Fourier dimension and Julia sets

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Content

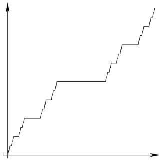
1. The Fourier dimension.
2. What is known ?
3. Outline of a technique to prove that a set has positive Fourier dimension.

On the Hausdorff "fractal" dimension

On the triadic cantor set $C := \left\{\sum_{n=1}^{\infty} \frac{2\varepsilon_n}{3^n} \ \ \varepsilon \in \{0,1\}^{\mathbb{N}^*}\right\}$, the cantor distribution is the law of the random variable

$$X := \sum_{n=1}^{\infty} \frac{2E_n}{3^n}$$

where $E_n \sim$ Bernouilli$(1/2)$ are iid. Its cumulative distribution function is the cantor's staircase:



Which is $\ln 2/\ln 3$ Hölder !

On the Hausdorff dimension

Theorem (Frostman lemma, 1935)

Let $E \subset \mathbb{R}^d$ be a compact set. Denote by $\mathcal{P}(E)$ the set of all (borel) probability measures with support in E. Then :

$$\dim_H(E) = \sup\{\alpha \in [0, d] \mid \exists \mu \in \mathcal{P}(E), \exists C > 0, \mu(B(x, r)) \leq Cr^\alpha\}$$

A measure that satisfies the earlier condition satisfies

$$\iint \frac{1}{|x-y|^\alpha} d\mu(x) d\mu(y) < \infty$$

Denoting $k_\alpha(t) := |t|^{-\alpha}$, the integral can be rewritten as

$$\langle \mu, k_\alpha * \mu \rangle$$

And then the Parseval formula gives

$$\langle \mu, k_\alpha * \mu \rangle = \int_{\mathbb{R}^d} |\widehat{\mu}(\xi)|^2 |\xi|^{\alpha-d} d\xi < \infty.$$

where

$$\widehat{\mu}(\xi) := \int_E e^{-2i\pi x\cdot\xi} d\mu(x).$$

On the Hausdorff dimension

Theorem (Frostman lemma, v2)

Let $E \subset \mathbb{R}^d$ be a compact set. Then :

$$\dim_H(E) = \sup \left\{ \alpha \in [0, d] \mid \exists \mu \in \mathcal{P}(E), \int_{\mathbb{R}^d} |\widehat{\mu}(\xi)|^2 |\xi|^{\alpha - d} d\xi < \infty \right\}$$

The condition reads: "$\widehat{\mu}(\xi)$ decays at least like $|\xi|^{-\alpha/2}$ <u>on average</u>"

A natural question

Can we get rid of the "on average" ? Is it true that

$$\dim_H(E) = \sup \left\{ \alpha \in [0, d] \mid \exists \mu \in \mathcal{P}(E),\ |\widehat{\mu}(\xi)| \leq C|\xi|^{-\alpha/2} \right\} ?$$

The case of the triadic Cantor set

Suppose that there exists $\mu \in \mathcal{P}(C)$ such that $\widehat{\mu}(\xi) \underset{\xi \to \infty}{\to} 0$. The Cantor set is invariant by the affine map $T(x) := 3x \mod 1$. The pushforward $T_* \mu$ satisfies

$$c_n(T_*\mu) = \int_0^{1/3} e^{2i\pi n(3x)} d\mu(x) + \int_{2/3}^{1} e^{2i\pi n(3x-2)} d\mu(x) = c_{3n}(\mu).$$

Hence $c_n(T_*^k \mu) = c_{3^k n}(\mu) \underset{k \to \infty}{\to} 0$ if $n \neq 0$. By considering a subsequence if necessary, $T_*^k \mu$ converges to a probability measure ν supported in C such that $c_n(\nu) = 0$ for any $n \neq 0$. This shows that ν is the lebesgue measure, contradiction.

An obstruction

The invariance of the triadic Cantor set by an affine map is an obstruction to the existence of probability measures with decaying Fourier transform, *even thought* $\dim_H C > 0$.

