Fourier transform of (fractal) measures and applications.

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Abstract

This paper is a Master thesis done in the end of June 2021, to conclude my scholarship at the Université de Rennes 1 and at the ENS Rennes. It was done after working for some months under the guidance of Frederic Naud, who helped me gather most of the material for this report.

The main goal of this thesis is to study the notion of Fourier transform of finite measures. We want to explain how geometric/arithmetic properties of the support of the measure changes the behavior of its Fourier transform.

The objective is to give an (incomplete) overview of the subject, without getting too much into the technical details, but while still being able to give some intuition and to state some recent results.

In particular, we are interested in three questions that will structure this thesis.

First of all, what is the Fourier transform of a finite measure, and what is the intuition behind its asymptotic behavior ?
Second, why is it interesting ? In what kind of problem do we use Fourier decay ?
And lastly, how can we actually prove that some Fourier transform decays ?

Those questions will lead us to state some classical links between Hausdorff dimension and Fourier transform. We will also talk a bit about additive combinatorics and the contributions of Bourgain to the subject. The second part will lead us to do a bit of harmonic analysis : we will talk about the uncertainty principle, the famous restriction problem and an applications to the Strichartz estimates. Finally, in the third part, we will see how we can have some estimates of the Fourier transform of a measure. The smooth case will be done with the method of the stationary phase. The fractal case will be discussed with the notion of transfer operator.

Contents

Chapter 1

A first overview

1.1 The Fourier transform

First of all, let's recall some standard facts about the Fourier transform and fix our conventions. For an introduction to the subject, see the first chapters of [Ma15]. It also contains some reminders on measure theory that can be useful. For the distributional point of view, see for example [Du01].

Definition 1. Let φ be in the Schwartz space $S(\mathbb{R}^d)$.
We define its Fourier transform $\widehat{\varphi} \in S(\mathbb{R}^d)$ by :

$$\widehat{\varphi}(\xi) := \int_{\mathbb{R}^d} e^{-2i\pi x\cdot\xi}\varphi(x)dx.$$

The Fourier transform as an operator is often denoted by $\mathcal{F}$.
let's recall some important properties of this operator, acting on $S(\mathbb{R}^d)$ first.

Theorem 1 (Inverse formula). *Let $\varphi \in S(\mathbb{R}^d)$. Then*

$$\varphi(x) = \int_{\mathbb{R}^d} e^{2i\pi x\cdot\xi}\widehat{\varphi}(\xi)d\xi$$

In other words, $\mathcal{F}^2 = \mathcal{S}$, where $\mathcal{S}(\varphi)(x) := \varphi(-x)$.

Theorem 2 (Parseval). *Let $\varphi \in S(\mathbb{R}^d)$. Then :*

$$\int_{\mathbb{R}^d} |\widehat{\varphi}(\xi)|^2 d\xi = \int_{\mathbb{R}^d} |\varphi(x)|^2 dx.$$

Theorem 3. *Let $\varphi, \psi \in S(\mathbb{R}^d)$. Define their convolution $\varphi * \psi \in S(\mathbb{R}^d)$ by :*

$$\varphi * \psi(x) := \int_{\mathbb{R}^d} \varphi(x-y)\psi(y)dy.$$

Then :

$$\widehat{\varphi * \psi} = \widehat{\varphi}\widehat{\psi}.$$

Now, to generalize the Fourier transform to a wide class of object, we proceed by duality. A quick use of the Fubini theorem allow us to see that

$$\forall \varphi, \psi \in S(\mathbb{R}^d),\ \int_{\mathbb{R}^d} \widehat{\varphi}\psi = \int_{\mathbb{R}^d} \varphi\widehat{\psi}$$

Which encourage us to define the Fourier transform on tempered distributions like so.

Definition 2. Let $T \in S'(\mathbb{R}^d)$. We define its Fourier transform $\widehat{T} \in S'(\mathbb{R}^d)$ by :

$$\forall \varphi \in S(\mathbb{R}^d),\ \langle \widehat{T}, \varphi\rangle := \langle T, \widehat{\varphi}\rangle.$$

(Warning : in this notation, the duality bracket is $\mathbb{C}$-bilinear.)

Like this, the Fourier transform is defined on a wide class of objects. The usual formulas that are true on the Schwartz class extends by duality on any tempered distribution. In particular, we still have that

$$\mathcal{F}^2 = \mathcal{S}$$

where S is conveniently defined by duality. With that, the Parseval formula gives us the fact that

$$\mathcal{F} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$$

is a bijective isometry.

Also, for a convenient definition of the convolution between tempered and compactly supported distributions, we still have the formula :

$$\widehat{T * S} = \widehat{T}\widehat{S}.$$

Notice that there is no problem in this formula, as the Fourier transform of a compactly supported distribution is a smooth function.

More generally, as soon as the Fourier transform and the convolution is well defined between some objects, one may hope that this formula is true.
It is the case, for example, for the convolution of finite measures. In this case we are also able to give a direct formula for the Fourier transform.

Definition 3. For a Borel set $E \subset \mathbb{R}^d$, denote by $\mathcal{M}(E)$ the space of all finite (Borel) measures with support on E.

Since a finite measure is a tempered distribution, we have a natural definition for the Fourier transform. Let's compute it to practice. Fix $\mu \in \mathcal{M}(E)$ and $\varphi \in S(\mathbb{R}^d)$. We have :

$$\langle \widehat{\mu}, \varphi \rangle = \langle \mu, \widehat{\varphi} \rangle = \int_E \widehat{\varphi}(x) d\mu(x)$$
$$= \int_{\mathbb{R}^d} \left(\int_E e^{-2i\pi x \cdot \xi} d\mu(x) \right) \varphi(\xi) d\xi.$$

by Fubini. This gives us the following formula :

$$\widehat{\mu}(\xi) = \int_E e^{-2i\pi x \cdot \xi} d\mu(x).$$

Notice that this defines a bounded and continuous function. In the case where $d\mu(x) = f(x)dx$ for some (positive) integrable function f, we recover the usual definition of the Fourier transform. In this case, the Riemann-Lebesgue theorem tells us that $\widehat{f}(\xi) \underset{\xi \to \infty}{\longrightarrow} 0$.
But in the general case, the Fourier transform of our measure may not vanish : for example

$$\widehat{\delta_a}(\xi) = \int_{\mathbb{R}^d} e^{-2i\pi x \cdot \xi} d\delta_a(x) = e^{-2i\pi a \cdot \xi}$$

has constant modulus 1.

Now, as promised, we discuss about the convolution of those measures. For a motivation about the definition, see [Ta13].

Definition 4. Consider two finite measures $\mu, \nu \in \mathcal{M}(\mathbb{R}^d)$. The convolution $\mu * \nu \in \mathcal{M}(\mathbb{R}^d)$ is defined by duality by the formula :

$$\forall f, \ \int_{\mathbb{R}^d} f(t) d\mu * \nu(t) := \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x+y) d\mu(x) d\nu(y).$$

Where f is a (borel) measurable and bounded function. Notice that if μ has support in some set E, and if ν has support in some set F, then $\mu * \nu$ is supported in the *sumset* $E + F$.

Now, let's play a bit with the convolution and the Fourier transform. Consider two finite measures $\mu, \nu \in \mathcal{M}(\mathbb{R}^n)$. We have, for any $\varphi \in S(\mathbb{R}^d)$:

$$\langle \widehat{\mu * \nu}, \varphi \rangle = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \widehat{\varphi}(x+y) d\mu(x) d\nu(y)$$
$$= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \varphi(\xi) e^{-2i\pi\xi\cdot(x+y)} d\xi d\mu(x) d\mu(y)$$
$$= \int_{\mathbb{R}^d} \varphi(\xi) \left(\int_{\mathbb{R}^d} e^{-2i\pi\xi\cdot x} d\mu(x) \right) \left(\int_{\mathbb{R}^d} e^{-2i\pi\xi\cdot y} d\nu(y) \right) d\xi = \langle\ \widehat{\mu}\ \widehat{\nu}, \varphi \rangle$$

And so our desired formula is true.
Notice that this formula holds in particular for absolutely continuous measures. And so, with the natural extensions of the definitions, we also have the formula :

$$\forall f \in L^1(\mathbb{R}^d), \forall \mu \in \mathcal{M}(\mathbb{R}^d),\ \widehat{f * \mu} = \widehat{f}\widehat{\mu}.$$

From now on, we will use the usual properties of the Fourier transform of measures without proof, the reader can refer to [Ma15] if details are needed.

For the rest of the report, we will try to investigate the link between properties of the support of our measure and the behavior of its Fourier transform. There is two worlds that are closely related to each other while still being quite different : the geometric properties of the support, and its arithmetic properties. We begin by the geometric part.

1.2 Decay in the L^2 sense

1.2.1 Hausdorff dimension and Hausdorff measure

In geometry, the notion of dimension and of surface measure is of crucial importance. It is the fundamental concept that is attached to all vector spaces, manifolds, schemes and so on. Every of these objects are somehow smooth. But here, we are interested in the study of much rougher objects: the support of a measure has no reason to be smooth. And, indeed, for measures that arises naturally in dynamical systems, the support can be quite... chaotic.

So we need adapted geometric tools to study those rough objects. Although there is a lot of different notion of "fractal dimension" that exists, one in particular stands among the others: the Hausdorff dimension.

For an interesting discussion on this topic, the reader can refer to the blog of Tao [Ta09]. They can also refer to the chapter 2 of [Ma15] or the appendix of [Zi96].

So, we wish to generalize the notion of dimension, and so we search for a characterizing property of it. For the Hausdorff dimension, the remark is the following.

Consider a d-dimensional object Ω. Then, its "volume" (the Lebesgue measure of dimension d) satisfy the following *scaling* property :

$$\lambda_d(t\Omega) = t^d \lambda_d(\Omega).$$

And I wish to insist on the fact that this is not an abstract mathematical artifact of something, it comes from the very idea of dimension. If we consider a d dimensional cube, and that we cut it in cubes that are half its side, we will end with 2^d cubes. This exponent that appears allows us to recover the dimension of the object, while only counting how many pieces we need of a given size to cover our object.

This is on this idea that we wish to base the notion of Hausdorff dimension. Intuitively, an object E of dimension α should be characterized by some scaling properties. If we consider its "total mass" $\mu(E)$ (to be defined soon), then we should expect a relation of the type

$$\mu(tE) = t^\alpha \mu(E).$$

This natural notion of mass depends itself of the intuitive dimension of the object, and so is linked to its definition. It will be the Hausdorff measure of dimension α.

Now, for the rigorous definitions.

Definition 5. Let $\alpha \in \mathbb{R}^+$. For $E \subset \mathbb{R}^d$, define the α-dimensional Hausdorff measure of E by :

$$H_\alpha(E) := \lim_{\varepsilon \to 0} \inf_{(U_i)_{i\in I} \in \mathcal{U}_E^\varepsilon} \sum_{i \in I} (\text{diam } U_i)^\alpha \quad \in [0, \infty].$$

Where $\mathcal{U}_E^\varepsilon := \left\{(U_i)_{i\in I} \mid \bigcup_{i\in I} U_i \supset E,\ \text{diam}(U_i) \leq \varepsilon\right\}$ is the set of all ε-coverings of E.

Some remarks about the definition.

First of all, the limit always exists, as the $\inf_{\mathcal{U}_E^\varepsilon}$ is decreasing in ε.
Consequently, to have a finite α-dimensional Hausdorff measure, we have to bound from above the inf, which can be done by giving an explicit covering of our set that works.

In the other hand, proving that an α-dimensional Hausdorff measure is nonzero can be a bit more technical. We will talk about it soon.

And finally, we see that our measure satisfy the desired scaling property. If we consider an ε-covering $(U_i)_i$ of E, then $(tU_i)_i$ is a $t\varepsilon$-covering of tE, and we have :

$$\sum_{i\in I} (\text{diam } tU_i)^\alpha = t^\alpha \sum_{i \in I} (\text{diam } U_i)^\alpha$$

And the scaling property follow : $H_\alpha(tE) = t^\alpha H_\alpha(E)$.

Now we still have to prove that this defines a borel measure. We will need some help from the Caratheodory theorem, which deals with outer measures. Recall :

Definition 6. An outer measure on $\mathbb{R}^d$ is a map $\mu : 2^{\mathbb{R}^d} \longrightarrow \mathbb{R}^+$, satisfying :

1. $E \subset F \Rightarrow \mu(E) \leq \mu(F)$
2. $\mu\left(\bigcup_i E_i\right) \leq \sum_i \mu(U_i)$.
   Moreover, if μ satisfy
3. $d(E, F) > 0 \Rightarrow \mu(E \cup F) = \mu(E) + \mu(F)$,
   then we say that μ is metric.

Theorem 4 (Caratheodory's extension)**.** *Let μ be an outer measure on $\mathbb{R}^d$. Define :*

$$\mathcal{T} := \left\{E \subset \mathbb{R}^d \mid \forall X \subset \mathbb{R}^d,\ \mu(X) = \mu(X \cap E) + \mu(X \setminus E)\right\}.$$

Then $\mathcal{T}$ is a σ-algebra, and μ is a measure on $\mathcal{T}$.
Moreover, if μ is metric, then $\mathcal{T}$ contains the borel sets.

From there, the proof is reduced to show that H_α is a metric outer measure. Let's do it.

Theorem 5. *H_α is a borel measure on $\mathbb{R}^d$.*

Proof. Define $H^\varepsilon_\alpha(E) := \inf_{(U_i)_{i\in I}\in\mathcal{U}^\varepsilon_E} \sum_{i\in I}(\text{diam } U_i)^\alpha$, so that $H_\alpha(E) = \lim_{\epsilon\to 0} H^\varepsilon_\alpha(E)$.

- If $E \subset F$, then any ε-covering of F is a ε-covering of E, and so $H^\varepsilon_\alpha(E) \le H^\varepsilon_\alpha(F)$. So $H_\alpha(E) \le H_\alpha(F)$.
- Consider a countable collection of sets $(E_i)_{i\in\mathbb{N}}$. Let $\delta > 0$. By the definition of H^ε_α, there exists an ε-covering $\left(U^{(i)}_j\right)_{j\in J_i}$ of E_i such that

  $$\sum_{j\in J_i}\left(\text{diam } U^{(i)}_j\right)^\alpha \le H^\varepsilon_\alpha(E_i) + \frac{\delta}{2^i}.$$

  Then, the collection $\left(U^{(i)}_j\right)_{i\in\mathbb{N}, j\in J_i}$ is a (countable) covering of $\bigcup_i E_i$, and :

  $$H^\varepsilon_\alpha(\bigcup_i E_i) \le \sum_i\sum_j \left(\text{diam } U^{(i)}_j\right)^\alpha \le \sum_i H^\varepsilon_\alpha(E_i) + 2\delta$$

  Letting δ and ε goes to zero gives us the desired inequality.
- Suppose that $d(E, F) > 0$. Take $\varepsilon < \frac{1}{4}d(E, F)$. To compute $H^\varepsilon_\alpha(E \cup F)$, we know that we can chose to consider only the ε-coverings (U_i) that all touch our set, that is, $U_i \cap (E\cup F) \neq \emptyset$ for all i. So, we consider one of those. The hypothesis made on ε ensures that a given U_i can only touch E or F, but not both at the same time.
  It allows us to see that this covering is the disjoint union of an ε-covering of E, and of one ε-covering of F. Hence :

  $$\sum_i (\text{diam } U_i)^\alpha \ge H^\varepsilon_\alpha(E) + H^\varepsilon_\alpha(F).$$

  Taking the inf on those coverings, and then letting ε goes to zero gives us the desired result.

So H_α is a metric outer-measure, and the Caratheodory's extension theorem allow us to conclude.

□

Now, we have equipped ourselves with a notion of α-dimensional measure. We can see that it generalizes the usual Lebesgue measure. Indeed, it is invariant by translations, and if $\alpha = d$, it assign a finite nonzero value to the cube $[0, 1]^d$, and so it is a multiple of our usual Lebesgue measure. Let's prove it quickly, to play a bit with the definitions.

Proof. So first of all, let's show that $H_d([0, 1]^d) < \infty$.

We cut the cube in 2^{Nd} little cubes (C_i) of size 2^{-N}. It gives us a 2^{-N}-covering of the cube. Now, the diameter a cube of this size is $2^{-N}\sqrt{d}$. So :

$$H^{2^{-N}}_\alpha\left([0, 1]^d\right) \le \sum_i (\text{diam}(C_i))^\alpha \le 2^{Nd}2^{-N\alpha}d^{\alpha/2}$$

Notice what happens here : the coefficient α that *have to* be chosen to stabilize the RHS is d. For other coefficients, it goes to zero or infinity.
Anyway, taking $\alpha = d$ and letting N goes to infinity gives us $H_d([0, 1]^d) \le d^{d/2} < \infty$.

Now, let's show that $H_d([0, 1]^d) > 0$.

We are going to use the fact that we know well another natural d-dimensional measure that assign a nonzero value to the cube.

For any ε-covering (U_i) of $[0,1]^d$, we consider (B_i) an associated covering with euclidean balls, such that $\text{diam}(B_i) = 2\text{diam}(U_i)$ and $U_i \subset B_i$. We know that there exists a constant $c_d > 0$ such that $\lambda_d(B_i) = c_d(\text{diam } B_i)^d$. Consequently,

$$\sum_i (\text{diam } U_i)^d = 2^{-d} c_d^{-1} \sum_i \lambda_d(B_i) \geq 2^{-d} {c_d}^{-1} \lambda_d(\cup_i B_i) \geq 2^{-d} {c_d}^{-1} \lambda_d\left([0,1]^d\right).$$

Taking the inf and then letting ε goes to zero gives us $H_d([0,1]^d) \geq 2^{-d} c_d > 0$. □

Finally, let's get to the definition of the Hausdorff dimension of a set.
The idea is the following : if you try to measure the length of a disk, you'll get an infinite length (because $1 \leq 2$), and if you try to measure its volume, you'll get zero (because $2 \leq 3$). This idea can be used to define the dimension.
The following remark makes this intuition rigorous.

Consider a set E, and let $\alpha < \beta$. We have the following relation :

$$H_\beta^\varepsilon(E) = \inf_{(U_i) \in \mathcal{U}_E^\varepsilon} \sum_i (\text{diam } U_i)^\beta \leq \varepsilon^{\beta - \alpha} \inf_{(U_i) \in \mathcal{U}_E^\varepsilon} \sum_i (\text{diam } U_i)^\alpha = \varepsilon^{\beta-\alpha} H_\alpha^\varepsilon(E)$$

Hence, if $H_\alpha(E) \in [0, \infty[$, then $H_\beta(E) = 0$, and if $H_\beta(E) \in]0, \infty]$, then $H_\alpha(E) = \infty$.

From this a natural definition of the dimension of a set arise.

Definition 7. For a set $E \subset \mathbb{R}^d$, define its Hausdorff dimension by :

$$\dim_H(E) := \inf\{\alpha \geq 0 \mid H_\alpha(E) = 0\} = \sup\{\alpha \geq 0 \mid H_\alpha(E) = \infty\}.$$

let's give some properties of the Hausdorff dimension. Then, to practice, we will do an explicit computation on the triadic Cantor set.

Proposition 6.

1. $E \subset F \Rightarrow \dim_H E \leq \dim_H F$
2. $\dim_H(\cup_n E_n) = \sup_n \dim_H E_n$.
3. Let $f : \mathbb{R}^d \to \mathbb{R}^d$ be lipschitz. Then $\dim_H f(E) \leq \dim_H E$.
4. Diffeomorphisms preserves the Hausdorff dimension.

Proof.

1. This follow from the fact that $H_\alpha(E) \leq H_\alpha(F)$.
2. Let $\alpha > \sup_n \dim_H(E)$. Then $H_\alpha(\cup_n E_n) \leq \sum_n H_\alpha(E_n) = 0$. Conversely, if $\alpha < \sup_n \dim_H(E_n)$, then there exists some n_0 such that $\alpha < \dim_H(E_{n_0})$. Hence, $H_\alpha(\cup_n E_n) \geq H_\alpha(E_{n_0}) = \infty$.
3. Denote by k the lipschitz constant of f. If $(U_i)_i$ is an ε-covering of E, then $(f(U_i))_i$ is an $k\varepsilon$-covering of $f(E)$. Moreover,

$$\sum_i (\text{diam} f(U_i))^\alpha \leq k^\alpha \sum_i (\text{diam} U_i)^\alpha.$$

Hence $H_\alpha^\varepsilon(f(E)) \leq k^\alpha H_\alpha^{k\varepsilon}(E)$, and so $H_\alpha(f(E)) \leq k^\alpha H_\alpha(E)$.
Hence, $\dim_H(f(E)) \leq \dim_H(E)$.

4. Consider a diffeomorphism $\varphi : U \to V$, with U open containing E. Consider (K_n) a sequence of compacts such that $\bigcup_n K_n = U$, and then define $E_n := K_n \cap E$. On K_n, our diffeomorphism is lipschitz, and its inverse is also lipschitz. So, by (3), $\dim_H E_n = \dim_H \varphi(E_n)$. Then, by (2), $\dim_H(E) = \dim_H(\varphi(E))$.

□

Example 1. The triadic Cantor set is :

$$C := \left\{ \sum_{n=1}^{\infty} \frac{2\varepsilon_n}{3^n},\ \varepsilon \in \{0,1\}^{\mathbb{N}^*} \right\}$$

We can notice that $C = (\frac{1}{3}C) \sqcup (\frac{1}{3}C + \frac{2}{3})$: we say that the triadic cantor set is self-similar. This relation can be though as (and *is*, in fact) the relation that defines the Cantor set. The fact that this relation is affine is of importance : we will say later that the cantor set has some "additive structure".

