W^u_{loc}(p)$. A corollary of all this discussion is the expansion:

$$\frac{d^s(q,r)}{d^s(s,p)} = 1 + O(\sigma)$$

In particular $d^s(p,s)^2 = d^s(q,r)^2(1+O(\sigma)) = d^s(q,r)^2 + O(\sigma^{1+2\beta_Z}) = d^s(q,r)^2 + O(\sigma^{1+\alpha+\beta_Z})$. Hence

$$\partial_s^2 \nabla_p(p) d^s(p,s)^2 - \partial_s^2 \nabla_r(r) d^s(r,q)^2 = \left(\partial_s^2 \nabla_p(p) - \partial_s^2 \nabla_r(r) \right) d^s(r,q)^2 + O(\sigma^{1+\beta_Z+\alpha}).$$

Finally, the map

$$\partial_s^2 \nabla_q(q) := \sum_{n=0}^{\infty} \partial_s^2 \tau(f^n(q))(\partial_s f^n(q))^2 + \partial_s \tau(f^n(q)) \partial_s^2(f^n(q))$$

being Hölder regular, we get $\partial_s^2 \nabla_p(p) - \partial_s^2 \nabla_r(r) = O(\sigma^\alpha)$. In particular, we find that $\left(\partial_s^2 \nabla_p(p) - \partial_s^2 \nabla_r(r) \right) d^s(r,q)^2 = O(\sigma^{2\beta_Z+\alpha}) = O(\sigma^{1+\alpha+\beta_Z})$. All of this discussion gives us the following expansion:

$$\Delta_p^+(q) = \pm(\partial_s \nabla_p(p) d^s(p,s) - \partial_s \nabla_r(r) d^s(r,q)) + O(\sigma^{1+\beta_Z+\alpha}).$$

We want to make $\partial_s \Delta_p^+(r)$ appear. We compute it and find out that

$$\partial_s \Delta_p^+(r) = \partial_s \nabla_p(p) \partial_s \pi_p^S(r) - \nabla_r(r).$$

We can make this term appear in our asymptotic expansion as follow:

$$\begin{aligned}\Delta_p^+(q) &= \pm\Big(\partial_s \nabla_p(p) \frac{d^s(p,s)}{d^s(r,q)} - \partial_s \nabla_r(r) \Big) d^s(r,q) + O(\sigma^{1+\alpha+\beta_Z}) \\ &= \pm\Big(\partial_s \Delta_p^+(s) + \varphi_{p,s}(r) \Big) d^s(r,q) + O(\sigma^{1+\alpha+\beta_Z})\end{aligned}$$

where

$$\varphi_{p,s}(r) := \partial_s \nabla_p(p) \left(\frac{d^s(p,s)}{d^s(r,q)} - \partial_s \pi_p^S(r) \right) = \partial_s \nabla_p(p) \left(\frac{d^s(\pi_p^S(r), \pi_p^S([r,s]))}{d^s(r,[r,s])} - \partial_s \pi_p^S(r) \right).$$

Now recall that we already proved that $\varphi_{p,s}(r) = O(\sigma^{1+\alpha})$. Hence $\varphi_{p,s}(r) d^s(r,q) = O(\sigma^{1+\alpha+\beta_Z})$, and we are done. □

Remark 4.7. Recall again that μ has a local product structure [Cl20], in the following sense. For all $x \in \Omega$, there exists μ_x^u and μ_x^s two measures supported on $U_x := \Omega \cap W^u_{loc}(x)$ and $S_x := \Omega \cap W^s_{loc}(x)$ such that, for all measurable $h : M \to \mathbb{C}$ supported in a small enough neighborhood of x, we have

$$\int_\Omega h d\mu = \int_{U_x} \int_{S_x} h([z,y]) d\mu_x^s(y) d\mu_x^u(z).$$

Notice in particular that, for some rectangle $R_p = [U_p, S_p] \ni p$, and for any borel set $I \subset \mathbb{R}$:

$$\mu(q \in R_p,\ h([q,p]) \in I) = \mu_p^s(S_p)\mu_p^u(z \in U_p, h(z) \in I)).$$

The family of measures $(\mu_x^u)_{x\in\Omega}$ satisfies some invariance properties under f that will prove usefull later. (Written like that, its interesting to see the possible similarities with the idea of templates... Since those measures lives on some local unstable manifold at x.) In the following lemma, we will denote $U_x^\sigma := B(x,\sigma) \cap U_x$ and $S_x^\sigma := B(x,\sigma) \cap S_x$.

Lemma 4.8. *Denote by* $\eta_x := (\Phi_x^u)^* \mu_x^u$*, the measure* μ_x^u *seen in the coordinates* Φ_x^u*. Suppose that the family* $(X_x)_x$ *satisfies the following (uniform) non-concentration estimates: there exists* $\alpha, \gamma, \sigma_0 > 0$ *such that, for all* $x \in \Omega$ *and for all* $0 < \sigma < \sigma_0$*, and for any* $a \in \mathbb{R}$*:*

$$\eta_x(z \in [-\sigma,\sigma],\ |X_x(z) - az| \leq \sigma^{1+\alpha/2}) \leq \sigma^\gamma \cdot \eta_x([-\sigma,\sigma]).$$

Then (NC) holds.

Proof. Recall that, by lemma 2.5, to check (NC), it suffices to establish the following bound:

$$\mu\left(q \in R_p^\sigma, |\Delta_p^+(q)| \leq \sigma^{1+\alpha+\beta Z}\right) \leq \sigma^\gamma \mu(R_p^\sigma)$$

where the bound is uniform in p, and where $R_p^\sigma = [U_p^{(\sigma)}, S_p^{(\sigma^{\beta Z})}]$ is a rectangle with p in its center of stable (resp. unstable) diameter $\sigma^{\beta Z}$ (resp. σ). Let us check this estimate by using the Taylor expansion of Δ_p^+. We can write, using the local product structure of μ:

$$\mu\left(q \in R_p^\sigma, |\Delta_p^+(q)| \leq \sigma^{1+\alpha+\beta Z}\right) = \int_{S_p^{(\sigma^{\beta Z})}} \mu_p^u\left(r \in U_p^\sigma\ ,\ |\Delta_p^+([r,s])| \leq \sigma^{1+\alpha+\beta Z}\right) d\mu_p^s(s)$$

$$\leq \int_{S_p^{(\sigma^{\beta Z})}} \mu_p^u\left(r \in U_p^\sigma,\ |\partial_s \Delta_p^+(r) d^s([r,s],r)| \leq C\sigma^{1+\beta Z+\alpha}\right) d\mu_p^s(s).$$

It is easy to see, using Gibbs estimates, that there exists $\delta_{reg} > 0$ such that

$$\mu_p^s(B(p,\sigma^{\beta Z+\alpha/2}) \cap S_p^{(\sigma^{\beta Z})}) \leq \sigma^{\alpha\delta_{reg}/2}\mu_p^s(S_p^{\sigma^{\beta Z}}).$$