Fourier dimension

We define the Fourier dimension:

$$\dim_F E := \sup\left\{\alpha \in [0,d] \mid \exists \mu \in \mathcal{P}(E),\ |\widehat{\mu}(\xi)| \leq C|\xi|^{-\alpha/2}\right\}.$$

The Fourier dimension can be seen as a measure of the "nonlinearity" of the fractal set E. We know that $\dim_F E \leq \dim_H E$.

- When does $\dim_F E = \dim_H E$?
- When does $\dim_F E > 0$?

What is known ?

We know how to construct Salem sets via random constructions, see for exemple:

- R. Salem, *On singular monotonic functions whose spectrum has a given Hausdorff dimension.* Ark. Mat., 1:353–365, 1951.
- J. P. Kahane, *Images d'ensembles parfaits par des séries de Fourier Gaussiennes* Acad. Sci. Paris 263 (1966), 678-681.
- Works of Shapiro, Bluhm, Laba and Pramanik, Chen and Seeger...

Before 2017, the only deterministic examples are given from arithmetic constructions:

- R. Kaufman, *On the theorem of Jarnik and Besicovitch*, Acta Arith. 39 (1981), 265-267.
- M. Queffélec and O. Ramaré, *Analyse de Fourier des fractions continues à quotients restreints*, Enseign. Math. (2), 49(3-4):335–356, 2003
- K. Hambrook, *Explicit Salem sets in $\mathbb{R}^2$.* Adv. Math. 311 (2017), 634–648.

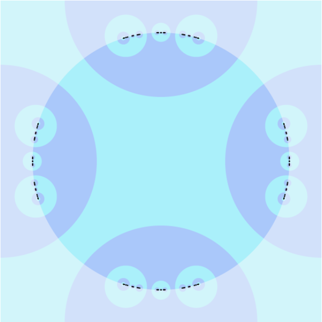
More recently:

- R. Fraser and K. Hambrook, *Explicit Salem sets in $\mathbb{R}^n$* Preprint 2020, arXiv:1909.04581

A breakthrough

In 2017, the first "concrete" example of fractal set with positive Fourier dimension is found.

- J. Bourgain, S. Dyatlov, *Fourier dimension and spectral gaps for hyperbolic surfaces*, Geom. Funct. Anal. 27, 744–771 (2017), arXiv:1704.02909



It can be seen as a "nonlinear cantor set" on the real line.

A breakthrough

Theorem (Bourgain, Dyatlov, 2017)

Let $\Gamma < Iso^+(\mathbb{H})$ be a Fuschian Shottky group, with a compact limit set $\Lambda_\Gamma \subset \mathbb{R}$ of dimension $\delta > 0$. Note μ an associated Patterson-Sullivan measure. Let $\varphi \in C^2(\mathbb{R}, \mathbb{R})$ and $g \in C^1(\mathbb{R}, \mathbb{C})$ such that:

$$\|\varphi\|_{C^2} + \|g\|_{C^1} \leq C_0 \ , \ \inf_{\Lambda_\Gamma} |f'| \geq C_0^{-1}$$

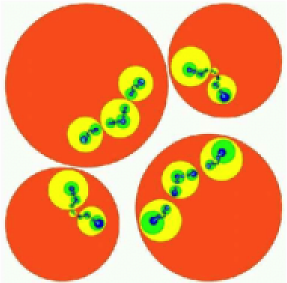
for some $C_0 > 0$. Then, there exists $\varepsilon > 0$ depending only on δ, and $C > 0$ depending on Γ and C_0, such that

$$\left| \int_{\Lambda_\Gamma} e^{i\xi\varphi(x)} g(x) d\mu(x) \right| \leq C|\xi|^{-\varepsilon}.$$

The Kleinian case

The result is generalized in dimension 2 in 2019.

