Riemannian geometry and geometric analysis

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1 Introduction

This paper is an internship report done at the end of my second year in the ENS Rennes, in the maths university of Freiburg (Germany), with Nadine Große.
The main goal was to learn as much differential geometry that I could, so I've discovered the area of Riemannian geometry and geometric analysis.
To be more specific, I worked on a course for some weeks to introduce myself to the subject (*An Introduction to Riemannian Geometry with Applications to Mechanics and Relativity*, Leonor Godinho and Jose Natario, 2014) , then I spent the rest of my time trying to understand a research paper of Justin Corvino, written in 1999 : *Scalar curvature Deformation and a gluing Construction for the Einstein Constraint Equations.*

Here, I'll try to give to the reader a short but clear introduction to Riemannian geometry, and then we'll explain the main ideas of the upper paper of Justin Corvino.

2 What is differential geometry ?

What is the goal of differential geometry ?

Let's give a motivation example. Suppose you are a physicist, trying to understand the properties of a given function defined on the surface of earth : say, $f : S^2 \longrightarrow \mathbb{R}$. For example, we are trying to find the extrema of this function. What can we do ? In the more usual case $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$, the first question that come in the mind of an analyst is : Is this function smooth ? Can I differentiate it ?

If the function is smooth, then studying it becomes way easier : We can find the extrema of it, most of the time, just by looking of the points where the differential vanishes; we can know if f is a local diffeo by checking if $(df)_x$ is invertible; we can also compute some nice polynomial approximations of f by using a taylor formula, bounding the error made.

One of the goals of differential geometry is to define a way for our physicist to differentiate maps that are not defined on an open subset of $\mathbb{R}^n$. More generally, differential geometry is a branch of geometry that nicely mixes algebra and analysis to study the so called *smooth manifolds* and the smooth maps we can define on it.

In the next part, I'll recall some elementary differential geometry, but since the main subject of this paper is to talk about *Riemannian manifolds*, I'll just quote the main definitions and the main results, without proving anything yet. All the missing proofs can be found in the course of Godinho and Natario.

2.1 Smooth manifolds

Definition 1. A topological manifold M of dimension n is a topological space such that:

- M is Hausdorff
- Each point $p \in M$ possesses a neighborhood V homeomorphic to an open subset $U \subset \mathbb{R}^n$.
- M has a countable basis for its topology.

Each pair (U, φ), where $U \subset \mathbb{R}^n$ is open and $\varphi : U \longrightarrow \varphi(U) \subset M$ is a homeomorphism, is called a parameterization of $\varphi(U)$. The map $x := \varphi^{-1}$ is called a coordinates system, or chart. Finally, $\varphi(U)$ is called a coordinate neighborhood.

Roughly speaking, a topological manifold is something that locally looks like a deformation of $\mathbb{R}^n$. For example, the cube $\partial[0,1]^n \subset \mathbb{R}^n$ is a topological submanifold of $\mathbb{R}^n$. Notice how our example possesses sharp edges : you don't need to be smooth to be a topological manifold. On the contrary, the euclidian sphere $S^n \subset \mathbb{R}^n$ is also a topological manifold, but it doesn't have sharp parts : it is a smooth submanifold of $\mathbb{R}^n$.

Definition 2. A smooth manifold M of dimension n is a topological manifold of dimension n together with a family of parameterizations $\varphi_\alpha : U_\alpha \longrightarrow M$, such that:

- $\bigcup\limits_\alpha \varphi_\alpha(U_\alpha) = M$
- For all α, β such that $W := \varphi_\alpha(U_\alpha) \cap \varphi_\beta(U_\beta) \neq \emptyset$,
  $\varphi_\alpha^{-1} \circ \varphi_\beta : \varphi_\beta^{-1}(W) \longrightarrow \varphi_\alpha^{-1}(W)$ is a diffeomorphism.
- The atlas $\mathcal{A} := \{(\varphi_\alpha, U_\alpha)\}$ is maximal in the following sense : if a parameterization ψ is such that $\psi \circ {\varphi_\alpha}^{-1}$ and $\varphi_\alpha \circ \psi^{-1}$ are smooth for all $\varphi_\alpha \in \mathcal{A}$, then $\psi \in \mathcal{A}$.

Notice how the definition of a manifold doesn't involve any surrounding euclidian space : This point of view is called "intrinsic". Manifolds that can be naturally understood as a subpart of $\mathbb{R}^n$ are called submanifolds.

Definition 3. $M \subset \mathbb{R}^N$ is said to be a smooth submanifold of dimension n of $\mathbb{R}^N$ iif, for each point $p \in M$, there exists an open (for $\mathbb{R}^N$) neighborhood of p, V, and a diffeomorphism
$\phi : U \subset \mathbb{R}^N \longrightarrow V$, such that $\phi(U \cap (\mathbb{R}^n \times \{0\}^{N-n})) = M \cap V$.

Of course, submanifolds are naturally seen as abstract manifolds, where the atlas is given by all the conveniently chosen restrictions of the diffeomorphism that appears in the upper definition.

From now on, "manifold" will stand for "smooth manifold".

Definition 4. Let M and N be two manifolds.
We say that $f : M \longrightarrow N$ is smooth iif, for all parameterizations φ of M and ψ of N,
$\widehat{f} := \psi^{-1} \circ f \circ \varphi$ is smooth.
We say that f is a diffeomorphism if f is bijective, smooth, and if f^{-1} is smooth.

Let's make a short remark here. The isomorphisms of the manifolds world are the diffeomorphisms. Meaning that, from the differentiable structure point of view, two diffeomorphic manifolds are considered the same. Hence, it seems important to me to actually be able to visualise what it means. For example, let's visualise S^n and play with it a bit. Every diffeomorphism of $\mathbb{R}^{n+1}$ will send S^n to a diffeomorphic sister of her. Which means that the relation "being diffeomorphic" is very jelly-ish. It is really important to visualise this correctly to be able to fully understand what will be going on when we'll add a metric on our manifolds, later in this paper.

2.2 Differentials

Now, we would like to be able to define actual differentials and derivatives on manifolds. On a submanifold of the euclidean space, there is a natural way to do it :

Definition 5. Let M be a submanifold of $\mathbb{R}^N$, and $p \in M$.
We define the *tangent space of M at p*, $T_pM := \{\dot\gamma(0) \mid \gamma : (-1,1) \to \text{M smooth s.t. } \gamma(0) = p\}$

If $f : M \longrightarrow N$ is a smooth map between two submanifolds, we define :

$$\begin{array}{rccl} (df)_p : & T_pM & \longrightarrow & T_{f(p)}N \\ & h & \longmapsto & \frac{d}{dt}(f\circ\gamma)(0) \end{array}$$

where γ is any smooth curve on M such that $\gamma(0) = p$ and $\dot\gamma(0) = h$.

If f is actually the corestriction to M and N of a smooth map $\tilde f$ defined on some open subset of $\mathbb{R}^N$, then the differential of f will just be the corestriction of the differential:
$\forall h \in T_pM,\ (df)_p(h) = (d\tilde f)_p(h)$.

Example 1. Consider the antipodal map :

$$\begin{array}{rccl} f : & S^n & \longrightarrow & S^n \\ & x & \longmapsto & -x \end{array}$$

Let γ a smooth curve on S^n, such that $\gamma(0) = p$ and $\dot\gamma(0) = h \in T_pS^n$. Then $f(\gamma(t)) = -\gamma(t)$, so $\frac{d}{dt}(f\circ\gamma)(0) = -\dot\gamma(0)$, and hence :

$$\begin{array}{rccl} (df)_p : & T_pS^n & \longrightarrow & T_{-p}S^n \\ & h & \longmapsto & -h \end{array}$$

This works well. Hence, in the case of abstracts manifolds, we are going to do the same. First of all, we have to define the derivative of smooth curves on a manifold. It is clear, in the submanifold case, that the operator

$$\begin{array}{rcl} C^\infty(p) & \longrightarrow & \mathbb{R} \\ f & \longmapsto & \frac{d}{dt}(f\circ\gamma)(0) \end{array}$$

where $C^\infty(p)$ is the set of all maps $f : M \to \mathbb{R}$ that are smooth around p,
characterise the vector $\dot\gamma(0)$. Indeed, if we choose $f = \pi_i : x = (x^1, x^2, \dots, x^n) \mapsto x^i$, then we are able to know all the coordinates of our vector. This allow us to identify $\dot\gamma(0)$ to the previous operator.

Definition 6. Let M be a manifold, and $\gamma : (-1,1) \to M$ a smooth curve on M. We define the tangent vector of γ at p :

$$\begin{array}{rccl} \dot\gamma(0) : & C^\infty(p) & \longrightarrow & \mathbb{R} \\ & f & \longmapsto & \frac{d}{dt}(f\circ\gamma)(0) \end{array}$$

A tangent vector to M at p is a tangent vector to some differentiable curve $\gamma : (-1,1) \to M$ with $\gamma(0) = p$. The tangent space of M at p is the space T_pM of all tangent vectors at p.
We also define $TM := \bigsqcup_p T_pM$, the tangent bundle of M.

Example 2. What we actually did is identifying a vector with the associated operator that takes the directional derivative. Let's choose a chart $\varphi : U \to \varphi(U)$ around $p \in M$. Let $x = \varphi^{-1}(p)$. Now let's consider the special curves defined by : $\gamma_i(t) = \varphi(x^1, \dots, x^i + t, \dots, x^n)$.
Then :

$$\dot\gamma_i(0)(f) = \frac{d}{dt}(f\circ\gamma_i)(0) = \frac{\partial f\circ\varphi}{\partial x^i}(\varphi^{-1}(p)) =: \left(\frac{\partial}{\partial x^i}\right)_p (f)$$

Hence, the tangent vector at p, $\left(\frac{\partial}{\partial x^i}\right)_p$, represents the speed of the curve γ_i when he goes around p.
Also, notice that

$$\frac{\partial\varphi}{\partial x^i}(x) = \left(\frac{\partial}{\partial x^i}\right)_p \in T_pM$$

Example 3. Consider the parameterization $\psi : (0,\pi) \times (-\pi,\pi) \to S^2$ given by

$$\psi(\theta,\varphi) = (\sin\theta\cos\varphi, \sin\theta\sin\phi, \cos\theta)$$

We have the corresponding tangent vectors, at $p = \psi(\theta,\varphi)$:

$$\left(\frac{\partial}{\partial\theta}\right)_p = (\cos\theta\cos\varphi, \cos\theta\sin\varphi, -\sin\theta) \quad ; \quad \left(\frac{\partial}{\partial\varphi}\right)_p = (-\sin\theta\sin\varphi, \sin\theta\cos\varphi, 0).$$

Theorem 1. *Let M be a n-dimensional smooth manifold. Consider a parameterization $\varphi : U \to \varphi(U)$ around $p \in M$. Then $\left(\frac{\partial}{\partial x^1}\right)_p, \dots, \left(\frac{\partial}{\partial x^n}\right)_p$ is a basis of T_pM.*

Example 4. Let $\gamma : (-\varepsilon,\varepsilon) \to M$ be a smooth curve in M, and $f \in C^\infty(p)$ Let's fix a chart φ around $\gamma(0)$: in coordinates, we have

$$x(t) := (\varphi^{-1}\circ\gamma)(t) = (x^1(t),\dots,x^n(t)) \in \mathbb{R}^n \quad , \text{ and } \quad \widehat{f}(x) := (f\circ\varphi)(x)$$

Let's compute $\dot\gamma$ in those coordinates.

$$\dot\gamma(t)(f) = \frac{d}{dt}(f\circ\gamma)(t) = \frac{d}{dt}(f\circ\varphi\circ\varphi^{-1}\circ\gamma)(t) = \sum_{i=1}^n \dot x^i(t)\frac{\partial\hat f}{\partial x^i}(x(t)) = \sum_{i=1}^n \dot x^i(t)\left(\frac{\partial}{\partial x^i}\right)_{\gamma(t)}(f)$$

Hence :

$$\dot\gamma(t) = \sum_{i=1}^n \dot x^i(t)\left(\frac{\partial}{\partial x^i}\right)_{\gamma(t)} \in T_pM$$

Definition 7. If $f : M \longrightarrow N$ is a smooth map between two manifolds, we define :

$$\begin{array}{rccc}(df)_p : & T_pM & \longrightarrow & T_{f(p)}N \\ & h & \longmapsto & \frac{d}{dt}(f\circ\gamma)(0)\end{array}$$

where γ is any smooth curve on M such that $\gamma(0) = p$ and $\dot\gamma(0) = h$.

Example 5. Let fix some local coordinates x around p and y around $f(p)$.
Let $h = \sum_i h^i \left(\frac{\partial}{\partial x^i}\right)_p \in T_pM$. we have :

$$(df)_p(h) = \frac{d}{dt}(f\circ\gamma)(0) = \sum_{j=1}^n \frac{d}{dt}(\widehat{f^j}\circ\widehat{\gamma})(0)\left(\frac{\partial}{\partial y^j}\right)_{f(\gamma(0))} = \sum_{i,j=1}^n h^i\frac{\partial\widehat{f^j}}{\partial x^i}\left(\frac{\partial}{\partial y^j}\right)_{f(p)} \in T_{f(p)}N$$

Hence, we verify that $(df)_p$ is a linear map and that its associated matrix is, in this choice of coordinates : $J_f(p) = \left(\frac{\partial\widehat{f^j}}{\partial x^i}\right)_{i,j}$.

Example 6. Let's fix a parameterization $\varphi : U \to \varphi(U)$ around $p \in M$. Take a smooth $f : M \to \mathbb{R}$, and define

$$\begin{array}{rccc}(dx^i)_p : & T_pM & \longrightarrow & \mathbb{R} \\ & h & \longmapsto & h^i\end{array}$$

where $h = \sum_i h^i\left(\frac{\partial}{\partial x^i}\right)_p$. Then :

$$(df)_p = \sum_i \frac{\partial\widehat{f}}{\partial x^i}(x(p))(dx^i)_p$$

We found out happily that the formulaes are the same as in the usual euclidean case. The usual properties of the differential are true and will be used without any justifications from now on : The differential vanishes on extrema, if the differential is invertible then f is a local diffeomorphism, the chain rule is satisfied, etc.
The maps $p \mapsto (dx^i)_p$ and $p \mapsto \left(\frac{\partial}{\partial x^i}\right)_p$ are the first examples of tensor fields. Tensor fields appears naturally in a lot of areas of physics and mathematics : they will be briefly introduce in the next section.

2.3 Vector fields

One of the simplest example of tensor field are the so-called vector fields.

Definition 8. A vector field on a smooth manifold M is a map that to each point $p \in M$ assigns a vector tangent to M at p:

$$\begin{array}{rrcl} X: & M & \longrightarrow & TM \\ & p & \longmapsto & X_p \in T_pM \end{array}$$

Locally, we can write :

$$X_p = X^1(p)\left(\frac{\partial}{\partial x^1}\right)_p + \cdots + X^n(p)\left(\frac{\partial}{\partial x^n}\right)_p$$

If the maps $X^i : U \subset M \to \mathbb{R}$ are smooth, we say that the vector field X is smooth. We denote by $\mathfrak{X}(M)$ the linear space of all smooth vector fields.

Example 7. Let $f : M \to N$ be a smooth map. Define the *push-forward* :

$$\begin{array}{rrcl} f_*: & TM & \longrightarrow & TN \\ & h \in T_pM & \longmapsto & (df)_p(h) \in T_{f(p)}N \end{array}$$

Then, if X is a smooth vector field on M, f_*X is a smooth vector field on N,
and : $\forall p \in M, (df)_p(X_p) = (f_*X)_{f(p)}$.

It is natural to visualise vector fields as a field of speed for particles moving on our manifold. Indeed, if we are given a point $p \in M$ and a smooth vector field $X \in \mathfrak{X}$, there locally exists a unique curve $\gamma(t)$ such that $\gamma(0) = p$, and $\dot{\gamma}(t) = X_{\gamma(t)}$.
The application $\psi_{t_0} : p \in M \mapsto \gamma(t_0) \in M$ is then called the flow of X.
Understanding the flow of a given vector field of a manifold is a problem that naturally arise when doing physics.

Definition 9. Let $f \in C^\infty(M)$ be a real valued map, and $X \in \mathfrak{X}(M)$.
We denote by

$$(X \cdot f)_p := X_p \cdot f = X_p(f) = \sum_i X^i(p)\frac{\partial \widehat{f}}{\partial x^i}(x(p)) = (df)_p(X_p)$$

the directional derivative of f at p in the direction X_p.

Definition 10. Let $X, Y \in \mathfrak{X}(M)$. There exist a unique $Z \in \mathfrak{X}(M)$, such that :

$$\forall f \in C^\infty(M, \mathbb{R}), Z \cdot f = X \cdot (Y \cdot f) - Y \cdot (X \cdot f)$$

We usually call $Z = [X, Y]$, the "Lie bracket" of X and Y.
If $[X, Y] = 0$, we say that X and Y commutes.
In coordinates, one can verify that :

$$[X, Y]_p = \sum_i (X_p \cdot Y^i - Y_p \cdot X^i)\left(\frac{\partial}{\partial x^i}\right)_p$$

The lie bracket will actually become quite important later, when we will define the Levi-Civita connection, so let's explain a bit more what it does represent. Take two vector fields X and Y. Now, we are going to follow the flow of X for a very short amount of time, and then we are going to follow Y for the same little amount of time. Next, we do the same, but following for Y and then X. The "vector" that infinitesimally represents the gap between the arrival of the first trajectory and the second one is the lie bracket of X and Y.
Actually, understanding this way what a lie bracket is, we should expect some link between the flow of two vector fields and the lie bracket of the two of them. Fortunately for us, such links exists.

Theorem 2. *Let X, Y be two vector fields on a compact manifold M.*
Then their flow commutes, ie $\psi_t^X \circ \psi_s^Y = \psi_s^Y \circ \psi_t^X$, iif $[X, Y] = 0$.