It follows that one can cut the integral over S_p in two parts: the part where r is $\sigma^{\beta Z+\alpha/2}$-close to p, and the other part. We get, using the aforementienned regularity estimates:

$$\mu\left(q \in R_p^\sigma, |\Delta_p^+(q)| \leq \sigma^{1+\alpha+\beta Z}\right)$$

$$\leq \sigma^{\delta_{reg}\alpha/2}\mu(R_p^\sigma) + \int_{S_p^{(\sigma^{\beta Z})}} \mu_p^u\left(r \in U_p^\sigma,\ |\partial_s \Delta_p^+(r)| \leq C\sigma^{1+\alpha/2}\right) d\mu_p^s(s). \quad (*)$$

We just have to control the integral term to conclude. To do so, notice that, for all s, we can write:

$$\mu_p^u\left(r \in U_p^\sigma,\ |\partial_s \Delta_p^+(r)| \leq C\sigma^{1+\alpha/2}\right)$$

$$= \left((\Phi_p^u)^*\mu_p^u\right)\left(z \in (\Phi_p^u)^{-1}(U_p^\sigma),\ |\partial_s \Delta_p^+(\Phi_p(z))| \leq C\sigma^{1+\alpha/2}\right)$$

$$\leq \left((\Phi_p^u)^*\mu_p^u\right)\left(z \in [-C\sigma, C\sigma],\ |\partial_s \Delta_p^+(\Phi_p(z))| \leq C\sigma^{1+\alpha/2}\right).$$

Now, since TM/E^u is a line bundle, and since ∂_s and ∂_s^p are $C^{1+\alpha}$, there exists a nonvanishing $C^{1+\alpha}$ function $a_p(z)$ such that $[\partial_s]_{\Phi_p^u(z)} = a_p(z)[\partial_s^p]_{\Phi_p^u(z)}$. We have $a_p(z) = e^{O(1)}$. Hence:

$$\left((\Phi_p^u)^*\mu_p^u\right)\left(z \in [-C\sigma, C\sigma],\ |\partial_s \Delta_p^+(\Phi_p(z))| \leq C\sigma^{1+\alpha/2}\right)$$

$$= \left((\Phi_p^u)^*\mu_p^u\right)\left(z \in [-C\sigma, C\sigma],\ |a_p(z)X_p(z)| \leq C\sigma^{1+\alpha/2}\right)$$

$$\left((\Phi_p^u)^*\mu_p^u\right)\left(z \in [-C\sigma, C\sigma],\ |X_p(z)| \leq C'\sigma^{1+\alpha/2}\right)$$

$$\leq C''\left((\Phi_p^u)^*\mu_p^u\right)([C'\sigma, C'\sigma])\,\sigma^\gamma,$$

where the last control is given by the nonconcentration hypothesis made on $(X_x)_{x\in\Omega}$. To conclude, notice that by regularity of the parametrizations Φ_p^u, and since the measure μ_x^u is doubling (thus constants can be neglected), we get $\left((\Phi_p^u)^*\mu_p^u\right)([-C\sigma, C\sigma]) \leq C\mu_p^u(U_p^\sigma)$. Injecting this estimate in $(*)$ yields

$$\mu\left(q \in R_p^\sigma, |\Delta_p^+(q)| \leq \sigma^{1+\beta_Z+\alpha}\right) \leq C\left(\sigma^{\delta_{reg}\alpha/2} + \sigma^\gamma\right)\mu(R_p^\sigma),$$

which is what we wanted. $\square$

We see that we are reduced to understand oscillations of $z \mapsto X_x(z)$ (modulo linear maps: this wasn't necessary here, but this claim will be natural after reading the next section). The next section will be devoted to proving a "blowup" result on the family $(X_x)_{x\in\Omega}$, which will help us understand deeper the oscillations of those functions. This "blowup" result will allow us to exhibit a rigidity phenomenon. The final section is devoted to proving the non-concentration estimates in the hypothesis of lemma 4.8, under our generic condition defined in remark 4.4.

5 Autosimilarity, polynomials, and rigidity

Let us recall our setting. We are given a family of Hölder maps $(X_x)_{x\in\Omega}$, where $X_x : \Omega_x^u \cap (-\rho,\rho) \longrightarrow \mathbb{R}$ is defined only on a (fractal) neighborhood of zero and vanish at $z = 0$. Recall that $\Omega_x^u := (\Phi_x^u)^{-1}(\Omega) \ni 0$. We have an autosimilarity relation: for any $x \in \Omega$, and any $z \in \Omega_x^u \cap (-\rho,\rho)$, we have

$$X_x(z) = \widehat{\tau}_x(z) + \mu_x X_{f(x)}(z\lambda_x),$$

where $\widehat{\tau}_x : \Omega_x^u \cap (-\rho,\rho) \to \mathbb{R}$ is a $C^{1+\alpha}$ map (in the sense of Whitney) that vanish at zero. Recall that, $C^{2+\alpha}$-generically in the choice of τ, we can suppose that the function

$$\Phi_\tau : x \in \Omega \mapsto (\widehat{\tau}_x)'(0)e^{h(x)} \in \mathbb{R}$$

is not cohomologous to zero (where $h : \Omega \to \mathbb{R}$ is such that $\mu_x\lambda_x = \exp(h(f(x)) - h(x))$). Let us call this cohomology condition (C). We will establish quantitative estimates on the oscillations of (X_x) under the cohomology condition (C). To do so, we start by proving a "blowup" result, directly inspired/taken from Appendix B in [TZ20]. The point of this lemma is to only keep, in the autosimilarity formula of $(X_x)_{x\in\Omega}$, the germ of $\widehat{\tau}_x$ (in the form of its Taylor expansion at zero at some order). Depending on the contraction/dilation rate on the dynamics, the order of this Taylor expansion is different: in our area-preserving case, it is enough to approximate $\widehat{\tau}_x(z)$ by $(\widehat{\tau}_x)'(0)z$.

Lemma 5.1 (Blowup ?). *There exists two families of functions $(Y_x)_{x\in\Omega}$, $(Z_x)_{x\in\Omega}$ such that:*

- *For all $x \in \Omega$ and $z \in \Omega_x^u \cap (-\rho,\rho)$,*

$$X_x(z) = Y_x(z) + Z_x(z).$$

- *The map $Y_x : \Omega_x^u \cap (-\rho,\rho) \to \mathbb{R}$ is $C^{1+\alpha}$, and there exists $C \geq 1$ such that, for all $x \in \Omega$ and $z \in \Omega_x^u \cap (-\rho,\rho)$:*

$$|Y_x(z)| \leq C|z|^{1+\alpha}.$$

- *The family $(Z_x)_{x\in\Omega}$ satisfies an autosimilarity formula: for any $x \in \Omega$, $z \in \Omega_x^u \cap (-\rho,\rho)$*

$$Z_x(z) = (\widehat{\tau}_x)'(0)z + \mu_x Z_{f(x)}(z\lambda_x).$$

Moreover, the dependence in x of $(X_x)_{x\in\Omega}$, $(Y_x)_{x\in\Omega}$ and $(Z_x)_{x\in\Omega}$ is Hölder.