Anyway, the given definition calls for a natural probability measure on it. The probabilistic point of view is the following : consider a sequence of independant random variables ε_n, such that ε_n is a uniform bernouilli variable. We then call the "cantor measure", or "selfsimilar cantor measure" the law μ of the random variable $\sum_{n=1}^{\infty} \frac{2\varepsilon_n}{3^n}$.

This measure is an interesting first exemple of non absolutly continuous measure without atoms. Its cumulative distribution function is the famous devils staircase. This function is often cited as an horrible exemple of function who is continuous (in fact, $\ln(2)/\ln(3)$-Hölder) and who admits a zero derivative almost everywhere while still being nonconstant. In particular, it is not the integral of its derivative in this (naive) sense.
However, this is just because this is not the good definition of the derivative, since the derivative *in the sense of distributions* of the Devil's staircase gives us back the natural measure on the Cantor set. So everything's fine in the end.

Another property of this measure is its autosimilarity, as the name suggested. If we denote (somewhat informally) by $d\mu(f^{-1}(x))$ the pushforward $f_*\mu$ of the measure for some map f, then we have the following relation :

$$\frac{d\mu(3x) + d\mu(3x-2)}{2} = d\mu(x).$$

And so the cantor measure reflects the additive structure of its support. We finish that little discussion about the cantor measure with an explicit computation.
Fix a finite word $\mathbf{a} \in \{0,1\}^N$. By definition, we have :

$$\mathbb{P}(\varepsilon_1 = a_1, \dots, \varepsilon_N = a_N) = 2^{-N}$$

On the other hand, this quantity is also equal to

$$\mathbb{P}\left(\sum_{n=1}^{\infty} \frac{2\varepsilon_n}{3^n} \in I_{\mathbf{a}} \right) = \mu(I_{\mathbf{a}})$$

Where $I_{\mathbf{a}} := \left[\sum_{n=1}^{N} \frac{2a_n}{3^n}, \sum_{n=1}^{N} \frac{2a_n}{3^n} + 3^{-N} \right]$.

In particular, we have the following relation :

$$\forall N, \forall \mathbf{a} \in \{0,1\}^N,\ \mu(I_{\mathbf{a}}) = \mathrm{diam}(I_{\mathbf{a}})^{\ln(2)/\ln(3)}.$$

Now we have all the tools to compute the Hausdorff dimension of the triadic cantor set.

Theorem 7. *The Hausdorff dimension of the triadic cantor set is* $\ln(2)/\ln(3)$.

Proof. Set $\alpha := \ln(2)/\ln(3)$.

There is a natural family of partitions for our Cantor set. For any N, we can consider the partition $C = \sqcup_{\mathbf{a} \in \{0,1\}^N} I_{\mathbf{a}}$. This is a 3^{-N}-covering, consisting of 2^N elements. Hence :

$$H_\alpha^{3^{-N}}(C) \le \sum_{\mathbf{a} \in \{0,1\}^N} (\mathrm{diam}\ I_{\mathbf{a}})^\alpha \le 2^N 3^{-\alpha N} = 1$$

And so $H_\alpha(C) < \infty$, which means that $\dim_H(C) \ge \alpha$. Now for the other inequality.

Consider (U_i), a 3^{-N}-covering of C.
For all i, there exists a word $\mathbf{a}_i$ (with length at least N) such that $C \cap U_i \subset I_{\mathbf{a}_i}$, with $\text{diam}(I_i) = \text{diam}(C \cap U_i)$. Hence :

$$\sum_i (\text{diam } U_i)^\alpha \geq \sum_i (\text{diam } C \cap U_i)^\alpha = \sum_i (\text{diam } I_{\mathbf{a}_i})^\alpha \geq \mu(\cup_i I_{\mathbf{a}_i}) = 1$$

And so, taking the inf on the covering and then letting N goes to infinity, we get $H_\alpha(C) > 0$, which means that $\dim_H(C) \leq \alpha$.

□

For some similar computations on modified Cantor sets, the interested reader can read the beginning of [KS94]. We finish this paragraph by quickly citing a celebrated result in the self-similar case. For a quick explanation of the topic, the reader can look at [Ei18].

Here, what helped us to compute the dimension of the Cantor set was its structure. Its self similarity allowed us to easily find some natural partitions, that helped bound the Hausdorff dimension from below, and then allowed us to define a natural associated self-similar measure, which helped us bound the Hausdorff dimension from above.
There is actually a general theorem that gives us the dimension of any self similar set.

Theorem 8. *Consider $g_1, \dots g_k$ some similarity transformation with ratio $r_1, \dots, r_k \in]0,1[$, and suppose that K is a (nonempty) compact set such that*

$$\bigsqcup_i g_i(K) = K.$$

Then $\dim_H(K) = \alpha_0$, where α_0 is the only solution of the equation $\sum_i r_i^\alpha = 1$.

Where does this formula comes from ? Just take the α-dimensionnal Hausdorff measure of the relation. It gives us :

$$H_\alpha(K) = \sum_i H_\alpha(g_i(K)) = \sum_i r_i^\alpha H_\alpha(K).$$

And so, the only α for which $H_\alpha(K)$ *may* be nonzero and finite is α_0.

Unfortunately, this reasoning is not enough to conclude, as some α_0-dimensionnal objects can still have infinite α_0-dimensionnal hausdorff measure.

A sketch of the actual proof is the following.
First, we have an obvious family of partition for K. Indeed, we can write :

$$K = \bigsqcup_{\mathbf{a} \in \{1,\dots,k\}^N} g_{\mathbf{a}}(K)$$

Where I denoted $g_{\mathbf{a}} := g_{a_1} \circ \cdots \circ g_{a_N}$. Estimating the various diameter of the $g_\alpha(K)$ allow us to give a first bound on the dimension. The other bound is obtained in a similar way as we did for the Cantor set : one uses the self-similarity of K to create an explicit self-similar measure μ on K, and use it to control the Hausdorff measure from below.

1.2.2 Frostman and the energy integral

This previous paragraph hints for a deeper link beetwen the existence of well behaved measured on a set E, and its Hausdorff dimension. This paragraph will make this hint more precise.

When we talked about the Devil's staircase, I mentioned the fact that it is $\ln(2)/\ln(3)$-hölder, which is exactly the dimension of the Cantor set. This is no coincidence. Indeed, in the case of fractals included in the real line, we can think of the Hausdorff dimension as the maximal parameter α for which there exists measures with support in E with cumulative distribution functions being β-Hölder, with $\beta < \alpha$.

What makes this sentence rigorous is the Frostman lemma. The proof is adapted from [KS94], in chapter 2. The reader can also look at [Ma15].

Theorem 9 (Frostman lemma)**.** *Let $E \subset \mathbb{R}^n$ be a compact set. Denote by $\mathcal{P}(E)$ the set of all (borel) probability measures with support in E. Then :*

$$\dim_H(E) = \sup\{\alpha \in [0, d] \mid \exists \mu \in \mathcal{P}(E), \exists C > 0, \mu(B(x, r)) \leq Cr^\alpha\}$$

Proof. We proceed by showing the two inequalities. One is easy (and we already did it), and the other is the Frostman lemma. We begin by the easy part first, and we will come back to the difficult part after.

Suppose that there exists $\mu \in \mathcal{P}(E)$ such that $\mu(B(x, R)) \leq Cr^\alpha$ for any x, r. Now consider (U_i), an ε-covering of E. There exists balls (B_i) such that $\text{diam}B_i = 2\text{diam}U_i$, and such that $U_i \subset B_i$. Hence :

$$\sum_i (\text{diam } U_i)^\alpha = \sum_i (\text{diam } B_i/2)^\alpha \geq \frac{1}{C} \sum_i \mu(B_i) \geq \frac{1}{C}.$$

Hence, taking the inf on the coverings and then letting ε goes to zero, we get that $H_\alpha(E) \geq \frac{1}{C} > 0$. So $\alpha \leq \dim_H(E)$.

□

Conversely, if $\alpha < \dim_H(E)$, the goal is to construct a measure that satisfy our desired estimate (we call them Frostman measures). To do so, we will need to recall some useful results of measure theory. The following results are standards and can be found, for example, in [Ga03].

Definition 8. Let (X, d) be a metric space. We say that a sequence $(\mu_n) \in \mathcal{M}(X)^\mathbb{N}$ converge weakly to a measure μ if, for any continuous and bounded function $f : X \to \mathbb{R}$, we have

$$\int f d\mu_n \longrightarrow \int f d\mu.$$

Notice that if $(\mu_n(X))$ is bounded, then $\mu \in \mathcal{M}(X)$.

Notice that we say "weakly", but we should actually say weakly*, as the dual of the space of all continuous and bounded functions are the finite signed measures. An important result is the fact that $\mathcal{P}(E)$ can be equipped by a distance d_P (the Lévy–Prokhorov metric) that gives us the topology of the weak convergence. The use of the Banach-Alaoglu theorem and the duality previously mentioned allow us to give the following result :

Theorem 10 (Prokhorov)**.** *Let $E \subset \mathbb{R}^n$ be compact. Then $(\mathcal{P}(E), d_P)$ is compact.*

Corollary 11. *Let $E \subset \mathbb{R}^n$ be compact. Let $(\mu_n) \in \mathcal{M}(E)^\mathbb{N}$ such that $\forall n,\ \mu_n(E) \in [c, C]$, with $0 < c \leq C < \infty$. Then there exists $\varphi : \mathbb{N} \to \mathbb{N}$ and $\mu \in \mathcal{M}(E)$ such that*

$$\forall f \in C^0(E, \mathbb{R}),\ \int_E f d\mu_{\varphi(n)} \longrightarrow \int_E f d\mu.$$

In particular, $\mu(E) \in [c, C]$.

Now that we are equipped with the right tools, let's prove the hard part of the Frostman lemma.

Proof. Let $\alpha < \dim_H(E)$.
Notice that this implies the existence of a constant $c > 0$ such that, for any cover of E,

$$\sum_i (\text{diam } U_i)^\alpha \geq c.$$

Now, we wish to construct a well behaved mesure supported on E.
To simplify, we can suppose that $E \subset [0,1]^d$. (Since E is supposed compact.)

For all $N \geq 1$, we cut this cube into 2^{dN} little cubes of size 2^{-N}. (So, diameter $2^{-N}\sqrt{d}$.) We denote them by $\left(C_k^{(N)}\right)_k$. For all cube such that $C_k^{(N)} \cap E \neq \emptyset$, we spread uniformly the mass $2^{-\alpha N}$, and we put zero mass everywhere else. This construct a measure ν_N supported near E, with total mass $\nu_N(E) = \#\{k \mid C_k^{(N)} \cap E \neq \emptyset\} 2^{-\alpha N}$.

Then, we look at the cubes $C_k^{(N-1)}$. If $\nu_N(C_k^{(N-1)}) \leq 2^{\alpha(N-1)}$, we don't do anything. In the other cubes, we replace ν_N by a proportional measure, but with total mass $2^{\alpha(N-1)}$. This way we get another measure $\nu_{N,1}$.

We then iterate this process. If $\nu_{N,k}$ is defined, then we construct $\nu_{N,k+1}$ by looking at the cubes $C^{(N-k-1)}$. If the mass is too large, we rescale, else we do nothing. (Notice that the sequence $(\nu_{N,k})_k$ is decreasing.)

At the end of the process, we have defined a measure $\mu_N := \nu_{N,N}$. By construction, for any little cube ($n \leq N$), we have

$$\mu_N(C_k^{(n)}) \leq 2^{-\alpha n}.$$

In particular, $\mu_N([0,1]^d) \leq 1$. In the other hand, by construction, we see that any point $x \in E$ is contained in some $C_k^{(n)}$ such that $\mu_N(C_k^{(n)}) = 2^{-\alpha n}$. This gives us a partition (U_i) of E, and we have :

$$\mu_N(E) = \sum_i \mu_N(U_i) = \frac{1}{d^{\alpha/2}} \sum_i (\text{diam } U_i)^\alpha \geq \frac{c}{d^{\alpha/2}}.$$

Hence, by the Prokhorov theorem since $[0,1]^d$ is compact, there exists a subsequence of this measure that converge weakly to a measure $\mu_\infty \in \mathcal{M}([0,1]^d)$. Moreover, $\mu_\infty([0,1]^d) \in [c, 1]$. Hence, letting $\mu := \mu_\infty/\mu_\infty([0,1]^d)$ defines a probability measure on $[0,1]^d$.

Now, since μ_N is supported on a $\sqrt{d}2^{-N}$-neighbourhood of E (which is compact), we easily see that μ is supported in E. Finally, we show the desired estimate. First of all, it suffices to prove it for balls of (uniformly) small enough radius, since we are working with a finite measure.

So let $x \in \mathbb{R}^n$ and $r \leq \delta := d(E, \partial[0,1]^d)/100$.

If $x \notin E + B(0,\delta)$, then $\mu(B(x,r)) = 0$, so there is nothing to prove.
Now, if $x \in E + B(0,\delta)$, then $B(x,r) \subset [0,1]^d$. This ball intersects at most 2^d cubes of size $\leq 4r$, let's denote them by $(C_k(x,r))_k$.
Consider a positive, continuous bump function $f \leq 1$ such that $f = 1$ on $B(x,r)$, and such that supp$f \subset \cup_k C_k(x,r)$. We have :

$$\mu(B(x,r)) \leq \int f d\mu = \frac{1}{\mu_\infty([0,1]^d)} \lim_{N\to\infty} \int f d\mu_N$$

$$\leq \frac{1}{\mu_\infty([0,1]^d)} \lim_N \sum_k \mu_N(C_k(x,r)) \leq \frac{2^d 4^\alpha}{\mu_\infty([0,1]^d)} r^\alpha.$$

$\square$

Frostman's lemma is an interesting result that allow us to compute the Hausdorff dimension of a set just by considering well behaved probability mesures on it. This result already gives us some insight about the link between the properties of a measure and the geometric properties of its support. We can't construct any measure on any set.

This lemma can be reformulated with the notion of energy integral : it allow us to say that a measure is Frostman iff it is, in some sense, in some L^2 space. Then, using the Parseval formula, we will rewrite the condition as $\widehat{\mu}$ being in some L^2 space, which means some decay "in average".

For details on the energy integral, see [Ma15] or [KS94].

Definition 9. Let $\mu \in \mathcal{M}(E)$. Let $0 < \alpha < d$. We define the α-energy of μ by :

$$I_\alpha(\mu) := \iint \frac{1}{|x-y|^\alpha} d\mu(x) d\mu(y).$$

We can rewrite this integral as

$$I_\alpha(\mu) = \int k_\alpha * \mu \; d\mu,$$

where $k_\alpha(x) := |x|^{-\alpha}$ is the *riesz kernel.*

Since everyone is positive, the convolution

$$(k_\alpha * \mu)(x) = \int \frac{d\mu(y)}{|x-y|^\alpha}$$

is well defined everywhere, we sometimes call it the potential.

Before going on, let's rewrite this potential in a form that will make obvious the link between finite energy measures and Frostman measures. We have :

$$\int_{\mathbb{R}^d} \frac{d\mu(y)}{|x-y|^\alpha} = \alpha \int_{\mathbb{R}^n} \int_{|x-y|}^\infty \frac{dr}{r^{\alpha+1}} d\mu(y) = \alpha \int_0^\infty \frac{\mu(B(x,r))}{r^{\alpha+1}} dr.$$

With this reformulation, it is easy to get the following :

Theorem 12. *Let $E \subset \mathbb{R}^d$ be compact. Then :*

$$dim_H(E) = \sup\{\alpha \in [0,d] \mid \exists \mu \in \mathcal{P}(E),\ I_\alpha(\mu) < \infty\}.$$

Proof. Fix $\alpha < \dim_H(E)$, then let $\alpha < \beta < \dim_H(E)$. By the Frostman lemma, there exists some probability measure $\mu \in \mathcal{P}(E)$ such that $\mu(B(x,r)) \leq Cr^\beta$. Hence, the potential is bounded by :

$$(k_\alpha * \mu)(x) \leq \alpha \int_0^\infty \min\left(\frac{C}{r^{1-(\beta-\alpha)}}, \frac{1}{r^{1+\alpha}}\right) dr < \infty.$$

And so $\int k_\alpha * \mu \; d\mu < \infty$.

Conversely, fix $0 < \alpha < d$, and suppose that there exists some measure $\mu \in \mathcal{P}(E)$ such that $I_\alpha(\mu) < \infty$. Fix $C > I_\alpha(\mu)$. By the Markov inequality, we have :

$$\mu\left(\{x \in E \mid (k_\alpha * \mu)(x) > C\}\right) \leq \frac{I_\alpha(\mu)}{C}.$$

And so

$$\mu\left(\{x \in E \mid (k_\alpha * \mu)(x) \leq C\}\right) \geq 1 - \frac{I_\alpha(\mu)}{C} > 0.$$

Denote this set by $\tilde{E}$. We have, for all x in $\tilde{E}$, and for all r :

$$C \geq \int_{\mathbb{R}^d} \frac{d\mu(y)}{|x-y|^\alpha} \geq \int_{B(x,r)} \frac{d\mu(y)}{|x-y|^\alpha} \geq \frac{1}{r^\alpha} \mu(B(x,r))$$

Hence, the probability measure $\mu(\tilde{E} \cap \cdot)/\mu(\tilde{E})$ has support in $\tilde{E} \subset E$, and is an α-Frostman measure. So $\alpha < \dim_H(E)$.

$\square$

This formulation is very useful, as from an information on the dimension of a set, one may gain the existence of some well behaved measures on it that can be used for computations.
We can compute the energy integral in another way. The Parseval formula allows us to hope for the following formula to be true :

$$I_\alpha(\mu) = \langle \mu, k_\alpha * \mu \rangle = \langle \overline{\widehat{\mu}}, \widehat{k_\alpha}\widehat{\mu} \rangle = \int |\widehat{\mu}|^2 \widehat{k_\alpha}$$

But we have to be a bit careful about who is $\widehat{k_\alpha}$. Let's carefully compute it.

Theorem 13. $\forall\ 0 < \alpha < d,\ \exists c_{\alpha,d} > 0,\ \widehat{k_\alpha} = c_{\alpha,d} k_{d-\alpha}$.

Proof. First of all, we see that if $\alpha > d/2$,

$$k_\alpha(x) = \frac{1}{|x|^\alpha} \mathbb{1}_{B(0,1)} + \frac{1}{|x|^\alpha} \mathbb{1}_{\mathbb{R}^d \setminus B(0,1)} \in L^1(\mathbb{R}^d) + L^2(\mathbb{R}^d).$$

In particular, k_α is a tempered distribution and so its Fourier transform is well defined. And by linearity, we even see that $\widehat{k_\alpha} \in L^\infty + L^2$ is a function (and not a distribution nor an abstract measure ! The following argument depends on this).
Now, to compute it, we do two remarks. First, k_α is a radial function, and so its Fourier transform is also radial. Second, k_α is α-homogeneous, and so its Fourier transform is $(d-\alpha)$-homogeneous. It follow that there exists a constant $c_{\alpha,d}$ such that

$$\widehat{k_\alpha} = c_{\alpha,d} k_{d-\alpha}.$$

This constant is nonzero by injectivity of the Fourier transform, and its sign will be determined in theorem 14.
Then, by the inverse Fourier transform formula, the results extend if $\alpha < d/2$.

□

Notice that this argument breaks for $\alpha = 0$. The constant 1 is 0-homogeneous, and is radial, but its Fourier transform isn't $1/|x|^n$. In fact, the latter is not a tempered distribution. But, in a sense, the argument is not completely false, as we can think of the Dirac distribution as being radial and homogeneous.

Anyway, this computation allows us to get to the pinnacle of this section : the formula of the energy integral with the Fourier transform of the measure, and a final reformulation of the Hausdorff dimension of a set.

Theorem 14. *Let $E \subset \mathbb{R}^d$ be a compact set, and let $\mu \in \mathcal{M}(E)$. We have :*

$$I_\alpha(\mu) = c_{\alpha,d} \int_{\mathbb{R}^d} |\widehat{\mu}(\xi)|^2 |\xi|^{\alpha-d} d\xi.$$

Corollary 15. *Let $E \subset \mathbb{R}^d$ be a compact set. Then :*

$$\dim_H(E) = \sup \left\{ \alpha \in [0,d] \mid \exists \mu \in \mathcal{P}(E),\ \int_{\mathbb{R}^d} |\widehat{\mu}(\xi)|^2 |\xi|^{\alpha-d} d\xi < \infty \right\}.$$

And now the link between the Hausdorff dimension of a set and the possible existence of decaying Fourier transform of measures on this set is made clear. We finish this part by explaining why we can think of this condition as $|\widehat{\mu}|$ decaying "on average".

Assume that we have $\mu \in \mathcal{P}(E)$ such that $I_\alpha(\mu) < \infty$.
Then, for any $\varepsilon > 0$ and $\beta < \alpha$, we have :

$$\frac{1}{R^d}\lambda_d\left(\left\{\xi \in B(0,R),\ |\widehat{\mu}(\xi)| > \varepsilon|\xi|^{-\beta/2}\right\}\right) = \frac{1}{R^d}\lambda_d\left(\left\{\xi \in B(0,R),\ |\widehat{\mu}(\xi)|^2|\xi|^{\alpha-d} > \varepsilon^2|\xi|^{\alpha-\beta-d}\right\}\right)$$

$$\leq \frac{1}{R^d}\lambda_d\left(\left\{\xi \in B(0,R),\ |\widehat{\mu}(\xi)|^2|\xi|^{\alpha-d} > \varepsilon^2 R^{\alpha-\beta-d}\right\}\right) \leq \frac{1}{\varepsilon R^{\alpha-\beta}} I_\alpha(\mu) \longrightarrow 0.$$

By Markov's inequality. Hence,

$$\frac{\lambda_d\left(\left\{\xi \in B(0,R),\ |\widehat{\mu}(\xi)| \leq \varepsilon|\xi|^{-\beta/2}\right\}\right)}{\lambda_d(B(0,R))} \underset{R\to\infty}{\longrightarrow} 1.$$

Which means that $|\widehat{\mu}(\xi)| = o(|\xi|^{-\beta/2})$ *on average.*

The question now is : what can we say about actual pointwise decay ? Is it always possible ?