2.4 Tensor fields

Definition 11. Let V be a finite dimensional vector space.
A multilinear map $T : V \times \cdots \times V \times V^* \times \cdots \times V^* \longrightarrow \mathbb{R}$ is called a (k,m)-tensor.
We say that T is k times covariant and m times contravariant.
The vector space of all the (k,m)-tensors on V is often denoted by $\mathcal{T}^{k,m}(V^*, V)$.

Definition 12. Let M be a manifold.
A (k,m)-tensor field on M is a map that, at each point $p \in M$, assigns a (k,m)-tensor on T_pM :
$T : p \in M \mapsto \mathcal{T}^{k,m}(T_pM^*, T_pM)$

Example 8. On a manifold of dimension n, tangent spaces are finite dimensional vector spaces, which implies a natural isomorphism $T_pM \simeq (T_pM)^{**}$.
For example, one can naturally identify the tangent vector $\left(\frac{\partial}{\partial x^i}\right)_p$ with a linear map $T_pM^* \to \mathbb{R}$, defined by :

$$\left(\frac{\partial}{\partial x^i}\right)_p ((dx^j)_p) := (dx_j)_p(\left(\frac{\partial}{\partial x^i}\right)_p) = \delta_{ij}$$

Hence, vector fields are naturally seen as 1-contravariant tensor fields :

$$X_p = X^1(p)\left(\frac{\partial}{\partial x^1}\right)_p + \cdots + X^n(p)\left(\frac{\partial}{\partial x^n}\right)_p \in T_pM^{**} = \mathcal{T}^{0,1}(T_pM^*, T_pM)$$

Example 9. Let $f : M \to \mathbb{R}$ be a smooth map.
Given local coordinates around p, we observe that :

$$df : p \in M \longmapsto (df)_p = \sum_i \frac{\partial \widehat{f}}{\partial x^i}(x(p))(dx^i)_p \in T_pM^*$$

is a 1-covariant tensor field.

Now, we can use those two examples to construct a lot of new tensor fields. Here is one way to do it :

Definition 13. Let T, S be two tensor fields over M.
We define : $(T \otimes S)_p(v_1, \ldots, v_K, w_1, \ldots, w_N) := T_p(v_1, \ldots, v_K)S_p(w_1, \ldots, w_N)$.
If T and S are covariant (or contravariant) tensor fields, then $T \otimes S$ is a (K+N)-covariant (contravariant) tensor field. Note that this operation is bilinear, and is NOT commutative.

Example 10. The usual inner product on $\mathbb{R}^n$ is a 2-covariant tensor field :

$$< \cdot, \cdot > = \sum_i dx^i \otimes dx^i$$

Example 11. The identity $TM \to TM$ is a (1,1)-tensor field.
Indeed, we can write, in coordinates :

$$(I)_p(h) = \sum_i h^i \left(\frac{\partial}{\partial x^i}\right)_p$$

Which (locally) identifies to :

$$I = \sum_i (dx^i) \otimes \left(\frac{\partial}{\partial x^i}\right)$$

Theorem 3. *Let T be a (k,m)-tensor field on M.*
Given a coordinate neighbourhood W around p, there exists $T^{j_1 \ldots j_m}_{i_1 \ldots i_k} : W \to \mathbb{R}$ a family of maps such that :

$$T_p = \sum T^{j_1 \ldots j_m}_{i_1 \ldots i_k}(p)(dx^{i_1})_p \otimes \cdots \otimes (dx^{i_k})_p \otimes \left(\frac{\partial}{\partial x^{j_1}}\right)_p \otimes \cdots \otimes \left(\frac{\partial}{\partial x^{j_m}}\right)_p$$

Roughly speaking, this means that one can understand tensor fields as maps that takes a bunch of vector fields and assigns them to another bunch of vector fields, like the differential above.
Again, we say that a tensor field is smooth if the $T^{j_1, \ldots, j_m}_{i_1, \ldots, i_k}$ are smooth.

2.5 Differential forms

There is a really important class of tensor fields that appears in differential geometry : the differential forms. The goal of differential n-forms is to be integrated over n-dimensional manifolds, like a measure. They are, in a sense, linked to the idea of ("smooth") measures, of the form $f(x)d\lambda_n(x)$ where λ_n is the usual Lebesgue measure on $\mathbb{R}^n$. Indeed, the definition of an abstract smooth manifold allow us to believe that we can (locally !) transport the measure from open subset of $\mathbb{R}^n$ to our manifold.

One might ask if one of those measures is more natural than others. What makes the Lebesgue measure so special in the euclidean case ? In the euclidean case, we have at our disposal a natural notion of lengths, provided by the usual inner product, and the lebesgue measure is the only measure that actually behave well with the natural idea of volume induced by the notion of length. In an abstract manifold, we don't have this chance : there is no natural idea of length, nor a natural idea of volume, and so no measures will be special in the same way that λ is.
A manifold that actually have a notion of lengths on it, like submanifolds of $\mathbb{R}^n$, are called Riemannian manifolds. For them, one special measure will actually rise.

Let's get more into the details. First of all, we define the pullback of a k-covariant tensor : this should be understood as a change of variables.

Definition 14. Let M and N be two smooth manifolds. Let α be a k-covariant tensor field over N, and $f : M \to N$ a smooth map. We define the pullback of α by f the following k-covariant tensor field on M :

$$(f^*\alpha)_p(h_1, \dots, h_n) := \alpha_{f(p)}((df)_p(h_1), \dots, (df)_p(h_n))$$

For all $h_1, \dots, h_n \in T_pM$.

Example 12. Let $f : M \to N$, and x,y be some coordinate charts.
We have :

$$\forall h \in TpM, f^*(dy^i)_p(h) = (dy^i)_{f(p)}((df)_p(h)) = (d\widehat{f^i})(h)$$

So : $f^*(a(y)dy^{i_1} \otimes \cdots \otimes dy^{i_k}) = a(f(x))d\widehat{f}^{i_1} \otimes \cdots \otimes d\widehat{f}^{i_k}$.

Example 13. Let $f(r, \theta) = (r\cos\theta, r\sin\theta) = (x, y) \in \mathbb{R}^2$. We have :

$$f^*(dx) = d(r\cos\theta) = \cos\theta dr - r\sin\theta d\theta$$

$$f^*(dy) = d(r\sin\theta) = \sin\theta dr + r\cos\theta d\theta$$

$$f^*(dx \otimes dy) = (\cos\theta dr - r\sin\theta d\theta) \otimes (\sin\theta dr + r\cos\theta d\theta)$$
$$= \cos\theta\sin\theta dr \otimes dr + r\cos^2\theta dr \otimes d\theta - r\sin^2\theta d\theta \otimes dr - r^2\sin\theta\cos\theta d\theta \otimes d\theta$$

Now, the goal is to define something that represents the dS that appears in physics when integrating over a surface. For example, one would like to be able to give some sense to : "$dS = dxdy = rdrd\theta$". Obviously, by the upper computations, the tensor product won't work. We have to find another product, the exterior product $\wedge$, on our forms which will behaves well with the change of variables.
In the usual case, we remember the formula :

$$\int_\Omega f(\varphi(x))|\det(d\varphi)_x|dx^1 \dots dx^n = \int_{\varphi(\Omega)} f(y)dy^1 \dots dy^n$$

Which naturally motivates us for a relation of the type :

$$\varphi^*(dy^1 \wedge \cdots \wedge dy^n) = \det(d\varphi)_x(dx^1 \wedge \cdots \wedge dx^n)$$

This kind of behavior, (factorizing a determinant) often occurs when one is working with alternating multilinear maps. Having the determinant in mind, this leads us to the following definitions :

Definition 15. Let T be a k-covariant tensor over a vector space V.
We say that T is alternating iff : $T(h_1, \dots, h_i, \dots, h_j, \dots, h_k) = -T(h_1, \dots, h_j, \dots, h_i, \dots, h_k)$
We note $\Lambda^k(V)$ the vector space of all alternating k-covariant tensors.

Definition 16. Let M be a smooth manifold.
We say that a tensor field α over M is a differential form iif $\alpha_p \in \Lambda^k(T_pM)$, for all $p \in M$.
We note $\Omega^k(M)$ the space of all smooth differential forms.

By definition, $\Omega^0(M) = C^\infty(M)$.
Note also that every 1-covariant tensor is trivially alternating.

Definition 17. Let α and β be k, l-differential forms. We define the exterior product of α and β the following (k+l)-differential form :

$$(\alpha \wedge \beta)_p(v_1, \dots, v_{k+l}) = \sum_{\sigma \in S_{k+l}} \varepsilon(\sigma)(\alpha \otimes \beta)_p(v_{\sigma(1)}, \dots, v_{\sigma(k+l)})$$

Where S_n is the group of permutations of $[|1, n|]$, and ε is the signature.
The wedge product $\wedge$ is bilinear, associative, and alternating (it was defined for it) in the following sense : $\alpha \wedge \beta = (-1)^{kl} \beta \wedge \alpha$.

One can also verify that :

$$(\alpha_1 \wedge \dots \wedge \alpha_n)_p(v_1, \dots, v_N) = \sum_{\sigma \in S_N} \varepsilon(\sigma)(\alpha_1 \otimes \dots \otimes \alpha_n)_p(v_{\sigma(1)}, \dots, v_{\sigma(N)})$$

Example 14. We have :

$$(dx^1 \wedge \dots \wedge dx^n)(h_1, \dots, h_n)_p = \sum_{\sigma \in S_n} \varepsilon(\sigma) h^1_{\sigma(1)}, \dots, h^n_{\sigma(k)} = \det((h^i_j))$$

We check that this is indeed an alternating tensor. Moreover, the requested formula is verified. Indeed, if one choose a smooth map $\varphi : U \to V$, with $U, V \subset \mathbb{R}^n$, we have :

$$\begin{aligned}\varphi^*(dy^1 \wedge \dots \wedge dy^n)_x(h_1, \dots, h_n) &= (dy^1 \wedge \dots \wedge dy^n)_{\varphi(x)}(d\varphi(h_1), \dots, d\varphi(h_n)) \\ &= \det(((d\varphi)_x(h_i))^j) = \det((d\varphi)_x)\det(h^j_i)\end{aligned}$$

i.e.

$$\varphi^*(dy^1 \wedge \dots \wedge dy^n)_x = \det(d\varphi)_x(dx^1 \wedge \dots \wedge dx^n)_x$$

Example 15. From the alternating behavior of the exterior product, some little computations rules appears : $dx \wedge dy = -dy \wedge dx$, and $dx \wedge dx = 0$.
Also : $dx \wedge dy = dx \otimes dy - dy \otimes dx$.

Example 16. Let's do an explicit computation with $f(r, \theta) = (r\cos\theta, r\sin\theta)$. We have :

$$\begin{aligned}f^*(dx \wedge dy) &= d(r\cos\theta) \wedge d(r\sin\theta) = (\cos\theta dr - r\sin\theta d\theta) \wedge (\sin\theta dr + r\cos\theta d\theta) \\ &= \cos\theta\sin\theta dr \wedge dr + r\cos^2\theta dr \wedge d\theta - r\sin^2\theta d\theta \wedge dr - r^2\sin\theta\cos\theta d\theta \wedge d\theta \\ &= r\cos^2\theta dr \wedge d\theta + r\sin^2\theta dr \wedge d\theta = r dr \wedge d\theta\end{aligned}$$

This is exactly what we wanted !

Theorem 4. *Let $\omega \in \Omega^k(M)$.*
Given a coordinate neighbourhood W around p, there exists a family of maps $w_{i_1,\dots,i_k} : W \to \mathbb{R}$ such that :

$$\omega_p = \sum_{i_1 < \dots < i_k} w_{i_1 \dots i_k}(p)(dx^{i_1})_p \wedge \dots \wedge (dx^{i_k})_p$$

Definition 18. Let ω be a n-differential form over $U \subset \mathbb{R}^n$, noted $\omega_x = f(x)dx^1 \wedge \dots \wedge dx^n$.
We suppose that f have a compact support. Then, we define :

$$\int_U \omega := \int_U f(x)dx^1 \dots dx^n$$

We say that a smooth map $\varphi : V \to U$ preserve the orientation iif $\det(d\phi) > 0$.
In this case, and if φ is a diffeomorphism, one can reformulate the formula for the change of variables in the following way :

$$\int_{\varphi(V)} \omega = \int_V \varphi^*\omega$$

2.6 Integration on orientable manifolds

We now have enough tools to define the integral of a n-form on a n-manifold. We will also define the exterior derivative, and, finally, quote the Stokes theorem.

The main idea to define the integral is to copy the formula from above and apply it to the case of an abstract manifold. Since the upper theorem only holds for orientation-preserving maps, we will have to define what it means for a chart φ to preserve the orientation (recall that "det($d\varphi$)" only make sense if $d\varphi$ is an endomorphism). A manifold that admits an atlas of orientation-preserving maps will called an orientable manifold.

Definition 19. Let V be a finite dimensional vector space, and consider two ordered basis $\mathcal{B}_1 = (u_1, \dots, u_n)$ and $\mathcal{B}_2 = (v_1, \dots, v_n)$. There exists a unique linear map such that $f(u_i) = v_i$.
If $\det f > 0$, we say that $\mathcal{B}_1$ and $\mathcal{B}_2$ are equivalent.
This defines an equivalence relation that divides the set of all ordered basis of V into two equivalence classes. An orientation for V is an assignment of a positive sign to the elements of one equivalence class and a negative sign to the elements of the other.

Example 17. In $\mathbb{R}^n$, the canonical basis is usually positively oriented.

Definition 20. Let M be a manifold. An orientation for M is the given of an orientation for all tangent spaces T_pM, and of an atlas $\mathcal{A} = \{(\varphi_\alpha, U_\alpha)\}$ such that all the maps φ_α preserve the orientation, i.e. : $(d\varphi_\alpha)_x : \mathbb{R}^n \to T_pM$ maps positively oriented basis to positively oriented basis.

Notice that, given an oriented manifold and two parameterizations φ_α and φ_β, the map $\varphi_\alpha^{-1} \circ \varphi_\beta$ is still orientation preserving, wich implies : $\det d(\varphi_\alpha^{-1} \circ \varphi_\beta)_x > 0$.

Definition 21. Let M be an oriented manifold.
Let $\omega \in \Omega^n(M)$ with compact support $S := \overline{\{p \in M | \ \omega_p \neq 0\}}$.
Let (V_i) be a finite open covering of S, where the V_i are coordinates neighbourhood associated to orientation preserving maps $\varphi_i : U_i \to V_i$.
Let $\chi_i : M \to \mathbb{R}$ be an associated partition of unity, i.e. :

- supp $\chi_i \subset V_i$
- $\sum_i \chi_i = 1$ on S.

We define :

$$\int_M \omega := \sum_i \int_{U_i} \varphi_i^*(\chi_i \omega)$$

We verify that this definition does not depend neither on the choice of the covering (V_i), the choice of the coordinates (φ_i), nor the choice of the partition of unity (χ_i).

Example 18. If the manifold is nearly covered, up to a (n-1)-submanifold, by one coordinate chart φ, then we can write :

$$\int_M \omega = \int_U \varphi^* \omega$$

For example, let $S^1 \subset \mathbb{R}^2$ be the circle, oriented anticlockwise.
Let dx, $dy \in \Omega^1(S^1)$ defined by :

$$(dx)_p \ , \ (dy)_p : \quad \begin{array}{ccc} T_pS^1 & \longrightarrow & \mathbb{R} \\ (h_1, h_2) & \longmapsto & h_1 \ , \ h_2 \end{array}$$

We want to compute the following integral :

$$\int_{S^1} xdy - ydx$$

For this, we have to parameterize the circle. Let $\gamma : (0, 2\pi) \to S^1$ defined by $\gamma(\theta) = (\cos(\theta), \sin(\theta))$.
We have :

$$\int_{S^1} xdy - ydx = \int_{(0,2\pi)} \gamma^*(xdy - ydx) = \int_0^{2\pi} \cos(\theta)^2 d\theta + \sin(\theta)^2 d\theta = 2\pi$$

Example 19. With the same notations than before, we want this time to compute the integral :

$$\int_{S^2} xdy \wedge dz - ydx \wedge dz + zdx \wedge dy$$

We consider the following parameterization of the sphere, oriented by $(\partial_\theta, \partial_\varphi)$:
$\varphi : (0, \pi) \times (0, 2\pi) \to S^2$, with $\varphi(\theta, \varphi) = (\sin(\theta)\cos(\phi), \sin(\theta)\sin(\phi), \cos(\theta))$.
Some computations show that

$$\varphi^*(xdy \wedge dz - ydx \wedge dz + zdx \wedge dy) = \sin(\theta) d\theta \wedge d\varphi$$

and hence :

$$\int_{S^2} xdy \wedge dz - ydx \wedge dz + zdx \wedge dy = \int_0^{2\pi} \int_0^{\pi} \sin(\theta) d\theta d\varphi = 4\pi$$

Now that we know how to compute integral of n-forms over our manifold, it is natural to search for a way to differentiate them. The difficulty resides (it often does) in the fact that we want the operation to be independent of the choice of coordinates.
It appears that the following, the exterior derivative, provides a way to do this.

Definition 22. Let $\omega \in \Omega^k(M)$. We define the exterior derivative $d\omega \in \Omega^k(M)$ in the following way : On a coordinate neighbourhood, given a choice of coordinates, we note :

$$\omega_p = \sum_{i_1 < \cdots < i_k} w_{i_1 \ldots i_k}(p)(dx^{i_1})_p \wedge \cdots \wedge (dx^{i_k})_p$$

We then define

$$(d\omega)_p = \sum_{i_1 < \cdots < i_k} (dw_{i_1 \ldots i_k})_p \wedge (dx^{i_1})_p \wedge \cdots \wedge (dx^{i_k})_p$$

We can show that this expression is independent of the choice of coordinates, and thus, that this is well defined on all M.