Proof. In the original proof, there is an implicit argument used, which is the fact that polynomials (of order one, here) are maps with vanishing (second order) derivative. In our fractal context, this is not true, as Ω_x^u may not be connected: so we have to replace this derivative with a notion adapted to our fractal context. For $\beta < \alpha$, define a "$(1+\beta)$-order fractal derivative" as follows: if $h : \Omega_x^u \cap (-\rho,\rho) \to \mathbb{R}$ is $C^{1+\alpha}$ in the sense of Whitney, then its Taylor expansion at zero makes sense, and we can consider the function:

$$\delta^{1+\beta}(h)(z) := \frac{h(z) - h(0) - h'(0)z}{z^{1+\beta}}.$$

This is a continuous function on $\Omega_x^u \cap (-\rho, \rho)$, and it is bounded and vanish at zero at order $|z^{(\alpha-\beta)-}|$. Moreover, notice that $\delta^{1+\beta}(h) = 0$ is equivalent to saying that h is affine. Notice further that

$$\delta^{1+\beta}\Big(\mu h(\lambda\cdot)\Big)(z) = (\mu\lambda^{1+\beta}) \cdot \delta^{1+\beta}(h)(z\lambda).$$

Now, let us begin the actual proof. Consider the autosimilarity equation of (X_x), and formaly take the $(1+\beta)$-th fractal derivative. We search for a $C^{1+\alpha}$ solution (Y_x) of this equation:

$$\delta^{1+\beta}(Y_x)(z) = \delta^{1+\beta}(\widehat{\tau}_x)(z) + \mu_x\lambda_x^{1+\beta} \cdot \delta^{1+\beta}(Y_x)(z\lambda_x).$$

Notice that $\kappa_x := \mu_x\lambda_x^{1+\beta}$ behaves like a greater-than-one multiplier. Indeed, if we denote, for $x \in \Omega$ and $n \in \mathbb{Z}$,

$$\kappa_x^{\langle n\rangle} := \kappa_x \dots \kappa_{f^{n-1}(x)}$$

(if $n \geq 0$, and similarly if $n \leq 0$ as in the definition of $\lambda_x^{\langle n\rangle}$), we see that $\kappa_x^{\langle n\rangle} \geq (\lambda_-^n)^\beta$ where $\lambda_- > 1$. We can wolve this equation by setting

$$\delta^{1+\beta}(Y_x)(z) := -\sum_{n=1}^{\infty} \delta^{1+\beta}(\widehat{\tau}_{f^{-n}(x)})(z\lambda_x^{\langle -n\rangle}) \cdot \kappa_x^{\langle -n\rangle} =: \tilde{Y}_x(z).$$

This is defines a continuous function that vanishes at zero. We then define Y_x as the only $C^{1+\alpha}$ function such that $Y_x(0) = Y_x'(0) = 0$ and $\delta^{1+\beta}(Y_x) = \tilde{Y}_x$. In other words, $Y_x(z) := z^{1+\beta}\tilde{Y}_x(z)$. Using the sum formula of $\tilde{Y}_x$, we find the autosimilarity formula:

$$Y_x(z) = \widehat{\tau}_x(z) - (\widehat{\tau}_x)'(0)z + \mu_x Y_{f(x)}(z\lambda_x).$$

We can then conclude by setting $Z_x := X_x - Y_x$. □

The idea now is to consider the distance from Z_x to the space of affine maps. By the autosimilarity formula of (Z_x), there is going to be some invariance that will prove usefull.

Definition 5.2. For any $\rho > 0$ small enough, consider the function $\mathcal{D}_\rho : \Omega \longrightarrow \mathbb{R}_+$ defined as

$$\mathcal{D}_\rho(x) := \inf_{a,b\in\mathbb{R}} \sup_{z\in\Omega_x^u\cap(-\rho,\rho)} |Z_x(z) - az - b|.$$

This function is continuous, since $x \in \Omega \mapsto Z_x \in C^0$ is, and since we are computing a distance to a finite-dimensional vector space.

Lemma 5.3. *We have the following criterion. Are equivalent, for some fixed $x \in \Omega$ and $\rho > 0$:*

- $\mathcal{D}_\rho(x) = 0$
- *For all $n \geq 0$, $Z_{f^{-n}(x)} \in C^{1+\alpha}\big(\Omega_{f^{-n}(x)}^u \cap (-\rho\lambda_x^{\langle -n\rangle}, \rho\lambda_x^{\langle -n\rangle}), \mathbb{R}\big)$, and there exists $C \geq 1$ such that, for all $n \geq 0$,*

$$\|\delta^{1+\alpha}(Z_{f^{-n}(x)})\|_{L^\infty(\Omega_{f^{-n}(x)}^u\cap(-\rho\lambda_x^{\langle -n\rangle},\rho\lambda_x^{\langle -n\rangle}))} \leq C.$$

Proof. Suppose that $\mathcal{D}_\rho(x) = 0$. Since $Z_x(0) = 0$, there exists $a \in \mathbb{R}$ such that $Z_x(z) = az$ on $(-\rho,\rho)$. The autosimilarity relation $Z_x(z) = (\widehat{\tau}_x)'(0)z + \mu_x Z_{f(x)}(\lambda_x z)$ gives, with a change of variable,

$$Z_{f^{-1}(x)}(z\lambda_x^{\langle -1\rangle})\mu_x^{\langle -1\rangle} = (\widehat{\tau}_{f^{-1}(x)})'(0)\lambda_x^{\langle -1\rangle}\mu_x^{\langle -1\rangle}z + Z_x(z).$$

Iterating this yields

$$Z_{f^{-n}(x)}(z\lambda_x^{\langle -n\rangle})\mu_x^{\langle -n\rangle} = \text{linear} + Z_x(z).$$