And the answer is no, as we shall soon see. The possible pointwise decay will be linked to the arithmetic structure of the support, which is an information more subtle than its Hausdorff dimension.

1.3 Pointwise decay

1.3.1 A (very) weak lemma and some examples

We already got some results about Fourier transforms decaying on average, but, to get some intuition of what will follow, I think that the following naive computation is helpful.

Consider μ a finite measure on $\mathbb{R}$. The goal is to feel what kind of conditions on μ we must have to get some pointwise decay on $\hat{\mu}$.

Since it is often easier to compute, we study the Cesaro average of $|\hat{\mu}|$.

We have (for $\xi > 0$) :

$$\begin{aligned}
\left(\frac{1}{\xi}\int_0^\xi |\hat{\mu}(\lambda)|d\lambda\right)^2 &\leq \frac{1}{\xi}\int_0^\xi |\hat{\mu}(\lambda)|^2 d\lambda \\
&= \frac{1}{\xi}\int_0^\xi \left|\int_{\mathbb{R}} e^{-2i\pi x\lambda}d\mu(x)\right|^2 d\lambda \\
&= \frac{1}{\xi}\int_0^\xi \int_{\mathbb{R}^2} e^{2i\pi(y-x)\lambda}d\mu(x)d\mu(y)d\lambda \\
&\leq \mu\otimes\mu\left(\{|x-y|\leq\sigma\}\right) + \left|\int_{\{|x-y|>\sigma\}} \frac{1}{\xi}\int_0^\xi e^{2i\pi(y-x)\lambda}d\lambda\ d\mu(x)d\mu(y)\right| \\
&\leq \mu\otimes\mu\left(\{\ |x-y|\leq\sigma\}\right) + \left|\int_{\{|x-y|>\sigma\}} \left(\frac{e^{2i\pi(y-x)\xi}-1}{2i\pi(y-x)\xi}\right) d\mu(x)d\mu(y)\right| \\
&\leq \mu\otimes\mu\left(\{\ |x-y|\leq\sigma\}\right) + \frac{1}{\pi\sigma\xi}\ \mu\otimes\mu\left(\{|x-y|>\sigma\}\right)
\end{aligned}$$

And so

$$\limsup_{\xi\to\infty} \frac{1}{\xi}\int_0^\xi |\hat{\mu}(\lambda)|d\lambda \ \leq\ \mu\otimes\mu(|x-y|<\sigma)\ ,$$

and this for any $\sigma > 0$. Hence, under the assumption that $\mu\otimes\mu(|x-y|<\sigma) \underset{\sigma\to 0}{\longrightarrow} 0$, we get that

$$\frac{1}{\xi}\int_0^\xi |\hat{\mu}(\lambda)|d\lambda \underset{\xi\to 0}{\longrightarrow} 0.$$

The hypothesis which has been made is called a *non-concentration hypothesis*. If we think of μ as a probability measure, our hypothesis means that two independent random variable following the distribution μ should have zero probability to be equal. It is a natural requirement for our measure to have decaying Fourier transform, as it ensures that we do not have to much the same phase contributing to the value of the integral.
This kind of hypothesis will come back, in a stronger form, in the much more involved theorem of Bourgain that we will soon cite.

This easy remark shows us (again) that for any Frostman measure of dimension $\alpha > 0$, we always have decay of the Fourier transform *on average*. In fact, it can be shown that is μ is a α-Frostman measure in $\mathbb{R}^d$, we have

$$\frac{1}{R^d}\int_{B(0,R)} |\widehat{\mu}(\xi)|^2 d\xi \lesssim R^{-\alpha},$$

see [Ma15], paragraph 3.8 ”Ball averages” for a quick proof. In the same time, they prove that the non-concentration hypothesis is necessary to have this kind of decay. It is also known that, for a measure to satisfy $\widehat{\mu}\to 0$ (we call them Rajchman measures), μ must not have any atoms. (See [Ly95] for a quick discussion about Rajchman measures.)

But it doesn't mean that the Fourier transform decays pointwisely in general, even for Frostman measures. So what can happen ?

As suggested by the previous computations, the question is linked to the properties of the set $\text{supp}(\mu) - \text{supp}(\mu) := \{x - y \mid x, y \in \text{supp}(\mu)\}$.

This, itself, is linked to the *additive structure* of the support of the measure. The mantra is the following : (microscopic) additive structure/invariances/periodicity in the space variable leads to (macroscopics) structure/invariances/periodicity in the frequency variable.

Let's give some example of this principle right away, to makes this clearer.

Example 2. Consider the uniform probability measure on $\{0, 1/N, \dots, (n-1)/N\}$: it is an arithmetic progression of length n and of common difference $1/N$. Its Fourier transform is :

$$\widehat{\mu}(\xi) = \int_{\mathbb{R}} e^{-2i\pi x\xi} d\mu(x) = \frac{1}{n} \sum_{k=0}^{n-1} e^{-2i\pi k\xi/N}$$

Hence :

$$|\widehat{\mu}(\xi)| = \frac{1}{n} \left| \frac{\sin(\pi n\xi/N)}{\sin(\pi\xi/N)} \right|$$

We notice that our Fourier transform is periodic ! In particular, it does not decay, and worse, it comes back to the maximal possible value 1 at $N\mathbb{Z}$. We notice that the parameter N is responsible for the periodicity : as announced, the smaller the common difference, the bigger the periodicity of the Fourier transform. On the other hand, the parameter n is responsible for the average decay/sharpness : we see that $\widehat{\mu} = 0$ at $\frac{N}{n}\mathbb{Z} \setminus N\mathbb{Z}$. We come back to another well known property of the Fourier transform: the bigger the standard deviation in space, the smaller the standard deviation in frequency.

While we are looking at this expression, notice what happens if we take $n = N$ and that we let $N \to \infty$. The measure weakly converge to the uniform measure on $[0, 1]$, which can be though as some sort of degenerated arithmetic sequence with common difference 0, and the Fourier transform converge to

$$|\widehat{\mu}(\xi)| = |\text{sinc}(\pi\xi)|$$

which is no longer periodic. And even better, we have gained some pointwise decay, of order ξ^{-1}.

Example 3. Now for a more subtle example, let's compute the Fourier transform of the Cantor measure. The Cantor set have some additive structure, and it will show in the properties of its Fourier transform.

Recall the functional equation of this measure : we have

$$d\mu(x) = \frac{d\mu(3x) + d\mu(3x - 2)}{2}.$$

Taking the Fourier transform of this relation gives us

$$\widehat{\mu}(\xi) = \frac{1 + e^{-4i\pi\xi/3}}{2} \widehat{\mu}(\xi/3)$$

And so, by continuity of the Fourier transform of a finite measure, and since $\widehat{\mu}(0) = 1$, we have :

$$\widehat{\mu}(\xi) = e^{-i\pi\xi} \prod_{k=1}^{\infty} \cos(2\pi\xi/3^k).$$

Now some remarks are needed. First of all, we see that $\cos(2\pi\xi/3^k) - 1 \sim 2\pi^2\xi^2/9^k$ is the term of an absolutely convergent series, and so our infinite product converges to something which is (most of the time) nonzero. So everything's fine.

The second remark is that there is no pointwise decay. Indeed, is we choose $\xi = 3^N$ for some N, then we get :

$$|\widehat{\mu}(3^N)| = |\widehat{\mu}(1)|,$$

and so no decay is possible. Notice that the situation is quite different from the previous example: there is no decay even though the Hausdorff dimension of our support is strictly positive. And we see that the responsible for this is the affine invariance of the Cantor measure.

Example 4. Now, one can wonder, what would have happened if we took another measure on the Cantor set? Could have we achieved some decay by choosing well our measure ?
The answer is no. Here is why : the Cantor set has some additive structure. One easy case where the additive structure of a set prevents the existence of decaying Fourier transform of measure is the following.

Consider a borel set A, and suppose that there exists some probability measure μ supported on A such that $\widehat{\mu}$ satisfies

$$|\widehat{\mu}(\xi)| \leq C(1 + |\xi|)^{-\varepsilon}$$

for some C and for some $\varepsilon > 0$.

Then, the measure $\mu * \cdots * \mu$ is a probability measure supported on the sumset $A + A + \cdots + A$. Moreover, we have :

$$|\mu \widehat{* \cdots *} \mu| = |\widehat{\mu}^k| \leq C^k (1 + |\xi|)^{-k\varepsilon}.$$

If we take k large enough, then $\mu \widehat{* \cdots *} \mu$ becomes a L^1 function, and so $\mu * \cdots * \mu$ can be identified to a continuous function. Therefore its support have nonempty interior, and so $A + \cdots + A$ has nonempty interior.

This little remark is central to understand the link between additive combinatorics and fourier decay. A set A such that $A + \cdots + A$ has nonempty interior is a good candidate to be "additively chaotic". It is those additively chaotic sets that will let us try to construct decaying fourier transform of measures on them, although the task of actually proving that the Fourier transform of a given measure decays is quite hard.

Example 5. Unfortunately, the argument does not work for the triadic cantor set, as $C + C = [0, 2]$. The example is quite instructive, as it make us see that the notion of "additive structure" is very subtle. Having a sumset with nonempty interior is still not enough in general to say that our set is really additively chaotic. In the cantor set, the real remark is the following.

Consider the map $T(x) := 3x \mod 1$. By construction of the Cantor set, we have $T(C) = C$, and this will obstruct the existence of measures with decaying Fourier transform. Indeed, assume that such a measure $\mu_0 \in \mathcal{P}(C)$ exists. Then, we would have :

$$\begin{aligned}\widehat{T_* \mu_0}(\xi) &= \int_0^{1/3} e^{2i\pi(3x)\xi} d\mu_0(x) + \int_{1/3}^{2/3} e^{2i\pi(3x-1)\xi} d\mu_0(x) + \int_{2/3}^{1} e^{2i\pi(3x-2)\xi} d\mu_0(x) \\ &= \int_0^{1/3} e^{2i\pi(3x)\xi} d\mu_0(x) + e^{4i\pi\xi} \int_{2/3}^{1} e^{2i\pi(3x)\xi} d\mu_0(x)\end{aligned}$$

Since $]1/3, 2/3[\cap C = \emptyset$. Hence, for any $n \in \mathbb{Z}$, we would get :

$$\widehat{T_* \mu_0}(n) = \widehat{\mu}(3n)$$

And so, by induction :

$$\widehat{(T_*)^k \mu_0}(n) = \widehat{\mu}(3^k n).$$

Then, the Prokhorov theorem ensures that we could extract a subsequence from $(T_*)^k \mu_0$ that converge weakly to a probability measure μ_∞ supported on the cantor set. But in this case, its Fourier transform would be $\widehat{\mu_\infty}(n) = \widehat{\mu_0}(\infty) = 0$ for $n \neq 0$, which means that our measure satisfy $\forall n \neq 0, c_n(\mu_\infty) = 0$ and $c_0(\mu_\infty) = 1$, which imply that μ_∞ is the Lebesgue measure on $[0, 1]$, which is a contradiction. (See the theorem 35.)

Our Cantor set is *not* additively chaotic.

To quantify this notion of additive chaos, we have the following definition.

Definition 10. The Fourier dimension of a borel set $E \subset \mathbb{R}^d$ is defined by :

$$\dim_F(E) = \sup \left\{ \alpha \in [0,d] \mid \exists \mu \in \mathcal{P}(E), \exists C > 0,\ |\widehat{\mu}(\xi)| \leq C(1+|\xi|)^{-\alpha/2} \right\}$$

We always have, by the Corollary 15, $\dim_F(E) \leq \dim_H(E)$. Sets with maximal Fourier dimension $\dim_F(E) = \dim_H(E)$ are called *Salem sets.*

On Salem sets. As said before, we can think of them as being sets that are extremely additively chaotic. Salem, in 1951, showed in [Sa51] the existence of a lot of Salem sets by the construction of random scaled cantor sets. A lot of random processes giving the existence of Salem sets was studied since then, but constructing deterministic Salem sets have been proven to be quite difficult, and such examples are much recent. For example, in 2020 (see [FH20]), R. Fraser and K. Hambrook showed that the following sets are Salem sets of dimension $2d/(1+\tau)$, for any $\tau > 1$:

$$\left\{ x \in \mathbb{R}^d\ ,\ |x - r/q| \leq |q|^{-(1+\tau)} \text{ for infinitely many } (r,q) \in \mathbb{Z}^{2d} \right\}.$$

This generalizes a previous construction of Kaufman.

Computing the Fourier dimension of a set is often very hard, but some special case are computable. Example 4 show that any countable set have Fourier dimension 0, and is Salem. The Cantor set also have Fourier dimension 0 : it has additive structure. In contrast, we will see at the end of this report that some nonlinear cantor sets have strictly positive Fourier dimension. A hyperplane in $\mathbb{R}^d$ have zero Fourier dimension, but the sphere $\mathbb{S}^{n-1}$ is a Salem set. The difference between those two hypersurfaces is the presence of curvature in the second one : we shall explore this deeper in the last chapter of this report. For now, let's just see what happens in the case of the circle.

Example 6. We consider the circle $\mathbb{S}^1 \subset \mathbb{R}^2$, endowed with the Lebesgue measure σ. We have :

$$\widehat{\sigma}(\xi) = \int_{\mathbb{R}^2} e^{-2i\pi x \cdot \xi} d\sigma(x)$$

Since the Lebesgue measure on the circle is its Haar measure, its Fourier transform is a radial function. We get :

$$\widehat{\sigma}(\xi) = \int_{\mathbb{R}^2} e^{-2i\pi|\xi| x\cdot(1,0)} d\sigma(x) = \int_0^{2\pi} e^{-2i\pi|\xi|\cos(\theta)} d\theta = 2\pi J_0(2\pi\xi)$$

Where J_0 is the zeroth Bessel function. In general, if σ_{d-1} denotes the surface measure on the sphere $\mathbb{S}^{d-1}$ (see, for example, [Be15] for the computations) :

$$\widehat{\sigma_{d-1}}(\xi) = 2\pi|\xi|^{1-d/2} J_{d/2-1}(2\pi\xi)$$

Where J_α is the Bessel function of order α. The usual estimate $|J_\alpha(\xi)| \leq C|\xi|^{-1/2}$ gives us the bound

$$|\sigma_{d-1}(\xi)| \leq C|\xi|^{-(d-1)/2}$$

Which proves that $\mathbb{S}^{d-1}$ is a Salem set.

1.3.2 Additive combinatorics

The central remark made in the last subsection was the following : if a borel set A has strictly positive Fourier dimension, then its k-th iterated sumset $A + \cdots + A$ has nonempty interior, for k large enough. Also, A may not be fixed by any map of the form $T(x) = nx \bmod 1$, with $n \geq 2$. We interpret this as A being "additively chaotic", or "additively pseudorandom". On the other hand, a set with zero fourier dimension is said to have some additive structure.

The field that is interested in the additive structure of sets is called additive combinatorics. It is here that some of the most recent results about the possible decay of Fourier transform have been found : the goal of this section is to explain the main ideas of the topic. For an introduction to the subject, see the expository article of Green, [Gr09a], from which this subsection is mostly inspired.

The results obtained by additive combinatorics can be better understood if we start by the following observation. Let's say that we are interested by the decay of the Fourier transform of some probability measure, that we will choose *randomly.* To do so, we consider the following toy example: our measure will be approximated by a sum of dirac masses, that will themselves be chosen randomly. So, in this toy example, we will consider $X_1, \dots, X_n \sim \mathcal{U}([-1/2, 1/2])$ some iid random variables, and we set :

$$S_n(\xi) := \frac{1}{n} \sum_{k=1}^{n} e^{2i\pi X_k \xi},$$

the Fourier transform of the random measure $\frac{1}{n}\sum_{k=1}^{n} \delta_{-X_k}$.

The expectation of one of those exponential is given by $\text{sinc}(\pi\xi)$. Hence, the law of large numbers gives us, at a fixed ξ, that

$$\frac{1}{n} \sum_{k=1}^{n} e^{2i\pi X_k \xi} \xrightarrow[n\to\infty]{a.s.} \text{sinc}(\pi\xi).$$

But this is not enough for our intuition, as, for a fixed n, it does not really help us understand our function $S_n(\xi)$. A more interesting result can be given by adapting a proof of a uniform version of the law of large numbers : this is the Glivenko-Cantelli theorem as stated in [Ha16]. As this isn't a probability thesis and since the details are a bit technical, I will not put any more detail on how I get this results, but the idea is to use the formula page 235 which gives a bound on the covering number for the set of the functions with total variations less than $V := n^{1-\varepsilon}$, plug it into the first inequality page 208, and then we get the following uniform result :

$$\mathbb{E}\left(\sup_{|\xi| \le n^{1-\varepsilon}} \left| \frac{1}{n} \sum_{k=1}^{n} e^{2i\pi X_k \xi} - \text{sinc}(\pi\xi) \right| \right) \lesssim n^{-\varepsilon/2}.$$

Which is already way more interesting. We wee that, at a fixed n, we have a good decay for $|\xi|$ that are not too large. Notice that, in this example, it was mandatory to stay away from too large ξ, as at a fixed n, the sum has no chance to decay at very large scales...

To summarize: measures with randomly chosen phase X_k enjoy good decay properties. Notice that the decay in ξ^{-1} is a bit artificial here, it comes from how randomly we picked our phases. But for any choice of randomness, as long as it is an absolutely continuous one, we'd still get some decay result.

So now, the challenge is to find a way to get this decay, that comes naturally from randomness, in a deterministic setting. How can we create "additive randomness" in a deterministic manner ?

And this were additive combinatorics comes to help. There is an interesting phenomenon that arise when studying how additive structure and multiplicative structure interacts with each other. The event is the following : sets that have a good multiplicative structure (like subgroups of $\mathbb{F}_p^\times$) have an extremely chaotic additive structure. A good use of this sum-product phenomenon allow us to create deterministicaly some additive pseudorandom settings, and from this we are able to mimic the results that we get in the random setting. The rest of this paragraph is devoted to understand this sum-product phenomenon.

To get an idea of what is going on, it is easier to begin with finite sets. As we are interested, in the end, to sums of exponential, it is relevant to first study the case of finite sets $A \subset \mathbb{F}_p$, where p is a large prime. (Here we think as $\mathbb{F}_p$ being the p-th roots of unity.)

A finite set $A \subset \mathbb{F}_p$ will be said to have some good additive structure if it contains a lot of arithmetic sequences, relatively to its total size. More precisely, we are interested in the size of $A - A := \{a - a' \mid a, a' \in A\}$: it is the set of all the common differences of arithmetic progressions in A. If A is taken randomly, we can expect $A - A$ to have $\sim |A|^2/2$ elements. On the other hand,

$A \hookrightarrow A - A$, and so $|A| \leq |A - A|$. The equality is obtained if A is of the form $a + H$ (a coset), where H is an additive subgroup of $\mathbb{F}_p$.

We can also consider the difference of two different sets A and B. In this case, $|A - B|$ is a number greater than $|A|^{1/2}|B|^{1/2}$, and the equality is achieved for sets of the form $a + H$ and $b + H$ respectfully, with H being the *same* subgroup of $\mathbb{F}_p$. So, this number being small means two things : first of all, A and B does have additive structure, and second, the structures are similar.

One useful result about the difference sets is the following.

Theorem 16 (Ruzsa's triangle inequality)**.** *Consider three finite sets $A, B, C \subset \mathbb{F}_p$. Then :*

$$|B||A - C| \leq |A - B||B - C|$$

Proof. By definition of the difference set, $\begin{array}{ccc} A \times C & \longrightarrow & (A - C) \\ (a, c) & \longmapsto & (a - c) \end{array}$ is surjective.

Choose $(\iota_A, \iota_C) : A - C \hookrightarrow A \times C$ a section. Then,

$$\begin{array}{cccc} f: & B \times (A - C) & \longrightarrow & (A - B) \times (B - C) \\ & (b, \alpha) & \longmapsto & (\iota_A(\alpha) - b, b - \iota_C(\alpha)) \end{array}$$

is injective.

□

This inequality leads to the notion of Rusza distance, which will measure how much sets are "additively similar". For some results about the Rusza distance, see for example [Sh16] or [Gr09b].

Definition 11. Define the Rusza distance :

$$d(A, B) := \log\left(\frac{|A - B|}{|A|^{1/2}|B|^{1/2}}\right).$$

Then $d(A, B) = d(B, A)$, and $d(A, C) \leq d(A, B) + d(B, C)$.

In the notation of the paper of Green, we fix a "roughness parameter" K, and we say that A and B have similar additive structure if $d(A, B) = O(\log(K))$. In this case, we note $A \sim B$. Be careful that the Rusza distance isn't a distance, as it is not true that $d(A, A) = 0$ for all A : in fact, $d(A, A) = 0$ means that A is a coset.

Now we have our first tool to study the additive structure of a set. To get where I'd like to get, we need to introduce another tool, the additive energy.