The exterior derivative plays an important role in geometry : it is the basic tool for studying the De Rham Cohomology, and is one of the central notions that appears in the Stokes formula. We recall some important formulae.

Theorem 5. *Let $\omega \in \Omega^k(M)$ and $\alpha \in \Omega^j(M)$. Let $f : N \to M$ be a smooth map.*

- $d(d\omega) = 0$
- $d(f^*\omega) = f^*(d\omega)$
- $d(\omega \wedge \alpha) = d\omega \wedge \alpha + (-1)^k \omega \wedge d\alpha$

Finally, we will quote the stokes theorem.
This theorem is a generalization of the fundamental theorem of analysis, $\int_a^b f'(t)dt = f(b) - f(a)$.
It has a lot of powerful corollaries, used everyday by analysts and physicists.
First, we have to define what a manifold with boundary is.

Definition 23. We note $\mathbb{H} = \{(x^1, \ldots, x^n) \in \mathbb{R}^n | x^n \geq 0\}$ the upper half space.
A smooth manifold with boundary is a topological manifold with boundary of dimension n and a family of parameterizations $\varphi_\alpha : U_\alpha \subset \mathbb{H} \to M$ such that :

- $\bigcup_\alpha \varphi_\alpha(U_\alpha) = M$
- For all α, β such that $W := \varphi_\alpha(U_\alpha) \cap \varphi_\beta(U_\beta) \neq \emptyset$,
  $\varphi_\alpha^{-1} \circ \varphi_\beta : \varphi_\beta^{-1}(W) \longrightarrow \varphi_\alpha^{-1}(W)$ is a diffeomorphism.
- The atlas $\mathcal{A} := \{(\varphi_\alpha, U_\alpha)\}$ is maximal in the following sense : if a parameterization ψ is such that $\psi \circ {\varphi_\alpha}^{-1}$ and $\varphi_\alpha \circ \psi^{-1}$ are smooth for all $\varphi_\alpha \in \mathcal{A}$, then $\psi \in \mathcal{A}$.

We say that $p \in M$ is a boundary point if it is in the image of $\partial\mathbb{H}$ by some parameterization. We denote by ∂M the $(n-1)$-manifold of all the boundary points of M.

Example 20. If $\Omega \subset M$ is a domain on a smooth manifold M, we sometimes says that Ω is a smooth domain if $\overline{\Omega}$ is a submanifold with boundary of dimension n.

If M is an oriented manifold with boundary, it's boundary ∂M is also oriented.
For $p \in \partial M$, the tangent space $T_p(\partial M)$ is a subspace of T_pM of codimension 1.
Given $p_0 \in \partial M$, there exists a coordinate chart $(x^1(p), \ldots, x^n(p))$ defined on $V \subset M$ such that $x^n(p) = 0$ for $p \in \partial M \cap V$. The vector $n_p = \left(\frac{\partial}{\partial x^n}\right)_p$ is called an outward pointing vector at p.
We then say that an ordered basis $\mathcal{B}$ of $T_p(\partial M)$ is positively oriented if $(n_p, \mathcal{B})$ is a positively oriented ordered basis of T_pM.

Example 21. The ball $\Omega = \{(x,y) \in \mathbb{R}^2 | x^2 + y^2 \leq 1\}$ is a smooth domain of $\mathbb{R}^2$, oriented by the usual orientation of $\mathbb{R}^2$. It's boundary, the circle, is oriented anticlockwise.

Now we are ready to quote the stokes theorem.

Theorem 6. *Let M be a smooth n-manifold with boundary.*
Let $\omega \in \Omega^{n-1}(M)$. Let $\iota : \partial M \hookrightarrow M$ be the inclusion map.
We have :

$$\int_M d\omega = \int_{\partial M} \iota^* \omega$$

As I said before, this powerful formula have a lot of useful corollaries.
We will quote some of them now.

Corollary 7. *Let Ω be a smooth compact domain of $\mathbb{R}^2$. Let $f, g : \mathbb{R}^2 \to \mathbb{R}$. We have :*

$$\int_{\partial\Omega} f dx + g dy = \iint_{\Omega} \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dx dy$$

Corollary 8. *Let Ω be a smooth compact domain of $\mathbb{R}^n$. Let $f, g : \mathbb{R}^n \to \mathbb{R}$. Note n_i the i-th component of the outward pointing vector. We have :*

$$\int_{\Omega} \frac{\partial f}{\partial x^i} g d\lambda = \int_{\partial\Omega} f g n_i d\sigma - \int_{\Omega} f \frac{\partial g}{\partial x^i} d\lambda$$

Where $d\sigma$ is the surface measure.

Let's prove this one, this will help us to understand a bit more what this "surface measure" is.
We begin with the case where $g = 1$. Let

$$\omega_x = \sum_j n_j(x)(-1)^{j+1} dx^1 \wedge \cdots \wedge dx^{j-1} \wedge dx^{j+1} \wedge \cdots \wedge dx^n$$

This is the surface measure usually associated to $\partial\Omega$.
Integrating f over this measure simply means the following :

$$\int_{\partial\Omega} f d\sigma := \int_{\partial\Omega} f\omega$$

Some computations allow us to see that

$$f(x) n_i \omega_x = f(x)(-1)^{i+1} dx^1 \wedge \cdots \wedge dx^{i-1} \wedge dx^{i+1} \wedge \cdots \wedge dx^n$$

Then the stokes formula gives :

$$\int_{\partial\Omega} f n_i \omega = \int_{\partial\Omega} f(-1)^{i+1} dx^1 \wedge \cdots \wedge dx^{i-1} \wedge dx^{i+1} \wedge \cdots \wedge dx^n = \int_{\Omega} \frac{\partial f}{\partial x^i}(x) dx$$

Finally, the desired result with $g \neq 1$ comes by replacing f by fg.

This corollary of the Stokes formula is often used when dealing with PDE (but only in $\mathbb{R}^n$, this becomes absolutly false in a general manifold).

3 Riemannian Manifolds

The goal here is to do analysis on manifolds.
What we did for now was more algebraically-minded, in the same way that an abstract vector space only offers us the possibility to do algebra. An analyst, to work, needs some way to measure lengths : on vector spaces, this give rise to the well known notion of Banach and Hilbert spaces. We are then going to add an extract "inner product" on our manifold.

A Riemannian manifold will be a manifold where measuring the speed of a curve $c(t)$ running on it will make sense, and where we will be able to talk about angles. This will naturally induce a special notion of length and distance on our manifold, and this will lead to a rigidification of the notion of manifold. The jelly-ish structure of an abstract manifold solidifies into a paper-ish structure of riemannian manifold.

The sphere of radius 1 and the sphere of radius 2 will no longer be seen as the same objects, as the lenghts on the first one are dilated when going to the second one.
A real piece of paper defines a riemannian manifold. By putting it flatly on a table, one can measure the distance between two points. This piece of paper will define the same riemannian manifold as long as we play with it without tearing it : we can roll it a bit for example.

3.1 Riemannian metrics

We are going to define the analog of the inner product on manifolds.

Definition 24. Let g be a covariant 2-tensor field over M We say that :

- g is symmetric iff $g_p(u,v) = g_p(v,u)$, $\forall u, v \in T_pM$
- g is positive definite iff $g_p(u,u) > 0$, $\forall u \in T_pM \setminus \{0\}$

Given some local coordinates, one can always write, for any covriant 2-tensor g :

$$g = \sum_{ij} g_{ij} dx^i \otimes dx^j$$

With

$$g_{ij} = g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})$$

One can then verify that g is symmetric iif the matrix (g_{ij}) is, and that g is symetric positive definite iif (g_{ij}) is too.

Definition 25. A Riemannian metric on a smooth manifold M is a symmetric positive definite covariant 2-tensor g. A smooth manifold M equipped with a Riemannian metric g is called a Riemannian manifold, and is denoted by (M, g).

A Riemannian metric is therefore a smooth assignment of an inner product to each tangent space. It is usually denoted by $g_p(u,v) = \langle u, v \rangle_p$.

Definition 26. Let N be a smooth manifold and $f : N \to M$ an immersion. (ie, $(df)_p$ is always injective). Then f^*g is a metric on N, called the induced metric.

It is clear that f^*g is still a symetric covariant 2-tensor :
$(f^*g)_p(u,v) = g_{f(p)}((df)_p u, (df)_p v) = (f^*g)_p(v,u)$.
The positive definite condition comes from the fact that f is supposed to be an immersion, keeping away $(df)_p u$ to be zero if u is not zero.

In the special case where $N \subset M$ and where the immersion $f = \iota : N \hookrightarrow M$ is the inclusion, we say that N is a submanifold of M. The induced metric, ι^*g, is just the restriction of the ambiant metric g.

Example 22. The euclidian space of dimension n, $(\mathbb{R}^n, g)$, with

$$g = \sum_i dx^i \otimes dx^i$$

is a riemannian manifold.

In the simplified case of the plane $\mathbb{R}^2$, it becomes :

$$g = dx \otimes dx + dy \otimes dy$$

Example 23. The 2-sphere $S^2 \subset \mathbb{R}^3$ is a submanifold of the riemannian manifold $\mathbb{R}^3$, and thus can naturally be seen as a riemannian manifold. The induced metric g is the restriction of the usual inner product of $\mathbb{R}^3$.

In spherical coordinates, with the usual $\psi(\theta, \varphi) = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$, recall that we have :

$$\left(\frac{\partial}{\partial\theta}\right)_p = (\cos\theta\cos\varphi, \cos\theta\sin\varphi, -\sin\theta) \quad ; \quad \left(\frac{\partial}{\partial\varphi}\right)_p = (-\sin\theta\sin\varphi, \sin\theta\cos\varphi, 0)$$

Let's compute the (g_{ij}) in those coordinates :

$$g_{\theta\theta} = g_p(\partial_\theta, \partial_\theta) = \langle\partial_\theta, \partial_\theta\rangle = 1$$
$$g_{\theta\varphi} = g_{\varphi\theta} = g_p(\partial_\theta, \partial_\varphi) = \langle\partial_\theta, \partial_\varphi\rangle = 0$$
$$g_{\varphi\varphi} = g_p(\partial_\theta, \partial_\theta) = \langle\partial_\varphi, \partial_\varphi\rangle = \sin(\theta)^2$$

We can then write, in those coordinates, the usual metric on the 2-sphere :

$$g = d\theta \otimes d\theta + \sin(\theta)^2 d\varphi \otimes d\varphi$$

Example 24. We note $\mathbb{H}^2 = \{(x, y) \in \mathbb{R}^2 |\ y > 0\}$. This manifold, equipped by the following metric :

$$g = \frac{1}{y^2}(dx \otimes dx + dy \otimes dy)$$

is called the hyperbolic plane.

Two riemannian manifolds will be regarded the same if they are isometric.

Definition 27. Let (M, g) and (N, h) be Riemannian manifolds. A diffeomorphism $f : M \to N$ is said to be an isometry if $f^*h = g$, i.e., $h_{f(p)}((df)_p u, (df)_p v) = g_p(u, v)$.
Similarly, a local diffeomorphism $f : M \to N$ is said to be a local isometry if $f^*h = g$.

Example 25. The map

$$\begin{array}{cccc} f : & \mathbb{H}^2 & \longrightarrow & \mathbb{H}^2 \\ & (x, y) & \longmapsto & (a + bx, by) \end{array}$$

is an isometry of the hyperbolic plane.

A riemannian metric allows us to measure the length $||u|| := g(u, u)^{1/2}$ of a vector (as well as the angle between two vectors with the same base point). Therefore, we can measure the length $l(c)$ of a piecewise smooth curve c : $[a, b] \to M$:

$$l(c) := \int_a^b ||\dot{c}(t)||dt$$

Finally, this induces a natural distance over our riemannian manifold : the distance between two points p and q is the length of the shortest path between those two.

$$d(p, q) = \inf\{\ l(c) \mid c \text{ is a piecewise differentiable curve connecting p to q } \}$$

The topology of (M, g) induced by this distance is the same as the initial topology on the manifold M.
Of course, an isometry between two riemannian manifolds must preserve this distance.

Now that we have a metric, we can choose a special n-form on our manifold to integrate with.

Definition 28. Let (M, g) be an orientable riemannian manifold. We say that $\omega \in \Omega^n(M)$ is a riemannian volume form iif :

$$\omega_p(v_1, \dots, v_n) = \pm 1$$

for all orthonormal bases $(v_1, \dots, v_n)$ of T_pM.

As we saw earlier, given some local coordinates, all n-forms ω must be of the following form :

$$\omega_p = f(p)(dx^1)_p \wedge \dots \wedge (dx^n)_p$$

with

$$f = \omega\left(\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}\right)$$

And we can choose the coordinates such that f is strictly positive. Then, since ω is n-linear and alternating, and because of the condition of normalization, we have :

$$f = \det S$$

where S is the matrix of all the components of the $\frac{\partial}{\partial x^i}$ in some positivey oriented orthonormal basis. But then :

$$f = \det S = \sqrt{{}^tSS} = \sqrt{\det g_{ij}}$$

Since $({}^tSS)_{ij} = g_{ij}$. Hence, we can quote the following result :

Theorem 9. *Let (M, g) be an oriented connected riemannian manifold. There exist a unique riemannian volume form ω on M such that, in positively oriented coordinates :*

$$\omega = \sqrt{\det g_{ij}} dx^1 \wedge \dots \wedge dx^n$$

This form is often noted "$dvol_g$".

Now that we have this result, we can freely integrate maps on our riemannian manifold in a natural way. If $f : M \to \mathbb{R}$ is a map of compact support, and if we note ω the riemannian volume form, we define :

$$\int_M f = \int_M f \ \mathrm{d}vol_g := \int_M f\omega$$

Note that, if $f > 0$, then $\int_M f > 0$.
This allows us to take a step further in our definition of integrals.

Let K_n be an increasing sequence of compacts such that $\bigcup K_n = M$. For any measurable and non-negative map $f : M \to \mathbb{R}^+$, we define :

$$\int_M f = \sup_n \int_{K_n} f \quad \in \mathbb{R}^+ \cup \{\infty\}$$

We then define the vector space of all integrable maps :

$$\mathcal{L}^1(M) = \{f : M \to \mathbb{R}, \text{ measurable} \mid \int_M |f| < \infty\}$$

That we, as usual, quotient by the subspace of all the maps that vanish almost everywhere (that is, $\int_M |f| = 0$), creating the well known normed space $L^1(M)$.

Finally, for all $f \in L^1(M)$, we define :

$$\int_M f := \int_M f^+ - \int_M f^-$$

Where $f^+ = \max(0, f)$ and $f^- = -\min(0, f)$.

Example 26. At the end of the last part, we introduced a "surface measure" on hypersurfaces of $\mathbb{R}^n$. This is the same measure that the one induced by it's associated riemann volume form.

Recall : We choose our surface measure to be

$$\omega_x = \sum_i n_i(x)(-1)^{i+1} dx^1 \wedge \dots \wedge dx^{i-1} \wedge dx^{i+1} \wedge \dots \wedge dx^n$$

With n_i the euclidian coordinates of the outward pointing vector at x. One can verify that, for all $v_1, \dots, v_n \in T_pM$, we have :

$$\omega_x(v_1, \dots, v_n) = \det(n\ v_1\ \dots\ v_n)$$

From this, we check that ω is indeed a riemannian volume form over our submanifold.

Example 27. We study the circle of radius r, noted $S^1(r)$, oriented anticlockwise.
Seen as a submanifold of $\mathbb{R}^2$, the unit normal vector is $n = (x/r, y/r)$, and so : $\mathrm{d}vol = \frac{1}{r}(ydx - xdy)$.
With the coordinates $\gamma(\theta) = (r\cos\theta, r\sin\theta)$, we can write it in the following way :

$\mathrm{d}vol = \frac{1}{r}(r\sin\theta d(r\cos\theta) - r\cos\theta d(r\sin\theta)) = rd\theta.$

This is coherent with the upper formula : in those coordinates, we have $\frac{\partial}{\partial\theta} = (-r\sin\theta, r\cos\theta)$, and so the metric can be written as :

$$g = \langle \partial_\theta, \partial_\theta \rangle d\theta \otimes d\theta = r^2 d\theta \otimes d\theta$$

And so : $\mathrm{d}vol = \sqrt{\det(g_{ij})} d\theta = rd\theta$, as wanted.
Hence, integrating $f : S^1(r) \to \mathbb{R}$ over the circle of radius r becomes :

$$\int_{S^1(r)} f = \int_0^{2\pi} f(r\cos\theta, r\sin\theta) rd\theta$$

3.2 Tensors and metric

Let's take some time to talk a bit more about tensors. Recall that any (k,m)-tensor field is a smooth map $T : p \in M \to T_p \in \mathcal{T}^{k,m}(T_pM)$, and that, given a coordinates neighbourhood V and a choice of coordinates x, we can write :

$$T_p = \sum T^{j_1 \dots j_m}_{i_1 \dots i_k}(p) (dx^{i_1})_p \otimes \dots \otimes (dx^{i_k})_p \otimes \left(\frac{\partial}{\partial x^{j_1}}\right)_p \otimes \dots \otimes \left(\frac{\partial}{\partial x^{j_m}}\right)_p$$

Now, we are going to understand what happens when we do a change of coordinates : let's say we switch from (x^i) to some (y^i).
More precisely, $x : U \subset M \to \mathbb{R}$ and $y : V \subset M \to \mathbb{R}$ are two coordinates chart, and we study what happens when changing coordinates on $U \cap V \neq \emptyset$.