In particular, if Z_x is linear on $\Omega_x^u \cap (-\rho,\rho)$, then $Z_{f^{-n}(x)}$ is linear on $\Omega_{f^{-n}(x)}^u \cap (-\rho\lambda_x^{\langle -n\rangle}, \rho\lambda_x^{\langle -n\rangle}))$. In particular, it is $C^{1+\alpha}$ and the bound on $\delta^{1+\alpha}(Z_{f^{-n}(x)}) = 0$ holds. Reciprocally, if the second point hold, then we can write, on $\Omega_x^u \cap (-\rho,\rho)$:

$$|\delta^{1+\alpha}(Z_x)(z)| = |\mu_x^{\langle -n\rangle}(\lambda_x^{\langle -n\rangle})^{1+\alpha} \cdot \delta^{1+\alpha}(Z_{f^{-n}(x)})(z\lambda_x^{\langle -n\rangle})| \leq C'(|\lambda_x|^{\langle -n\rangle})^\alpha \underset{n\to\infty}{\longrightarrow} 0,$$

where we used the fact that $\mu_x^{\langle n\rangle}\lambda_x^{\langle n\rangle} = e^{O(1)}$ by our area-preserving hypothesis made on the dynamics f. Hence $\delta^{1+\alpha}(Z_x) = 0$ on $\Omega_x^u \cap (-\rho,\rho)$, which means that Z_x is linear on this set. □

Lemma 5.4 (Rigidity lemma). *Suppose that there exists $x_0 \in \Omega$ such that $\mathcal{D}_\rho(x_0) = 0$. Then $\mathcal{D}_\rho = 0$ on Ω.*

Proof. The proof is in three steps. Suppose that $\mathcal{D}_\rho(x_0) = 0$ for some $\rho > 0$ and $x_0 \in \Omega$.

- We first show that there exists $0 < \rho' < \rho$ and a set $\omega \subset W^u_{loc}(x_0) \cap \Omega$ which is an open neighborhood of x for the topology of $W^u_{loc}(x_0) \cap \Omega$, such that $\mathcal{D}_{\rho'}(\tilde{x}) = 0$ if $\tilde{x} \in \omega$.

So suppose that $\mathcal{D}_\rho(x_0) = 0$. Since Z_{x_0} vanish at zero, this means that Z_{x_0} is linear on $(-\rho, \rho) \cap \Omega^u_{x_0}$. In particular, since Y_{x_0} is $C^{1+\alpha}$, we know that X_{x_0} is $C^{1+\alpha}$ on $(-\rho, \rho) \cap \Omega^u_{x_0}$. Now, recall that, by Lemma 4.5, we know that if $\tilde{x} \in W^u_{loc} \cap \Omega$ is close enough to x, we can write

$$\mathrm{Aff}_{x_0,\tilde{x}}\Big(X_{\tilde{x}}(z)\Big) = X_{x_0}(\mathrm{aff}_{x_0,\tilde{x}}(z)),$$

where $\mathrm{Aff}_{x_0,\tilde{x}}$ and $\mathrm{aff}_{x_0,\tilde{x}}$ are affine function (that gets close to the identity as $\tilde{x} \to x_0$). If follows that $X_{\tilde{x}}$ is $C^{1+\alpha}$ on some (smaller) open neighborhood of zero, $(-\rho', \rho') \cap \Omega^u_{\tilde{x}}$. In particular, $Z_{\tilde{x}}$ is also $C^{1+\alpha}$. Let us show that $\mathcal{D}_{\rho'}(\tilde{x}) = 0$ by checking the criterion given in the previous lemma. We have:

$$Z_{\tilde{x}}(z) = X_{\tilde{x}}(z) - Y_{\tilde{x}}(z) = \mathrm{Aff}^{-1}_{x_0,\tilde{x}}\Big(X_{x_0}(\mathrm{aff}_{x_0,\tilde{x}}(z))\Big) - Y_{\tilde{x}}(z)$$

$$= \mathrm{Aff}^{-1}_{x_0,\tilde{x}}\Big(Z_{x_0}(\mathrm{aff}_{x_0,\tilde{x}}(z))\Big) + \mathrm{Aff}^{-1}_{x_0,\tilde{x}}\Big(Y_{x_0}(\mathrm{aff}_{x_0,\tilde{x}}(z))\Big) - Y_{\tilde{x}}(z)$$

By hypothesis, $\mathrm{Aff}^{-1}_{x_0,\tilde{x}}\Big(Z_{x_0}(\mathrm{aff}_{x_0,\tilde{x}}(z))\Big)$ is affine in z, and so its $(1+\alpha)$-th derivative is zero. We can then write, for all $n \geq 0$ and $z \in (-\rho'\lambda_x^{\langle -n\rangle}, \rho'\lambda_x^{\langle -n\rangle}) \cap \Omega^u_{f^{-n}(x)}$:

$$\delta^{1+\alpha}(Z_{f^{-n}(\tilde{x})})(z) = \alpha_{f^{-n}(x_0),f^{-n}(\tilde{x})}\delta^{1+\alpha}(Y_{f^{-n}(x_0)})(\mathrm{aff}_{f^{-n}(x_0),f^{-n}(\tilde{x})}(z)) - \delta^{1+\alpha}(Y_{f^{-n}(\tilde{x})})(z),$$

where $\alpha_{x_0,\tilde{x}} := (\mathrm{Aff}^{-1}_{\tilde{x},x_0})'(0)(\mathrm{aff}_{\tilde{x},x_0})'(0)^{1+\alpha} = 1 + O(d^u(x_0,\tilde{x}))$. Since $|\delta^{1+\alpha}(Y_x)| \leq \|Y_x\|_{C^{1+\alpha}}$, the criterion applies.

- Second, we show that if $\mathcal{D}_{\rho'}(x) = 0$ for some x and small ρ', then $\mathcal{D}_{\min(\rho'\lambda_x,\rho)}(f(x)) = 0$.

This directly comes from the autosimilarity formula. We have, for $z \in (-\rho,\rho) \cap \Omega^u_x$:

$$Z_x(z) = (\widehat{\tau}_x)'(0)z + \mu_x Z_{f(x)}(z\lambda_x).$$

In particular, if Z_x is linear on $(-\rho', \rho') \cap \Omega^u_x$, then $Z_{f(x)}$ is linear on $(-\rho'\lambda_x, \rho'\lambda_x) \cap (-\rho,\rho) \cap \Omega^u_{f(x)}$.

- We conclude, using the transitivity of the dynamics and the continuity of $\mathcal{D}_\rho$.