Definition 12. Let $A, B \subset \mathbb{F}_p$.
We define $\omega_+(A, B)$, the additive energy of the pair (A, B), by the formula:

$$\omega_+(A, B) := \frac{1}{|A|^{3/2}|B|^{3/2}} \# \left\{(a_1, a_2, b_1, b_2) \in A^2 \times B^2 \mid a_1 + b_1 = a_2 + b_2\right\}.$$

Some remarks : if we know a_1, a_2 and b_1, then b_2 is known, and so we have the bound $\omega_+(A, B) \leq |A|^{1/2}|B|^{-1/2}$. Symmetrically, we also have $\omega_+(A, B) \leq |A|^{-1/2}|B|^{1/2}$, and so $\omega_+(A, B) \leq 1$. Reciprocally, we see that for any $(a, b) \in A \times B$, we have the empty relation $a + b = a + b$, and so $|A|^{-1/2}|B|^{-1/2} \leq \omega_+(A, B)$.

When we compute it for cosets $A := a + H$ and $B := b + H$, we get the following :

$$\begin{aligned} \omega_+(A, B) &= \frac{1}{|H|^3} \# \left\{(h_1, h_2, k_1, k_2) \in H^2 \times H^2 \mid a + b + h_1 + k_1 = a + b + h_2 + k_2\right\} \\ &= \frac{1}{|H|^3} \# \left\{(h_1, h_2, k_1, k_2) \in H^2 \times H^2 \mid h_1 + k_1 = h_2 + k_2\right\} = 1. \end{aligned}$$

So the idea of this number is the following : if $\omega_+(A, B) \simeq 1$, then A and B have some good additive structure that are compatible (but in a sense slightly different from before), and if $\omega_+(A, B) \simeq |A|^{-1/2}|B|^{-1/2}$, then A and B do not have compatible additive structure.

We have the following result that help us compare the two notions. It is given in a somewhat informal way, but it is sufficient for the intuition. For a more precise statement, see the article of Green [Gr09a], and for a proof, see [Gr09c].

Theorem 17 (Balog-Szemerédi-Gowers). *Let $A, B \subset \mathbb{F}_p$. Then :*

- *If $A \sim B$ then $\omega_+(A,B) \simeq 1$.*
- *If $\omega_+(A,B) \simeq 1$ then there are subsets $A' \subset A$ and $B' \subset B$ with $|A'| \simeq |A|$ and $|B'| \simeq |B|$ such that $A' \sim B'$.*

The first item is easy, we will prove it right away. The second is way more difficult, and is the real Balog-Szemerédi-Gowers theorem. This theorem is a key result in additive combinatorics.

Proof. For any $x \in A - B$, define $r_-(x) := \#\{(a,b) \in A \times B \mid a - b = x\}$. Then we have :

$$\begin{aligned}\omega_+(A,B) &= \frac{1}{|A|^{3/2}|B|^{3/2}} \# \left\{(a_1,a_2,b_1,b_2) \in A^2 \times B^2 \mid a_1 - b_2 = a_2 - b_1\right\} \\ &= \frac{1}{|A|^{3/2}|B|^{3/2}} \sum_{x \in A-B} r_-(x)^2 \\ &\geq \frac{1}{|A|^{3/2}|B|^{3/2}} \left(\sum_{x\in A-B} r_-(x)\right)^2 \frac{1}{|A-B|}\end{aligned}$$

Hence,

$$\frac{|A|^{1/2}|B|^{1/2}}{|A-B|} \leq \omega_+(A,B)$$

which proves the first point : if $A \sim B$, then $|A - B| \simeq |A|^{1/2}|B|^{1/2}$, and so $\omega_+(A,B) \simeq 1$.

□

From this, we have the right tools to study what it means for a set to be additively pseudorandom, or to have some additive structure. Now, let's get to the point of the sum-product phenomenon.

Lemma 18. *Let $A \subset \mathbb{F}_p$ be any set. Then, there always exists some $\xi \in \mathbb{F}_p^\times$ such that*

$$|A - \xi A| \geq \frac{1}{2}\min(|A|^2, p).$$

Now THIS is extremely interesting. It implies the following : if we take $A = H$ an additive subgroup of $\mathbb{F}_p$, then for some $\xi \in \mathbb{F}_p^\times$, the cardinal $|H - \xi H|$ reaches (nearly) the maximum possible value, which is obtained when, in $|H - B|$, B is chosen randomly ! This means that the action $\mathbb{F}_p^\times \curvearrowright \mathbb{F}_p$ destroy the additive structure of any set, it shuffle well any additive subgroup, in other words : multiplication creates additive pseudorandomness. This will be used to our advantage soon.

Proof. Let $A \subset \mathbb{F}_p$ We have :

$$\begin{aligned}\sum_{\xi \in \mathbb{F}_p^\times} \omega_+(A, \xi A) &= \frac{1}{|A|^3} \# \left\{(a_1,a_2,a_3,a_4,\xi) \in A^4 \times \mathbb{F}_p^\times \mid a_1 + \xi a_2 = a_3 + \xi a_4\right\} \\ &= \frac{1}{|A|^3} \sum_{a_1,a_2,a_3,a_4} \# \left\{\xi \in \mathbb{F}_p^\times \mid (a_1 - a_3) = \xi(a_4 - a_2)\right\} \\ &= \frac{1}{|A|^3}\left(\sum_{a_1\neq a_3, a_2 \neq a_4} 1 + \sum_{a_1 = a_3, a_2\neq a_4} 0 + \sum_{a_1\neq a_3, a_2 = a_4} 0 + \sum_{a_1 = a_3, a_2 = a_4} (p-1)\right) \\ &= \frac{1}{|A|^3}\left((|A|^2 - |A|)^2 + |A|^2(p-1)\right) = \frac{1}{|A|}\left((|A|-1)^2 + p - 1\right).\end{aligned}$$

From this, we deduce that there exists some $\xi \in \mathbb{F}_p^\times$ such that

$$\omega_+(A, \xi A) \leq \frac{(|A|-1)^2 + p - 1}{|A|(p-1)} \leq 2\max(\frac{1}{|A|}, \frac{|A|}{p}).$$

Hence, by the easy part of the BSG theorem, we get :

$$|A - \xi A| \geq \frac{1}{2}\min(|A|^2, p).$$

□

From this, we can work out the following mantra : multiplicative subgroups of $\mathbb{F}_p^\times$ are additively pseudorandom. The proofs are done in [Gr09a], we won't do them, but before going on let's cite two of the more striking result that we get at the end of this paper. The first one is a reformulation of this "sum product" phenomenon, and the second one is a powerful consequence for sums of exponential.

Theorem 19 (Bourgain-Katz-Tao)**.** *Let $\delta > 0$. There exists $\delta' > 0$ and $c > 0$ such that the following holds. For any $A \subset \mathbb{F}_p$ such that $p^\delta \leq |A| \leq p^{1-\delta}$, we have*

$$\max(|A + A|, |A|) \geq c|A|^{1+\delta'}.$$

Theorem 20 (Bourgain-Gilibichuk-Konyagin)**.** *Suppose that $H \subset \mathbb{F}_p^\times$ is a multiplicative subgroup of size at least p^δ, where $\delta > 0$ and where p is a large enough prime. Then, uniformly in $\xi \in \mathbb{F}_p^\times$, we have*

$$\frac{1}{|H|}\left|\sum_{x\in H} e^{2i\pi x\xi/p}\right| \lesssim p^{-\delta'}$$

where $\delta' > 0$ depends only on δ.

As said by Green in the paper : this theorem "is something of a triumph for additive combinatorics, for the question had previously been extensively studied by quite sophisticated number-theoretical arguments".

1.3.3 Consequences of the sum-product phenomenon

From this sum-product phenomenon, we can modify the Fourier transform of our measures so that the phases lives in something that have more multiplicative structure, (so more additively chaotic) and hence improve our chances to get some decay.
A way to do this is to define the multiplicative convolution of two measures. In the real case, for $\mu \in \mathcal{M}(E)$, $\nu \in \mathcal{M}(F)$, it is defined by duality via this natural formula :

$$\int_{\mathbb{R}} f(z)d\mu \odot \nu(z) := \int_E \int_F f(xy)d\mu(x)d\nu(y).$$

Notice that $\mu \odot \nu$ have support in $EF := \{xy \mid x \in E, y \in F\}$.

The idea is, being given a measure μ that satisfy some necessary non-concentration properties, to consider the Fourier transform of $\mu^{k\odot}$, for k large enough.

And, incredibly, it works. We have the following theorems of Bourgain, extracted from [BD17] and [Bo10].

Theorem 21. *For all $\delta > 0$, there exist $\varepsilon_1, \varepsilon_2 > 0$ and $k \in \mathbb{N}$ such that the following holds. Let μ be a probability measure on $[1/2, 1]$ and let N be a large integer. Assume that for all $\sigma \in \left[N^{-1}, N^{-\varepsilon_1}\right]$,*

$$\sup_x \mu([x - \sigma, x + \sigma]) < \sigma^\delta.$$

Then for all $\eta \in \mathbb{R}$, $|\eta| \sim N$:

$$\left|\int \exp(2i\pi\eta x_1 \dots x_k)d\mu(x_1)\dots d\mu(x_k)\right| \leq N^{-\varepsilon_2}.$$

Theorem 22. *Fix $\delta > 0$. Then there exist $k \in \mathbb{N}$, $\varepsilon > 0$ depending only on δ such that the following holds. Let $C_0 > 0$ and $\mu_1, ..., \mu_k$ be Borel measures on $\left[C_0^{-1}, C_0\right] \subset \mathbb{R}$ such that $\mu_j(\mathbb{R}) \leq C_0$.*
Let $\eta \in \mathbb{R}$, $|\eta| \geq 1$, and assume that for all $\sigma \in \left[C_0|\eta|^{-1}, C_0^{-1}|\eta|^{-\varepsilon}\right]$ and $j = 1, \dots, k$, we have

$$\mu_j \otimes \mu_j \left(\{(x, y) \in \mathbb{R}^2 \mid |x - y| \leq \sigma\}\right) \leq C_0 \sigma^\delta$$

Then there exists a constant C_1 depending only on C_0, δ such that

$$\left| \int \exp(2i\pi\eta x_1 \dots x_k) d\mu_1(x_1)...d\mu_k(x_k) \right| \leq C_1 |\eta|^{-\varepsilon}$$

Theorem 23. *Fix $\delta > 0$. Then there exist $k \in \mathbb{N}$, $\varepsilon > 0$ depending only on δ such that the following holds. Let $C_0, N_\mathcal{Z} \geq 0$ and $\mathcal{Z}_1, ..., \mathcal{Z}_k$ be finite sets such that $\#\mathcal{Z}_j \leq C_0 N_\mathcal{Z}$.*

Let $|\eta| \geq 1$. Take some maps $\zeta_j : \mathcal{Z}_j \to [C_0^{-1}, C_0]$, $j = 1, \dots, k$, such that

$$\forall j, \forall \sigma \in \left[|\eta|^{-1}, |\eta|^{-\varepsilon}\right], \ \# \left\{(b, c) \in \mathcal{Z}_j^2 \mid |\zeta_j(b) - \zeta_j(c)| \leq \sigma\right\} \leq C_0 N_\mathcal{Z}^2 \sigma^\delta.$$

Then there exists some constant C_1 depending only on C_0, δ, such that :

$$\frac{1}{N_\mathcal{Z}^k} \left| \sum_{b_1 \in \mathcal{Z}_1, \dots, b_k \in \mathcal{Z}_k} \exp(2i\pi\eta\zeta_1(b_1) \dots \zeta_k(b_k)) \right| \leq C_1 |\eta|^{-\varepsilon}.$$

There is some things that are worth noticing in those theorems : the k and the decay rate only depends on the non-concentration property of the measure/maps that we wish to study. This is formulated in a different way that we are used to, but in fact this is the type of estimate that we studied when we talked about Frostmann measure. In particular, the existence of measures satisfying those non-concentration hypothesis can only be obtained if the support of our measures have a sufficiently high Hausdorff dimension. (Everything's linked to each other !)

Those theorems have been generalized past the dimension 1, in a paper of J. Li, [LI18]. The non-concentration hypothesis still appears, but in a bit different form.
In those theorems, we define the product of two vectors $x, y \in \mathbb{R}^d$ by $xy := (x_i y_i)_i$. The scalar product is still denoted $x \cdot y$.

Theorem 24. *Given $\kappa_0 > 0$, there exists $\varepsilon, \varepsilon_1 > 0$ and $k \in \mathbb{N}$ such that the following holds for any $\delta > 0$ small enough. Let μ be a probability measure on $[1/2, 1]^d \subset \mathbb{R}^d$ such that :*

$$\forall \rho \geq \delta, \quad \sup_{a \in \mathbb{R}, v \in \mathbb{S}^{d-1}} \mu\{x \mid v \cdot x \in [a - \rho, a + \rho]\} \leq \delta^{-\varepsilon} \rho^{\kappa_0}.$$

Then, for all $\xi \in \mathbb{R}^n$ with $|\xi| \in [\delta^{-1}/2, \delta^{-1}]$,

$$\left| \int \exp(2i\pi\xi \cdot (x^1 \dots x^k)) d\mu(x^1) \dots d\mu(x^k) \right| \leq \delta^{\varepsilon_1}.$$

Theorem 25. *Fix $\kappa > 0$. Then there exists ε and $k \in \mathbb{N}$ such that the following holds for any τ large enough. Let λ be a borel probability measure on $[1/2, 1]^d \subset \mathbb{R}^d$ such that :*

$$\forall \rho \in [\tau^{-1}, \tau^\varepsilon], \quad \sup_{a \in \mathbb{R}, v \in \mathbb{S}^{d-1}} \lambda\{x \mid v \cdot x \in [a - \rho, a + \rho]\} \leq \rho^\kappa.$$

Then, for all $\xi \in \mathbb{R}^n$ with $|\xi| \in [\tau/2, \tau]$,

$$\left| \int \exp(2i\pi\xi \cdot (x^1 \dots x^k)) d\mu(x^1) \dots d\mu(x^k) \right| \leq \tau^{-\varepsilon}.$$

The author also state a version for measures in $\mathbb{C}$, for which the natural multiplication is different from the one on $\mathbb{R}^2$. It uses a complex version of the sum-product estimates that comes from [BG12], proposition 2. The result is the following.

Theorem 26. *Fix $\kappa_0 > 0$. Then there exists $\varepsilon, \varepsilon_1 > 0$ and $k \in \mathbb{N}$ such that the following holds for $\delta > 0$ small enough. Let μ be a Borel probability measure on the annulus $\{z \in \mathbb{C} \mid 1/2 \leq |z| \leq 1\}$ such that :*

$$\forall \rho \geq \delta, \sup_{a \in \mathbb{R}, \theta \in \mathbb{R}} \mu\{z \in \mathbb{C} \mid Re(e^{i\theta} z) \in [a-\rho, a+\rho]\} \leq \delta^{-\varepsilon} \rho^{\kappa_0}.$$

Then, for all $\xi \in \mathbb{C}$ with $|\xi| \in [\delta^{-1}/2, \delta^{-1}]$,

$$\left| \int \exp(2i\pi Re(\xi z_1 \dots z_k)) d\mu(z_1) \dots d\mu(z_k) \right| \leq \delta^{\varepsilon_1}.$$

And I think that's enough theorems for this part.

To conclude this first part, let's recall what we have learned. The decay of the Fourier transform of a measure gives us information about its support. Decay in L^2 average is directly linked to the Hausdorff dimension of the support, while pointwise decay can only be achieved if the support is additively chaotic. Then, a phenomenon from additive combinatorics comes to help : the sum product phenomenon, which says that additive and multiplicative structures hardly lives together. Hence, to create additive chaos, one can generate more multiplicative structure on our measures, by looking at some multiplicative convolution. Under some natural hypothesis of non concentration, the idea prove to be fruitful, as the theorems of Bourgain highlight.

But still, from there, the reader could complain that being given a measure, we are still unable to prove that its Fourier transform decay. We only found principles and necessary conditions. But don't worry, we will come back to this question in the third part of this thesis : we will see how, in a dynamical setting, the notion of transfer operator can help us gets informations on μ when we have informations on $\mu^{\odot k}$. A lot of measures comes as invariant measures from some dynamical system, and in this case, we have some tools to help us.

From now on, let's take a break, and explore a bit what we can do if we know that the Fourier transform of some measure decays.

Chapter 2

Some applications of Fourier decay

2.1 Geometric applications

In this section, we will be mainly interested in stating some examples where the results related to the energy integral are useful. As I said before, the fact that one can reformulate the statement $\dim_H E < \alpha$ as "there exists some well behaved measure with support on E" allow us to do some computations with the said measure. This can be really helpful for the study of some geometric problems. Let's state some of them.
This paragraph is heavily inspired by the chapter 4 of [Ma15].

The question that will occupy us here is : given a set E of known Hausdorff dimension, and given a function f, what can I say about the Hausdorff dimension of $f(E)$?

For example :

Theorem 27. *Let $E, F \subset \mathbb{R}^d$ be borel sets. Then :*

$$\dim_H(E \times F) \geq \dim_H(E) + \dim_H(F).$$

Proof. Let $\alpha < \dim_H(E)$ and $\beta < \dim_H(F)$. By Frostman's lemma, there exists adapted Frostman measures $\mu \in \mathcal{P}(E)$ and $\nu \in \mathcal{P}(F)$. Now, consider the product measure $\mu \otimes \nu \in \mathcal{P}(E \times F)$. It satisfies the estimate :

$$\mu \otimes \nu(B(x,r)) \leq \mu \otimes \nu([x-r,x+r]^{2d}) = \mu([x-r,x+r]^d)\nu([x-r,x+r]^d) \leq dC_\mu C_\nu r^{\alpha+\beta}$$

And so $\alpha + \beta < \dim_H(E \times F)$. □

The idea will be the same in all the paragraph : if we can pushforward measures on the set that is of interest, then we can often say something about its dimension.

A bit more involved example is the case of projection sets. Being given a vector $u \in \mathbb{S}^{d-1}$, define $P_u(x) := v \cdot x$. It is essentially the projection on the line $u\mathbb{R}$. The question is, knowing the dimension of some $E \subset \mathbb{R}^d$, what can we say about the dimension of $P_u(E)$?

Theorem 28. *Let $E \subset \mathbb{R}^d$ be a borel set. If $\dim_H(E) \leq 1$, then*

$$\dim_H(P_u(E)) = \dim_H(E) \text{ , for almost all } u \in \mathbb{S}^{d-1}.$$

If $\dim_H(E) > 1$, then

$$\lambda_1(P_u(E)) > 0 \text{ , for almost all } u \in \mathbb{S}^{d-1}.$$

Proof. Being given a measure $\mu \in \mathcal{M}(E)$, define $\mu_u \in \mathcal{M}(P_u(E))$ by $\mu_u(A) := \mu(P_u^{-1}(A))$. We then have, for all $r \in \mathbb{R}$:

$$\widehat{\mu_u}(r) = \int_{\mathbb{R}} e^{-2i\pi rx} d\mu_u(x) = \int_{\mathbb{R}^d} e^{-2i\pi ru\cdot x} d\mu(x) = \widehat{\mu}(ru).$$

Assume that $\dim_H(E) \leq 1$. Let $0 < \alpha < \dim_H(E)$ and let $\mu \in \mathcal{P}(E)$ such that $I_\alpha(\mu) < \infty$. We have :

$$I_\alpha(\mu) = c_{\alpha,d} \int_{\mathbb{R}^d} |\widehat{\mu}(\xi)|^2 |\xi|^{\alpha-d} d\xi = c_{\alpha,d} \int_0^\infty \int_{\mathbb{S}^{d-1}} |\widehat{\mu}(ru)|^2 r^{\alpha-1} d\sigma(u) dr = \frac{c_{\alpha,d}}{2c_{\alpha,1}} \int_{\mathbb{S}^{d-1}} I_\alpha(\mu_u) d\sigma(u) < \infty.$$

Therefore, for almost all $u \in \mathbb{S}^{d-1}$, $\alpha < \dim_H(P_u(E))$. Choosing a sequence $\alpha_n \nearrow \dim_H(E)$ allow us to get the desired result.

Now, suppose that $\dim_H(E) > 1$. Then, there is $\mu \in \mathcal{P}(E)$ such that $I_1(\mu) < \infty$. Consequently:

$$\int_{\mathbb{R}^n} |\widehat{\mu}(\xi)|^2 |\xi|^{1-d} d\xi = \frac{1}{2} \int_{\mathbb{S}^{d-1}} \int_{-\infty}^\infty |\widehat{\mu}_u(r)|^2 dr d\sigma(u) < \infty$$

So $\widehat{\mu_u} \in L^2(\mathbb{R})$, for almost all $u \in \mathbb{S}^{d-1}$. This is equivalent to say that, for those u, $\mu_u \in L^2(\mathbb{R})$, and since the support of a nonzero L^2 function has strictly positive Lebesgue measure, we get our desired result.