First of all, let's try to understand how the vectors $\frac{\partial}{\partial x^i}$ and $\frac{\partial}{\partial y^i}$ are linked. We have, for any smooth $f : M \to \mathbb{R}$:

$$\left(\frac{\partial}{\partial x^i}\right)_p \cdot f = \frac{\partial f \circ x^{-1}}{\partial x^i}(x(p)) = \frac{\partial f \circ y^{-1} \circ y \circ x^{-1}}{\partial x^i}(x(p))$$

$$= \sum_j \frac{\partial y^j \circ x^{-1}}{\partial x^i}(x(p)) \frac{\partial f \circ y^{-1}}{\partial y^j}(y(p)) = \sum_j \frac{\partial y^j}{\partial x^i}(p) \left(\frac{\partial}{\partial y^j}\right)_p \cdot f$$

Hence, we can write :

$$\frac{\partial}{\partial x^i} = \sum_j \frac{\partial y^j}{\partial x^i} \frac{\partial}{\partial y^j}$$

This is the usual chain rule. Now, let's see what happens for the (dx^i). We have :

$$(dx^i)_p = d(x^i \circ y^{-1} \circ y)_p = \sum_j \frac{\partial x^i \circ y^{-1}}{\partial y^j}(y(p))(dy^j)_p$$

Hence :

$$dx^i = \sum_j \frac{\partial x^i}{\partial y^j} dy^j$$

It seems here that covariant and contravariant tensors behave nicely when a change of variable occurs. For example, let's study what happens to the metric when we do this change of coordinates. Let's note g_{ij} the coordinates of the metric on the chart x and $\tilde{g}_{ab}$ the coordinates of the metric on the chart y. We have, using the multilinearity of the tensor product :

$$g = \sum_{ij} g_{ij} dx^i \otimes dx^j = \sum_{ijab} g_{ij} \frac{\partial x^i}{\partial y^a} \frac{\partial x^j}{\partial y^b} dy^a \otimes dy^b$$

Hence :

$$\tilde{g}_{ab} = \sum_{ij} g_{ij} \frac{\partial x^i}{\partial y^a} \frac{\partial x^j}{\partial y^b}$$

More generally, we have :

Theorem 10. *Let T be a (k,m)-tensor field over M. With the same notations as before, we can write under a change of coordinates the following :*

$$\tilde{T}^{b_1 \dots b_m}_{a_1 \dots a_k} = \sum T^{j_1 \dots j_m}_{i_1 \dots i_k} \frac{\partial x^{i_1}}{\partial y^{a_1}} \cdots \frac{\partial x^{i_k}}{\partial y^{a_k}} \frac{\partial y^{b_1}}{\partial y^{j_1}} \cdots \frac{\partial y^{b_m}}{\partial y^{j_m}}$$

This is how tensor fields behave under a change of coordinates.
Then, we can see that a familly of maps $(T^{j_1 \dots j_m}_{i_1 \dots i_k})$ define a tensor if and only if this familly of maps behaves like above under a change of coordinates. This is often what people mean when, in the litterature, we can read that something "behaves like a tensor".

For example, we know that the family of maps (g_{ij}) defines a 2-covariant tensor field, because we verified that it behaves nicely under a change of coordinates. Let's define a new tensor field using this one.

We denote by g^{ij} the (ij)-th coefficient of the inverse of the natrix (g_{ij}), i.e. we have :

$$\sum_j g^{ij} g_{jk} = \delta^j_k$$

where δ^j_k is the kronecker symbol.
Does those coefficients define a 2-contravariant tensor field ? I.e., does there exists a 2-contravariant tensor field, T, such that, in coordinates :

$$T = \sum g^{ij} \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j} \text{ ?}$$

If we want this to be true, we have to verify that those coefficients behave nicely under a change of coordinates. In fact, we can verify easily that $\tilde{g}^{ab} = \sum g^{ij} \frac{\partial y^a}{\partial x^i} \frac{\partial y^b}{\partial x^j}$, because those are the coefficients of the inverse of the matrix $\tilde{g}_{ab} = \sum g_{ij} \frac{\partial x^i}{\partial y^a} \frac{\partial x^j}{\partial y^b}$.

Hence, the (g^{ij}) defines a 2-contravariant tensor field over M.

This nice behavior of the inverse of the metric allows us to create a lot of new tensors by "uppering and lowering" the indices. For example, take a vector field X, denoted by $X = \sum_i X^i \frac{\partial}{\partial x^i}$. We can define a new 1-covariant tensor field, denoted by $X^\flat = \sum_i X_i dx^i$, whose coordinates will be defined by

$$X_i := \sum_j X^j g_{ji}.$$

Because of the behavior of the metric under a change of coordinates, one can verify that the family of maps (X_i) behave well under a change of coordinates, and thus, defines a 1-covariant tensor field.

Another example : Given a 2-covariant tensor with coordinates R_{ij}, one can define a new (1,1)-tensor field defined by the following coordinates :

$$R_i{}^j := \sum_k R_{ik} g^{kj}.$$

We can even make all the indices go up if we want : the following coordinates defines a 2-contravariant tensor field :

$$R^{ij} = \sum_a R_a{}^j g^{ai} = \sum_{ab} R_{ab} g^{ai} g^{bj}$$

This operation is often used in more advanced tensor calculus.

Now, let's use this new formalism to generalise a bit the domain of definition of the metric. We already know how to measure the norm of vectors : $|X|^2 = \langle X, X\rangle = \sum_{ij} X^i X^j g_{ij}$.

We can as well generalize this metric to more general tensors : since $g_{ij} = \langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\rangle$, and because of the wanted behavior under a change of coordinates, it seems natural to define :

$$\langle dx^i, dx^j\rangle := g^{ij}$$

This gives us the following formula, for any 1-covariant tensor T :

$$|T|^2 = \sum_{ij} T_i T_j g^{ij}$$

Let's generalize it further : because of the very idea of a tensor product, it seems natural to define

$$\langle R \otimes S, T \otimes U\rangle := \langle R, T\rangle\langle S, U\rangle$$

This generalize the metric on all tensor fields.

For two (k,m)-tensor fields denoted in coordinates by $T_{i_1 \dots i_k}{}^{j_1 \dots j_m}(p)$ and $S_{i_1 \dots i_k}{}^{j_1 \dots j_m}(p)$, this gives us the following formula :

$$\langle T, S\rangle = \sum T_{a_1 \dots a_k}{}^{b_1 \dots b_m} S_{c_1 \dots c_k}{}^{d_1 \dots d_m} g^{a_1 c_1} \dots g^{a_k c_k} g_{b_1 d_1} \dots g_{b_m d_m}$$

Which can be reformulated in the following way, using the previous formalism :

$$\langle T, S\rangle = \sum T_{a_1 \dots a_k}{}^{b_1 \dots b_m} S^{a_1 \dots a_k}{}_{b_1 \dots b_m}$$

For example, the norm of the metric can be computed easily :

$$|g|^2 = \sum_{ij} g_{ij} g^{ij} = \sum_i \delta_i^i = n.$$

We can also verify by the symmetry of the formula that the operation of uppering and lowering the indices becomes an isometry. For example, given a vector field X :

$$|X|^2 = \sum X^i X^j g_{ij} = \sum X^i X_i = \sum X_i X_j g^{ij} = |X^\flat|^2$$

Moreover, this norm is the same as the norm found naturally by considering tensors as multilinear maps. Indeed, for any (k,m)-tensor T, and for any vector fields $X_1, \dots, X_k$ and 1-covariant tensor fields $Y_1, \dots, Y_m$, one can verify that :

$$|T(X_1, \dots, X_k, Y_1, \dots, Y_m)| \leq |T||X_1| \dots |X_k||Y_1| \dots |Y_m|$$

3.3 Affine Connections

Take a smooth manifold M.
We do know how to compute the speed of a smooth curve $c(t)$ on M: on each time t, this will be a vector $\dot{c}(t)$ defined on the tangent plane $T_{c(t)}M$. But what if we want to compute the acceleration on our curve c ? We are going to need an operator that allows us to derive vector fields.

Let's try to understand what it could mean on submanifolds of $\mathbb{R}^N$. Take $c(t)$ a smooth curve on $M \subset \mathbb{R}^N$. This represents a little human moving on our manifold.
The vector $\dot{c}(t)$ is the speed of our human. This vector is tangent to the manifold at the point where our human is.
Now, let's take a look at the acceleration. We can write

$$\ddot{c}(t) = \ddot{c}^{\|}(t) + \ddot{c}^{\perp}(t)$$

, where $\ddot{c}^{\|} \in T_{c(t)}M$ and $\ddot{c}^{\perp}(t) \in T_{c(t)}M^{\perp}$. In this decomposition, $\ddot{c}^{\|}$ is the only acceleration that the human sees. The orthogonal contribution only serve to stay on the manifold. (Imagine going around the world walking on the equator : The human seems to go at a constant speed, this is because all the acceleration point toward the center of earth.)

Hence, denoting by $\pi_p : \mathbb{R}^N \to T_pM$ the orthogonal projection, we can define the perceived acceleration on M by the following :

$$\frac{D\dot{c}}{dt}(t) := \pi_{c(t)}(\ddot{c}(t)) \in T_{c(t)}M$$

This is called the covariant derivative of $\dot{c}(t)$.

More generally, for two vector fields X and Y defined on M, one can define the covariant derivative of Y in the direction X, denoted by $\nabla_X Y$. $\nabla_X Y$ is a vector field on M defined in the following way :

$$(\nabla_X Y)_p := \pi_p\,(D_X Y) \in T_pM$$

Where $D_X Y$ is the usual directional derivative of Y in the direction X.
Studying a bit this operator leads us to the following definition :

Definition 29. Let M be a smooth manifold.
An affine connection on M is a map $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ such that

- $\nabla_{fX+gY} Z = f\nabla_X Z + g\nabla_Y Z$
- $\nabla_X(Y + Z) = \nabla_X Y + \nabla_X Z$
- $\nabla_X(fY) = (X \cdot f)Y + f\nabla_X Y$

for all $X, Y, Z \in \mathfrak{X}(M)$ and $f, g \in \mathcal{C}^\infty(M, \mathbb{R})$.

We, then, can easily verify that in the case of submanifolds, our covariant derivative defines indeed a connection. In the general case, connections are the operator that will represent derivatives of vector fields.

Theorem 11. *Let ∇ be an affine connection on M , let $X, Y \in \mathfrak{X}(M)$ and $p \in M$. Then $(\nabla_X Y)_p \in T_pM$ depends only on X_p and on the values of Y along a curve tangent to X at p. I we choose some local coordinates $x : W \to \mathbb{R}^n$ around p, then noting*

$$X = \sum_i X^i(p)\frac{\partial}{\partial x^i} \quad , \quad Y = \sum_i Y^i(p)\frac{\partial}{\partial x^i} \ ,$$

there exists n^3 maps $\Gamma^k_{ij} : W \to \mathbb{R}$ such that :

$$\nabla_X Y = \sum_i \left(X \cdot Y^i + \sum_{jk} \Gamma^i_{jk} X^j Y^k \right) \frac{\partial}{\partial x^i}$$

This is a straighforward use of the properties of a connection :

$$\nabla_X Y = \nabla_X \left(\sum_i Y^i \frac{\partial}{\partial x^i} \right) = \sum_i \left((X \cdot Y^i) \frac{\partial}{\partial x^i} + Y^i \nabla_X \frac{\partial}{\partial x^i} \right)$$

$$= \sum_i \left((X \cdot Y^i) \frac{\partial}{\partial x^i} + \sum_j Y^i X^j \nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^i} \right).$$

Defining $\Gamma_{ij}^k := dx^k \left(\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} \right)$, (ie $\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \sum_k \Gamma_{ij}^k \frac{\partial}{\partial x^k}$) and relabeling the indices gives :

$$\nabla_X Y = \sum_i \left(X \cdot Y^i + \sum_{jk} \Gamma_{jk}^i X^j Y^k \right) \frac{\partial}{\partial x^i}.$$

The local maps Γ_{ij}^k are called the christoffel symbols associated to the connection ∇. Locally, an affine connection is uniquely determined by specifying its Christoffel symbols on a coordinate neighborhood. However, the choices of Christoffel symbols on different charts are not independent, as the covariant derivative must agree on the overlap.

Let's compute what happens to the christoffel symbols when one apply a change of coordinates: we note

$$\Gamma_{ij}^k := dx^k \left(\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} \right) \quad \text{and} \quad \tilde{\Gamma}_{ab}^c := dy^c \left(\nabla_{\frac{\partial}{\partial y^a}} \frac{\partial}{\partial y^b} \right).$$

We have :

$$\tilde{\Gamma}_{ab}^c = dy^c \left(\nabla_{\frac{\partial}{\partial y^a}} \frac{\partial}{\partial y^b} \right) = \sum_k \frac{\partial y^c}{\partial x^k} dx^k \left(\nabla_{\frac{\partial}{\partial y^a}} \left(\sum_j \frac{\partial x^j}{\partial y^b} \frac{\partial}{\partial x^j} \right) \right)$$

$$= \sum_{jk} \frac{\partial y^c}{\partial x^k} \frac{\partial^2 x^j}{\partial y^a \partial y^b} \delta_j^k + \sum_{ijk} \frac{\partial y^c}{\partial x^k} \frac{\partial x^j}{\partial y^b} dx^k \left(\nabla_{\frac{\partial}{\partial y^a}} \left(\frac{\partial}{\partial x^j} \right) \right).$$

Hence :

$$\tilde{\Gamma}_{ab}^c = \sum_j \frac{\partial y^c}{\partial x^j} \frac{\partial^2 x^j}{\partial y^a \partial y^b} + \sum_{ijk} \frac{\partial y^c}{\partial x^k} \frac{\partial x^j}{\partial y^b} \frac{\partial x^i}{\partial y^a} \Gamma_{ij}^k.$$

Thus, the christoffel symbols does not behave like a tensor under a change of coordinates, and thus does NOT defines a tensor field. Ie, there is no (2,1)-tensor field T defined on all M such that, in coordinates, ${T_{ij}}^k = \Gamma_{ij}^k$.

Example 28. The usual connection connection on $\mathbb{R}^n$ is the usual directionnal derivative. In this case,

$$\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \frac{\partial e_j}{\partial x^i} = 0$$

where e_j is the j-th vector of the canonical basis. Hence, we find that $\Gamma_{ij}^k = 0$, and then, in coordinates, we verify that the upper formula gives us the usual definition of the directionnal derivative on vector fields :

$$\nabla_X Y = \sum_i (X \cdot Y^i) \frac{\partial}{\partial x^i}.$$

Hence, we verify the following symmetry formula :

$$\nabla_X Y - \nabla_Y X = \sum_i (X \cdot Y^i - Y \cdot X^i) \frac{\partial}{\partial x^i} = [X, Y].$$

All connections that satisfy this last equality will be called symmetric. The same computations show that all the natural connections on submanifolds of $\mathbb{R}^n$ are symmetric. Symmetry of the connection is equivalent to $\Gamma_{ij}^k = \Gamma_{ji}^k$.

Connections give us a way to differentiate vector fields. But once again, the same "problem" that appeared with n-form occurs : the choice of a connection on a general manifold, even symmetric ones, is fairly arbitrary.
But again, things get better when dealing with a riemannian manifold.

In the case of submanifolds M of $\mathbb{R}^n$ again, we see that, for any vector field X, Y, Z on M :

$$X \cdot \langle Y, Z\rangle = \langle D_X Y, Z\rangle + \langle Y, D_x Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$$

We then say that the connection ∇ respects the metric.
We then have the following theorem.

Theorem 12. *Let (M, g) be a riemannian manifold.*
There exists a unique connection ∇ on M, called the Levi-Civita connection, such that :

- *The connection is symmetric : $\nabla_X Y - \nabla_Y X = [X, Y]$*
- *The connection respects metric : $X \cdot \langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$*

Its christoffels symbols are given by :

$$\Gamma^k_{ij} = \sum_l g^{kl} \left(\frac{\partial g_{jl}}{\partial x^i} + \frac{\partial g_{il}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^l} \right)$$

As we already saw, the natural connection on submanifolds satisfies those properties, and hence is the unique Levi-Civita connection on it.

Example 29. Recall the metric of S^2 in spherical coordinates :

$$g = d\theta \otimes d\theta + (\sin\theta)^2 d\varphi \otimes d\varphi$$

Applying the previous formula for the christoffels symbols gives us, for the usual connection :

$$\Gamma^{\phi}_{\theta\phi} = \Gamma^{\phi}_{\phi\theta} = \cot\theta$$

$$\Gamma^{\theta}_{\varphi\varphi} = -\sin\theta\cos\theta$$

All the other 5 christoffels symbols vanishes.
In particular, it means that :

$$\nabla_\theta \left(\frac{\partial}{\partial\theta} \right) = 0 \quad ; \quad \nabla_\varphi \left(\frac{\partial}{\partial\varphi} \right) = -\sin\theta\cos\theta \frac{\partial}{\partial\theta}$$

$$\nabla_\theta \left(\frac{\partial}{\partial\varphi} \right) = \nabla_\varphi \left(\frac{\partial}{\partial\theta} \right) = \cot\theta \frac{\partial}{\partial\varphi}$$

Example 30. Recall the metric on the hyperbolic plane $\mathbb{H}^2 = \{(x, y) \in \mathbb{R}^2 |\ y > 0\}$:

$$g = \frac{1}{y^2}(dx \otimes dx + dy \otimes dy)$$

The formula gives us all the nonvanishing christoffels symbols for its Levi-Civita connection :

$$\Gamma^y_{xx} = 1/y$$

$$\Gamma^y_{yy} = -1/y$$

$$\Gamma^x_{xy} = \Gamma^x_{yx} = -1/y$$

Hence :

$$\nabla_x \left(\frac{\partial}{\partial x} \right) = \frac{1}{y} \frac{\partial}{\partial y} \quad ; \quad \nabla_y \left(\frac{\partial}{\partial y} \right) = -\frac{1}{y} \frac{\partial}{\partial y}$$

$$\nabla_x \left(\frac{\partial}{\partial y} \right) = \nabla_y \left(\frac{\partial}{\partial x} \right) = -\frac{1}{y} \frac{\partial}{\partial x}$$

Now that we have a nice objects to compute directionnal derivatives on our manifolds, we can define the covariant derivative.