We know that $\mathcal{D}_\rho(x_0) = 0$, by hypothesis. By step one, there exists ω, some unstable neighborhood of x_0, and $\rho' < \rho$ such that $\mathcal{D}_{\rho'} = 0$ on ω. Step 2 then ensures that $\mathcal{D}_{\min(\rho'\lambda_x^{\langle n\rangle},\rho)} = 0$ on $f^n(\omega)$. Choosing N large enough, we conclude that

$$\forall x \in \bigcup_{n\geq N} f^n(\omega),\ \mathcal{D}_\rho(x) = 0.$$

Since the dynamics f is transitive on Ω, we know that $\bigcup_{n\geq N} f^n(\omega)$ is dense in Ω. The function $\mathcal{D}_\rho$ being continuous, it follows that $\mathcal{D}_\rho = 0$ on Ω. □

Lemma 5.5 (Oscillations everywhere in x). *Under the $C^{2+\alpha}$-generic condition $\Phi_\tau \nsim 0$, the following hold. There exists $\kappa \in (0, \rho/10)$ such that, for all $x \in \Omega$, for all $a, b \in \mathbb{R}$, there exists $z_0 \in \Omega^u_x \cap (-\rho/2, \rho/2)$ such that*

$$\forall z \in \Omega^u_x \cap (z_0 - \kappa, z_0 + \kappa),\ |Z_x(z) - az - b| \geq \kappa.$$

Proof. Our previous lemma gives us the following dichotomy: either $\mathcal{D}_\rho > 0$ on Ω, or either $\mathcal{D}_\rho = 0$ on Ω. Suppose the later. In this case, for all x, $Z_x \in C^{1+\alpha}$. Write the autosimilarity relation and take the (usual) first derivative in z. We find:

$$Z'_x(0) = (\widehat{\tau}_x)'(0) + \mu_x\lambda_x Z'_{f(x)}(0).$$

Recall that, since f is area preserving, we can write $\mu_x \lambda_x = \exp(h(f(x)) - h(x))$ for some Hölder function $h : \Omega \to \mathbb{R}$. Our previous relation can then be rewritten as

$$Z_x'(0) e^{h(x)} = e^{h(x)} (\widehat{\tau}_x)'(0) + Z_{f(x)}'(0) e^{h(f(x))},$$

which implies that $\Phi_\tau \sim 0$. So our generic condition $\Phi_\tau \nsim 0$ ensures that $\mathcal{D}_\rho > 0$ on Ω. By continuity of $\mathcal{D}_\rho$, and by compacity of Ω, there exists some $\kappa > 0$ such that $\mathcal{D}_\rho(x) \geq \kappa$ for all $x \in \Omega$. One can do the same proof replacing ρ with $\rho/2$, so we can directly says that $\mathcal{D}_{\rho/2}(x) \geq \kappa$, taking κ smaller if necessary. Now, this means the following: for every $a, b \in \mathbb{R}$, for every $x \in \Omega$, there exists $z_0(x, a, b) \in \Omega_x^u \cap (-\rho/2, \rho/2)$ such that

$$|Z_x(z_0) - a z_0 - b| \geq \kappa.$$

We still have to show that this doesn't only hold for some point z_0, but on a whole small interval. The proof is different, depending if a, b are small or large.

First of all, since Ω_x^u is perfect (and by compacity of Ω), there exists $\kappa_0 > 0$ such that $\Omega_x^u \cap \{2\kappa_0 \leq |z| < \rho\} \neq \emptyset$. Define $M := \sup_{x \in \Omega} \|Z_x\|_{C^\alpha(\Omega_x^u \cap (-\rho,\rho), \mathbb{R})}$, and then define $\tilde{M} := 4\kappa_0^{-1}(\kappa + M)$. In the case where $|a| \leq \tilde{M}$, then we consider the associated $z_0 \in \Omega_x^u$ from before, and we define $\tilde{\kappa} := \min((\kappa/4M)^{1/\alpha}, \kappa/4\tilde{M})$. We then find:

$$\forall z \in \Omega_x^u \cap (z_0 - \tilde{\kappa}, z_0 + \tilde{\kappa}), \quad |Z_x(z) - az - b|$$
$$= |(Z_x(z_0) - a z_0 - b) + (Z_x(z) - Z_x(z_0)) + a(z_0 - z)|$$
$$\geq \kappa - |Z_x|_{C^\alpha} \tilde{\kappa}^\alpha - a\tilde{\kappa} \geq \kappa/2.$$

If $|a| \geq \tilde{M}$, we look at the value of b. If $|b| \leq |a|\kappa_0/2$, then:

$$\forall z \in \Omega_x^u, |z| \geq \kappa_0, \quad |Z_x(z) - az - b| \geq |a|\kappa_0 - |b| - \|Z_x\|_\infty \geq \kappa.$$

If $|b| \geq |a|\kappa_0/2$, then:

$$\forall z \in \Omega_x^u,\ |z| \leq \kappa_0/4,\ |Z_x(z) - az - b| \geq |b| - |a|\kappa_0/4 - \|Z_x\|_\infty \geq \kappa.$$

This proves what we wanted: for all $a, b \in \mathbb{R}$, there exists some open interval of positive diameter (bounded from below uniformly in a, b and x), centered at a point lying in $\Omega_x^u \cap (-\rho/2, \rho/2)$, on which $Z_x(z)$ is far away from $az + b$. □

Lemma 5.6 (Oscillation everywhere in x, at all scales in z)**.** *Under the condition $\Phi_\tau \nsim 0$, the following hold. There exists $\kappa > 0$ such that, for all $x \in \Omega$, for all $a, b \in \mathbb{R}$, for all $n \geq 0$, there exists $z_0 \in \Omega_x^u \cap (-\rho/2, \rho/2)$ such that*

$$\forall z \in \Omega_x^u \cap (z_0 - \kappa, z_0 + \kappa),\ |Z_x^{\langle n \rangle}(z) - az - b| \geq \kappa,$$

where $Z_x^{\langle n \rangle}(z) := \mu_x^{\langle -n \rangle} Z_{f^{-n}(x)}(z \lambda_x^{\langle -n \rangle})$. The family $(Z_x^{\langle n \rangle})$ is a (n-th times) zoomed-in and rescaled version of (Z_x).

Proof. We know that $Z_x^{\langle n \rangle}(z) = \mathit{linear} + Z_x(z)$ on $(-\rho, \rho) \cap \Omega_x^u$. The result follows from the previous lemma. □

Lemma 5.7. *Under the generic condition $\Phi_\tau \nsim 0$, the following hold. There exists $\kappa > 0$ and $n_0 \geq 0$ such that, for all $x \in \Omega$, for all $a, b \in \mathbb{R}$, for all $n \geq n_0$, there exists $z_0 \in \Omega_x^u \cap (-\rho/2, \rho/2)$ such that*

$$\forall z \in \Omega_x^u \cap (z_0 - \kappa, z_0 + \kappa),\ |X_x^{\langle n \rangle}(z) - az - b| \geq \kappa,$$

where $X_x^{\langle n \rangle}(z) := \mu_x^{\langle -n \rangle} X_{f^{-n}(x)}(z \lambda_x^{\langle -n \rangle})$.

Proof. Recall that $X_x = Y_x + Z_x$, and that $|Y_x(z)| \leq C|z|^{1+\alpha}$. Zooming in, we find, for all $n \geq 0$:

$$X_x^{\langle n \rangle}(z) = Y_x^{\langle n \rangle}(z) + Z_x^{\langle n \rangle}(z),$$

where $Y_x^{\langle n \rangle}(z) := \mu_x^{\langle -n \rangle} Y_{f^{-n}(x)}(z \lambda_x^{\langle -n \rangle}) = O((\lambda_x^{\langle -n \rangle})^\alpha)$. Taking n_0 large enough so that this is less than $\kappa/2$ for the κ given by the previous lemma allows us to conclude. □

We conclude this section by establishing what we will call the "Uniform Non Integrability condition" (UNI) in our context.