□

Theorem 29. *Let $E \subset \mathbb{R}^d$ be a borel set such that $\dim_H(E) > 2$. Then, for almost all $u \in \mathbb{S}^{d-1}$, $P_u(E)$ has nonempty interior, .*

Proof. Let $2 < \alpha < \dim_H(A)$ and choose $\mu \in \mathcal{P}(E)$ such that $I_\alpha(\mu) < \infty$. Then :

$$\int_{\mathbb{S}^{d-1}} \int_{\mathbb{R}} |\widehat{\mu_u}(r)| dr d\sigma(u) \leq 2 \int_{\mathbb{S}^d} \int_1^\infty |\widehat{\mu_u}(r)| dr d\sigma(u) + 2\sigma(\mathbb{S}^{d-1})$$

$$\leq 2 \left(\int_{\mathbb{S}^{d-1}} \int_1^\infty |\widehat{\mu}(ru)|^2 r^{\alpha-n+n-1} \right)^{1/2} \left(\int_{\mathbb{S}^{d-1}} \int_1^\infty r^{1-\alpha} dr d\sigma(u) \right)^{1/2} + 2\sigma(\mathbb{S}^{d-1})$$

$$\leq 2 \left(\frac{\sigma(\mathbb{S}^{d-1})}{\alpha - 2} \right)^{1/2} \int_{\mathbb{R}^d} |\widehat{\mu}(\xi)|^2 |\xi|^{\alpha-d} d\xi + 2\sigma(\mathbb{S}^{d-1}) < \infty$$

Hence, $\widehat{\mu_u}$ is in $L^1(\mathbb{R})$ for almost every $u \in \mathbb{S}^{d-1}$. This imply that, for those u, μ_u is a continuous function, and so its support have nonempty interior, which proves the desired result.

□

Another setting where this idea is fruitful is in the study of distance sets. If we consider $E \subset \mathbb{R}^d$, we define its distance set by

$$D(E) := \{|x - y| \mid x, y \in E\}.$$

We are interested, again, by the dimension of $D(E)$. We have the following theorem :

Theorem 30. *Let $E \subset \mathbb{R}^d$ be a Borel set.*

1. *if $\dim_H(E) > (d+1)/2$, then $D(E)$ has nonempty interior.*
2. *If $(d-1)/2 \leq A \leq (d+1)/2$, then $\dim_H(E) \geq \dim_H E - (n-1)/2$.*

Proof. We will only give ideas of the proof for the first point : the interested reader can find the whole proof in the chapter 4 of [Ma15]. The idea is the same as before : we construct measures on $D(E)$ from measures on E. Namely, being given a measure $\mu \in \mathcal{M}(E)$, define $\delta(\mu) \in \mathcal{M}(D(E))$ by:

$$\int g d\delta(\mu) := \int_{\mathbb{R}^{2d}} g(|x - y|) d\mu(x) d\mu(y)$$

If we consider the case of an absolutely continuous measure $d\mu = f dx$ (here, the support is a neighborhood of E and $D(E)$), the distance measure can be computed explicitly. Namely, we have:

$$\int g d\delta(f) = \int_{\mathbb{R}^{2d}} g(|x - y|) f(x) f(y) dx dy$$

$$= \int_{\mathbb{R}^d} \int_0^\infty \int_{\mathbb{S}^{d-1}} g(r) f(x) f(x + ru) r^{d-1} dr d\sigma(u) dx$$

$$= \int_0^\infty g(r) \left(\int_{\mathbb{R}^d} f(x) \int_{\mathbb{S}^{d-1}} f(x+ru) r^{d-1} d\sigma(u) dx \right) dr$$
$$= \int_0^\infty g(r) \left(\int f(f * \sigma_r) \right) dr.$$

Hence, $\delta(f)$ is identified to the continuous function $\delta(f)(r) = \int f(f * \sigma_r)$, where σ_r is the surface measure on $r\mathbb{S}^{d-1}$. Then, the plancherel formula then gives :

$$\delta(f)(r) = \int |\widehat{f}(\xi)|^2 \widehat{\sigma_r}(\xi) d\xi$$

Now, the goal is to extend this formula for some non absolutely continuous measures. Recall from example 5 that we have the following estimate : $|\widehat{\sigma_r}(\xi)| = |r^{d-1}\widehat{\sigma}(r\xi)| \leq Cr^{(d-1)/2}|\xi|^{(1-d)/2}$. We will use this estimate to use a dominated convergence theorem.

Let $\mu \in \mathcal{P}(E)$ such that $I_{(d+1)/2}(\mu) < \infty$. Let Ψ be a positive smooth function with compact support in $\mathbb{R}^d$ such that $\int \Psi = 1$. Let $\Psi_\varepsilon(x) := \varepsilon^{-d}\Psi(x/\varepsilon)$ and $\mu_\varepsilon := \mu * \Psi_\varepsilon$. By the previous formula, we have :

$$\delta(\Psi_\varepsilon * \mu)(r) = \int |\widehat{\Psi}_\varepsilon(\xi)|^2 |\widehat{\mu}(\xi)|^2 \widehat{\sigma_r}(\xi) d\xi$$

First, we see that $\Psi_\varepsilon * \mu$ converges weakly to μ as $\varepsilon \to 0$, and so $\delta(\Psi_\varepsilon * \mu)$ also converge weakly to $\delta(\mu)$. Then, we have

$$|\widehat{\Psi}_\varepsilon|^2 |\widehat{\mu}(\xi)|^2 |\widehat{\sigma_r}(\xi)| \leq C|\widehat{\mu}(\xi)|^2 |\xi|^{(1-d)/2} \in L^1(\mathbb{R}^d)$$

And so, by the dominated convergence theorem,

$$\delta(\mu)(r) = \int |\widehat{\mu}(\xi)|^2 \widehat{\sigma_r}(\xi) d\xi.$$

In particular, it means that $\delta(\mu)$ is identified to a continuous function with support in $D(E)$, and so $D(E)$ have nonempty interior.

□

The preceding results are informative but are far from what is expected to be optimal. The interested reader can look at the Falconer's conjecture. Those theorems have an interesting application that is proved in the same chapter of [Ma15].

Theorem 31. *Let E be a Borel subring of $\mathbb{R}$. Then either $\dim_H(E) = 0$, or either $E = \mathbb{R}$.*

The idea of the proof is the following. Consider a borel subring of $\mathbb{R}$ such that $\dim_H(E) > 0$. Since $\dim_H(E^k) \geq k \dim_H(E)$, there exists some $k \in \mathbb{N}$ such that $\dim_H(E) > 2$. Then, for this k, there exists $u \in \mathbb{S}^{d-1}$, such that $P_u(E^k)$ have non empty interior. Since it is a subring of $\mathbb{R}$, we get that $P_u(E^k) = \mathbb{R}$.

So there exists some k and some linear map $\varphi : \mathbb{R}^k \longrightarrow \mathbb{R}$ such that $\varphi(E^k) = \mathbb{R}$. The second step is to prove that, if we take the minimal k for which such a φ exists, then φ must be bijective. The last step is to see that, for such φ to be bijective, k must be equal to 1, and then $E = \mathbb{R}$.

Let's finish by mentioning the fact that this result is true in a much more general setting.

Theorem 32. *[Sa17] Let G be a connected simple real Lie group endowed with a riemannian metric. There is no Borel measurable dense subgroup of G with Hausdorff dimension strictly between 0 and $\dim_H G$.*

2.2 A unicity problem on the circle

In one of the first book to study the Fourier transform of measures, *Ensembles Parfaits et Séries Trigonometriques* [KS94], (first edition in 1964 !) Kahane and Salem linked the decay of the fourier tranform of some measure to a unicity problem on Fourier series. To problem is the following : on which kind of set does it suffice to know the value of a Fourier series for us to know it entirely? Historically, this is a really important problem. The study of Fourier series was what motivated Cantor to create set theory ($\sim$ 1870). It also motivated the study of convolution and helped to convince people of the interest of the modern formulation of the Lebesgue integral ($\sim$1900-1910).

The following paragraph is adapted from the chapter 4 of [KS94]. In the book, they get a characterization of the multiplicity set in term of *pseudofunctions.* let's state the result.

Definition 13. We work on $\mathbb{R}$ mod 2π, and we think of it as the circle $\mathbb{S}^1$. We sometimes also identify it to $[-\pi, \pi]$. We are interested in the convergence of Fourier series.

Let $E \subset [-\pi, \pi]$. We say that E is a set of unicity if, for any sequence $(c_k) \in \mathbb{C}^{\mathbb{Z}}$:

$$\left(\forall x \notin E, \sum_{k=-n}^{n} c_k e^{2i\pi kx} \underset{n\to\infty}{\longrightarrow} 0 \right) \Longrightarrow \left(\forall x \in E, \sum_{k=-n}^{n} c_k e^{2i\pi kx} \underset{n\to\infty}{\longrightarrow} 0. \right)$$

In the other case, we say that E is a set of multiplicity.

Kahane and Salem related the property of being a set of unicity with the existence of some well behaved distribution, that they call pseudofunctions.

Definition 14. Consider T a distribution on the circle. We can think of it as distribution with support in $[-\pi, \pi]$, which is compact. In particular, for all $n \in \mathbb{Z}$, the following quantity is well defined :

$$c_n(T) := \langle T, e_{-n} \rangle$$

Where the bracket is the duality bracket of the circle : if $T \in L^1([-\pi, \pi])$, it is

$$\langle T, e_{-n} \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-inx} T(x) dx.$$

We then say a distribution is a *pseudofunction* if the riemann-lebesgue theorem is still true, that is, if $c_n(T) \underset{|n|\to\infty}{\longrightarrow} 0$

We have the following result.

Theorem 33. *Let $E \subset [-\pi, \pi]$. Then E is a set of multiplicity if and only if there exists a nonzero pseudofunction with support in E.*

To dodge all the small technicalities that comes with working with distributions and to stay as elementary as possible, I'm only going to prove a weaker result stated with measures that are pseudofunctions. The idea of the complete proof is the same as what will come, and moreover, I find the following results quite interesting on their own.

The point of this paragraph is to study the convergence of the Fourier series associated to such objects. If μ is a finite measure on the circle, denote by $S_n(\mu) := \sum_{k=-n}^{n} c_n(\mu) e^{inx}$ its associated Fourier series. Notice that $S_n(\mu) \to \mu$ in the sense of distribution, but not in the weak sense in general ! So we see that the study of the pointwise convergence or not of these objects might be a bit subtle. let's get to it.

Lemma 34. *Consider $\mu \in \mathcal{M}(\mathbb{S}^1)$ a pseudofunction, and $\varphi \in \mathcal{C}^\infty(\mathbb{S}^1)$. Then $\varphi d\mu$ is a pseudofunction.*

Proof. Let $\varepsilon > 0$. By hypothesis,

$$c_n(\mu) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-inx} d\mu(x) \underset{|n|\to\infty}{\longrightarrow} 0,$$

so there exists $N > 0$ such that, for all $|n| > N$, $|c_n(\mu)| \leq \varepsilon$. Then, we have :

$$c_n(\varphi d\mu) = \frac{1}{2\pi}\int e^{-inx}\varphi(x)d\mu(x)$$

$$= \frac{1}{2\pi}\int \sum_{k\in\mathbb{Z}} e^{-i(n-k)x}c_k(\varphi)d\mu(x) = 2\pi \sum_{k\in\mathbb{Z}} c_k(\varphi)c_{n-k}(\mu)$$

by Fubini. Hence :

$$|c_n(\varphi d\mu)|/2\pi \leq \sum_{k\in\mathbb{Z}} |c_{n-k}(\varphi)c_k(\mu)| \leq \sum_{|k|\leq N} |c_{n-k}(\varphi)c_k(\mu)| + \sum_{|k|>N} |c_{n-k}(\varphi)c_k(\mu)|$$

$$\leq \mu([-\pi,\pi]) \sum_{|k|\leq N} c_{n-k}(\varphi) + \varepsilon \|c(\varphi)\|_{l^1(\mathbb{Z})}$$

And so $\limsup_{|n|\to\infty} |c_n(\varphi d\mu)| \leq \varepsilon 2\pi \|c(\varphi)\|_{l^1(\mathbb{Z})}$, and this for any $\varepsilon > 0$, so $c_n(\varphi d\mu) \to 0$.

□

Theorem 35. *Let $\mu \in \mathcal{M}(\mathbb{S}^1)$ a pseudofunction. We have the following result :*

$$\forall x \notin Supp(\mu),\ S_n(\mu)(x) \underset{|n|\to\infty}{\longrightarrow} 0.$$

Proof. Without loss of generality, we can translate our measure so that $0 \notin \text{Supp}(\mu)$. We are then interested in the series $S_n(\mu)(0) = \sum_{k=-n}^{n} c_n(\mu)$.
We have :

$$S_n(\mu)(0) = \sum_{k=-n}^{n} \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-inx}d\mu(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{k=-n}^{n} e^{-inx}d\mu(x) = \langle \mu, D_n\rangle$$

Where $D_n(x) = \frac{\sin((n+1/2)x)}{\sin(x/2)} = \sin(nx)\cot(x/2) + \cos(nx)$ is the Dirichlet kernel. The idea is the following : since $0 \notin \text{supp}\mu$, we can modify $\cot(x/2)$ in the neighbourhood of 0 to get rid of the singularity without changing the value of the integral. We call this new function $\varphi \in \mathcal{C}^\infty(\mathbb{S}^1)$. We obtain:

$$S_n(\mu)(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \sin(nx)\varphi(x)d\mu(x) + \int_{-\pi}^{\pi}\cos(nx)d\mu(x) \longrightarrow 0$$

since μ and $\varphi d\mu$ are pseudofunctions !

□

Then we are interested in a reciprocal. This is a bit more tricky, but still true.

Theorem 36. *Let $\mu \in \mathcal{M}(\mathbb{S}^1)$ a pseudofunction. Suppose that I is an open interval such that $\forall x \in I,\ S_n(\mu)(x) \to 0$. Then $I \cap Supp\ \mu = \emptyset$.*

Proof. This would be easier to prove if $S_n(\mu)$ would converge weakly to μ. But, as stated previously, this is not the case. We will bypass this technical difficulty by integrating two times our measure.

Let $\varphi \in \mathcal{C}^\infty(\mathbb{S}^1)$ with support in I. We want to show that $\int \varphi d\mu = 0$. Since $S_n(\varphi)$ converges uniformly to φ, we have :

$$\int \varphi d\mu(x) = \lim_{n\to\infty}\int S_n(\varphi)d\mu.$$

Fixing n, we get :

$$\int S_n(\varphi)d\mu = \int \sum_{k=-n}^{n} c_k(\varphi)e^{ikx}d\mu(x) = 2\pi \sum_{k=-n}^{n} c_k(\varphi)c_{-k}(\mu)$$

$$= \sum_{k=-n}^{n}\int e^{-ikx}\varphi(x)dx\ c_{-k}(\mu) = \int \varphi S_n(\mu).$$

Then, integrating by parts two times gives us :

$$\int S_n(\varphi)d\mu = \int \varphi'' F_n(\mu)$$

With

$$F_n(\mu)(x) := c_0(\mu)\frac{x^2}{2} - \sum_{k=-n}^{n} \frac{c_k(\mu)}{k^2}e^{ikx}.$$

The interest of doing this is that, now, F_n converges uniformly to the full sum F, and so we can write :

$$\int \varphi d\mu = \int \varphi'' F(\mu)$$

With

$$F(x) = c_0(\mu)\frac{x^2}{2} - \sum_{k=-\infty}^{\infty} \frac{c_k(\mu)}{k^2}e^{ikx}.$$

Then, showing that $\int \varphi d\mu = 0$ reduce to show that F is affine on I. To do so, we will carefully compute its second derivative. We consider $\Delta_2 F(x,h) := F(x+h) + F(x-h) - 2F(x)$. We get, for $x \in I$:

$$\frac{\Delta_2 F(x,2h)}{4h^2} = c_0(\mu) + \sum_{k=-\infty}^{\infty} c_k(\mu)e^{ikx}\text{sinc}^2(kh)$$

Then an Abel transform gives :

$$\frac{\Delta_2 F(x,2h)}{4h^2} = \sum_{n=0}^{\infty} S_n(\mu)(x)\left(\text{sinc}^2(nh) - \text{sinc}^2((n+1)h)\right)$$

hence :

$$\left|\frac{\Delta_2 F(x,2h)}{4h^2}\right| \leq \sum_{n<h^{-1/2}} |S_n(\mu)(x)\left(\text{sinc}^2(nh) - \text{sinc}^2((n+1)h)\right)|$$

$$+ \sum_{n>h^{-1/2}} |S_n(\mu)(x)\left(\text{sinc}^2(nh) - \text{sinc}^2((n+1)h)\right)|$$

$$\leq \max_{n\geq 0}|S_n(x)| \int_0^{\sqrt{h}} \left|\frac{d}{dx}\text{sinc}^2(x)\right| dx + \max_{n>h^{-1/2}} |S_n(\mu)(x)| \int_0^{\infty} \left|\frac{d}{dx}\text{sinc}^2(x)\right| dx \underset{h\to 0}{\longrightarrow} 0.$$

And from this we can conclude.

Consider, for $\varepsilon > 0$, $F_\varepsilon(x) := F(x) + \varepsilon x^2$. Then, for any $x \in I$ and then for any h small enough, $\Delta_2 F_\varepsilon(x,h) \geq \varepsilon$. From this we see that F_ε must be convex: if we choose two points $x_0, x_1 \in I$ and if f is the affine map such that $f(x_0) = F_\varepsilon(x_0)$ and $f(x_1) = F_\varepsilon(x_1)$, then $g := F_\varepsilon - f$ must be negative on $[x_0, x_1]$. Indeed, if it were to have a non negative value on $]x_0, x_1[$, then it would have a maximum $\widehat{x}$ on $]x_0, x_1[$. In particular, we would have, for h small enough, $\varepsilon \leq \Delta_2 F_\varepsilon(\widehat{x}, h) = \Delta_2 g(\widehat{x}, h) \leq 0$, which is an obvious contradiction.
Hence, F_ε is convex, and so, by letting ε goes to zero, we see that F is also convex.

The same reasoning apply to $-F$: so F is convex and concave on I, which means that F is affine on I. The proof is done.

□

Before going to the next part, let's summarize what we've proved :

Theorem 37. *Let $\mu \in \mathcal{M}(\mathbb{S}^1)$, a finite measure such that $c_n(\mu) \underset{|n|\to\infty}{\longrightarrow} 0$.*
We have, for any I open set :

$$\left(I \cap supp\ \mu = \emptyset\right) \iff \left(\forall x \in I, \sum_{k=-n}^{n} c_k(\mu)e^{ikx} \underset{n\to 0}{\longrightarrow} 0\right).$$

2.3 The fractal uncertainty principle

Fourier decay of measures give us the ability to prove a Fractal Unicity Principle (FUP). The FUP has itself some important applications, some of them are exposed in [Da21] (some notes taken from a Bourbaki Seminar in April 2021 by N.V. Dang). The interested reader should also have a look at the survey of Dyatlov [Dy19].

First of all, let's recall the usual uncertainty principle: it's a phenomenon that is quite famous in quantum physics. One way to understand it is from the usual Fourier tradeoff : localization in space implies diffusion in frequency. For example : any function $f \in L^1(\mathbb{R}^d)$ with compact support such that $\widehat{f}$ have compact support is the zero function.

A quantitative version of this idea is the following. We will study the semi-classical Fourier transform, defined on the Schwartz class by :

$$\mathcal{F}_h(f)(\xi) := h^{-d/2} \int_{\mathbb{R}^d} e^{-2i\pi x\cdot\xi/h} f(x) dx.$$

Where h is represent the Planck constant. In a mathematical setting, we will often be interested to look at what happens when $h \to 0$: the goal is often to make the link between classical and quantum mechanics. Notes that the definition is scaled so that it remains an isometry on L^2.

Then, consider a ball $B(0,h) \subset \mathbb{R}^d$. We look at the quantity $\|\mathbb{1}_{B(0,h)}\mathcal{F}_h\mathbb{1}_{B(0,h)}\|_{L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^d)}$. Notice what happens with the scalings : $\mathbb{1}_{B(0,h)}$ is localized on a radius of h, then the semiclassical Fourier transform re-scales it at a macroscopic scale. Another important remark is the following : if we replace $\mathcal{F}_h$ by another isometry of L^2 that doesn't satisfy an uncertainty principle, we expect the operator norm to be constant in h. (This is what we obtain if we replace $\mathcal{F}_h$ by the identity for example.) Instead, we have :

$$\|\mathbb{1}_{B(0,h)}\mathcal{F}_h\mathbb{1}_{B(0,h)}\|_{L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^d)} \leq \|\mathbb{1}_{B(0,h)}\|_{L^\infty\to L^2}\|\mathcal{F}_h\|_{L^1\to L^\infty}\|\mathbb{1}_{B(0,h)}\|_{L^2\to L^1}$$
$$\leq \left(C_d h^d\right)^{1/2} h^{-d/2} \left(C_d h^d\right)^{1/2} = C_d h^{d/2}.$$

The total mass is not there : most of it escape at $B(0,h)^c$.

In a more general setting, we consider a measure $\mu \in \mathcal{M}(E)$, for some borel set E. Denote by $E(h)$ the h-approximation of E : $E + B(0,h)$. We will say that μ satisfies a Fractal Uncertainty Principle if there exists some $\varepsilon > 0$ such that :

$$\|\mathbb{1}_{E(h)}\mathcal{F}_h\mathbb{1}_{E(h)}\|_{L^2\to L^2} \lesssim h^\varepsilon.$$

For measures that satify some positive Fourier decay condition, we can prove a FUP. The following proofs are adapted from [BD17]. Before getting to the main theorems, we need two small lemmas, that are often used in the study of integral operators : the TT^* lemma, and the Schur test.