A vector field defined along a differentiable curve $c : I \to M$ is a differentiable map $V : I \to TM$ such that $V(t) \in T_{c(t)}M$ for all $t \in I$. An obvious example is the tangent vector $\dot{c}(t)$. If V is a vector field defined along the differentiable curve c with $\dot{c} \neq 0$, its covariant derivative along c is the vector field defined along c given by

$$\frac{DV}{dt}(t) := \nabla_{\dot{c}(t)} V$$

In coordinates, noting $V(t) = \sum_i V^i(t) \frac{\partial}{\partial x^i} \in T_{c(t)}M$, it becomes :

$$\frac{DV}{dt}(t) = \sum_i \left(\dot{V}^i(t) + \sum_{jk} \Gamma^i_{jk}(c(t)) V^j(t) \dot{x}^k(t) \right) \left(\frac{\partial}{\partial x^i} \right)_{c(t)}.$$

If this is zero, then the map $V(t)$ will feel like not turning around when $c(t)$ moves : We call this the parallel transport. V being parrallel transported along c exactly means that, on coordinates :

$$\forall i, \quad \dot{V}^i(t) + \sum_{jk} \Gamma^i_{jk}(c(t)) V^j(t) \dot{x}^k(t) = 0.$$

This is a first order system of ODE. Hence, being given an initial vector $V_0 \in T_{c(0)}$ and a smooth curve c, usual ODE theory tells us that there exists a unique $V(t)$, defined for t small enough, such that $V(t)$ is parallel transported along c.

One can also define the acceleration of a curve c by the following :

$$\frac{D\dot{c}}{dt} = \sum_i \left(\ddot{x}^i(t) + \sum_{jk} \Gamma^i_{jk} \dot{x}^j(t) \dot{x}^k(t) \right) \left(\frac{\partial}{\partial x^i} \right)_{c(t)}$$

If this vanishes, then the curve has the feeling of going forward at a constant speed. Curves with null acceleration will be called geodesics. Then, the equations of the geodesics are :

$$\forall i, \quad \ddot{x}^i + \sum_{jk} \Gamma^i_{jk} \dot{x}^j \dot{x}^k = 0.$$

This is a second order system of ODE. So this time, usual theorems tell us that each geodesics is characterised by a given starting point $c(0)$ and an initial speed $\dot{c}(0)$.

Example 31. On the euclidian space $\mathbb{R}^n$, the christoffels symbols are null. Hence, geodesics are actual straight lines.

Example 32. Imagine living on earth. You decide to go straight forward and see what happens. Your trajectory will describe a great circle around earth : this is actually the geodesics of S^2. Let's verify this by computing the acceleration of the following curve, moving along the equator : $c(t) = (\cos t, \sin t, 0)$. In coordinates, it corresponds to $\theta(t) = \pi/2$ and $\varphi(t) = t$. Thus : $\dot{c}(t) = (\frac{\partial}{\partial \varphi})_{c(t)}$, and hence :

$$\frac{D\dot{c}}{dt} = \nabla_\varphi \left(\frac{\partial}{\partial \varphi} \right) = \sin(\pi/2)\cos(\pi/2) \frac{\partial}{\partial \theta} = 0,$$

making our equator a geodesic. More generally, the equation of geodesics on S^2 are :

$$\begin{cases} \ddot{\theta} - \sin(\theta)\cos(\theta)\dot{\varphi}^2 = 0 \\ \ddot{\varphi} - 2\cot(\theta)\dot{\theta}\dot{\varphi} = 0 \end{cases}$$

Example 33. On the hyperbolic plane, geodesics satisfies the equations :

$$\begin{cases} \ddot{x}y - 2\dot{x}\dot{y} = 0 \\ \ddot{y}y - \dot{y}^2 + \dot{x}^2 = 0 \end{cases}$$

Geodesics are then vertical lines and half-circles.

3.4 Connection and metric

On a riemannian manifold, we now know how to compute the derivatives of vector fields. Our goal now is to generalize this construction to all tensor fields.

First of all, for any 0-tensors, i.e., for any smooth map $f : M \to \mathbb{R}$, there is a natural generalization of the notation. Given local coordinates (x^i), we note :

$$\nabla_i f = \frac{\partial f}{\partial x^i} = \partial_i f$$

With this notation, the differential of f is the covariant tensor ∇f, with coordinates $\nabla_i f$.

Similarly, for a given vector field X, one can define a (1,1) tensor ∇X that, in coordinates, is defined by :

$$\nabla_i X^j := (\nabla_{\frac{\partial}{\partial x^i}} X)^j = \partial_i X^j + \sum_k \Gamma^j_{ik} X^k.$$

Now, let's generalize this to 1-forms. Let ω be a 1-covariant tensor field over M, noted, in coordinates : $\omega = \sum_i \omega_i dx^i$. By bilinearity of the map

$$\begin{array}{ccc} \Omega^1(M) \times \mathfrak{X}(M) & \longrightarrow & \mathcal{C}^\infty(M, \mathbb{R}) \\ (\omega, X) & \longmapsto & \omega(X) \end{array},$$

it is natural to define $\nabla_Y \omega$ via the desired relation :

$$Y \cdot \omega(X) = (\nabla_Y \omega)(X) + \omega(\nabla_Y X)$$

In particular, for the forms dx^j, this gives us : $(\nabla_{\frac{\partial}{\partial x^i}} dx^j)(X) = \partial_i X^j - dx^j(\nabla_i X) = -\sum_k \Gamma^j_{ik} X^k$ Hence :

$$\nabla_i dx^j = -\sum_k \Gamma^j_{ik} dx^k$$

Then, linearity of the derivative and formula for deriving a product gives us a formula for

$$\nabla_{\frac{\partial}{\partial x^i}} \omega = \sum_i \left(\partial_i \omega^j dx^j + \omega^j \nabla_i dx^j\right) = \sum_i \left(\partial_i \omega^j + \sum_k \Gamma^j_{ik} \omega^k\right) dx^j.$$

In fact, we can once again define $\nabla \omega$ a 2-covariant tensor field, defined in coordinates by :

$$\nabla_i \omega_j = (\nabla_{\frac{\partial}{\partial x^i}} \omega)_j = \partial_i \omega_j - \sum_k \Gamma^j_{ik} \omega^k$$

Finally, the tensor product being naturally bilinear, we can naturally generalize our derivative to all tensors T. Let's take T a (k, m)-tensor field over M, noted :

$$T = \sum T^{j_1 \ldots j_m}_{i_1 \ldots i_k} dx^{i_1} \otimes \cdots \otimes dx^{i_k} \otimes \left(\frac{\partial}{\partial x^{j_1}}\right) \otimes \cdots \otimes \left(\frac{\partial}{\partial x^{j_m}}\right)$$

We can naturally define $\nabla_i T$ by just deriving it like a product of (1+k+m)-terms. This defines ∇T as a (k+1,m)-tensor field, and some computations allow us to see that, in coordinates :

$$\nabla_i T_{i_1 \ldots i_k}{}^{j_1 \ldots j_m} = \partial_i T_{i_1 \ldots i_k}{}^{j_1 \ldots j_m} - \sum_l \Gamma^l_{i i_1} T_{l \ldots i_k}{}^{j_1 \ldots j_m} - \cdots - \sum_l \Gamma^l_{i i_k} T_{i_1 \ldots l}{}^{j_1 \ldots j_m}$$

$$+ \sum_l \Gamma^{j_1}_{il} T_{i_1 \ldots i_k}{}^{l \ldots j_m} + \cdots + \sum_l \Gamma^{j_m}_{il} T_{i_1 \ldots i_k}{}^{j_1 \ldots l}$$

Notice that this definition is coherent with what we wanted with our 1-forms : For any tensor ω, and vector fields $X_1, \ldots, X_n$, we still have :

$$Y \cdot \omega(X_1, \ldots, X_n) = (\nabla_Y T)(X_1, \ldots, X_n) + \omega(\nabla_Y X_1, \ldots, X_n) + \cdots + \omega(X_1, \ldots, \nabla_Y X_n)$$

We have defined the covariant derivative of any tensor. This actually helps in some computations, and allows us to reformulate some statement.

For example, the Levi-Civita connection is supposed to be compatible with the metric, i.e. :

$$X \cdot g(Y,Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z),$$

wich exacly means $\nabla g = 0$. In coordinates, this gives us the following useful formula :

$$\partial_i g_{jk} = \sum_l \Gamma_{ij}^l g_{lk} + \sum_l \Gamma_{ik}^l g_{jl}.$$

It is then natural to ask if ∇g^{ij} also vanishes. Let's do the computations. We know that :

$$\sum_j g_{ij} g^{jk} = \delta_i^k,$$

and δ_i^k is the coordinates of a (1,1)-tensor : the identity $I : TM \to TM$.

Before going on, let's explain a bit more what's under this formula.
Creating a (1,1)-tensor by using a (2,0)-tensor and a (0,2)-tensor by this sort of formula is an example of a process called contraction. A contraction of two tensors T and S, noted in coordinates ${T_{i_1 \dots i_k}}^{j_1 \dots j_m}$ and ${S_{I_1 \dots I_K}}^{J_1 \dots J_M}$, is a tensor R defined in coordinates by a relation of the type :

$${R_{i_1 \dots i_k I_1 \dots I_K}}^{j_1 \dots j_m J_1 \dots J_M} = \sum_s {T_{i_1 \dots i_k}}^{j_1 \dots s} {S_{s \dots I_K}}^{J_1 \dots J_M}.$$

This special use of the coordinates of T and S allows us to verify that this family of maps indeed defines a tensor. Hence, ∇R makes sense, and one can verify through a bit of computations that the following natural formula is satisfied :

$$\nabla_l {R_{i_1 \dots i_k I_1 \dots I_K}}^{j_1 \dots j_m J_1 \dots J_M} = \sum_s \nabla_l {T_{i_1 \dots i_k}}^{j_1 \dots s} {S_{s \dots I_K}}^{J_1 \dots J_M} + \sum_s {T_{i_1 \dots i_k}}^{j_1 \dots s} \nabla_l {S_{s \dots I_K}}^{J_1 \dots J_M}.$$

It is important to realize that this formula is not trivial. The first derivative acts on a (k+K,m+M)-tensor, while the second and third acts on (k,m) and (K,M) tensors. Strictly speaking, those are not the same objects, and thus does not has any apparent simple reason to behave like this. One have to do the computations, making the christoffel symbols appears, to convince himself of this formula.

Now, let's get back to our g^{ij}. Differentiating δ_i^k, as a (1,1)-tensor, we find :

$$\nabla_l \delta_i^k = \partial_l \delta_i^k + \sum_m \Gamma_{li}^m \delta_m^k - \sum_m \Gamma_{lm}^k \delta_i^m = 0.$$

Hence, by our previous comment, we find that :

$$0 = \nabla_l \delta_i^k = \sum_j \nabla_l g_{ij} g^{jk} + \sum_j g_{ij} \nabla_l g^{jk}.$$

Hence : $\nabla_l g^{ij} = 0$.

Those relations, $\nabla_i g_{jk} = 0$ and $\nabla_i g^{jk} = 0$, will actually help us to define a lot of operators. For example, the contravariant derivative (for example on a map f, but it can be defined analogously for all tensor fields) :

$$\nabla^i f := \sum_j g^{ij} \nabla_j f = \sum_j \nabla_j (f g^{ji})$$

Or the so called connection laplacian, defined by contraction :

$$\Delta f := \sum_i \nabla^i \nabla_i f = \sum_i \nabla_i \nabla^i f$$

Careful that, in those kind of expression, the first derivative is deriving a tensor. A more explicit formula is :

$$\sum_i \nabla^i \nabla_i f = \sum_i g^{ij} \nabla_j \nabla_i f = \sum_i g^{ij} \left(\partial_j \partial_i f + \sum_k \Gamma_{ji}^k \partial_k f \right).$$

One can proove, after some computation, the following expression for the connection laplacian :

$$\Delta f = \frac{1}{\sqrt{|\det g|}} \sum_{ij} \partial_i(\sqrt{|\det g|} g^{ij} \partial_j f).$$

Again, recall that all of those complicated objects ∇ exists to provide sense to derivatives in such sense that it does not depends on the choice of coordinates. Boringly defining a laplacian by $\sum_i \partial_{ii}^2 f$ would not have defined the same formula depending on our coordinates. Our connection laplacian, here, has the good property that the formula does not depend on the choice of the coordinates.

Example 34. Take the usual euclidian space $(\mathbb{R}^n, g)$, with

$$g = \sum^i dx^i \otimes dx^i,$$

i.e., (g_{ij}) is the identity matrix. We know that, in cartesian coordinates, the christoffel symbols are zero. Then, we can write down the connection laplacian associated to this metric :

$$\Delta f = \sum_i \frac{\partial^2 f}{\partial x_i^2}$$

Now let's change coordinates. We decide to go on spherical coordinates, (r, θ, φ), defined by :

$$(r \sin\theta \cos\varphi, r \sin\theta \sin\varphi, r\cos\theta) = (x, y, z)$$

In those coordinates, the metric becomes :

$$g = dr \otimes dr + r^2 d\theta \otimes d\theta + r^2 (\sin\theta)^2 d\varphi \otimes d\varphi$$

Hence, we find that $\sqrt{\det(g_{ij})} = r^2 \sin\theta$, and thus, the general formula for the connection laplacian gives us the following expression :

$$\Delta f = \frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2(\sin\theta)^2} \frac{\partial^2 f}{\partial \varphi^2}$$

Which is the usual formula for the laplacian in spherical coordinates.

Example 35. Let's compute the connection laplacian on S^2, equipped with the induced metric from $\mathbb{R}^3$:

$$g = d\theta \otimes d\theta + (\sin\theta)^2 d\varphi \otimes d\varphi$$

Hence, $\sqrt{\det(g_{ij})} = \sin\theta$, and thus, the formula gives us the spherical laplacian :

$$\Delta f = \frac{1}{\sin\theta} \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial f}{\partial \theta}\right) + \frac{1}{(\sin\theta)^2} \frac{\partial^2 f}{\partial \varphi^2}$$

This is a natural expression to obtain, since this is the upper one but without taking derivative over r and putting $r = 1$ in the expression.

Example 36. On the hyperbolic plane, recall that the metric is given by :

$$g = \frac{1}{y^2}(dx \otimes dx + dy \otimes dy)$$

Hence, we have the following connection laplacian :

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + y \frac{\partial}{\partial y}\left(\frac{1}{y} \frac{\partial f}{\partial y}\right)$$

4 Curvature

Now that we know how to do a bit of analysis on riemannian manifolds, let's take a break to talk about more geometric aspects : the curvature.

Given a riemannian manifold (M, g), there exists a lot of different ways to describe its curvature. Here is one way to understand it. Suppose you, and a friend, decide to go north at a constant speed, starting from the equator. You will describe geodesics on earth, with the same initial direction. At first, it will seem like your two trajectories are parallel, but as the time goes on, you will actually approach each other, and finally meet at the north pole.

This is because the earth is curved. If you were living on $\mathbb{R}^2$ instead, you would have never met up, and your trajectory would have stayed parallel. Instead, you experimented a deflect from your original parallel-ness and approached your friend : this is a sign that the earth is positively curved. On the hyperbolic plane, instead of approaching your friend, you would have been moving away from each others. This is because of the negative curvature on the hyperbolic plane.
Another way to understand curvature is to look at triangles. On $\mathbb{R}^2$, the sum of all the angles of a triangle will always add to π. But on the earth, triangles mades from geodesics will always have angles that adds up to more than π. For example, one can easily construct a triangle on earth that have 3 right angles.
On the hyperbolic plane, all geodesic triangle will have angles that will adds to less than π.

Those way of measuring curvature are by essence intrinsic, and will stay unchanged under an isometry. For example, a piece of paper is flat, and so will be a piece of paper rolled to a cylinder. (Recall that an isometry is everything that does not tear our paper appart). This can sound counter-intuitive to think that a cylinder is called flat, but think about it this way : if you try to roll a piece of paper, there will always be a direction that will stay straight. You can not turn your paper into a sphere.

4.1 The Riemann tensor

Definition 30. Let (M, g) be a riemannian manifold, and denote by ∇ it's Levi-Civita connection. The Riemann tensor is a $(3,1)$-tensor $R : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$, defined by :

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

for all vector field $X, Y, Z \in \mathfrak{X}(M)$.

The Riemann tensor is a way of measuring the non-commutativity of the connection. The fact that this indeed defines a tensor is not clear at first. First of all, recall that the lie bracket $[X, Y]$ is defined as the only vector field such that, for all smooth real valued map f :

$$[X,Y] \cdot f = X \cdot (Y \cdot f) - Y \cdot (X \cdot f),$$

and that, in coordinates, we have :

$$[X,Y] = \sum_i \left(X \cdot Y^i - Y \cdot X^i \right) \frac{\partial}{\partial x^i}.$$

This formula allow us to easily prove the following properties :

- $[X, Y] = -[Y, X]$
- $[X + Y, Z] = [X, Z] + [Y, Z]$
- $[fX, Y] = f[X, Y] - (Y \cdot f)X$.

Now we can try to prove that R defines a tensor. As a map who takes three vector fields and maps them to one vector field, we only have to check the $C^\infty-$multilinearity of R.