Proposition 5.8 (UNI)**.** Under the generic condition $\Phi_\tau \not\sim 0$, the following hold. There exists $0 < \kappa < 1/10$ and $\sigma_0 > 0$ such that, for all $x \in \Omega$, for all $a, b \in \mathbb{R}$, for all $0 < \sigma < \sigma_0$, there exists $z_0 \in \Omega_x^u \cap (-\sigma/2, \sigma/2)$ such that

$$\forall z \in \Omega_x^u \cap (z_0 - \kappa\sigma, z_0 + \kappa\sigma) \subset (-\sigma, \sigma),\ |X_x(z) - az - b| \geq \kappa\sigma.$$

Proof. For each σ small enough, define $n_x(\sigma)$ as the largest positive integer such that $\sigma \leq \rho\lambda_x^{\langle -n_x(\sigma)\rangle}$. We then have $\sigma \simeq \lambda_x^{\langle -n_x(\sigma)\rangle} \simeq (\mu_x^{\langle -n_x(\sigma)\rangle})^{-1}$, and we see that we can deduce our statement written with σ by our statement written with $n_x(\sigma)$. □

6 Nonconcentration under (UNI)

In this last section, we establish our last estimate, given by the following lemma. We call (UNI) the estimate given by Lemma 5.8.

Proposition 6.1. Under (UNI), there exists $\gamma > 0$ and $\alpha > 0$ such that for all $\sigma > 0$ small enough, for all $x \in \Omega$, for all $a \in \mathbb{R}$, we have

$$\eta_x\Big(z \in [-\sigma, \sigma],\ |X_x(z) - az| \leq \sigma^{1+\alpha}\Big) \leq \sigma^\gamma \eta_x([-\sigma, \sigma]),$$

where $\eta_x := (\Phi_x^u)^* \mu_x^u$.

Once this proposition is proved, lemma 4.8 will ensure that the nonconcentration estimates (NC) are true under the generic condition $\Phi_\tau \not\sim 0$. Let us begin by strenghtening a bit the conclusion of (UNI): we will go from a statement about oscillations of (X_x) at zero to a statement about oscillations of (X_x) everywhere.

Lemma 6.2 (Oscillations everywhere !)**.** *Under the generic condition $\Phi_\tau \not\sim 0$, the following hold. There exists $0 < \kappa < 1/10$ and $\sigma_0 > 0$ such that, for all $x \in \Omega$, for all $a, b \in \mathbb{R}$, for all $0 < \sigma < \sigma_0$, for any $z_0 \in \Omega_x^u \cap (-\rho, \rho)$, there exists $z_1 \in \Omega_x^u \cap (z_0 - \sigma/2, z_0 + \sigma/2)$ such that*

$$\forall z \in \Omega_x^u \cap (z_1 - \kappa\sigma, z_1 + \kappa\sigma) \subset (z_0 - \sigma, z_0 + \sigma),\ |X_x(z) - az - b| \geq \kappa\sigma.$$

Proof. Let us fix the κ and σ_0 from Proposition 5.8. Let $x \in \Omega$, let $\sigma < \sigma_0$, and let $z_0 \in \Omega_x^u$. Define $\tilde{x} := \Phi_x^u(z_0)$. Recall that, by lemma 4.5, there exists $\mathrm{Aff}_{x,\tilde{x}}$ and $\mathrm{aff}_{x,\tilde{x}}$, two affine functions with $e^{O(1)}$ linear coefficients, such that

$$\mathrm{Aff}_{x,\tilde{x}}(X_{\tilde{x}}(z)) = X_x(\mathrm{aff}_{x,\tilde{x}}(z)).$$

Furthermore, $\mathrm{aff}_{x,\tilde{x}} = (\Phi_x^u)^{-1} \circ (\Phi_{\tilde{x}}^u)$. Notice that $\mathrm{aff}_{x,\tilde{x}}(0) = z_0$. Since $(\mathrm{Aff}_{x,\tilde{x}})'(0) = e^{O(1)}$, the previous lemma applied to $X_{\tilde{x}}$ gives us some $z_2 \in (-\sigma/2, \sigma/2)$ such that:

$$\forall z \in (z_2 - \kappa\sigma, z_2 + \kappa\sigma),\ |\mathrm{Aff}_{x,\tilde{x}}(X_{\tilde{x}}(z)) - az - b| \geq \kappa\sigma,$$

choosing κ smaller if necessary. Setting $z_1 := (\mathrm{aff}_{x,\tilde{x}})^{-1}(z_2)$ yields the desired result. □

We do a little break by proving a clean cutting lemma (as clean as I can right now). The difficulty here is a to cut Ω_x^u into "equal" pieces, but the fractal nature of Ω_x^u makes it a bit subtle (especially because I prefer not to use Markov partitions). This is nothing new, though.

Lemma 6.3. *Denote by ρ, κ the constant given by Lemma 6.2. There exists $\eta_{cut,1}, \eta_{cut,2} \in (0,1)$, with $\eta_{cut,1} \leq \kappa/100$, such that the following hold. Let $x \in \Omega$, $\sigma \in]0, \rho]$ and $n \geq 1$. Let $U_x^{(\sigma)} \subset W_{loc}^u(x) \cap \Omega$ be such that $B(x,\sigma) \cap W_{loc}^u(x) \cap \Omega \subset U_x^{(\sigma)} \subset B(x, 10\sigma) \cap W_{loc}^u(x)$. There exists a finite set $\mathcal{A}$ and a family of intervals $(I_{\mathbf{a}})_{\mathbf{a} \in \bigcup_{k=0}^n \mathcal{W}_k}$, where $\mathcal{W}_k \subset \mathcal{A}^k$ is a set of words on the alphabet $\mathcal{A}$, and where $I_{\mathbf{a}} \subset W_{loc}^u(x)$ are such that:*

1. *$I_\emptyset \cap \Omega = U_x^{(\sigma)}$*

2. *For all* $\mathbf{a} \in \mathcal{A}^k$*,* $0 \leq k \leq n$*, we have* $I_{\mathbf{a}} \cap \Omega = \bigcup_{b \in \mathcal{A}} (I_{\mathbf{a}b} \cap \Omega)$*, and this union is disjoint modulo a zero-measure set.*

3. *For all* $\mathbf{a} \in \mathcal{A}^k$*,* $0 \leq k \leq n$*, there exists* $x_{\mathbf{a}} \in \Omega \cap I_{\mathbf{a}}$ *such that*

$$W^u_{loc}(x) \cap B(x_{\mathbf{a}}, 10^{-1} \, diam^u(I_{\mathbf{a}})) \subset I_{\mathbf{a}}$$

4. *For all* $\mathbf{a} \in \mathcal{A}^k$*,* $0 \leq k \leq n-1$*, we have for any* $b \in \mathcal{A}$*:*

$$\eta_{cut,1}^2 \, diam^u(I_{\mathbf{a}}) \leq diam^u(I_{\mathbf{a}b}) \leq \eta_{cut,1} \, diam^u(I_{\mathbf{a}}) \quad , \quad \mu^u_x(I_{\mathbf{a}b}) \geq \eta_{cut,2} \mu^u_x(I_{\mathbf{a}}).$$

Proof. The proof is done in a couple of steps.