Lemma 38 (TT^* lemma)**.** *Let E be a Banach space and H be a Hilbert space, and let $T : H \to E$ be a continuous linear operator. We define $T^* : E' \to H$ via the relation :*

$$\forall \varphi \in E', \forall y \in H, \ \langle T^*\varphi, y\rangle_H := \langle \varphi, Ty\rangle_{E',E}.$$

Then we have the following relation :

$$\|T\|^2_{H\to E} = \|TT^*\|_{E'\to E} = \|T^*\|^2_{E'\to H}.$$

Proof. First of all, we have, by the Hahn-Banach theorem :

$$\|T\|_{H\to E} = \sup_{\|y\|_H=1} \|Ty\|_E = \sup_{\|y\|_H=1} \sup_{\|\varphi\|_{E'}=1} \langle \varphi, Ty\rangle_{E',E}$$

$$= \sup_{\|\varphi\|_{E'}=1} \sup_{\|y\|_H=1} \langle T^*\varphi, y\rangle_H = \sup_{\|\varphi\|_{E'}=1} \|T^*\varphi\|_H = \|T^*\|_{E'\to H}$$

Notice that the previous computation holds even if H was just a Banach space. Then, we have :

$$\|T^*\|_{E'\to H}^2 = \sup_{\|\varphi\|_{E'}=1} \|T^*\varphi\|_H^2 = \sup_{\|\varphi\|_{E'}=1} \langle \varphi, TT^*\varphi\rangle_{E',E} \leq \|TT^*\|_{E'\to E} \leq \|T\|_{H\to E}\|T^*\|_{E'\to H}.$$

And we conclude as the inequalities are equalities, since $\|T\| = \|T^*\|$.

□

This lemma is much more interesting that it may seems. Being given an operator T that we wish to study,we can study the operator TT^* instead, and it is often well better behaved than T. It have the same image as T and often gain in injectivity if T have a large image. Moreover, it have the excellent property of being autoadjoint and positive.

Also, the computations allow us to see that if TT^* is continuous, then T and T^* must be continuous too.

In the study of integral operator, we often combine the TT^* lemma with the Schur test, which allow us to easily bound the norm of an operator in L^2.

Lemma 39 (Schur test). *Let $(X, \mu, \mathcal{A})$ and $(Y, \nu, \mathcal{B})$ be two measured space, with σ-finite measures. Let $K : X \times Y \longrightarrow \mathbb{C}$ be a (measurable) kernel for the following integral operator :*

$$T_K f(x) := \int_Y K(x, y) f(y) d\nu(y).$$

Suppose that there exists some positive (and measurable) functions $p : X \to \mathbb{R}_+^*$ and $q : Y \to \mathbb{R}_+^*$, and some $\alpha, \beta > 0$ such that

$$\int_Y |K(x, y)| q(y) d\nu(y) \leq \alpha p(x)$$

and

$$\int_X p(x) |K(x, y)| d\mu(x) \leq \beta q(y).$$

Then T_K is a well defined and continuous operator $L^2(Y, \nu) \to L^2(X, \mu)$, and

$$\|T_K\|_{L^2(Y,\mu)\to L^2(X,\mu)} \leq \sqrt{\alpha\beta}.$$

Proof. Let $f \in L^2(Y, \nu)$. We have :

$$\left(\int_Y |K(x, y) f(y)| d\nu(y)\right)^2 \leq \int_Y |K(x, y)| q(y) d\nu(y) \int_Y |K(x, y)| \frac{|f(y)|^2}{q(y)} d\nu(y)$$

$$\leq \alpha p(x) \int_Y |K(x, y)| \frac{|f(y)|^2}{q(y)} d\nu(y)$$

Hence, by Fubini-Tonelli :

$$\int_X \left(\int_Y |K(x, y) f(y)| d\nu(y)\right)^2 d\mu(x) \leq \alpha \int_X p(x) \int_Y |K(x, y)| \frac{|f(y)|^2}{q(y)} d\nu(y) d\mu(x)$$

$$= \alpha \int_Y \left(\int_X p(x) |K(x, y)| d\mu(x)\right) \frac{|f(y)|^2}{q(y)} d\nu(y)$$

$$\leq \alpha\beta \int_Y |f(y)|^2 d\nu(y).$$

Hence, the function $x \mapsto \int_Y |K(x, y) f(y)| d\nu(y)$ is finite μ-almost everywhere. On the set where it converge, the function $x \mapsto \int_Y K(x, y) f(y) d\nu(y)$ is well defined, and then the triangle inequality ensure us that $T_K f$ is a well defined element of $L^2(X, \mu)$. Moreover, we have the desired inequality:

$$\|T_K f\|_{L^2(X,\mu)}^2 \leq \alpha\beta \|f\|_{L^2(Y,\nu)}^2.$$

□

The Schur test is often used in the setting where T_K is autoadjoint, with the same space $X = Y$. In this case, we have $K(x, y) = \overline{K(y, x)}$. If we take $p = q = 1$, then we get in this case the following bound :

$$\|T_K\|_{L^2(\mu)\to L^2(\mu)} \leq \sup_x \int_X |K(x, y)| d\mu(y).$$

Now we are equipped with the right lemmas. let's get to the FUP : the proof is in two parts.

Theorem 40. *Let $E \subset \mathbb{R}^d$ be a borel set with strictly positive Fourier dimension. Suppose that there exists some $\mu \in \mathcal{P}(E)$ such that the two following conditions holds:*

- $\forall \xi \in \mathbb{R}^d, \; |\widehat{\mu}(\xi)| \leq C|\xi|^{-\beta}$
- $\forall x \in \mathbb{R}^d, \; \forall r \geq 0, \; \mu(B(x, r)) \leq C r^\alpha$

For some $\alpha/2 \geq \beta > 0$.

Consider the following operator :

$$\begin{array}{cccc} B(h) : & L^2(\mu) & \longrightarrow & L^2(\mu) \\ & u & \longmapsto & B(h)u \end{array}$$

Defined by :

$$B(h)u(\xi) := \int_{\mathbb{R}^d} e^{-ix\cdot\xi/h} u(x) d\mu(x).$$

Then $B(h)$ is well defined and continuous, and we have:

$$\|B(h)\|_{L^2(\mu)\to L^2(\mu)} \lesssim h^{\beta/4}.$$

Proof. First of all, By the TT^* lemma, we have $\|B(h)B(h)^*\|_{L^2(\mu)\to L^2(\mu)} = \|B(h)\|^2_{L^2(\mu)\to L^2(\mu)}$. We compute $B(h)^*$ by hand to practice a bit :

$$\langle v, B(h)u\rangle_{L^2(\mu)} = \int_E v(\xi)\overline{B(h)u(\xi)} d\mu(\xi).$$

$$= \int_E v(\xi) \int_E e^{ix\cdot\xi/h} \overline{u}(x) d\mu(x) d\mu(\xi)$$

$$= \int_E \int_E e^{ix\cdot\xi/h} v(\xi) d\xi \overline{u(x)} d\mu(x) = \langle B(h)v(-\xi), u(\xi)\rangle_{L^2(\mu)}.$$

Hence, we get the following formula :

$$B(h)B(h)^* u(x) = \int_E e^{-i\xi\cdot x/h} B(h)^* u(\xi) d\mu(\xi)$$

$$= \int_E e^{-i\xi\cdot x/h} \int_E e^{i\xi\cdot y/h} u(y) d\mu(y) d\mu(\xi) = \int_E K(x, y) u(y) d\mu(y)$$

Where

$$K(x, y) := \int_E e^{-i(x-y)\cdot\xi/h} d\mu(\xi)$$

By the Schur Test, we then know that :

$$\|B(h)\|^2_{L^2(\mu)\to L^2(\mu)} \leq \sup_x \int_E |K(x, y)| d\mu(y)$$

And from this we can conlude, thanks to the Fourier decay hypothesis made on μ. Indeed, we have:

$$\int_E |K(x, y)| d\mu(y) \leq \mu(B(x, \sqrt{h}) + \int_{|x-y|>h^{1/2}} |\widehat{\mu}((x-y)/2\pi h)| d\mu(y)$$

$$\leq C h^{\alpha/2} + C(2\pi h)^\beta \int_{|x-y|>h^{1/2}} |x-y|^{-\beta} d\mu(y) \lesssim h^{\beta/2}.$$

□

The next version require a bit more regularity on the measure μ. We say that a measure μ with support E is Ahlfors-David regular of dimension α if there exist some constants $c, C > 0$ such that

$$\forall x \in E,\ \forall r \leq 1,\ cr^\alpha \leq \mu(B(x,r)) \leq Cr^\alpha.$$

If μ is a probability measure that is Ahlfors-David regular of dimension α, then its support enjoy good regularity properties : in particular, it has Hausdorff dimension α and finite Hausdorff measure.

One of the interests of Ahlfors-David regular measures is that they can help us transfert informations from E to informations on $E(h)$, as the measure will behave nicely under convolution. Let's state a lemma about that.

Lemma 41. *Let $\mu \in \mathcal{P}(E)$ be an Ahlfors-David regular measure of dimension α. Define*

$$G_h(x) := \frac{1}{h^\alpha}\mu(B(x,2h)).$$

Then $supp(G_h) \subset E(2h)$, and for any $x \in E(h)$, $G_h(x) \geq c$.

Proof. Notice that

$$G_h(x) = h^{-\alpha}\mu * \mathbb{1}_{B(0,2h)}.$$

Hence the support statement is true. Then, consider $x \in E(h)$. There exists some $x' \in E$ such that $|x - x'| < h$. Since $B(x', h) \subset B(x, 2h)$, we get :

$$G_h(x) \geq \frac{1}{h^\alpha}\mu(B(x',h)) \geq c.$$

□

Theorem 42. *Let $E \subset \mathbb{R}^d$ be a compact set with strictly positive Fourier dimension. Suppose that there exists some $\mu \in \mathcal{P}(E)$ that satisfy the two following conditions :*

- $\forall \xi \in \mathbb{R}^d,\ |\widehat{\mu}(\xi)| \leq C|\xi|^{-\beta}$
- $\forall x \in E, \forall r \leq 1,\ cr^\alpha \leq \mu(B(x,r)) \leq Cr^\alpha$

for some $\alpha/2 \geq \beta > 0$.

Then

$$\|\mathbb{1}_{E(h)}\mathcal{F}_h\mathbb{1}_{E(h)}\|_{L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^d)} \lesssim h^{d/2-\alpha+\beta/4}.$$

We see that this theorem ensure a FUP only if $\alpha < d/2 + \beta/4$. If μ has the best decay possible (ie, if μ is a Salem measure), then we have a FUP as long as $\alpha < 4d/7$. Its less than $d-1$ as soon as $d \geq 3$. In [BD17], the theorem was proved in dimension 1, this is probably why it seems more suited for low dimensional settings. In fact, according to Dyatlov, the FUP is not well understood past the dimension 1.

Proof. First of all, we see that our problem reduce to proving the bound

$$\forall u \in L^2(\mathbb{R}^d),\ \|\sqrt{G_h}\mathcal{F}_h G_h u\|_{L^2(\mathbb{R}^d)} \leq Ch^{d/2-\alpha+\beta/4}\|\sqrt{G_h}u\|_{L^2(\mathbb{R}^d)}.$$

Indeed, if this is true, then

$$\|\mathbb{1}_{E(h)}\mathcal{F}_h\mathbb{1}_{E(h)}u\|^2_{L^2(\mathbb{R}^d)} = \int_{E(h)} |\mathcal{F}_h(\mathbb{1}_{E(h)}u)|^2$$

$$\leq \frac{1}{c}\int_{E(h)} G_h|\mathcal{F}_h(\mathbb{1}_{E(h)}u)|^2 \leq \|\sqrt{G_h}\mathcal{F}_h G_h u\mathbb{1}_{E(h)}/G_h\|^2_{L^2(\mathbb{R}^d)}$$

$$\leq (C^2/c)h^{2(d/2-\alpha+\beta/4)}\int_{E(h)} G_h^{-1}|u|^2 \leq (Ch^{d/2-\alpha+\beta/4})^2\|u\|^2_{L^2(\mathbb{R}^d)}.$$

So let's prove the announced bound. Define the shift operator ω_t by

$$\omega_t v(x) := v(x-t).$$

We have, by Fubini-Tonelli :

$$\|\sqrt{G_h}v\|^2_{L^2(\mathbb{R}^d)} = \int_{\mathbb{R}^d} \frac{1}{h^\alpha} \int_{\mathbb{R}^d} \mathbb{1}_{B(0,2h)}(t-x)d\mu(x)|v(t)|^2 dt$$

$$= \frac{1}{h^\alpha} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mathbb{1}_{B(0,2h)}(t-x)|v(t)|^2 dt d\mu(x)$$

$$= \frac{1}{h^\alpha} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mathbb{1}_{B(0,2h)}(t)|v(-t+x)|^2 dt d\mu(x) = \frac{1}{h^\alpha} \int_{B(0,2h)} \|\omega_t v\|^2_{L^2(\mu)} dt.$$

Moreover, by Fubini :

$$\omega_t \mathcal{F}_h G_h u(x) = \frac{1}{h^{d/2}} \int_{\mathbb{R}^d} e^{-2i\pi s\cdot(x-t)/h} G_h(s)u(s)ds$$

$$= \frac{1}{h^{\alpha+d/2}} \int_{\mathbb{R}^d} e^{-2i\pi s\cdot(x-t)/h} \int_{\mathbb{R}^d} \mathbb{1}_{B(0,2h)}(s-y)d\mu(y)u(s)ds$$

$$= \frac{1}{h^{\alpha+d/2}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{-2i\pi s\cdot(x-t)/h} \mathbb{1}_{B(0,2h)}(s-y)u(s)ds d\mu(y)$$

$$= \frac{1}{h^{\alpha+d/2}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{-2i\pi(y-s)\cdot(x-t)/h} \mathbb{1}_{B(0,2h)}(s)u(y-s)ds d\mu(y)$$

$$= \frac{1}{h^{\alpha+d/2}} \int_{B(0,2h)} B_{t,s}(h)u(x)ds$$

With

$$B_{t,s}(h)u(x) := \int_{\mathbb{R}^d} e^{-2i\pi(y-s)\cdot(x-t)/h} u(y-s)d\mu(y).$$

Using theorem 38, we can conclude as follow:

$$\|\sqrt{G_h}\mathcal{F}_h G_h u\|^2_{L^2(\mathbb{R}^d)} = \frac{1}{h^\alpha} \int_{B(0,2h)} \|\omega_t \mathcal{F}_h G_h u\|^2_{L^2(\mu)} dt$$

$$\lesssim h^{d-\alpha} \sup_{|t|\le 2h} \int_E \left| \frac{1}{h^{\alpha+d/2}} \int_{B(0,h)} B_{t,s}(h)u(x)ds \right|^2 d\mu(x)$$

$$\lesssim h^{d-3\alpha} \sup_{|t|\le 2h} \int_E \int_{B(0,h)} |B_{t,s}(h)u(x)|^2 ds d\mu(x)$$

$$\lesssim h^{d-3\alpha} \sup_{|t|\le 2h} \int_{B(0,2h)} \|B_{t,s}(h)u\|^2_{L^2(\mu)} ds$$

$$\lesssim h^{d-2\alpha+\beta/2} \frac{1}{h^\alpha} \int_{B(0,2h)} \|\omega_t u\|^2_{L^2(\mu)} dt = h^{d-2\alpha+\beta/2} \|\sqrt{G_h}v\|^2_{L^2(\mathbb{R}^d)}.$$

□

The use of Fourier decay is not the only way to prove some FUP. In [Dy19] and in [Da21] it is explain in detail how one can get a FUP from some porosity asumptions on the support. They also take the time to explain various applications of the FUP.

One of the applications lies in the spectral theory of hyperbolic surfaces. In [DZ16], Dyatlov and Zahl where interested in the study of the pole of the scattering resolvant. By the use of a Fractal Uncertainty Principle, they were able to prove the existence of a spectral gap on convex cocompact hyperbolic surfaces.

Let's turn to our last application of some Fourier decay : The restriction problem.

2.4 The restriction problem

2.4.1 Preliminary results

Before explaining the topic of the restriction problem, we will explain one more result about the Fourier transform that I didn't explained at the beginning of this thesis. The result is the following: functions in $L^p(\mathbb{R}^d)$, with $p \in [1,2]$, have a Fourier transform in $L^{p'}(\mathbb{R}^d)$, where $\frac{1}{p'} + \frac{1}{p} = 1$. This result is cited in [Ma15].

To prove this, we need to explain how complex interpolation work. Let's begin with a lemma.

Lemma 43 (Hadamar's three line theorem).
Let F be a holomorphic function in $0 < Re(z) < 1$, that is also continuous and bounded on $0 \leq Re(z) \leq 1$. If $|F(it)| \leq M_0$ and $|F(1+it)| \leq M_1$ for all $t \in \mathbb{R}$, then we have the following bound :

$$\forall \theta \in [0,1], |F(\theta + it)| \leq M_0^{1-\theta} M_1^{\theta}$$

Proof. Let $\varepsilon > 0$ and $\lambda \in \mathbb{R}$. Let

$$F_\varepsilon(z) = e^{\varepsilon z^2 + \lambda z} F(z).$$

Then, we have

$$|F_\varepsilon(z)| = e^{\varepsilon \mathrm{Re}(z^2) + \lambda \mathrm{Re}(z)} |F(z)| \underset{|\mathrm{Im}(z)| \to \infty}{\longrightarrow} 0$$

And moreover, $|F_\varepsilon(it)| \leq M_0$ and $F_\varepsilon(1+it) \leq M_1 e^{\varepsilon + \lambda}$. Hence, by the maximum principle,

$$|F_\varepsilon(z)| \leq \max(M_0, M_1 e^{\varepsilon+\lambda})$$

Letting $\varepsilon \to 0$ gives us :

$$|F(\theta + it)| \leq \max(M_0 e^{-\lambda\theta}, M_1 e^{(1-\theta)\lambda})$$

Finally, choosing $\lambda = \ln(M_0/M_1)$ gives us the desired bound.

□

Theorem 44. *Let $1 \leq p_0, q_0, p_1, q_1 \leq \infty$. Let $\Omega, U \subset \mathbb{R}^d$ be open sets, and μ, ν two borel measures on Ω, U that are σ-finite. Consider an operator T defined on $L^{p_0}(\Omega) + L^{p_1}(\Omega)$ with value in the space of measurable functions on U such that*

$$\forall f \in L^{p_0}(\mu), \ \|Tf\|_{L^{q_0}(\nu)} \leq C_0 \|f\|_{L^{p_0}(\mu)}$$

and

$$\forall f \in L^{p_1}(\mu), \ \|Tf\|_{L^{q_1}(\nu)} \leq C_1 \|f\|_{L^{p_1}(\mu)}$$

Then, for all $\theta \in [0,1]$,

$$\forall f \in L^{p}(\mu), \ \|Tf\|_{L^{q}(\nu)} \leq C_0^{1-\theta} C_1^{\theta} \|f\|_{L^{p}(\mu)}$$

Where

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \quad and \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

Just a quick remark about the formulation of the theorem before explaining the proof : If $p \in [p_0, p_1]$, then $L^p \subset L^{p_0} + L^{p_1}$. Indeed, we can always write $f = \mathbb{1}_{|f| \leq 1} f + \mathbb{1}_{|f|>1} f$. We first see that $\mathbb{1}_{|f|\leq 1} f \in L^{p_1}$. And then, since $\{|f| > 1\}$ has finite measure otherwise f would not be in L^p, we see that $\mathbb{1}_{|f|>1} f \in L^{p_0}$. So the formulation of our theorem makes sense.

Proof. If $p_0 = p_1$ or $q_0 = q_1$ the statement is obvious. So let's suppose that $p_0 \neq p_1$ and $q_0 \neq q_1$. Also, fix some $\theta \in]0,1[$. Define p, q like in the theorem. With our hypothesis, $p, q \in]1, \infty[$.

By duality, we have the following formula :

$$\|T\|_{L^p \to L^q} = \sup\{\langle Tf, g \rangle \mid \|f\|_{L^p(\mu)} = 1, \|g\|_{L^{q'}(\nu)} = 1\}.$$

So, to prove that T is continuous from L^p to L^q, we will fix some $f \in C^0(\Omega)$, $g \in C^0(U)$ with compact support such that $\|f\|_{L^p(\mu)} = 1$,$\|g\|_{L^{q'}(\nu)} = 1$, and prove a bound of the form

$$\langle Tf, g\rangle \leq C$$

with C independant of the choice of f, g. By density of the space of continuous functions with compact support in L^p and $L^{q'}$, it will prove that $T : L^p \to L^q$ is well defined and continuous, and give us the desired bound on $\|T\|_{L^p \to L^q}$.

So we fix those f, g. For $0 \leq \mathrm{Re}(z) \leq 1$, define

$$\frac{1}{p(z)} := \frac{1-z}{p_0} + \frac{z}{p_1} \quad , \quad \frac{1}{q(z)} := \frac{1-z}{q_0} + \frac{z}{q_1}$$

In particular, $p(\theta) = p$ and $q(\theta) = q$.
We also define :

$$\varphi(z) := |f|^{p/p(z)} \frac{f}{|f|} \in L^{p_0}(\mu)$$
$$\psi(z) := |g|^{q'/q(z)'} \frac{g}{|g|} \in L^{q_0'}(\nu)$$

And finally, we set, for any $0 \leq \mathrm{Re}(z) \leq 1$:

$$F(z) := \langle T\varphi(z), \psi(z)\rangle.$$

We see that F is analytic on the open strip, and bounded and continuous on the closed strip. Indeed, we see that

$$\forall 0 \leq \mathrm{Re}(z) \leq 1, |\varphi(z)| = \left||f|^{p/p(z)}\right| \leq |f|^{p/p_0 + p/p_1} \in L^{p_0}$$

And for any x, $z \mapsto \varphi(z)(x)$ is holomorphic on the open strip and continuous on the closed strip. Hence, by the dominated convergence theorem, $\varphi : \Omega \to L^{p_0}$ is holomorphic on the open strip, continuous on the closed strip, and is bounded. Since T is linear and continuous, $T\varphi : \Omega \to L^{q_0}$ also is. Finally, the same argument applies to ψ. And so, by bilinearity and continuity of the duality bracket, we get that F also is. So the Hadamar three line theorem can be applied.