- J. Li, F. Naud, W. Pan, *Kleinian Schottky groups, Patterson-Sullivan measures, and Fourier decay* Duke Math. J. 170 (4) 775 - 825, 2021. arXiv:1902.01103



This is a Cantor set on the complex plane.

The Kleinian case

Theorem (Li, Naud, Pan, 2019)

Assume that Γ is a Zariski dense Schottky group in $PSL_2(\mathbb{C})$, and let μ be a Patterson-Sullivan measure on Λ_Γ. Fix any neighborhood $\mathcal{U}$ of Λ_Γ. Let g be in $C^1(\mathcal{U},\mathbb{C})$ and φ be in $C^2(\mathcal{U},\mathbb{R})$, with

$$M := \inf_{z\in\mathcal{U}} |\nabla_z \varphi| > 0.$$

Assume that $\|g\|_{C^1} + \|\varphi\|_{C^2} \leq M'$. Then there exists $C := C(M, M', \Gamma) > 0$ and $\varepsilon > 0$ depending only on μ, such that for all $t \in \mathbb{R}$ with $|t| \geq 1$,

$$\left| \int_{\Lambda_\Gamma} e^{it\varphi(z)} g(z) d\mu(z) \right| \leq C|t|^{-\varepsilon}$$

An explicit "nonlinearity condition"

In 2020, the work of Bourgain and Dyatlov is generalized by Sahlsten and Steven in the context of any "totally nonlinear" cantor set on the real line. Under some natural hyperbolicity assumptions, they show that any equilibrium measure enjoys polynomial Fourier decay.

- T. Sahlsten, C. Stevens, *Fourier transform and expanding maps on Cantor sets*. Preprint, 2020. arXiv:2009.01703

An explicit nonlinearity condition

Theorem (Sahlsten, Stevens, 2020)

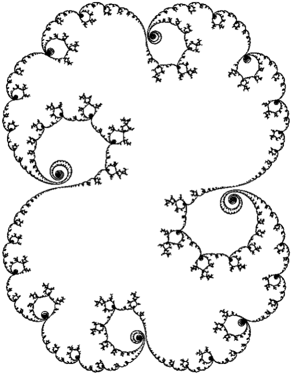
Let $I_1, \dots, I_N$, be closed, disjoint, bounded intervals, and write $I := \cup_a I_a$. Let $T : I \to \mathbb{R}$ be such that each restriction $T_{|I_a}$ is real analytic and assume T is conjugated to the shift on $\mathcal{A}^{\mathbb{N}}$, where $\mathcal{A} = \{1, \dots, N\}$. Suppose:

1. *Hyperbolicity:* $\exists \gamma > 1,\ |(T^n)'(x)| \geq C\gamma^n$.
2. *Markov property:* $T(I_b) \cap \mathring{I}_a \neq \emptyset \Rightarrow T(I_b) \supset I_a$.
3. *Bounded distortions:* $\|T''/T'\|_\infty < \infty$.
4. *Total non-linearity: There exists no* $g \in C^1(I)$ *such that* $\log T' = \psi_0 + g \circ T - g$ *where* $\psi_0 : I \to R$ *is locally constant.*

Let $K := \bigcap_{n=0}^{\infty} T^{-n}(I)$ be the T-invariant cantor set. Let φ be a Hölder potential, and let μ be the equilibrium measure associated to φ.
Then $\widehat{\mu}$ vanish with a polynomial rate. In particular, $\dim_F(K) > 0$.

The case of hyperbolic Julia sets

In 2021, "concrete" sets with positive Fourier dimension which are not Cantor sets are found. The Julia set of any hyperbolic rational map has positive Fourier dimension.