- Step 1: We cut, once ($n = 1$), intervals $[-\sigma, \sigma]$ when $\sigma \in [\rho \lambda_+^{-2}, \rho]$.

First of all, recall that for all ε, there exists $\delta > 0$ such that for any $z \in W^u_{loc}(x) \cap \Omega$, $\mu^u_x(B(z,\varepsilon) \cap W^u_{loc}(x)) > \delta$. Since, moreover, the measure μ^u_x is upper regular, that is:

$$\forall U \subset W^u_{loc}(x), \ \mu^u_x(U) \leq C \mathrm{diam}(U)^{\delta_{reg}},$$

It follows that reciprocally, if $\mu^u_x(U) \geq \varepsilon$, then $\mathrm{diam}(U) \geq \delta$.

Consider a finite cover of $U_x^{(\sigma)}$ by a set of balls E of the form$B(x_i, \varepsilon)$ with $x_i \in U_x^{(\sigma)}$, for ε small enough (for example, $\varepsilon := \kappa \rho \lambda_+^{-2} \cdot 10^{-10}$ should be enough). By Vitali's lemma, there exists a subset $D \subset E$ such that the balls in D are disjoints, and such that $\cup_{B \in D} 3B \supset U_x^{(\sigma)}$. Since $W^u_{loc}(x)$ is one-dimensionnal, we can set $\mathcal{A} := D$, and choose $B \subset I_a \subset 3B$ such that the (I_a) have disjoint union and covers $U^{(\sigma)}$. By construction, the diameter of those intervals is small, but in a controlled way, the measure of them is greater than some constant, and they all contain a ball of radius ε for some ε. This construct our $(I_a)_{a \in \mathcal{A}}$ in this case. Notice that the cardinal of $\mathcal{A}$ is uniformly bounded. We see that it suffice to take ε small enough to find other constants $\eta_{cut,1}, \eta_{cut,2}$ with $\eta_{cut,1} \leq \kappa/100$ that satisfies what we want in this macroscopic context.

- Step 2: We cut, once (n=1), intervals $[-\sigma, \sigma]$ when $\sigma > 0$ is small.

The idea is to use the properties of μ^u_x. If we denote by $\varphi : \Omega \to \mathbb{R}$ the potential defining our equilibrium state μ, then we know [Cl20] that $f_* d\mu^u_x = e^{\varphi \circ f^{-1} - P(\varphi)} d\mu^u_{f(x)}$. Iterating yields $f^n_* d\mu^u_x = e^{S_n \varphi \circ f^{-n} - nP(\varphi)} d\mu^u_{f^n(x)}$. The Hölder regularity of φ allows us to see that, for all intervals $J \subset I := W^u_{loc}(f^n(x)) \cap B(f^n(x), \rho)$, we have

$$\frac{\mu^u_x(f^{-n}(J))}{\mu^u_x(f^{-n}(I))} = \frac{\mu^u_x(J)}{\mu^u_x(I)} \Big(1 + O(\rho^\alpha) \Big) \quad , \quad \frac{\mathrm{diam}(f^{-n}(J))}{\mathrm{diam}(f^{-n}(I))} = \frac{\mathrm{diam}(J)}{\mathrm{diam}(I)} \Big(1 + O(\rho^\alpha) \Big).$$

So we see that, using the dynamics, (and reducing ρ if necessary) we can reduce our setting to the setting of step 1. This might deform a bit the constants though, which is way we chosed the 10^{-1} in point three instead of 6^{-1} (which was the constant given by Vitali's lemma), for example.

- Step 3: We iterate this construction to the subintervals I_b.

We consider $U^{(\sigma)}$ as in the statement of the lemma. We use step 2 to construct $(I_a)_{a \in \mathcal{A}}$. Then, we can iterate our construction on each of the I_a, since they satisfy the necessary hypothesis to do so. This gives us intervals $(I_{a_1 a_2})_{a_1 a_2 \in \mathcal{W}_2}$, with $\mathcal{W}_2 \subset \mathcal{A}^2$ (taking $\mathcal{A}$ with more letters if necessary). Doing this again and again yield the desired construction. □

Using this partition lemma, we can prove proposition 6.1. Notice that, looking at this lemma in coordinates Φ^u_x, one can get the same construction replacing subintervals of W^u_{loc} by intervals of $\mathbb{R}$, Ω by Ω^u_x, μ^u_x by η_x, etc.

Proof (Proposition 6.1). Let $\sigma > 0$ be small enough, let $a, b \in \mathbb{R}$. Let $x \in \Omega$. Define $k(\sigma) \in \mathbb{N}$ as the largest integer such that

$$\sigma^\alpha \leq (\kappa/20) \cdot \Big(\eta_{cut,2} \Big)^{2k(\sigma)}.$$

We have $k(\sigma) \simeq C|\ln \sigma|$. By applying the the previous construction to $\Phi_x^u([-\sigma,\sigma])$, with depth $k(\sigma)$, we find a family of intervals $(I_{\mathbf{a}})_{\mathbf{a}\in\cup_{j\le k_x(\sigma)}\mathcal{A}^j}$ that satisfies some good partition properties. Then, the heart the proof is as follow: we want to construct (up to renaming some of the $I_{\mathbf{a}}$) a word $\mathbf{b} \in W_{k(\sigma)} \subset \mathcal{A}^{k_x(\sigma)}$ such that

$$\{z \in [-\sigma,\sigma],\ |X_x(z) - az - b| \le \sigma^{1+\alpha}\} \subset \bigcup_{\substack{\mathbf{a}\in\mathcal{W}_{k(\sigma)} \\ \forall i, a_i \neq b_i}} I_{\mathbf{a}}.$$