Let $X, \tilde{X}, Z, T$ be vector fields over M, and f a smooth real valued map. We have :

$$R(\tilde{X}+fX,Y)Z = \nabla_{\tilde{X}+fX}\nabla_Y Z - \nabla_Y \nabla_{\tilde{X}+fX} Z - \nabla_{[\tilde{X}+fX,Y]}Z$$

$$= \nabla_{\tilde{X}}\nabla_Y Z + f\nabla_X\nabla_Y Z - \nabla_Y(\nabla_{\tilde{X}}Z + f\nabla_X Z) - \nabla_{[\tilde{X},Y]}Z - f\nabla_{[X,Y]}Z - (Y\cdot f)\nabla_X Z$$

$$= R(\tilde{X},Y)Z + fR(X,Y)Z$$

Thus, R is linear with respects to its first variable. Since $R(X,Y)Z = -R(Y,X)Z$, we see immediately that R is also C^∞−linear with respects to its second variable. Finally, for any vector fields $X, Y, Z, \tilde{Z}$ and real valued map f :

$$R(X,Y)(\tilde{Z}+Z) = R(X,Y)(\tilde{Z}) + R(X,Y)(Z)$$

is clear, and :

$$R(X,Y)(fZ) = \nabla_X\nabla_Y fZ - \nabla_X\nabla_Y fZ - \nabla_{[X,Y]}fZ$$

$$= \nabla_X\left((Y\cdot f)Z + f\nabla_Y Z\right) + \nabla_Y\left((X\cdot f)Z + f\nabla_X Z\right) - ([X,Y]\cdot f)Z - f\nabla_{[X,Y]}Z$$

$$= (Y\cdot f)\nabla_X Z + (X\cdot f)\nabla_Y Z + f\nabla_X\nabla_Y Z - (X\cdot f)\nabla_Y Z - (Y\cdot f)\nabla_X Z - f\nabla_Y\nabla_X Z - f\nabla_{[X,Y]}Z$$

$$= fR(X,Y)Z$$

Hence, R defines a (3,1)-tensor on M. Given some coordinates, we can then write :

$$R(X,Y)Z = \sum R_{ijk}{}^l dx^i \otimes dx^j \otimes dx^k \otimes \frac{\partial}{\partial x^l}$$

With

$$R_{ijk}{}^l = dx^l\left(R\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)\frac{\partial}{\partial x^k}\right).$$

We can actually compute those coefficients. First of all, remark that $\left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right] = 0$. Indeed, for all smooth real valued map, we have :

$$\left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right]\cdot f = \frac{\partial^2 f}{\partial x^i \partial x^j} - \frac{\partial^2 f}{\partial x^j \partial x^i} = 0$$

Hence, we have :

$$R\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)\frac{\partial}{\partial x^k} = \nabla_{\frac{\partial}{\partial x^i}}\nabla_{\frac{\partial}{\partial x^j}}\frac{\partial}{\partial x^k} - \nabla_{\frac{\partial}{\partial x^j}}\nabla_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^k}$$

$$= \nabla_{\frac{\partial}{\partial x^i}}\left(\sum_m \Gamma^m_{jk}\frac{\partial}{\partial x^m}\right) - \nabla_{\frac{\partial}{\partial x^j}}\left(\sum_m \Gamma^m_{ik}\frac{\partial}{\partial x^m}\right)$$

$$= \sum_m \left(\frac{\partial \Gamma^m_{jk}}{\partial x^i} - \frac{\partial \Gamma^m_{ik}}{\partial x^j}\right)\frac{\partial}{\partial x^m} + \sum_{ml}\left(\Gamma^m_{jk}\Gamma^l_{mi} - \Gamma^m_{ik}\Gamma^l_{mj}\right)\frac{\partial}{\partial x^l}$$

$$= \sum_l \left(\frac{\partial \Gamma^l_{jk}}{\partial x^i} - \frac{\partial \Gamma^l_{ik}}{\partial x^j} + \sum_m \Gamma^m_{jk}\Gamma^l_{mi} - \Gamma^m_{ik}\Gamma^l_{mj}\right)\frac{\partial}{\partial x^l}$$

Hence :

$$R_{ijk}{}^l = \frac{\partial \Gamma^l_{jk}}{\partial x^i} - \frac{\partial \Gamma^l_{ik}}{\partial x^j} + \sum_m \Gamma^m_{jk}\Gamma^l_{mi} - \Gamma^m_{ik}\Gamma^l_{mj}$$

Example 37. In the euclidian space $(\mathbb{R}^n, g)$, since all the christoffel symbols vanishes, the Riemann tensor vanishes too.

This tensor appears naturally when one is dealing with the idea mentioned before : deflection of geodesics. One of the most well knowned example is the case of the so called Jacobi-field. Let's quickly explain this.

Recall that a geodesic is a smooth curve c such that $\frac{D\dot{c}}{dt} = 0$, i.e., such that :

$$\forall i, \ddot{x}^i + \sum_{jk} \Gamma^i_{jk} \dot{x}^j \dot{x}^k = 0$$

This is a second order equation, hence, the local solution is uniquely determined by an initial point and an initial velocity. Being given a point $p \in M$, we can then define the exponential map $\exp_p : U \subset T_pM \to M$ as the map that takes $v \in T_pM$ sufficiently small, and maps it to $c(1)$, where c is the geodesic that starts from p at initial velocity v.
With this notation, we check that $c(t) = \exp_p(tv)$. In other words, another way to write the geodesic that starts at p at intial velocity v is $c(t) = \exp_p(tv)$.

Now, we can study what happens when, given $p \in M$, $t \in \mathbb{R}$, and v_0 sufficiently small, one do a little deflect on v_0. Namely, take a smooth map $v : (-\epsilon, \epsilon) \to T_pM$ such that $v(0) = v_0$, and define

$$f(t, s) := \exp_p (tv(s))$$

This is the point where p is sent when following for t seconds the geodesic with initial velocity $v(s)$. The local deviation from $c(t) = f(t, 0)$ is :

$$J(t) := \frac{\partial}{\partial s} \exp_p(tv(s))|_{s=0}$$

This vector field J along c is called a Jacobi field. This field is naturally linked to the curvature by the following theorem :

Theorem 13. *The Jacobi field J defined earlier satisfy the following equation :*

$$\frac{D^2 J}{dt^2} = R(\dot{c}, J)\dot{c}.$$

Let's prove this, to familiarize ourself a bit more with covariant derivatives and this new curvature tensor. First of all, since $t \mapsto \exp_p(tv(s))$ is a geodesic, we have by definition :

$$\frac{D}{dt}\frac{\partial f}{\partial t} = 0.$$

Recall that $\frac{DV}{dt}$ is defined as $\nabla_{\dot{\gamma}}$, where γ is the path taken by $V(t)$, in the sense that $V(t) \in T_{\gamma(t)}M$. Hence :

$$0 = \frac{D}{ds}\frac{D}{dt}\frac{\partial f}{\partial t} = \nabla_{\frac{\partial f}{\partial s}} \nabla_{\frac{\partial f}{\partial t}} \frac{\partial f}{\partial t} = \frac{D}{dt}\frac{D}{ds}\frac{\partial f}{\partial t} + R\left(\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right)\frac{\partial f}{\partial t}$$

Where we have used the fact that $[\frac{\partial f}{\partial t}, \frac{\partial f}{\partial s}] = 0$ (this will be proved soon).
Exchanging derivatives of vector fields can only be made by making this curvature term appear. However, as we will see, exchanging derivative can still be made freely when dealing with maps $\mathbb{R}^2 \to M$. Indeed :

$$\frac{D}{ds}\frac{\partial f}{\partial t} = \frac{D}{ds}\sum_i \frac{\partial \hat{f}^i}{\partial t}\frac{\partial}{\partial x^i} = \sum_i \frac{\partial^2 \hat{f}^i}{\partial s \partial t}\frac{\partial}{\partial x^i} + \sum_i \frac{\partial \hat{f}^i}{\partial t} \nabla_{\frac{\partial f}{\partial s}} \frac{\partial}{\partial x^i}$$

ie

$$\frac{D}{ds}\frac{\partial f}{\partial t} = \sum_i \frac{\partial^2 \hat{f}^i}{\partial s \partial t} + \sum_{ij} \frac{\partial \hat{f}^i}{\partial t}\frac{\partial \hat{f}^j}{\partial s} \nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^i} = \frac{D}{dt}\frac{\partial f}{\partial s}$$

This also proves that $[\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}] = \frac{D}{ds}\frac{\partial f}{\partial t} - \frac{D}{ds}\frac{\partial f}{\partial t} = 0$.
Hence, we can finish our computations :

$$0 = \frac{D}{dt}\frac{D}{ds}\frac{\partial f}{\partial t} + R\left(\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right)\frac{\partial f}{\partial t} = \frac{D^2}{dt^2}\frac{\partial f}{\partial s} + R\left(\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right)\frac{\partial f}{\partial t}$$

Taking $s = 0$ gives the desired result.

Hence, we see that the Riemann tensor appears when studying how geodesics deflect. Let's study a bit some more properties of this tensor.

Theorem 14. *(Bianchi identity) If M is a riemannian manifold then the associated curvature satisfies*

$$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0.$$

This result is a direct consequence of the jacobi identity for vector fields, namely :

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0.$$

I won't prove these results for the clarity of this paper.

Sometimes, the riemann tensor is more convenient to use if we lower the last index, creating the curvature tensor. More precisely :

Definition 31. Let X, Y, Z, W be 4 vector fields on (M, g) a riemannian manifold. We denote by R the riemann tensor. We then define :

$$R(X,Y,Z,W) := \langle R(X,Y)Z, W\rangle.$$

In coordinates, we have :

$$R_{ijkl} = R\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l}\right) = \left\langle \sum_m R_{ijk}{}^m \frac{\partial}{\partial x^m}, \frac{\partial}{\partial x^l}\right\rangle = \sum_m R_{ijk}{}^m g_{ml}$$

Hence, this definition is a special case of what we previously called "lowering an index".

This curvature tensor inherit from the symmetries of the Riemann tensor and the Levi-Civita curvature. I'll just quote the main symmetries.

Theorem 15. *If X, Y, Z, W are vector fields on (M, g) we have :*

- $R(X,Y,Z,W) + R(Y,Z,X,W) + R(Z,X,Y,W) = 0$
- $R(X,Y,Z,W) = -R(Y,X,Z,W)$
- $R(X,Y,Z,W) = -R(X,Y,W,Z)$
- $R(X,Y,Z,W) = R(Z,W,X,Y)$

These symmetries show us that this 4-tensor is a bit too large to countain all its information. An equivalent way of encoding the riemann tensor is the following definition :

Definition 32. Let Π be a 2-dimensional subspace of T_pM and let X_p, Y_p be two linearly independent elements of Π. Then, the sectional curvature of Π is defined as

$$K(\Pi) = -\frac{R(X_p, Y_p, X_p, Y_p)}{|X_p|^2|Y_p|^2 - \langle X_p, Y_p\rangle^2}$$

Note that $|X_p|^2|Y_p|^2 - \langle X_p, Y_p\rangle^2$ is the square of the area of the paral- lelogram in T_pM spanned by X_p, Y_p , and thus the above definition of sectional curvature does not depend on the choice of the linearly independent vectors X_p, Y_p . Indeed, when we change the basis of Π, both $R(X_p, Y_p, X_p, Y_p)$ and $|X_p|^2|Y_p|^2 - \langle X_p, Y_p\rangle^2$ change by the square of the determinant of the change of basis matrix. It is possible to show that all the values of all the sectional curvatures indeed characterize the riemann tensor.

In the case of 2-dimensionnal manifolds, "sectional curvature" is often what someone means when talking about curvature. In this case, the sectional curvature at a point p is often denoted by K_p. Let's compute it on some examples.

Example 38. The sectional curvature of $\mathbb{R}^2$ is zero. Indeed, for $p \in \mathbb{R}^2$, denoting by $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$ the canonical basis, one has : $K_p = -\langle R(\frac{\partial}{\partial x}, \frac{\partial}{\partial y})\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\rangle = -\langle 0, \frac{\partial}{\partial y}\rangle = 0$.

Example 39. Let's compute the sectional curvature of $S^2(r)$, the sphere of radius r. The spherical coordinates here give us the following coordinates on $S^2(r)$:

$$\psi(\theta,\varphi) = (r\sin\theta\cos\varphi, r\sin\theta\sin\varphi, r\cos\theta)$$

Which gives us the following induced metric from $\mathbb{R}^3$:

$$g = r^2 d\theta\otimes d\theta + r^2(\sin\theta)^2 d\varphi\otimes\varphi.$$

Then, using the formula giving the christoffel symbol for the l Levi-Civita connection from the metric, we find the nonvanishing christoffel symbols :

$$\Gamma^{\phi}_{\theta\phi} = \Gamma^{\phi}_{\phi\theta} = \cot\theta \quad ; \quad \Gamma^{\theta}_{\varphi\varphi} = -\sin\theta\cos\theta.$$

In particular, it means that :

$$\nabla_\theta\left(\frac{\partial}{\partial\theta}\right) = 0 \quad ; \quad \nabla_\varphi\left(\frac{\partial}{\partial\varphi}\right) = -\sin\theta\cos\theta\frac{\partial}{\partial\theta} \quad ; \quad \nabla_\theta\left(\frac{\partial}{\partial\varphi}\right) = \nabla_\varphi\left(\frac{\partial}{\partial\theta}\right) = \cot\theta\frac{\partial}{\partial\varphi},$$

and so we can write :

$$\begin{aligned} K_p &= -\frac{1}{g_{\theta\theta}g_{\varphi\varphi} - g^2_{\theta\varphi}}\left\langle \nabla_\theta\nabla_\varphi\frac{\partial}{\partial\theta} - \nabla_\varphi\nabla_\theta\frac{\partial}{\partial\theta}, \frac{\partial}{\partial\varphi}\right\rangle \\ &= -\frac{1}{r^4(\sin\theta)^2}\left\langle \nabla_\theta\left(\cot\theta\frac{\partial}{\partial\varphi}\right), \frac{\partial}{\partial\varphi}\right\rangle \\ &= -\frac{1}{r^4(\sin\theta)^2}\left\langle -\frac{1}{(\sin\theta)^2}\frac{\partial}{\partial\varphi} + (\cot\theta)^2\frac{\partial}{\partial\varphi}, \frac{\partial}{\partial\varphi}\right\rangle = \frac{1}{r^2}. \end{aligned}$$

For example, S^2 has a sectional curvature of 1 everywhere. Since the sectional curvature depends only on the metric, this is invariant under all isometries. In particular, this means that there is not open subset of $S^2(r_1)$ that is isometric to any open subset of $S^2(r_2)$ if $r_1 \neq r_2$. Also, $K(S^2(r)) \underset{r\to\infty}{\to} K(\mathbb{R}^2)$, which is geometrically coherent with the intuition.

Example 40. We define the hyperbolic plane of radius r the manifold $\mathbb{H}^2(r) := \{(x,y)\in\mathbb{R}^2 | y > 0\}$ equipped with the following metric :

$$g := \frac{r^2}{y^2}(dx\otimes dx + dy\otimes dy)$$

The nonvanishing christoffel symbols asociated to it's Levi-civita connection are :

The formula gives us all the nonvanishing christoffels symbols for it's levi-civita connection :

$$\Gamma^y_{xx} = 1/y \quad ; \quad \Gamma^y_{yy} = \Gamma^x_{xy} = \Gamma^x_{yx} = -1/y$$

Hence :

$$\nabla_x\left(\frac{\partial}{\partial x}\right) = \frac{1}{y}\frac{\partial}{\partial y} \quad ; \quad \nabla_y\left(\frac{\partial}{\partial y}\right) = -\frac{1}{y}\frac{\partial}{\partial y} \quad ; \quad \nabla_x\left(\frac{\partial}{\partial y}\right) = \nabla_y\left(\frac{\partial}{\partial x}\right) = -\frac{1}{y}\frac{\partial}{\partial x}$$

We can then compute the sectionnal curvature of the hyperbolic plane of radius r :

$$\begin{aligned} K(\mathbb{H}^2(r)) &= -\frac{y^4}{r^4}\left\langle \nabla_x\nabla_y\frac{\partial}{\partial x} - \nabla_y\nabla_x\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right\rangle \\ &= -\frac{y^4}{r^4}\left\langle -\nabla_x\left(\frac{1}{y}\frac{\partial}{\partial y}\right) - \nabla_y\left(\frac{1}{y}\frac{\partial}{\partial y}\right), \frac{\partial}{\partial y}\right\rangle \\ &= -\frac{y^2}{r^4}\left\langle \frac{1}{y^2}\frac{\partial}{\partial y} + \frac{1}{y^2}\frac{\partial}{\partial y} - \frac{1}{y^2}\frac{\partial}{\partial y}, \frac{\partial}{\partial y}\right\rangle = -\frac{1}{r^2} \end{aligned}$$

Hence, the standard hyperbolic plane has a sectional curvature of -1. Notice that, with all these examples, we just proved that there exist manifolds of constant curvature κ for all $\kappa\in\mathbb{R}$.

4.2 Ricci tensor, scalar curvature

Let's talk a bit about general relativity. In general relativity, we often say that mass and energy "deform spacetime". What does it mean actually ?

In general relativity, we think about space and time as $M = \mathbb{R}^4$ equipped with a (pseudo)-metric g. In pure vaccum, without any presence of mass anywhere, the metric g is given by :

$$g = -c^2 dt \otimes dt + dx \otimes dx + dy \otimes dy + dz \otimes dz$$

Where c is the speed of light.
Stricto sensu, this is not a metric because it does not satisfy the definite positive axiom. Instead, we call it a pseudo-metric, and $(\mathbb{R}^4, g)$ is called a pseudoriemannian manifold. Nearly everything that was defined on riemannian manifolds can still be defined on riemannian manifolds, such as the christoffel symbols (via the explicit formula that involve the metric for example), the associated connection, the riemann tensor...

A vector u is said to be timelike if $\langle u, u \rangle < 0$, null if $\langle u, u \rangle = 0$, and spacelike if $\langle u, u \rangle > 0$. For example, the vector $(1, 0, 0, 0) \in T_{(t,x,y,z)}M$ is timelike and represents a jump in time of 1 second, without moving in space. The vector $(0, 1, 1, 1) \in T_{(t,x,y,z)}M$ is spacelike and represents a jump, at $t = cste$, from where we are to the point $(x + 1, y + 1, z + 1)$.