- L - *Julia sets of hyperbolic rational maps have positive Fourier dimension*. Preprint, 2021. arXiv:2112.00701

The case of hyperbolic Julia sets

Theorem (L, 2021)

Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a hyperbolic rational map of degree $d \geq 2$. Denote by $J \subset \mathbb{C}$ its Julia set, and suppose that J is not included in a circle. Let V be an open neighborhood of J, and consider any potential $\varphi \in C^1(V, \mathbb{R})$. Let $\mu_\varphi \in \mathcal{P}(J)$ be its associated equilibrium measure. Then:

$$\exists \varepsilon > 0,\ \exists C > 0,\ \forall \xi \in \mathbb{C},\ |\widehat{\mu}_\varphi(\xi)| \leq C(1 + |\xi|)^{-\varepsilon}.$$

What is a Julia set ?

Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a rational map of degree $d \geq 2$. ($f = P/Q$ where P and Q are polynomials). The Fatou set of f is the largest open set in $\widehat{\mathbb{C}}$ where the family of iterates $\{f^n,\ n \in \mathbb{N}\}$ is a normal family. Its complement is called the Julia set and is denoted by J.

- J is f invariant, so (J, f) is a well-defined dynamical system.
- The system is "chaotic": for example it is topologically mixing.
- Without loss of generality, we may suppose $J \subset \mathbb{C}$.

We way that f is hyperbolic if it is eventually expanding:

$$\exists \kappa > 1,\ \exists c > 0,\ \forall x \in J,\ \forall n \geq 0,\ |(f^n)'(x)| \geq c\kappa^n.$$

What is an equilibrium measure ?

For any $\varphi \in C^1(J, \mathbb{R})$, define the associated **transfer operator** $\mathcal{L}_\varphi : C^0(J, \mathbb{C}) \to C^0(J, \mathbb{C})$ by the formula:

$$\forall h \in C^0(J, \mathbb{R}),\ \forall x \in J,\ \mathcal{L}_\varphi h(x) := \sum_{y \in f^{-1}(x)} e^{\varphi(y)} h(y)$$

Suppose that $\mathcal{L}_\varphi 1 = 1$.
Then there exists a unique f-invariant measure $\mu \in \mathcal{P}(J)$ such that

$$\forall h \in C^0(J, \mathbb{R}),\ \int_J (\mathcal{L}_\varphi h) d\mu = \int_J h\ d\mu$$

It is called the equilibrium measure associated to φ. It has full support and the system (f, J, μ) is ergodic.

A first tool: Markov partitions

The hyperbolicity assumptions implies the existence of **Markov partitions**. There exists small sets $(P_a)_{a\in\mathcal{A}}$ such that:

- $J = \cup_{a\in\mathcal{A}} P_a$
- $\overline{\mathrm{int}_J P_a} = P_a$,
- $\mathrm{int}_J P_a \cap \mathrm{int}_J P_b = \emptyset$ if $a \neq b$,
- each $f(P_a)$ is a union of sets P_b.

When $P_b \subset f(P_a)$, we can find a holomorphic section of f that we denote $g_{ab} : P_b \to P_a$. These sections are "eventually contracting", ie, whenever that makes sense, and for n large enough, the maps $g_{a_1 \dots a_n} := g_{a_1 a_2} \circ \cdots \circ g_{a_{n-1} a_n}$ are contracting.

Transfer operator v2

Using those notations, we may rewrite the transfer operator as follow:

$$\forall x \in P_b,\ \mathcal{L}_\varphi h(x) := \sum_a e^{\varphi(g_{ab}(x))} h(g_{ab}(x))$$

Where the sum runs on all $a \in \mathcal{A}$ for which $P_b \subset f(P_a)$. Iterating this expression gives:

$$\forall x \in J,\ \mathcal{L}_\varphi^n h(x) := \sum_{\underline{a}} e^{S_n\varphi(g_{\underline{a}}(x))} h(g_{\underline{a}}(x))$$

Where the sum runs on all words $\underline{a} = a_1 \dots a_n$ on which the expression $g_{a_1 \dots a_n}(x)$ makes sense, and where

$$S_n\varphi(x) := \sum_{k=0}^{n-1} \varphi(f^k(x))$$

is a Birkhoff sum.