This is nothing more than a technical way of keeping track of the oscillations happening at all scales between σ and $\sigma^{1+\alpha}$. Notice that, once this is proved, then the conclusion is easy: we will have

$$\begin{aligned}
\eta_x\Big(\bigcup_{\substack{\mathbf{a}\in\mathcal{W}_{k(\sigma)} \\ \forall i, a_i \neq b_i}} I_{\mathbf{a}}\Big) &= \sum_{\substack{\mathbf{a}\in\mathcal{W}_{k(\sigma)} \\ \forall i, a_i \neq b_i}} \eta_x(I_{\mathbf{a}}) \\
&\le (1-\eta_{cut,2}) \sum_{\substack{\mathbf{a}\in\mathcal{W}_{k(\sigma)-1} \\ \forall i, a_i \neq b_i}} \eta_x(I_{\mathbf{a}}) \\
&\le (1-\eta_{cut,2})^2 \sum_{\substack{\mathbf{a}\in\mathcal{W}_{k(\sigma)-2} \\ \forall i, a_i \neq b_i}} \eta_x(I_{\mathbf{a}}) \\
&\le \dots \le (1-\eta_{cut,2})^{k(\sigma)} \eta_x([-\sigma,\sigma]) \simeq \sigma^\gamma \eta_x([-\sigma,\sigma])
\end{aligned}$$

for some $\gamma > 0$, since $k(\sigma) \simeq \ln(\sigma^{-1})$. Now, let us construct this word $\mathbf{b}$: the idea is that since $2\sigma^{1+\alpha} \simeq \sigma(\kappa/10)\cdot\Big(\eta_{cut,2}\Big)^{2k(\sigma)}$, for each i, we can find oscillations at scales $\sim \sigma\eta_{cut,1}^i$ of magnitude $\sim \sigma(\kappa/10)(\eta_{cut,1})^{2i} \ge 2\sigma^{1+\alpha}$, and conclude.

Let us begin by the case $i = 0$. By Lemma 6.2, we know that there exists a point $z_\emptyset \in \Omega_x^u \cap (-\sigma/2, \sigma/2)$ such that:

$$\forall z \in (z_\emptyset \pm \kappa\sigma),\ |X_x(z) - az - b| \ge \kappa\sigma.$$

There exists $b_1 \in \mathcal{A}$ such that $z_\emptyset \in I_{b_1}$. Since $\mathrm{diam}(I_b) \le \eta_{cut,1}\sigma \le \frac{\kappa}{100}\sigma$, we find

$$\forall z \in I_{b_1},\ |X_x(z) - az - b| \ge \kappa\sigma.$$

Hence:

$$\{z \in [-\sigma,\sigma],\ |X_x(z) - az - b| \le \sigma^{1+\alpha}\} \subset \{z \in [-\sigma,\sigma],\ |X_x(z) - az - b| \le \sigma\kappa/2\} \subset \bigcup_{a_1\in\mathcal{A}\setminus\{b_1\}} I_{a_1}.$$

Now, let $a_1 \in \mathcal{A} \setminus \{b_1\}$, and work on I_{a_1}. We know that there exists $z_{a_1} \in \Omega_x^u$ such that $]z_{a_1} \pm 10^{-1}\mathrm{diam}(I_{a_1})[\subset I_{a_1}$. Now, (UNI) applied at this interval yields:

$$\exists \tilde{z}_{a_1} \in I_{a_1} \cap \Omega_x^u, \forall z \in]\tilde{z}_{a_1} \pm (\kappa/10)\mathrm{diam}(I_{a_1})[,\ |X_x(z) - az - b| \ge (\kappa/10)\mathrm{diam}(I_{a_1}) \ge (\kappa/10)\eta_{cut,1}^2\sigma.$$

Since $\mathrm{diam}(I_{a_1 b}) \le \eta_{cut,1}\mathrm{diam}(I_{a_1})$, with $\eta_{cut,1} \le \kappa/100$, we find, for some $b_2 \in \mathcal{A}$ (we can say that this is the same b_2 for any $a_1 \in \mathcal{A}$, up to renaming the intervals),

$$\forall z \in I_{a_1 b_2}, |X_x(z) - az - b| \ge (\kappa/10)\eta_{cut,1}^2\sigma.$$

Hence:

$$\begin{aligned}
&I_{a_1} \cap \{z \in [-\sigma,\sigma],\ |X_x(z) - az - b| \le \sigma^{1+\alpha}\} \\
&\subset I_{a_1} \cap \{z \in [-\sigma,\sigma],\ |X_x(z) - az - b| \le (\kappa/20)\eta_{cut,1}^2\sigma\} \subset \bigcup_{a_2\neq b_2} I_{a_1 a_2}.
\end{aligned}$$

Hence

$$\{z \in [-\sigma,\sigma],\ |X_x(z) - az - b| \le \sigma^{1+\alpha}\} \subset \bigcup_{a_1\neq b_1} \bigcup_{a_2\neq b_2} I_{a_1 a_2}.$$

Let us do the next step, and then we will stop here because the construction will be clear enough. We could formaly conclude by induction. Take $a_1 \neq b_1$ and $a_2 \neq b_2$. We know that there exists $z_{a_1a_2} \in \Omega_x^u$ such that $]z_{a_1a_2} \pm 10^{-1}\mathrm{diam}(I_{a_1a_2})[\subset I_{a_1a_2}$. Applying UNI to this interval yield

$$\exists \tilde{z}_{a_1a_2} \in I_{a_1a_2} \cap \Omega_x^u, \forall z \in]\tilde{z}_{a_1a_2} \pm (\kappa/10)\mathrm{diam}(I_{a_1a_2})[,$$

$$|X_x(z) - az - b| \geq (\kappa/10)\mathrm{diam}(I_{a_1a_2}) \geq (\kappa/10)\eta_{cut,1}^4\sigma.$$

There exists $b_3 \in \mathcal{A}$ such that $\tilde{z}_{a_1a_2} \in I_{a_1a_2b_3}$. Since $\mathrm{diam}(I_{a_1a_2b_3}) \leq \eta_{cut,1}\mathrm{diam}(I_{a_1a_2})$, with $\eta_{cut,1} \leq \kappa/100$, we have

$$\forall z \in I_{a_1a_2b_3},\ |X_x(z) - az - b| \geq (\kappa/10)\eta_{cut,1}^4\sigma \geq 2\sigma^{1+\alpha}.$$

Hence:

$$I_{a_1a_2} \cap \{z \in [-\sigma,\sigma],\ |X_x(z) - az - b| \leq \sigma^{1+\alpha}\} \subset \bigcup_{a_3 \neq b_3} I_{a_1a_2a_3},$$

and so

$$\{z \in [-\sigma,\sigma],\ |X_x(z) - az - b| \leq \sigma^{1+\alpha}\} \subset \bigcup_{a_1 \neq b_1} \bigcup_{a_2 \neq b_2} \bigcup_{a_3 \neq b_3} I_{a_1a_2a_3}.$$

This algorithm is done until the $k(\sigma)$-th step. This conclude this construction, hence the proof, (hence our notes !). □

Theorem 1.2 is then proved using Proposition 6.1, Proposition 5.8, and Lemma 4.8.

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