We then have $F(it) \leq C_0$, $F(1 + it) \leq C_1$, and so $F(\theta) \leq C_0^{1-\theta} C_1^{\theta}$. The proof is done.

□

The complex interpolation theorem is very useful tool, as its proof is quite elementary and since it has a lot of powerful corollary. Let's state two of them.

Theorem 45. *For any $p \in [1, 2]$, the Fourier transform define a continuous map $L^p(\mathbb{R}^d) \to L^{p'}(\mathbb{R}^d)$, with norm ≤ 1.*

Proof. We know that $\|\widehat{f}\|_{L^\infty(\mathbb{R}^d)} \leq \|f\|_{L^1(\mathbb{R}^d)}$, and we also know that $\|\widehat{f}\|_{L^2(\mathbb{R}^d)} = \|f\|_{L^2(\mathbb{R}^d)}$. let's compute the exponents that comes from complex interpolation : For $\theta \in [0, 1]$, $1/p = (1-\theta)/1 + \theta/2 = 1 - \theta/2$ gives us $\theta = 2(1 - 1/p)$ and $1/q = (1-\theta)/\infty + \theta/2 = 1 - 1/p$. Hence $q = p'$, and we get our bound :

$$\|f\|_{L^{p'}(\mathbb{R}^d)} \leq \|f\|_{L^p(\mathbb{R}^d)}$$

□

Theorem 46 (Young inequality)**.** *Let $p, q, r \geq 1$ be such that $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$. Let $f \in L^p(\mathbb{R}^d)$ and $g \in L^q(\mathbb{R}^d)$. Then $f * g \in L^r(\mathbb{R}^d)$, and we have :*

$$\|f * g\|_{L^r(\mathbb{R}^d)} \leq \|f\|_{L^p(\mathbb{R}^d)} \|g\|_{L^q(\mathbb{R}^d)}.$$

Proof. let's fix $g \in L^q$, and look at the linear map $f \mapsto f * g$. If $f \in L^{q'}$, then $f * g$ is well defined everywhere, and we have $\|f * g\|_{L^\infty} \leq \|f\|_{L^{q'}} \|g\|_{L^q}$. Then, if $f \in L^1$, then $f * g$ is well defined almost everywhere and we have $\|f * g\|_{L^q} \leq \|f\|_{L^1} \|g\|_{L^q}$ by the integral triangle inequality (Minkowski).

Let's compute the exponents given by the complex interpolation : if $1/p = (1-\theta)/1 + \theta/q' = 1 - \theta/q$, then $1/r = (1-\theta)/q + \theta/\infty = 1/q + (1/p - 1)$ ie : $1 + 1/r = 1/p + 1/q$.

□

2.4.2 The restriction problem

Consider a function $f \in L^1(\mathbb{R}^d)$. We then know that $\widehat{f} \in C^0_{\to 0}(\mathbb{R}^d)$, and so $\widehat{f}$ can be meaningfully restricted to any points. However, if $f \in L^2(\mathbb{R}^d)$, then $\widehat{f} \in L^2(\mathbb{R}^d)$, and since moreover the fourier transform is surjective in this setting, $\widehat{f}$ can in general only be restricted to sets that have positive lebesgue measure.

The question is : what happens between those case ? If $f \in L^p(\mathbb{R}^d)$ with $p \in]1,2[$, what can we say about the regularity of $\widehat{f}$? More precisely, on what kind of sets can we meaningfully restrict $\widehat{f}$? Does $\widehat{f}$ has more regularity than a typical element of $L^{p'}$?

To formalise this question, consider a borel set E (of d-Lebesgue measure 0), equipped with a measure $\mu \in \mathcal{M}(E)$. We are interested in bounds of the form :

$$\|\widehat{f}\|_{L^q(\mu)} \leq C\|f\|_{L^p(\mathbb{R}^d)}.$$

If we can prove a bound of this form for $f \in S(\mathbb{R}^d)$, then by density, it will mean that the Fourier transform operator can be meaningfully restricted to a continuous operator $L^p(\mathbb{R}^d) \to L^q(E,\mu)$. This kind of result is already interesting on its own, but it also has various applications, we will briefly explain one of them at the end of the section.

For some various discussion on the restriction problem, see for example some notes of Tao : [Ta03a], [Ta03b], [Ta20]. The chapter 19 of [Ma15] is also a very good reference, from which this section will be mostly inspired.

First of all, to study this problem, it might seems relevant to look at what the TT^* lemma has to say in this setting. We have the following theorem, extracted from [Ma15] :

Theorem 47. *Let $1 \leq p, q \leq \infty$ and let $\mu \in \mathcal{M}(E)$. The following are equivalent for any $0 < C < \infty$:*

1. $\|\widehat{f}\|_{L^q(\mu)} \leq C\|f\|_{L^p(\mathbb{R}^d)}$ *for all* $f \in S(\mathbb{R}^d)$
2. $\|\widehat{fd\mu}\|_{L^{p'}(\mathbb{R}^d)} \leq C\|f\|_{L^{q'}(\mu)}$ *for all* $f \in S(\mathbb{R}^d)$
   In the case $q = 2$, (1) and (2) are equivalent to
3. $\|\widehat{\mu} * f\|_{L^{p'}(\mathbb{R}^d)} \leq C^2\|f\|_{L^p(\mathbb{R}^d)}$ *for all* $f \in S(\mathbb{R}^d)$.

Proof. Let's follow the proof of the TT^* lemma in our setting, to see what happen explicitly. So, suppose (1). Let $g, f \in S(\mathbb{R}^d)$. We have :

$$\int g\widehat{fd\mu} = \int \widehat{g}fd\mu \leq \|\widehat{g}\|_{L^q(\mu)}\|f\|_{L^{q'}(\mu)} \leq C\|g\|_{L^p(\mathbb{R}^d)}\|f\|_{L^{q'}(\mu)}.$$

Taking the supremum in $\|g\|_{L^p(\mathbb{R}^d)} = 1$ gives us (2).

Then suppose (2). Let $f, g \in S(\mathbb{R}^d)$. We have :

$$\int \widehat{f}gd\mu = \int f\widehat{gd\mu} \leq \|f\|_{L^p(\mathbb{R}^d)}\|\widehat{gd\mu}\|_{L^{p'}(\mathbb{R}^d)} \leq C\|f\|_{L^p(\mathbb{R}^d)}\|g\|_{L^{q'}(\mu)}$$

Taking the supremum in $\|g\|_{L^{q'}(\mu)} = 1$ gives (1).

Now suppose that (1) and (2) are satisfied with $q = 2$. Then, for $f, g \in S(\mathbb{R}^d)$, we have :

$$\left|\int(\widehat{\mu} * \overline{f})g\right| = \left|\int \overline{\widehat{f}}\widehat{g}d\mu\right| \leq \|\widehat{g}\|_{L^2(\mu)}\|\widehat{f}\|_{L^2(\mu)} \leq C^2\|f\|_{L^p(\mathbb{R}^d)}\|g\|_{L^p(\mathbb{R}^d)}$$

Taking the sup in $\|g\|_{L^p(\mathbb{R}^d)} = 1$ gives (3).

Finally, if (3) is true, then for $f \in S(\mathbb{R}^d)$, we have :

$$\|\widehat{f}\|_{L^2(\mu)} = \int(\widehat{\mu} * \overline{f})f \leq \|\widehat{\mu} * f\|_{L^{p'}(\mathbb{R}^d)}\|f\|_{L^p(\mathbb{R}^d)} \leq C^2\|f\|^2_{L^p(\mathbb{R}^d)}.$$

□

From this reformulation we can already get a first, weak restriction result for measures that have positive Fourier dimension. Indeed, if $|\widehat{\mu}(\xi)| \lesssim |\xi|^{-\beta}$, then $\widehat{\mu} \in L^r$ for r large enough, and so a well chosen Young inequality gives us the bound (3) for some p. We will do way better than that in the next subsection.

2.4.3 A fractal Stein-Tomas restriction theorem

This section is devoted to the proof of a theorem that ensure some restriction property under some assumptions on the measure. It is a fractal version of a more classic theorem called the Stein-Tomas restriction theorem that we will cite right after. The theorem is extracted from [LW16].

Theorem 48. *Let $\mu \in \mathcal{P}(E)$ such that :*

- $\forall x \in \mathbb{R}^d,\ \forall r > 0,\ \mu(B(x,r)) \leq Cr^\alpha$
- $\forall \xi \in \mathbb{R}^d,\ |\widehat{\mu}(\xi)| \leq C|\xi|^{-\beta}$

For some $\alpha/2 \geq \beta \geq 0$.

Then, for all $p \leq \frac{2(d-\alpha+\beta)}{2(d-\alpha)+\beta}$, we have :

$$\forall f \in S(\mathbb{R}^d),\ \|\widehat{f}\|_{L^2(\mu)} \lesssim_p \|f\|_{L^p(\mathbb{R}^d)}.$$

In the case of the sphere, we can consider σ the surface measure on $\mathbb{S}^{d-1}$. Recall that we said that this measured enjoyed the maximum decay rate possible : $\mathbb{S}^{d-1}$ is a Salem set. Namely, in this case, we have $\alpha = d-1$ and $\beta = (d-1)/2$. This give us the following nice corollary, which is the usual Stein Tomas theorem :

Theorem 49. *For all $p \leq 2\frac{d+1}{d+3}$, we have*

$$\forall f \in S(\mathbb{R}^d),\ \|\widehat{f}\|_{L^2(\mathbb{S}^{d-1})} \lesssim_p \|f\|_{L^p(\mathbb{R}^d)}.$$

Being able to restrict a Fourier transform on some hypersurface (the sphere, a parabola, a cone...) will be useful in the study of some PDE. We will explain briefly this is the next subsection. For now, let's turn onto the proof of this fractal Stein-Thomas theorem.

The proof with the endpoint $p = \frac{2(d-\alpha+\beta)}{2(d-\alpha)+\beta}$ is pretty hard, so we will only prove it without the endpoint. I think the proof of this part is already interesting in itself, as it use a lot of previously mentioned techniques of harmonic analysis. The following proof can be found in [Ma15].

Proof. First of all, if $p = 1$, we have the trivial bound $\|\widehat{f}\|_{L^2(\mu)} \leq \|\widehat{f}\|_{L^\infty(\mathbb{R}^d)} \leq \|f\|_{L^1(\mathbb{R}^d)}$.
So, now, consider $1 < p < \frac{2(d-\alpha+\beta)}{2(d-\alpha)+\beta}$. Notice that it means that $2(d-\alpha+\beta) < p' < \infty$.
We are going to use a recurrent idea in harmonic analysis: the dyadic decomposition.

Let $\chi \in C^\infty(\mathbb{R}^d)$ be a radial function such that $\chi \geq 0$, $\chi = 1$ if $|x| \geq 1$ and $\chi(x) = 0$ if $|x| \leq 1/2$. Then, set $\varphi(x) = \chi(2x) - \chi(x)$, and finally, for any $j \geq 1$, set $\varphi_j(x) := \varphi(2^{-j}x)$.

We see that $\sum_{j=1}^\infty \varphi_j(x) = \chi(x)$. So, if we denote $\varphi_0(x) = 1 - \chi(x)$, we see that we have constructed a family of bump functions such that $\sum_{j=0}^\infty \varphi_j = 1$, with supp $\varphi_0 \subset B(0,1)$, and supp $\varphi_j \subset B(0,2^j) \setminus B(0,2^{j-2})$ if $j \geq 1$. Moreover, the fact that they are linked through the scaling relation $\varphi_j(x) = \varphi(2^{-j}x)$ will be of great help to see how our terms will decrease in j.

let's move on. By the TT^* lemma, the desired estimate is equivalent to $\|\widehat{\mu} * f\|_{L^{p'}(\mathbb{R}^d)} \lesssim_p \|f\|_{L^p(\mathbb{R}^d)}$ for $f \in S(\mathbb{R}^d)$. For any $f \in S(\mathbb{R}^d)$, we can write :

$$\widehat{\mu} * f = \sum_{j=0}^{\infty} (\varphi_j \widehat{\mu}) * f.$$

Now the idea is the following : instead of brutally applying Young's inequality to the LHS, we will consider, for each j, the linear map $f \mapsto \varphi_j \widehat{\mu} * f$. The use of complex interpolation will give us a nice bound for each j, and then we will sum everything.

First, we look at the term in $j = 0$. In this case, Young's inequality is enough. We get

$$\|(\varphi_0 \widehat{\mu}) * f\|_{L^{p'}(\mathbb{R}^d)} \leq \|\varphi_0 \widehat{\mu}\|_{L^{p'/2}(\mathbb{R}^d)} \|f\|_{L^p(\mathbb{R}^d)}.$$

since $\frac{2}{p'} + \frac{1}{p} = 1 + \frac{1}{p'}$. Now fix some $j \geq 1$. We look at the map $f \mapsto (\varphi_j \widehat{\mu}) * f$. If $f \in L^1(\mathbb{R}^d)$, then we have, by Young inequality again, and by the decay hypothesis made on $\widehat{\mu}$:

$$\|(\varphi_j \widehat{\mu}) * f\|_{L^\infty} \leq \|\varphi_j \widehat{\mu}\|_{L^\infty} \|f\|_{L^1(\mathbb{R}^d)} \lesssim 2^{-\beta j} \|f\|_{L^1(\mathbb{R}^d)}.$$

Then, if $f \in L^2(\mathbb{R}^d)$, we have :

$$\|(\varphi_j \widehat{\mu}) * f\|_{L^2(\mathbb{R}^d)} = \|(\widehat{\varphi_j} * \mu)\widehat{f}\|_{L^2(\mathbb{R}^d)} \leq \|\widehat{\varphi_j} * \mu\|_{L^\infty(\mathbb{R}^d)} \|f\|_{L^2(\mathbb{R}^d)}$$

And : $\widehat{\varphi_j}(\xi) = 2^{dj} \widehat{\varphi}(2^j \xi)$. Hence :

$$|\widehat{\varphi_j} * \mu(\xi)| = 2^{dj} \int_{\mathbb{R}^d} \widehat{\varphi}(2^j(\xi - \eta)) d\mu(\eta)$$

Then, recall that χ is radial, so φ_j also is and so $\widehat{\varphi}$ also is. Hence, we can write $\widehat{\varphi}(\xi) = \zeta(|\xi|)$ for some ζ in the schwartz class, and so :

$$|\widehat{\varphi_j} * \mu(\xi)| = 2^{dj} \left| \int_{\mathbb{R}^d} \zeta(2^j |\xi - \eta|) d\mu(\eta) \right|$$

$$= \left| 2^{dj} \int_0^\infty 2^j \zeta'(2^j r) \mu(B(\xi, r)) dr \right|$$

$$= 2^{dj} \left| \int_0^\infty \zeta'(r) \mu(B(\xi, 2^{-j} r)) dr \right|$$

$$\lesssim 2^{j(d-\alpha)} \int_0^\infty |\zeta'(r)| r^\alpha dr.$$

Hence, we get the following bound :

$$\|(\varphi_j \widehat{\mu}) * f\|_{L^2(\mathbb{R}^d)} \lesssim 2^{j(d-\alpha)} \|f\|_{L^2(\mathbb{R}^d)}.$$

With those results, we can use complex interpolation on the maps $f \mapsto (\varphi_j \widehat{\mu}) * f$ to get a bound if $f \in L^p$. Let's compute it : for $\theta \in [0, 1]$, if $1/p = (1 - \theta)/1 + \theta/2$, then $\theta = 2/p'$. The space of arrival is then $(1 - \theta)/\infty + \theta/2 = 1/p'$. So we get :

$$\forall f \in L^p(\mathbb{R}^d),\ \|(\varphi_j \widehat{\mu}) * f\|_{L^{p'}(\mathbb{R}^d)} \lesssim 2^{j((1-\theta)(-\beta)+\theta(d-\alpha))} \|f\|_{L^p(\mathbb{R}^d)}.$$

$$= 2^{j(2(d-\alpha+\beta)/p'-\beta)} \|f\|_{L^p(\mathbb{R}^d)}.$$

Now, recall that $p' < 2(d - \alpha + \beta)/\beta$. Hence, $2(d - \alpha + \beta)/p' - \beta < 0$, and so we can sum all of those inequalities ! So, now, back to the beginning. Let $f \in S(\mathbb{R}^d)$. We have :

$$\|\widehat{\mu} * f\|_{L^{p'}(\mathbb{R}^d)} \leq \sum_{j=0}^\infty \|(\varphi_j \widehat{\mu}) * f\|_{L^{p'}(\mathbb{R}^d)} \lesssim \left(1 + \sum_{j=1}^\infty 2^{j(2(d-\alpha+\beta)/p'-\beta)}\right) \|f\|_{L^p(\mathbb{R}^d)}$$

And the proof is done.

□

2.4.4 Strichartz estimate

One of the most useful corollaries of the restriction results is an application to PDE. Let's explain briefly the case of the Schrodinger equation : it will also motivate us for the study of the Fourier transform of surface measures in our next chapter.

So we consider the following equation :

$$2i\pi\partial_t u(x,t) - \Delta_x u(x,t) = 0 \quad , \quad u(\cdot,0) = f \in S(\mathbb{R}^d).$$

And suppose that $\widehat{f}$ is supported on some compact set. Its solution can be found by taking the Fourier transform in x of the equation. We get :

$$2i\pi\partial_t \widehat{u}(\xi,t) + 4\pi^2|\xi|^2\widehat{u}(\xi,t) = 0 \quad , \quad \widehat{u}(\cdot,0) = \widehat{f}$$

Which gives us

$$\widehat{u}(\xi,t) = \widehat{f}(\xi)e^{2i\pi|\xi|^2 t}.$$

Taking the inverse Fourier transform gives us the solution of the equation :

$$u(x,t) = \int_{\mathbb{R}^d} \widehat{f}(\xi)e^{2i\pi(\xi\cdot x+|\xi|^2 t)}d\xi.$$

Now, the point is to recognise the Fourier transform of a well chosen measure. In our case, consider the parabola $P = \{(\xi,|\xi|^2) \mid \xi \in \mathbb{R}^d\}$. If we denote by σ the pushforward of the Lebesgue measure on the parabola smoothly cut on a large enough compact, if we set $s := (x,t)$, and if we define $g(\xi,\lambda) := \widehat{f}(\xi)$, we get :

$$u(s) = \int_P e^{2i\pi s\cdot\eta}g(\eta)d\sigma(\eta) = \widehat{gd\sigma}(-s).$$

Then, as we will see in the next chapter, $\widehat{\sigma}$ enjoys some good decay property. Hence we have, by the previous theorem, a restriction result that is valid for p large enough :

$$\|\widehat{gd\sigma}\|_{L^p(\mathbb{R}^d\times\mathbb{R})} \lesssim \|g\|_{L^2(\sigma)}.$$

And since

$$\|g\|^2_{L^2(\sigma)} = \int_{\mathbb{R}^d} |\widehat{f}(\xi)|d\xi = \|f\|^2_{L^2(\mathbb{R}^d)},$$

we get :

$$\|u\|_{L^p(\mathbb{R}^d\times\mathbb{R})} \lesssim \|f\|_{L^2(\mathbb{R}^d)}.$$

Of course, those kind of estimates are of tremendous importance in the study of PDE.
For some more details about how we can get the Strichartz estimates for the Schrodinger equation, the interested reader can look at [Ta20].

This reasoning can also be adapted for the wave equation :

$$\partial_t^2 u(x,t) - \Delta_x u(x,t) = 0 \quad , \quad u(x,0) = 0 \ , \ \partial_t u(x,0) = f \in S(\mathbb{R}^d).$$

But this time, the hypersurface considered will be the cone.

All of those Strichartz estimates can be true only if the duo measure/hypersurface associated to our differential operator enjoy some good Fourier decay properties. But, as we will see in the next section, this behavior is not general : in the case of surface measures, Fourier decay depends on the curvature of the surface. We will explain this in detail in the next chapter.

Chapter 3

Old and new techniques to prove Fourier decay

3.1 The smooth case

3.1.1 One variable

In this chapter, we will present some strategies to show that the fourier transform of some measure have polynomial decay. The case of smooth measures is known since some time know : our references for this part will be [Ma15], and mostly the classical work of Stein, [SM93], especially the chapter 8: "Oscillatory integral of the first kind". We will also use some notes of Tao, [Ta20b]. The fractal case will be studied in the second part of this chapter.

So, following Stein, we begin by the one variable case. The setting is the following : we are interested in the asymptotic study of integrals of the form

$$I(\lambda) := \int_a^b e^{i\lambda\phi(x)}\psi(x)dx,$$

Where ϕ is a real valued smooth function that we call the phase, and where ψ is complex valued and smooth. We will often suppose that ψ have compact support in $]a, b[$.

As we already saw previously, the behavior of this integral will depend on how much the phase accumulates. If it doesn't accumulates at all, (we talk about non-stationary phase) then the integral will decay very quickly. Else, the asymptotic will be drived by how much and where the phase accumulate. Let's make this rigorous.