An approximation of the Fourier transform

Fix $\xi \in \mathbb{C}$ and let $h(x) := e^{i\mathrm{Re}(\bar{\xi}x)}$. Then:

$$\widehat{\mu}(\xi) = \int_J h \, d\mu = \int_J (\mathcal{L}_\varphi^n h) d\mu$$

$$= \sum_{\underline{a}} \int_{P_{a_n}} e^{S_n\varphi(g_{\underline{a}}(x))} h(g_{\underline{a}}(x)) d\mu(x)$$

1. Since the system is ergodic, $S_n\varphi \approx n \times \int \varphi d\mu$ for most of the words $\underline{a}$. We can discard the other words (the measure μ neglects them).
2. Once this is done, the sum runs over a set of "well behaved words" $\mathcal{W}_n$ that has a cardinal of $\simeq \exp n \times \int \varphi$.

We get:

$$\widehat{\mu}(\xi) \approx \frac{1}{\#\mathcal{W}_n} \sum_{\underline{a} \in \mathcal{W}_n} \int_{P_{a_n}} e^{i\mathrm{Re}(\bar{\xi} g_{\underline{a}}(x))} \, d\mu(x)$$

Cauchy-Schwartz

From

$$|\widehat{\mu}(\xi)|^2 \approx \left| \frac{1}{\#\mathcal{W}_n} \sum_{\underline{a}\in\mathcal{W}_n} \int_{P_{a_n}} e^{i\mathrm{Re}(\bar{\xi} g_{\underline{a}}(x))} \, d\mu(x) \right|^2$$

The Cauchy-Schwartz inequality gives

$$|\widehat{\mu}(\xi)|^2 \lesssim \frac{1}{\#\mathcal{W}_n} \sum_{\underline{a}\in\mathcal{W}_n} \left| \int_{P_{a_n}} e^{i\mathrm{Re}(\bar{\xi} g_{\underline{a}}(x))} \, d\mu(x) \right|^2$$

$$\simeq \frac{1}{\#\mathcal{W}_n} \sum_{\underline{a}\in\mathcal{W}_n} \int_{P_{a_n}^2} e^{i\mathrm{Re}(\bar{\xi}(g_{\underline{a}}(x) - g_{\underline{a}}(y)))} \, d\mu(x) d\mu(y)$$

Linearizing

We then linearize the expression

$$|\widehat{\mu}(\xi)|^2 \leq \frac{1}{\#\mathcal{W}_n} \sum_{\underline{a}\in\mathcal{W}_n} \int_{P^2_{a_n}} e^{i\mathrm{Re}(\overline{\xi}(g_{\underline{a}}(x)-g_{\underline{a}}(y))} \, d\mu(x)d\mu(y).$$

Writing $g_{\underline{a}}(x) - g_{\underline{a}}(y) \approx g'_{\underline{a}}(x_{\underline{a}})(x-y)$, and neglecting the part of the integral where $(x-y)$ is too small, we get an inequality that roughly looks like

$$|\widehat{\mu}(\xi)|^2 \lesssim \frac{1}{\#\mathcal{W}_n} \left| \sum_{\underline{a}\in\mathcal{W}_n} e^{i\mathrm{Re}(\overline{\xi} g'_{\underline{a}}(x_{\underline{a}}))} \right|.$$

More words

Choosing $n = 2kN$ with N large , and k some integer, we can write $g_{\underline{a}}$ as the composition of some $g_{\underline{a}_1 \underline{b}_1}, \ldots, g_{\underline{a}_k \underline{b}_k}$, where $\underline{a}_k$ and $\underline{b}_k$ are words of length N. With this choice of n, we get:

$$|\widehat{\mu}(\xi)|^2 \lesssim \frac{1}{\#\mathcal{W}_{n/2}} \sum_{\underline{a} \in \mathcal{W}_{n/2}} \left| \frac{1}{(\#\mathcal{W}_N)^k} \sum_{\underline{b}_1, \ldots \underline{b}_k \in \mathcal{W}_N} e^{i \ \mathrm{Re}\left(\overline{\xi} g'_{\underline{a}_1 \underline{b}_1} \cdots g'_{\underline{a}_k \underline{b}_k}\right)} \right|$$