With those definition, let's represent a ray of light moving through space and time. Say that we shoot a laser through the x-axis. A chosen photon in this light ray will have the following trajectory through space time : $\gamma(t) = (t, ct, 0, 0)$. He will have the following speed in spacetime : $\dot{\gamma}(t) = (1, c, 0, 0)$. Hence :

$$\langle \dot{\gamma}, \dot{\gamma} \rangle = -c^2 + c^2 = 0.$$

In particular, we see that, in the case of an absolutly empty space time with no mass nor gravity nor energy anywhere, light rays describe straight lines through spacetime and have null length. Notice that, in this particular case, the christoffel symbols associated to the metric vanishe (because the g_{ij} are constant), and thus, straight line are the geodesics of this manifold. The Riemann tensor also vanishes, making our manifold "flat". We then say that the trajectory of light describe null geodesics.

But it is well known that, most of the time, light doesn't go straightforward. Most of the time, light trajectory is curvy. For example, when approaching a star, a light curve will be deflected from it's initial straigh path. (This phenomenon is used by astrophysicists and is called gravitationnal lensing.)

This is because the upper metric is deformed when we add massive objects in our spacetime, like the sun. In general, the trajectory of light still describes null geodesics, but with a curved metric instead.

For example, around the sun, the metric of spacetime is often approximated by the Schwarzschild metric. Given spherical coordinates centered in the sun, we can express it like this (it is valid only outside the sun) :

$$g = -\left(1 - \frac{r_s}{r}\right) c^2 dt \otimes dt + \left(1 - \frac{r_s}{r}\right)^{-1} dr \otimes dr + r^2 d\theta \otimes d\theta + r^2 (\sin\theta)^2 d\varphi \otimes \varphi$$

Where r_s is a constant called the Schwarzschild radius (it is way smaller than the radius of the sun).
The trajectory of light in this curved spacetime will be null geodesics, ie : $\nabla_{\dot{\gamma}} \dot{\gamma} = 0$ and $\langle \dot{\gamma}, \dot{\gamma} \rangle = 0$. Hence, one (theoretically) can compute the christoffel symbols of the Schwarzschild metric, write down the equations of geodesics and the condition of nullity of the norm of $\dot{\gamma}$, and solve this set of equation to determine the trajectory of light under these conditions.

So now, we know that in general relativity, we represent the spacetime as a (pseudo)-riemannian manifold, and that we use geodesics to compute trajectory of light. We also know that, in presence of mass, this metric is deformed. But how exactly ?

The Einstein Equation tells us, given a set of mass and of energy in space, what are the possible metrics that can exists. To quote them when need to define the so called Ricci tensor, and the scalar curvature.

Definition 33. Let (M, g) be a (pseudo-)riemannian manifold. We denote by R the riemann tensor associated to the levi-civita connection on M. Given some coordinates, we define Ric the following symmetric 2-tensor :

$$Ric(X,Y) := \sum_k dx^k \left(R\left(\frac{\partial}{\partial x^k}, X\right) Y \right)$$

In coordinates, we see that it correspond to a contraction of R with himself :

$$Ric_{ij} = \sum_k R_{kij}{}^k$$

And hence,we know that Ric is well defined. (i.e., does not depend of the chosen coordinates.)

The fact that this tensor is symmetric comes directly from the symmetries of the Riemann tensor that we previously mentioned. This Ricci tensor is another information that express some notions about the curvature of a riemannian manifold.

Then, by contracting the Ricci tensor with himself again, one can define the so called scalar curvature :

$$S := \sum_i R_i{}^i = \sum_j R_{ij} g^{ji} = \langle Ric, g \rangle$$

Those informations about curvature appears in the following Einstein equation, the "master equation" of general relativity :

Theorem 16. *The general Einstein field equation is*

$$Ric - \frac{S}{2} g = E$$

where E is the so-called energy-momentum tensor of the matter content of the spacetime.

This equation links the amount of mass and energy in spacetime to its curvature. A massive object will deform spacetime around him more than a little object.
In the vaccum, the energy-momemtum tensor vanishes, and the Einstein field equation becomes :

$$Ric = 0.$$

Remember that this is just a fancy notation to talk about a huge system of equations that only depends on the metric g. Indeed, one can write the coordinates of the Ricci tensor using the christoffel symbols :

$$Ric_{ij} = \sum_k R_{kij}{}^k = \sum_k \left(\frac{\partial \Gamma^k_{ij}}{\partial x^k} - \frac{\partial \Gamma^k_{kj}}{\partial i} + \sum_m \left(\Gamma^m_{ij} \Gamma^k_{mk} - \Gamma^m_{kj} \Gamma^k_{mi} \right) \right),$$

Which themselves depends only of the metric via the formula :

$$\Gamma^k_{ij} = \sum_l g^{kl} \left(\frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^l} \right)$$

Hence, if we underline the dependence of Ricci from the metric, we see that $Ric(g) = 0$ is nothing more than a system of $\frac{n(n+1)}{2}$ highly non-linear differential equations, with $\frac{n(n+1)}{2}$ variables g_{ij}. In the case of special relativity, $n = 4$, and hence we have a huge system of 10 equations, which are very difficult to solve.

In the case of riemannian manifolds of dimension 2, the scalar curvature and the sectional curvature actually encode the same information. Indeed, one can prove that, in this case, we have $S = 2K$.

4.3 Theorema egregium

Historically, the riemann tensor, the sectional curvature, the Ricci tensor and the scalar curvature were not how we talked first about curvature of surfaces. If we take an embbeded riemannian 2-manifold in $\mathbb{R}^3$, that is, a submanifold of dimension 2 in $\mathbb{R}^3$, one natural way to see if something is curved is to look if the normal vector moves.
More precisely, take $M \subset \mathbb{R}^3$ a smooth surface. Choose $\varphi(x, y)$ a local parameterization of our surface in a coordinate neighbourhood W. Then, in our coordinates chart, there exists a unique normalized vector $n_p \in T_pM^\perp$ such that $(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, n_p)$ is a positively oriented basis of $\mathbb{R}^3$.
This induces a local map, called the gauss map, defined by :

$$G : p \in W \subset M \mapsto n_p \in S^2.$$

This map is smooth, and its differential can be identified to the following symmetric tensor :

$$(dG)_p : h \in T_pM \mapsto (dG)_p h \in T_pS^2 \simeq T_pM$$

Hence, there exist orthonormal direction $(u, v) \in T_pM$, and eigenvalues λ, μ, such that :

$$(dG)_p u = \lambda u \quad ; \quad (dG)_p v = \lambda v$$

The values λ and μ are called the principal curvatures at p. They, of course, depend on the immersion of the manifold M into $\mathbb{R}^3$: i.e., they can change under the action of an isometry. We call $H := \frac{1}{2}(\lambda + \mu)$ the mean curvature at p, and $\kappa = \lambda\mu$ the Gauss Curvature at p.

Naturally, the mean curvature depends on the embedding. But surprisingly, the Gauss curvature can be proved to not depend on the chosen embbeding, but only on the intrinsic geometry of the surface. In fact, one can proove that the gauss curvature of a surface is equal to it's sectionnal curvature, which is invariant under the action of an isometry.
Gauss was so pleased by this discovery that he called this result *theorema egregium* (remarkable theorem).

Example 41. We take a sheet of paper : $M :=]0, 1[^2 \times \{0\} \subset \mathbb{R}^3$, equipped with the induced metric $g = dx \otimes dx + dy \otimes dy$. The gauss map is given by :

$$G(p) = \frac{\partial}{\partial z} \in S^2.$$

It is a constant map, hence, it's differential is null, and we have :

$$\lambda = \mu = 0 \quad ; \quad \lambda\mu = 0.$$

Now, we roll our piece of paper to a cylinder using the following isometry :

$$\varphi(x, y) := (x, \cos y, \sin y).$$

We check that this is indeed an isometry and that we haven't teared our paper :

$$\langle (d\varphi)_p u, (d\varphi)_p v\rangle = \langle (u_x, -\sin(y)u_y, \cos(y)u_y), (v_x, -\sin(y)v_y, \cos(y)v_y u)\rangle = u_x v_x + u_y v_y = \langle u, v\rangle.$$

Hence, our (almost) cylinder $\varphi(M)$ is our same sheet of paper, but isometrically rolled. We have:

$$\frac{\partial \varphi}{\partial x} = (1, 0, 0) \quad ; \quad \frac{\partial \varphi}{\partial y} = (0, -\sin y, \cos y).$$

Hence, the Gauss map is given by :

$$G : p \in \varphi(M) \mapsto \frac{\partial \varphi}{\partial x} \times \frac{\partial \varphi}{\partial y} = (0, -\cos y, -\sin y) \in S^2$$

We easily find the eigenvalues of the differential :

$$\partial_x G = 0 \times \frac{\partial \varphi}{\partial x} \quad ; \quad \partial_y G = -\frac{\partial \varphi}{\partial y}$$

Hence, $\lambda = 0; \mu = -1$, and so $\lambda\mu = 0$.
In this example, we indeed verified that the gauss curvature stays unchanged under an isometry, which is not the case for the principal curvature values.

5 Geometric analysis

Now that we introduced the concept of riemannian manifolds and that we have discussed curvature and a bit of physics, let's try to do a bit more of analysis. My goal here will be to introduce Sobolev spaces on riemannian manifolds, for maps and tensors. Then, I will briefly explain the topic of the paper of Justin Corvino, quote the theorem that I studied, and present some used techniques.

5.1 The Sobolev spaces

Let (M, g) be an orientable riemannian manifold. Being given an integrable measurable map $f : M \to \mathbb{R}$, I will denote by $\int_M f$ the integral $\int_M f dvol_g$, where $dvol_g$ is the riemannian volume form.
We will denote by $[f]$ the equivalence class of f under the equivalence relation :
$f \sim g \Leftrightarrow$ "$f = g$ almost everywhere".

Let $\Omega \subset M$ be a smooth open domain of M. We define :

$$L^2(\Omega) = \left\{ [f] \mid f : M \to \mathbb{R} \text{ measurable }, \int_\Omega |f|^2 < \infty \right\}$$

This vector space, equipped with the following inner product :

$$\langle f, g \rangle_{L^2(\Omega)} := \int_\Omega fg$$

defines a Hilbert space.
If desired, one can add a weight $\rho : \Omega \to \mathbb{R}$, defining the weighted L^2 spaces :

$$L^2_\rho(\Omega) := \left\{ [f] \mid f : M \to \mathbb{R} \text{ measurable }, \int_\Omega |f|^2 \rho < \infty \right\}$$

equipped with the obvious inner product :

$$\langle f, g \rangle_{L^2_\rho} := \int_\Omega fg\rho$$

Those spaces can be naturally generalized to countain tensors instead of maps.

If T be a measurable (k,m)-tensor field over Ω, we say that T is in $L^2(\Omega)$ if $|T|$ is in $L^2(\Omega)$. Then, for T and S two L^2-tensor, we define :

$$\langle T, S \rangle_{L^2(\Omega)} = \int_\Omega \langle T, S \rangle.$$

Then, with (k,m) fixed, the set of all (equivalence classes of) (k,m)-tensors in $L^2(\Omega)$, equipped with the previous inner product, is an Hilbert space.
Those notations can seems ambiguous, but actually it will always be clear from the context in wich L^2-space we work on.

Now we are going to define the sobolev spaces. First of all, let's denote by $C_k(\Omega)$ the following vector space :

$$C_k(\Omega) := \{\varphi \in C^\infty(\Omega) \mid \forall l \leq k, \nabla^{(l)}\varphi \in L^2(\Omega)\},$$

where we have noted

$$\nabla^{(k)}\varphi := \nabla \dots \nabla\varphi.$$

For example, in the particular case of the space $C_1(\Omega)$, and if Ω is covered entirely by one coordinate chart,the condition $\nabla f \in L^2(\Omega)$ can be written the more explicit way :

$$\int_\Omega |\nabla f|^2 = \sum_i \int_\Omega (\nabla_i f)(\nabla^i f) = \sum_{ij} \int_\Omega (\nabla_i f)(\nabla_j f) g^{ij} < \infty$$

If our domain Ω is compactly contained in M, then we have : $C_k(\Omega) = C^\infty(\Omega)$.

Definition 34. For all $\varphi, \psi \in C_k(\Omega)$, we define :

$$\langle \varphi, \psi \rangle_{H^k(\Omega)} = \int_\Omega \varphi\psi + \int_\Omega \langle \nabla\varphi, \nabla\psi \rangle + \cdots + \int_\Omega \langle \nabla^{(k)}\varphi, \nabla^{(k)}\psi \rangle$$

This defines an inner product on $C_k(\Omega)$.

Definition 35. We define the so-called Sobolev space $H^k(\Omega)$ as the completion of $(C_k(\Omega), ||.||_{H^k(\Omega)})$. Equipped with the induced norm, this defines a Hilbert space.

This space is the natural generalization of the euclidian Sobolev space. By construction, $C_k(\Omega)$ is a dense subset of $H^k(\Omega)$.

Definition 36. Analogously, we can define the set of all (K,M)-tensors in $H^k(\Omega)$ by the following construction : define

$$C_k^{(K,M)} := \{T \text{ smooth (K,M)-tensor field over } \Omega \mid \forall l \leq K, \nabla^{(l)}T \in L^2(\Omega)\},$$

then define the following inner product on $C_k^{(K,M)}(\Omega)$:

$$\langle T, S \rangle_{H^k(\Omega)} = \int_\Omega \langle T, S \rangle + \int_\Omega \langle \nabla T, \nabla S \rangle + \cdots + \int_\Omega \langle \nabla^{(k)}T, \nabla^{(k)}S \rangle$$

We then define $H^k(\Omega)$ as the completion of $C_k^{(K,M)}(\Omega)$ with respect to this inner product.

The Sobolev spaces arise naturally when we try to solve differential equations. A lot of operators, that exists in $C_k(\Omega)$, can be extended by continuity to $H^k(\Omega)$. Differential operators, such as the connection laplacian, will naturally be studied in these spaces.

Example 42. We are going to define the notion of weak derivative.
The linear map

$$\nabla : C_1(\Omega) \to L^2(\Omega)$$

is clearly continous, since

$$\int_\Omega |\nabla\varphi|^2 \leq ||\varphi||^2_{H^1(\Omega)}.$$

Then, as $C_1(\Omega)$ is dense in $H^1(\Omega)$, and since $L^2(\Omega)$ is an Hilbert, we know that there exists a unique continuous extension

$$\nabla : H^1(\Omega) \to L^2(\Omega)$$

This is called the weak derivative of a map $f \in H^1(\Omega)$. Analogously, there exists a unique way to continously extends all the maps $\nabla^{(l)} : C_{k+l}(\Omega) \subset H^{(k+l)}(\Omega) \to C_k(\Omega) \subset H^k(\Omega)$.

By continuity, weak derivatives of maps still behave like we are used to. For example :

$$\forall f, g \in H^1(\Omega), \nabla(fg) = (\nabla f)g + f(\nabla g).$$

This notion of weak derivative allow us to give an explicit formula for the induced norm over $H^k(\Omega)$ by just taking the natural continuation :

$$\langle f, g \rangle_{H^k(\Omega)} = \int_\Omega fg + \int_\Omega \langle \nabla f, \nabla g \rangle + \cdots + \int_\Omega \langle \nabla^{(k)}f, \nabla^{(k)}g \rangle.$$

With this formula for the norms $||.||_{H^k(\Omega)}$, it is not difficult to see that the weak derivative

$$\nabla : H^{k+1}(\Omega) \to H^k(\Omega)$$

is also a continuous map.

Example 43. Computing integrals assumes to have coordinates-free objects at our disposal. Only tensors that are defined globally will have a chance to behave well.

For example, suppose that our subset Ω is covered by one coordinate neighbourhood. We choose x^i some coordinates over Ω. Therefore, and this is only because we supposed so, for $f \in H^1(\Omega)$, $\nabla_i f$ is defined globally.
Most of the time, only the tensor ∇f would be defined globally, and the expression $\nabla f = \sum_i \nabla_i f \frac{\partial}{\partial x^i}$ would only make sense locally.

But let's suppose that our coordinates allow us to compute $\nabla_i f$ on all Ω. Since $f \in H^1(\Omega)$, we know that $\nabla f \in L^2(\Omega)$, which means exactly that :

$$\int_\Omega (\nabla_i f)(\nabla_j f) g^{ij} < \infty$$

Notice that this condition does not imply that $\nabla_i f \in L^2(\Omega)$, because we don't know how the g^{ij} behave !
Still, in the special case where $\overline{\Omega}$ is compact, we can prove that $f \in H^1(\Omega)$ implies that $\nabla_i f \in L^2(\Omega)$, for all i. (But once again, this only works if we can cover our domain entirely by one coordinate chart, since in the other case it won't even makes any sense.)

Example 44. Choose a smooth vector field X on Ω. Given some coordinates, we define :

$$\mathrm{div} X := \sum_i \nabla_i X^i.$$

This is the contraction of the (1,1)-tensor ∇X with himself, and hence this quantity does not depends on the choice of coordinates.

Making the christoffel symbols appear, we can write it like this :

$$\mathrm{div} X = \sum_i \partial_i X^i + \sum_{ij} \Gamma^i_{ij} X^j$$

It seems natural to hope that this can be extend into a continuous map div : $H^1(\Omega) \to L^2(\Omega)$.

Indeed, this is the case. We are going to prove it by using some special coordinates.
First of all, choose $X \in C_1^{(0,1)}(\Omega)$ a smooth vector field. Fix $p \in \Omega$. We admit that there exist coordinates $(x^i)_i$ around p, called the normal coordinates, such that : $g_{ij}(p) = g^{ij}(p) = \delta_i^j$. Of course, this only works at the point p, even in these normal coordinates, we wont have $g_{ij}(q) = g^{ij}(q) = \delta_i^j$ for $q \neq p$ in general.
In these coordinates, at the point p, we have :

$$|(\mathrm{div} X)(p)| = \left|\sum_i (\nabla_i X^i)(p)\right| = \left|\sum_i (\nabla X)_p \left(\left(\frac{\partial}{\partial x^i}\right)_p, (dx^i)_p\right)\right| \leq n|\nabla X| \times \left|\left(\frac{\partial}{\partial x^i}\right)_p\right| \times |(dx^i)_p| = n|(\nabla X)_p|$$

And now, we see that the resulting inequality is coordinates free, and hence is valid for all $p \in \Omega$. Integrating over Ω gives :

$$\int_\Omega |\mathrm{div} X|^2 \leq n \int_\Omega |\nabla X|^2 \leq n||X||^2_{H^1(\Omega)}$$

Hence, div : $C_1^{(0,1)}(\Omega) \to L^2(\Omega)$ defines a continuous operator.
Thus, we can extend it continuously to a map div : $H^1(\Omega) \to L^2(\Omega)$.