Normalization

The quantities $g'_{\underline{a}_j \underline{b}_j}$ all having the same order of magnitude $e^{-2\lambda n}$ (where λ is a Lyapunov exponent), we may renormalise the expression like so:

$$|\widehat{\mu}(\xi)|^2 \lesssim \frac{1}{\#\mathcal{W}_{n/2}} \sum_{\underline{a} \in \mathcal{W}_{n/2}} \left| \frac{1}{(\#\mathcal{W}_N)^k} \sum_{\underline{b}_1, \dots \underline{b}_k \in \mathcal{W}_N} e^{i \ \mathrm{Re}\left(\eta \zeta_{\underline{a},1}(\underline{b}_1) \dots \zeta_{\underline{a},k}(\underline{b}_k)\right)} \right|$$

Where $\zeta_{\underline{a},j}(\underline{b}) := e^{2\lambda n} g'_{\underline{a}_{j-1}\underline{b}}(x_{\underline{a}_j})$, and $\eta \sim \overline{\xi} e^{-2\lambda k n}$.
We may relate ξ and n so that $|\eta| \sim |\xi|^{\varepsilon_0}$.

The sum product phenomenon.

Theorem (J. Li 2018, generalized from Bourgain, $\sim$ 2010)

Fix $0 < \gamma < 1$. *There exist* $k \geq 1$ *and* $\varepsilon_1 > 0$ *such that the following holds for* $|\eta|$ *large enough. Let* $R > 1$, $N > 1$ *and* $\mathcal{Z}_1, ..., \mathcal{Z}_k$ *be finite sets such that* $\#\mathcal{Z}_j \leq RN$. *Consider some maps* $\zeta_j : \mathcal{Z}_j \to \mathbb{C}$ *such that* $\zeta_j(\mathcal{Z}_j) \subset \{z \in \mathbb{C} \, , \; R^{-1} \leq |z| \leq R\}$ *and such that*

$$\forall \sigma \in [|\eta|^{-2}, |\eta|^{-\varepsilon_1}], \sup_{R^{-1} \leq |a| \leq R} \#\{b \in \mathcal{Z}_j, \, \zeta_j(b) \in B(a, \sigma)\} \leq N\sigma^{1+\gamma}.$$

Then there exists a constant $c > 0$ *depending only on* γ *and* R *such that*

$$\left| N^{-k} \sum_{\mathrm{b}_1 \in \mathcal{Z}_1, ..., \mathrm{b}_k \in \mathcal{Z}_k} e^{iRe(\eta \zeta_1(\mathrm{b}_1) ... \zeta_k(\mathrm{b}_k))} \right| \leq c|\eta|^{-\varepsilon_1}$$

The non concentration hypothesis

To conclude the proof, it suffices to check the non-concentration hypothesis for some $\gamma > 0$:

$$\forall \sigma \in [|\eta|^{-2}, |\eta|^{-\varepsilon_1}],\ \sup_{a} \#\{\underline{b},\ \zeta_{\underline{a},j}(\mathrm{b}) \in B(a,\sigma)\} \leq \#\mathcal{W}_N\ \sigma^{1+\gamma}.$$

The difficulty is then to check this condition.
In our case, we can use the contraction of a complex transfer operator similar to the one found in the work of Dolgopyat (*on decay of correlations in Anosov flows*, Ann. of Maths, 1998) to conclude.

- H. Oh, D. Winter, *Prime number theorem and holonomies for hyperbolic rational maps*. Inventiones mathematicae, 2017. 208. arXiv:1603.00107

The end

Thank you for your attention.