Analogously, we can define a continuous operator div : $H^{k+1}(\Omega) \to H^k(\Omega)$.

Example 45. Define, for any map $f \in H^1(\Omega)$, the following vector field, in local coordinates :

$$\mathrm{grad} f := \sum_i \nabla^i f \frac{\partial}{\partial x^i}$$

grad$f$ is just the vector field obtained when uppering the indices of the tensor ∇f. Hence, $|\mathrm{grad} f| = |\nabla f|$ and thus grad defines a continuous map $H^{k+1}(\Omega) \to H^k(\Omega)$.

Now, let f be a smooth map $\Omega \to \mathbb{R}$. Remember that we defined the connection laplacian of f by :

$$\Delta f := \sum_i \nabla_i \nabla^i f.$$

We can re-write it in the following way :

$$\Delta f = \mathrm{div}(\mathrm{grad} f).$$

Hence,we see that the connection laplacian can be naturally extended, in the following way :

$$\begin{array}{rccc} \Delta : & H^{k+2}(\Omega) & \longrightarrow & H^k(\Omega) \\ & f & \longmapsto & \mathrm{div}(\mathrm{grad} f) \end{array}.$$

5.2 Differential Equations

As said before, the Sobolev spaces will be one of the most suited places to deal with linear differential operators. One strategy that is often used when we are trying to solve an equation $L(f) = g$ is, first, to search solutions to this equations in a weak sense, where we will be able to find one more easily. Then, we try to show that our weak solution is indeed a classical solution. To explain this a bit more, let's define the notion of formal adjoint.

Definition 37. Let L be a linear differential operator of order k. In particular, $L : H^k(\Omega) \to L^2(\Omega)$ is a continuous linear map. There exists a unique linear differential operator of order k, denoted by $L^* : H^k(\Omega) \to L^2(\Omega)$, such that :

$$\forall \varphi, \psi \in C_c^\infty(\Omega) \ , \ \langle L(\varphi), \psi \rangle_{L^2(\Omega)} = \langle \varphi, L^*(\psi) \rangle_{L^2(\Omega)}$$

We call L^* the formal adjoint of L.

Formal adjoints are used to define the notion of weak solutions to differential equations.

Definition 38. Let $L : H^k(\Omega) \to L^2(\Omega)$ be a linear differential operator of order k. We say that $f \in L^2(\Omega)$ is a weak solution of the equation $L(f) = g$, for $g \in L^2(\Omega)$, if and only if :

$$\forall \varphi \in C_c^\infty(\Omega), \ \langle f, L^*(\varphi) \rangle_{L^2(\Omega)} = \langle g, \varphi \rangle$$

Example 46. Let's try to compute the formal adjoint of $\mathrm{div} : H^1(\Omega) \to L^2(\Omega)$. Let Φ be a smooth vector field, with compact support in Ω. Let ψ be a smooth, real valued map, with compact support in Ω. It is a classic corollary of the Stokes formula that, for any smooth vector field X with compact support :

$$\int_\Omega \mathrm{div} X = 0$$

(This is the divergence theorem.) Replacing X by $\psi\Phi$ in the divergence gives, given some local coordinates :

$$\mathrm{div}(\psi\Phi) = \sum_i \nabla_i(\psi\Phi^i) = \sum_i \left(\psi \nabla_i \Phi^i + (\nabla_i \psi)\Phi^i\right) = \psi \mathrm{div}(\Phi) + \langle \Phi, \mathrm{grad}\psi \rangle.$$

Hence, plugging this into the integral gives :

$$\langle \mathrm{div}(\Phi), \psi \rangle_{L^2(\Omega)} = -\langle \Phi, \mathrm{grad}\psi \rangle_{L^2(\Omega)}.$$

Thus, we see that the formal adjoint of $\mathrm{div} : H^1(\Omega) \to L^2(\Omega)$ is $\mathrm{grad} : H^1(\Omega) \to L^2(\Omega)$.

Example 47. From this previous example, we are able to easily compute the formal adjoint of the laplacian. We have, for any smooth φ, ψ with compact support :

$$\int_\Omega (\Delta\varphi)\psi = -\int_\Omega \langle \nabla\varphi, \nabla\psi \rangle = \int_\Omega \varphi(\Delta\psi)$$

Hence, Δ is formally self-adjoint. Moreover, these computations allow us to check that it also is a negative operator, in the following sense :

$$\forall \varphi \in C_c^\infty(\Omega), \langle \varphi, \Delta\varphi \rangle \leq 0.$$

5.3 Deformation of the scalar curvature

Now, we have all the tools to talk quickly about the paper of Justin Corvino : "Scalar curvature Deformation and a Gluing Construction for the Einstein Constraint Equations."

First of all, let's explain what this is all about.
Choose a riemannian manifold (M, g_0), and choose $\Omega \subset M$ a compactly contained smooth domain. Denote by $R(g_0) : M \to \mathbb{R}$ the associated scalar curvature. The question is the following : if we choose a small and localized deformation of the scalar metric, that is, a map $S : M \to \mathbb{R}$ such that $S = R(g_0)$ outside Ω, and such that $S - R(g_0)$ is small enough in Ω, can we find a metric g, close to g_0, such that $R(g) = S$?
The paper of Justin Corvino gives an answer to this question. Before quoting the theorem, we will have to define some notations. Denote by d the distance to the boundary : $d(x) := d(x, \partial\Omega)$.
We choose a smooth weight, $\rho : \Omega \to \mathbb{R}$, such that :

- $\rho = 1$ when $d \geq d_0$
- $\rho \sim_{d\to 0} d^N$ sharply, for N large enough
- $|\nabla^{(l)}\rho| \sim_{d\to 0} d^{N-l}$

And then, we define $L^2_{\rho^{-1}}(\Omega)$ the ρ^{-1}-weighted L^2 space. Since $f \in L^2_{\rho^{-1}}(\Omega)$ means :

$$||f||^2_{L^2_{\rho^{-1}}(\Omega)} = \int_\Omega |f|^2\rho^{-1} < \infty,$$

all the maps in this space will be forced to decay quickly when approaching the boundary.

We also define the banach space $C^{k,\alpha}(\Omega)$, normed by the following :

$$||f||_{k,\alpha} = \sum_{l\leq k} \sup_{x\in\Omega} |\nabla^{(l)} f(x)| + \sup_{x\neq y\in\Omega} \frac{|\nabla^{(k)} f(x) - \nabla^{(k)} f(x)|}{d(x,y)^\alpha}$$

And finally, we define $C^{k,\alpha}_{\rho^{-1}}(\Omega) := C^{k,\alpha}(\Omega) \cap L^2_{\rho^{-1}}(\Omega)$: this is the set of all maps that are "smooth enough" and that "decay quickly enough" when approaching the boundary. The following norm gives it a structure of a Banach space : $||f||_{k,\alpha,\rho^{-1}} := ||f||_{L^2_{\rho^{-1}}(\Omega)} + ||f||_{k,\alpha}$.

We are now ready to quote the theorem.

Theorem 17. *(Justin Corvino, 2000)*

Let (M, g_0) be a riemannian manifold, where g_0 is a $C^{k+4,\alpha}$ metric.
Denote by $L_{g_0} : C^{k+4,\alpha}(\Omega) \to C^{k+2,\alpha}(\Omega)$ the linearization of the scalar curvature $R : g \mapsto R(g)$ at g_0. If its formal adjoint $L^*_{g_0} : H^2_{loc}(\Omega) \to L^2_{loc}(\Omega)$ is injective, then there exists $\varepsilon > 0$, such that :
For any $S \in C^{k,\alpha}$ such that

- *$supp(S - R(g_0)) \subset \overline{\Omega}$*
- *$S - R(g_0) \in C^{k,\alpha}_{\rho^{-1}}(\Omega)$*
- *$||S - R(g_0)||_{k,\alpha,\rho^{-1}} \leq \varepsilon$*

there exists g, a $C^{k+2,\alpha}$-metric, such that $R(g) = S$.

First of all, let's explain the strategy. How can one come up with this theorem ?
Let's take a closer look at what we are trying to do. Our goal, here, is to find a metric g, such that $R(g) = S$. This is nothing more than a 2nd order, non linear, differential equation, whose unknown is the metric.

One classical approach to solve this kind of equations is to use the Newton algorithm. It goes like this :

- First of all, linearize the problem. Instead of studying $R(g) = S$, we linearize at g_0 and study the following problem :

$$L_{g_0}(h) = S - R(g_0)$$

- Then, we solve the linearized equation, with care. To ensure that our solution to this linearized problem has nice properties (like h being small and smooth), we will have to solve it via a variationnal approach, making the adjoint appear here.
- Once having solved $L_{g_0}(h_0) = S - R(g_0)$, we define $g_1 = g_0 + h_0$. And then we iterate the process : we linearize at g_1, solve the linearized problem, define $g_2 = g_1 + h_1$, etc.
- To finish, we have to prove that the created sequence of metrics g_n converges to a metric g that solves the problem.

We will try to see, roughly, what happens when we try to do this.

5.4 The linearization of the scalar curvature

Theorem 18. *The scalar curvature $R : C^{k+2,\alpha}(\Omega) \to C^{k,\alpha}$ is a differentiable map around g_0, and we have the following expansion :*

$$R(g_0 + h) = R(g_0) + L_{g_0}(h) + O(||h||_{k+2,\alpha})$$

with :

$$L_{g_0}(h) = -\Delta(\langle g, h\rangle) + div\ div\ h - \langle Ric, h\rangle$$

That is, in coordinates notations :

$$L_{g_0}(h) = -\sum_{ij} \Delta(g^{ij}h_{ij}) + \sum_{ij} \nabla^i\nabla^j h_{ij} - \sum_{ij} Ric^{ij}h_{ij}.$$

If we can easily see that the scalar curvature is indeed differentiable, the computations to provide a formula for L_{g_0} are quite long and technical, but are doable with all the tools that we have developped here. The computations are clearly presented in the blog of Terence Tao, in a post called "285G, Lecture 1: Flows on Riemannian manifolds".

Notice that the scalar curvature takes a smooth symmetric 2-tensor h and maps it to a real valued map f. Hence, its formal adjoint will take a real valued map f, and maps it to a symmetric 2-tensor h.

Theorem 19. *The formal adjoint of L_{g_0} can be written as :*

$$L^*_{g_0}(f) = -\Delta(f)g_0 + Hessf - fRic(g_0)$$

That is, in coordinates :

$$L^*_{g_0}(f)_{ij} = -\Delta(f)(g_0)_{ij} + \nabla_i\nabla_j f - fRic_{ij}$$

In the case where this formal adjoint is injective, one can prove the following estimate :

Theorem 20. *There exists $C > 0$ such that :*

$$\forall f \in H^2_\rho(\Omega), ||f||_{H^2_\rho(\Omega)} \leq C||L^*_{g_0}f||_{L^2_\rho(\Omega)}$$

where :

$$||f||^2_{H^2_\rho(\Omega)} := \int_\Omega |f|^2\rho + \int_\Omega |\nabla f|^2\rho + \int_\Omega |\nabla\nabla f|^2\rho.$$

$H^2_\rho(\Omega)$ is a weighted sobolev space : it is defined analogously than $H^2(\Omega)$, but with respects to the upper norm.

5.5 Solving the linearized problem

Now, we will explain briefly how to solve conveniently our linearized problem $L^*_{g_0}(f) = S - R(g_0)$. Since $(S - R(g_0)) \in L^2_{\rho^{-1}}(\Omega)$ by hypothesis, we can write it in the form : $S - R(g_0) = f\rho$, with $f \in L^2_\rho(\Omega)$. Then, define the following functional :

$$\begin{array}{rrcl} J: & H^2_\rho(\Omega) & \longrightarrow & \mathbb{R} \\ & u & \longmapsto & \frac{1}{2}\int_\Omega |L^*_{g_0}u|^2\rho - \int_\Omega fu\rho \end{array}$$

The estimates quoted previously allow us to show that this map admits a minimum in $H^2_\rho(\Omega)$. Indeed, we have :

$$|J(u)| \geq \frac{1}{2}||L^*_{g_0}u||^2_{L^2_\rho(\Omega)} - |\langle f, u\rangle_{L^2_\rho(\Omega)}| \geq \frac{1}{2C^2}||u||^2_{H^2_\rho(\Omega)} - ||f||_{L^2_\rho(\Omega)}||u||_{H^2_\rho(\Omega)}$$

Hence, $J(u) \to \infty$ when $||u||_{H^2_\rho(\Omega)} \to \infty$. This proves that there exists a bounded subset $A \subset H^2_\rho(\Omega)$, such that $\inf_{H^2_\rho(\Omega)} J = \inf_A J$.
Then, by taking a minimizing sequence (u_i) which will be bounded, one can extract a subsequence that converge weakly to a map $u \in H^2_\rho(\Omega)$, and then prove that this u is a minimum of J.

Once we know that there exists a $u_0 \in H^2_\rho(\Omega)$ such that $J(u_0) = \inf J$, we can compute the Euler-Lagrange equations associated to this functional. We have, for all $\eta \in C^\infty_c(\Omega)$:

$$\frac{d}{dt}|_{t=0} J(u_0 + t\eta) = 0$$

i.e. :

$$\int_\Omega \langle L^*_{g_0}f, L^*_{g_0}\eta\rangle\rho - \int_\Omega f\eta\rho,$$

which is exactly the weak formulation of

$$L_{g_0}(\rho L^*_{g_0}u_0) = f\rho = S - R(g_0).$$

Hence, we have found $u_0 \in H^2_\rho(\Omega)$ such that $h_0 := \rho L^*_{g_0}u_0 \in L^2_{\rho^{-1}}(\Omega)$ solves our linearized equation weakly. But we want more.

It appears that, conveniently, the operator $\mathcal{P} := L_{g_0}(\rho L^*_{g_0})$ is an elliptic operator of degree 4. Moreover, its coefficients are smooth enough ($C^{k+4,\alpha}$), and $S - R(g_0)$ is also smooth enough ($C^{k,\alpha}$). Hence, our weak solution $u_0 \in H^2_\rho(\Omega)$ is upgraded, via elliptic regularity, to a classical solution $u_0 \in C^{k+4,\alpha}$.

Finally, defining $h_0 := \rho L^*_{g_0}u_0$ gives us a solution to $L_{g_0}h_0 = S - R(g_0)$ that is in $C^{k+2,\alpha}_{\rho^{-1}}(\Omega)$.

5.6 The final step

Now, recall that our goal is to do the Newton Algorithm. Hence, we need to check some things : that h_0 is small enough, and that $R(g_1)$ is closer to S than $R(g_0)$ already was.
Fortunately enough, one can prove something that looks like this.

Theorem 21. *We have the following estimates :*

$$||h_0||_{k+2,\alpha,\rho^{-1}} = O(||S - R(g_0)||_{k,\alpha,\rho^{-1}})$$

$$||S - R(g_1)||_{k,\alpha,\rho^{-1}} = O(||S - R(g_0)||^2_{k,\alpha,\rho^{-1}})$$

So, things seem to go well for our algorithm. A remark shall be made here : the technique used to obtain these inequalities uses the so-called Schauder Theory. In Schauder theory, one can only have control over the regularity if we know that our maps decrease quickly enough when approaching the boundaries. This is here, in those inequalities, that our weight ρ is really mandatory. If we forget the decreaseness information, inequalities of the form

$$||S - R(g_1)||_{k,\alpha} = O(||S - R(g_0)||^2_{k,\alpha})$$

become false.

So, our newton algorithm seems to have a healthy start. But what happens next ?

We want to linearize again, but at g_1 this time. We then solve, weakly, the following equation :

$$L_{g_1}(\rho L_{g_1} u_1) = S - R(g_1)$$

But here, an important problem appear. Our operator is still a 4-order elliptic operator, but now, its coefficients are only $C^{k+2,\alpha}$! Hence, our weak solution is only upgraded to a classical solution $u_1 \in C^{k+2,\alpha}_\rho$, and so, h_1 becomes only $C^{k,\alpha}$. We lost two degrees of differentiability !

It seems, then, that the newton algorithm won't do well. To overcome this difficulty, we modify a bit the newton algorithm, using the so called picard approach. The idea is, simply, to continue linearizing at g_0 at each time. i.e.,to define h_1 through :

$$L_{g_0}(h_1) = S - R(g_1).$$

By doing so, we ensure that, as before, h_1 stays a $C^{k+2,\alpha}$-metric.

Let's see what happens for the second step of iteration. We suppose that $||S-R(g_0)||_{k,\alpha,\rho^{-1}} \leq \epsilon$. We can then write :

$$R(g_2) = R(g_1) + L_{g_1}(h_1) + O(||h_1||_{k+2,\alpha}) = S + (L_{g_1} - L_{g_0})(h_1) + O(||h_1||^2_{k+2,\alpha})$$

Since $g \mapsto L_g$ is continuous, we can write :

$$R(g_2) = S + O(||h_0||_{k+2,\alpha}||h_1||_{k+2,\alpha}) + O(||h_1||^2_{k+2,\alpha}) = S + O(\varepsilon^3)$$

And hence, this time, things seem to be going well for our algorithm.
And then, one can proove that, indeed, if we follow this algorithm, the constructed sequences of metrics g_n will actually geometrically converge to a $C^{k+2,\alpha}$-metric g, such that $R(g) = S$.
This finishes the sketch of the proof of the theorem of Justin Corvino.

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