Theorem 50 (Non-stationary phase)**.** *Let ϕ and ψ be smooth functions so that ψ has compact support in $]a, b[$, and $\phi'(x) \neq 0$ for all $x \in [a, b]$. Then, for any $N \geq 0$, we have*

$$I(\lambda) = O_N(\lambda^{-N}).$$

Proof. The proof is just some integration by parts. For λ fixed, define the differential operator

$$Df(x) := \frac{f'(x)}{i\lambda\phi'(x)}$$

and denote by ${}^tD$ its transpose :

$${}^tDf = -\frac{d}{dx}\left(\frac{f}{i\lambda\phi'}\right)$$

Then $D^N(e^{i\lambda\phi}) = e^{i\lambda\phi}$ for every N, and so we get :

$$I(\lambda) = \int_a^b D^N(e^{i\lambda\phi})\psi = \int_a^b e^{i\lambda\phi}({}^tD)^N(\psi)$$

Hence we have

$$|I(\lambda)| \leq C(N, \psi, \phi)\lambda^{-N}.$$

Notice that the constant does not depend on a, b, and can be controlled by the C^N norm of ϕ and $1/\psi'$.

□

Some remark on those results : if we don't ask for ψ to vanish near the endpoint, then we'd still get some decay, but at the rate $O(\lambda^{-1})$ only. In the special case where the functions ϕ and ψ are smooth and $b-a$ periodic on $\mathbb{R}$, then we get back the rapid decay.

The non-stationnary phase principle is used to show that we can localize the behavior of $I(\lambda)$ around the points where ϕ accumulates.

In the case where φ' vanish, we can still have some decay under some suitable hypothesis on the further derivatives of φ.

Theorem 51 (Van der Corput)**.** *Suppose ϕ is real valued and smooth in $]a, b[$. Let $k \geq 1$ and suppose that $|\phi^{(k)}(x)| \geq 1$ for all $x \in]a, b[$. Then*

$$\left| \int_a^b e^{i\lambda\phi(x)} dx \right| \leq c_k \lambda^{-1/k}$$

holds when:

1. *$k = 1$ and ϕ' is monotonic, or*
2. *$k \geq 2$.*

The bound c_k is independent of a,b, ϕ and λ in this case.

Proof. The proof is done by induction. We begin by the case where $k = 1$ and ϕ' is monotonic. We have :

$$\int_a^b e^{i\lambda\phi} dx = \int_a^b e^{i\lambda\phi}\ {}^tD(1) dx + \left[(i\lambda\varphi')^{-1} e^{i\lambda\phi}\right]_a^b.$$

The boundary terms are bounded from above by $2/\lambda$, while

$$\left| \int_a^b e^{i\lambda\phi}\ {}^tD(1) dx \right| = \left| \int_a^b e^{i\lambda\phi} (i\lambda)^{-1} \frac{d}{dx} \left(\frac{1}{\phi'} \right) dx \right|$$

$$\leq \lambda^{-1} \int_a^b \left| \frac{d}{dx} \left(\frac{1}{\phi'} \right) \right| dx$$

$$= \lambda^{-1} \left| \int_a^b \frac{d}{dx} \left(\frac{1}{\phi'} \right) \right|$$

since ϕ' is monotonic. The last expression equals

$$\lambda^{-1} \left| \frac{1}{\phi'(b)} - \frac{1}{\phi'(a)} \right| \leq 1/\lambda.$$

Which gives the desired bound with $c_1 = 3$.

Now, suppose the results known for some $k \geq 1$, and assume that $|\phi^{(k+1)}| \geq 1$ on $[a, b]$. Without loss of generality, we may assume that $\phi^{(k+1)} \geq 1$. Hence, $\phi^{(k)}$ is strictly increasing, and so $|\phi^{(k)}|$ admits a minimum on $[a, b]$ which is obtained at a unique point $c \in [a, b]$. There is two cases.

The first case is if $c \in]a, b[$, and so $\phi^{(k)}(c) = 0$. In this case, for some $\delta > 0$, we cut our integral like this:

$$\int_a^b e^{i\lambda\phi} = \int_a^{c-\delta} e^{i\lambda\phi} + \int_{c-\delta}^{c+\delta} e^{i\lambda\phi} + \int_{c+\delta}^b e^{i\lambda\phi}$$

In the first and third integral, by our hypothesis on $\phi^{(k+1)}$, we see that $|\phi^{(k)}| \geq \delta$. Hence, by our inductive hypothesis, we get :

$$\left| \int_a^b e^{i\lambda\phi(x)} dx \right| \leq c_k (\lambda\delta)^{1/k} + 2\delta + c_k (\lambda\delta)^{1/k}.$$

Choosing $\delta \leq \lambda^{-1/(k+1)}$, we get the desired estimate with $c_{k+1} = 2c_k + 2$.
The second case is when $c \in \{a, b\}$, and the proof in this case is similar.

And the proof is done. Notice that we have an explicit computation for the constants: we have $c_k = 5 \cdot 2^{k-1} - 2$.

□

We have the following consequence.

Corollary 52. *Let ϕ be as in the theorem 49. Let $\psi \in C^1([a, b])$. Then, we have :*

$$\left| \int_a^b e^{i\lambda\phi(x)} \psi(x) \right| \leq c_k \lambda^{-1/k} \left(|\psi(b)| + \int_a^b |\psi'(x)| dx \right)$$

Proof. We have :

$$\int_a^b e^{i\lambda\phi} \psi = \psi(b) \int_a^b e^{i\lambda\phi(x)} dx - \int_a^b \left(\int_a^x e^{i\lambda\phi(t)} dt \right) \psi'(x) dx$$

And the result follow. □

The conclusion of those results is that, as long as some minimal non-concentration hypothesis are made on the phase, we can always hope for some decay in the smooth case. As we are only interested here in fourier decay, more sophisticated results are not needed here, but let's state one classical case anyway.

Theorem 53. *Let ϕ and ψ be smooth, with ψ with compact support such that it contain a unique point x_0 such that $\phi'(x_0) = 0$, and suppose moreover that $\phi''(x_0) > 0$. Then we have, for $\lambda > 0$:*

$$\int_{\mathbb{R}} e^{i\lambda\phi} \psi = \psi(x_0) e^{i\lambda\phi(x_0)} e^{i\pi/4} \sqrt{\frac{2\pi}{\lambda\phi''(x_0)}} + O(\lambda^{-3/2}).$$

The proof is done by first studying the special case where $\phi(x) = x^2$, and then reducing the general problem to this one. So let's first study this particular case.

Proof. Let $\varepsilon > 0$ and let $\alpha := \varepsilon - i\lambda$. Remember that we have :

$$\int_{-\infty}^{\infty} e^{-\alpha x^2} dx = \sqrt{\frac{\pi}{\alpha}}$$

Where the square root is the standard branch. It can be proved by computing the square of this integral in the case where $\alpha > 0$, and then by observing that the integral is holomorphic on the set $\{\mathrm{Re}(\alpha) > 0\}$.

With this we can write :

$$\int_{-\infty}^{\infty} \psi(x) e^{-\alpha x^2} dx = \psi(0) \sqrt{\frac{\pi}{\alpha}} + \int_{-\infty}^{\infty} \tilde{\psi}(x) x e^{-\alpha x^2} dx$$

Where $\tilde{\psi}(x) := \frac{\psi(x) - \psi(0)}{x}$, which is again a smooth function. But here we have to be careful that it is no longer integrable. To continue our computations, we do an integration by parts :

$$\int_{-\infty}^{\infty} \psi(x) e^{-\alpha x^2} dx = \psi(0) \sqrt{\frac{\pi}{\alpha}} - \frac{1}{2\alpha} \int_{-\infty}^{\infty} \frac{d\tilde{\psi}}{dx}(x) e^{-\alpha x^2} dx$$

And now everything's fine, since for x large enough, $\frac{d}{dx}\tilde{\psi}(x) = -1/x^2$ is integrable near infinity. This allows us to finally use the dominated convergence theorem to make ε vanish. We get :

$$\int_{-\infty}^{\infty} \psi(x) e^{i\lambda x^2} dx = \psi(0) e^{i\pi/4} \sqrt{\frac{\pi}{\lambda}} + \frac{1}{2i\lambda} \int_{-\infty}^{\infty} \frac{d\tilde{\psi}}{dx}(x) e^{i\lambda x^2} dx$$

Then, we can conclude by using the Van der Corput theorem on the integral on the right, and we get :

$$\int_{-\infty}^{\infty} \psi(x) e^{i\lambda x^2} dx = \psi(0) e^{i\pi/4} \sqrt{\frac{\pi}{\lambda}} + O(\lambda^{-3/2}).$$

□

Notice that this method allows us to get, by hand, the following terms in the asymptotic expansion. The reader interested in the computation of some really precise expansions should look up at [SM93] and [Ta20b].

Let's finish our proof : it uses the Morse lemma. See [Ma15], lemma 14.6 for a proof.

Proof. Around x_0, we have : $\phi(x) \simeq \phi(x_0) + \frac{\phi''(x_0)}{2}(x - x_0)^2 + O((x - x_0)^3)$. The Morse lemma state that, since $\phi''(x_0) > 0$, there exists a suitable choice of coordinates $\Phi(y)$ near x_0 (a diffeomorphism), such that, in those coordinates, $\phi(\Phi(y)) = \phi(x_0) + y^2$.
Moreover, we have the expansion $\Phi(y) = x_0 + \sqrt{\frac{2}{\phi''(x_0)}} y + O(y^2)$.

So we can apply this to our integral. By the non stationnary phase, we can consider a bump function χ localized around x_0 such that

$$\int_{\mathbb{R}} e^{i\lambda\phi}\psi = \int e^{i\lambda\phi(x)}\psi(x)\chi(x)dx + O(\lambda^{-N}).$$

Then, applying our change of variables, we get

$$\int_{\mathbb{R}} e^{i\lambda\phi}\psi = \int e^{i\lambda(\phi(x_0)+y^2)}\psi(\Phi(y))\chi(\Phi(y))\sqrt{\frac{2}{\phi''(x_0)}}dy$$

Which give us the desired result by the previous part of the proof.

□

3.1.2 Several variables

Now we look at what happen in the case where we have several variables. Most of the preceding results have a multidimensional counterpart. In this subsection, we consider $\phi : \mathbb{R}^d \to \mathbb{R}$ a smooth phase and $\psi : \mathbb{R}^d \to \mathbb{C}$ smooth with compact support, and we are interested in integrals of the form:

$$I(\lambda) = \int_{\mathbb{R}^d} e^{i\lambda\phi(x)}\psi(x)dx.$$

We begin by the non-stationary phase.

Theorem 54 (Non-stationary phase)**.** *Let $\phi : \mathbb{R}^d \to \mathbb{R}$ a smooth phase and $\psi : \mathbb{R}^d \to \mathbb{C}$ smooth with compact support. Suppose that, for any $x \in supp(\psi)$, $\nabla\phi(x) \neq 0$. Then, for any $N \geq 0$, we have :*

$$\int_{\mathbb{R}^d} e^{i\lambda\phi(x)}\psi(x)dx \leq C(N, \phi, \psi)\lambda^{-N}.$$

Proof. By hypothesis, for any $x_0 \in \mathrm{Supp}(\psi)$, $\nabla\phi(x_0) = {}^t(\partial_1\phi(x_0)\dots\partial_d\phi(x_0)) \neq 0$. Hence, there exists an index $i(x_0)$ for which, for any x close enough to x_0, we have $\partial_{i(x_0)}\phi(x) \neq 0$.

Since the support is supposed to be compact, we can extract a finite subcover from this. We can then consider some bump functions $\chi_1, \dots, \chi_n \geq 0$ such that $\sum_{k=1}^n \chi_k = 1$ on a neighborhood of suppψ, and such that, for any k, there exists an index i_k such that for any $x \in \mathrm{supp}(\chi_i)$, $\partial_{i_k}\phi \neq 0$.

Also, choose $R > 0$ such that supp$\psi \subset [-R, R]^d$. We get :

$$I(\lambda) = \sum_{k=1}^{n} \int_{[-R,R]^d} e^{i\lambda\phi(x)}\psi(x)\chi_k(x)dx$$

$$= \sum_{k=1}^{n} \int_{[-R,R]^{d-1}} \left(\int_{-R}^{R} e^{i\lambda\phi(x)}\psi(x)\chi_k(x)dx_{i_k} \right) dx_1 \dots dx_{i_{k-1}} dx_{i_{k+1}} \dots dx_d$$

$$\leq C(N, \phi, \psi)\lambda^{-N}$$

By the one dimensionnal nonstationnary phase applied on the inner integral. Notice that the bound obtained in this inner integral can be made uniform in $[-R, R]^{d-1}$ since the implied constants given by the one dimensionnal case are bounded by the C^k norm of ψ and of $(1/\partial_i\phi)_i$, that is why it works.

□

We also have a counterpart for the Van-der-Corput theorem. It will be a weaker version of it as, unfortunately, we will not be able to find constants that doesn't depend on the support of ψ in this case.
First, we need a lemma.

Lemma 55. *Let $k \geq 1$. Then $\{(u \cdot x)^k \mid u \in \mathbb{S}^{d-1}\}$ spans the linear space of k-homogeneous polynomials.*

Proof. Consider the inner product

$$\langle P, Q \rangle := \sum_{|\alpha|=k} \alpha! a_\alpha b_\alpha,$$

where $P = \sum a_\alpha x^\alpha$ and $Q = \sum b_\alpha x^\alpha$ are k-homogeneous polynomials. (Here, α is a multi-index.) We see that we can rewrite it as

$$P(\partial)Q = \sum_{|\alpha|=|\beta|=k} a_\alpha b_\beta \partial^\alpha x^\beta = \langle P, Q \rangle,$$

which is very convenient. Suppose that P is a k-homogeneous polynomial which is orthogonal to all the $(u \cdot x)^k$. It means that

$$\forall u, \ \frac{d^k}{dt^k} P(x + ut)|_{t=0} = (d^k P)_x(u, \dots, u) = 0.$$

And so, the k-th differential of P is zero. But since P it is a k-homogeneous polynomial, it must be zero.

□

Theorem 56. *Let $\phi : \mathbb{R}^d \to \mathbb{R}$ a smooth phase and $\psi : \mathbb{R}^d \to \mathbb{C}$ smooth supported on the unit ball. Suppose that there exists a multi-index α with $|\alpha| > 0$ such that for any $x \in supp(\psi)$, $|\partial^\alpha \phi(x)| \geq 1$. Then, we have :*

$$\left| \int_{\mathbb{R}^d} e^{i\lambda\phi}\psi \right| \leq c_k(\phi)\lambda^{-1/k} \left(\|\psi\|_{L^\infty} + \|\nabla\psi\|_{L^1} \right).$$

The constant $c_k(\phi)$ is independent of ψ and λ, and is bounded as long as the C^{k+1} norm of ϕ is bounded.

Proof. The idea is the same as before : we localize with partitions of unity, and then we use the one dimensional case.

First of all, we extract a basis from $\{(u \cdot x)^k \mid u \in \mathbb{S}^{d-1}\}$, and denote its cardinal by N_k. It is the dimension of the k-homogeneous polynomials, so it also depends on d in reality. We have two basis for this space: the obvious one, (x^α) for all multi index $|\alpha| = k$, and our new basis $(u_i \cdot x)^k_{i=1,\dots,N_k}$.

Since we are considering a finite dimensional and real vector space, all the norm are equivalent. In particular, the sup norm with respect to our two basis are equivalent.

It implies the following : let α be a multi index. We decompose it in our basis :

$$x^\alpha = \lambda_1 (u_1 \cdot x)^k + \cdots + \lambda_{N_k}(u_{N_k} \cdot x)^k$$

And so there exists a constant $A_k > 0$ such that, for all i, $|\lambda_i| \leq A_k$.

Now we begin the real proof.
We consider the α from the hypothesis that satisfy $|\partial^\alpha \phi| \geq 1$ on the unit ball.

For any $|x_0| \leq 1$, there must be some index $i(x_0)$ such that $|(u_{i(x_0)} \cdot \nabla)^k \phi(x_0)| \geq 1/(N_k A_k)$. Indeed, in the other case, we would have $|\partial^\alpha \phi(x_0)| \leq \sum_{i=1}^{N_k} |\lambda_i||(u_{i(x_0)} \cdot \nabla)^k \phi(x_0)| < 1$, which is false.

Now, since we have supposed $\|\phi\|_{C^{k+1}}$ bounded by some constant, there exists a small radius r_k independent of x_0 such that

$$\forall x \in B(x_0, r_k\sqrt{d}),\ |(u_{i(x_0)} \cdot \nabla)^k \phi(x)| \geq 1/(2N_k A_k) > 0.$$

By compacity of the closed unit ball, we can find a finite covering with balls of the form $B(x^{(j)}, r_k)$. let's say that it contain n_k balls. Also, consider an adapted family of bump functions χ_j. We can write :

$$I(\lambda) = \sum_{j=1}^{n_k} \int_{B(x^{(j)}, r_k)} e^{i\lambda\phi(x)} \phi(x) \chi_j(x) dx.$$

Then, for all j, we consider a change of variable $x =: x^{(j)} + \Omega_j y$, with $\Omega_j \in SO_d(\mathbb{R})$ such that our first coordinates is now in the same direction as the unit vector $u_{i(x^{(j)})}$. In particular, with those coordinates, we have $\partial_{y_1}^k = (u_{i(x^{(j)})} \cdot \nabla)^k$.

So we have :

$$\left| \int_{B(x^{(j)}, r_k)} e^{i\lambda\phi(x)} \phi(x) \chi_j(x) dx \right|$$

$$= \left| \int_{[-r_k, r_k]^{d-1}} \left(\int_{r_k}^{r_k} e^{i\lambda\phi(x^{(j)} + \Omega_j y)} \psi(x^{(j)} + \Omega_j y) \chi_j(x^{(j)} + \Omega_j y) dy_1 \right) dy_2 \dots dy_d \right|$$

$$\leq (2r_k)^{d-1} c_k (2N_k A_k \lambda)^{-1/k} \left(\|\psi\chi_j\|_\infty + \|\nabla(\psi\chi_j)\|_{L^1} \right)$$

$$\leq B_k \lambda^{-1/k} \left(\|\psi\|_\infty + \|\nabla\psi\|_{L^1} \right)$$

For some constant B_k that only depends on k (and d). Summing those inequalities finishes the proof.

□

Finally, let's state a multidimensional analog of the theorem 51. We say that ϕ have a critical point at x_0 if $\nabla\phi(x_0) = 0$. We say that this critical point is nondegenerate if, moreover, the Hessian $\nabla^2\psi(x_0)$ is invertible. Notice that, in this case, the Morse lemma apply, and this is basically why we do those assumptions.
We have the following result.

Theorem 57. *Let ϕ and ψ be smooth, ψ with compact support such that it contains a unique critical point x_0 for ϕ. Suppose moreover that this critical point is nondegenerate, with strictly positive determinant. Then we have, for $\lambda > 0$:*

$$\int_{\mathbb{R}^d} e^{i\lambda\phi}\psi = \psi(x_0) e^{i\lambda\phi(x_0)} e^{i\pi sgn(\nabla^2\phi(x_0))/4} \sqrt{\frac{(2\pi)^d}{|\det \nabla^2\phi(x_0)|}} \frac{1}{\lambda^{d/2}} + O(\lambda^{-3d/2})$$

Where sgn denote the signature of $\nabla^2\phi(x_0)$.

The proof is analog to the one dimensional case. First we study the case where the phase is a quadratic form, and then we use the morse lemma to conclude. So first, let's prove the theorem for $\phi(x) := \sum_i \varepsilon_i x_i^2 = Q(x)$, where $\varepsilon_i \in \{-1, 1\}$.

Proof. We proceed by induction on the dimension d. The one dimensional case is already proved. Supposed that the asymptotic is known, for this kind of quadratic form, in dimension $d-1$. Then we can do the following.

$$\int_{\mathbb{R}^d} e^{i\lambda Q(x)} \psi(x) dx = \int_{\mathbb{R}^{d-1}} e^{i\lambda \tilde{Q}(\tilde{x})} \left(\int_{\mathbb{R}} e^{i\lambda \varepsilon_1 x_1^2} \psi(x) dx_1 \right) d\tilde{x}$$

Where $x = (x_1, \tilde{x})$ and $Q(x) = \varepsilon_1 x_1^2 + \tilde{Q}(\tilde{x})$. Carefully inspecting the proof of the one dimensional case, we see that there exists a smooth function $\omega_\lambda(\tilde{x})$ with compact support independent of λ, such that

$$\int_{\mathbb{R}} e^{i\lambda \varepsilon_1 x_1^2} \psi(x) dx_1 = e^{i\pi\varepsilon_1/4} \sqrt{\frac{\pi}{\lambda}} \psi(0, \tilde{x}) + \frac{\omega_\lambda(\tilde{x})}{\lambda^{3/2}}$$

and such that $\forall \lambda, \|\omega_\lambda\|_{C^1} \leq C$ for some constant $C > 0$.
Thus we can write :

$$\int_{\mathbb{R}^d} e^{i\lambda Q(x)} \psi(x) dx = e^{i\pi\varepsilon_1/4} \sqrt{\frac{\pi}{\lambda}} \int_{\mathbb{R}^{d-1}} e^{i\lambda \tilde{Q}(\tilde{x})} \psi(0, \tilde{x}) d\tilde{x} + \frac{1}{\lambda^{3/2}} \int_{\mathbb{R}^{d-1}} e^{i\lambda \tilde{Q}(\tilde{x})} \omega_\lambda(\tilde{x}) d\tilde{x}.$$

The last integral is a $O(\lambda^{-(d-1)/2})$ by the theorem 54. Notice that we used the bound $\|\omega_\lambda\|_{C^1} \leq C$ here. Then we can use our induction hypothesis on the first integral of the RHS and we get :

$$\int_{\mathbb{R}^d} e^{i\lambda Q(x)} \psi(x) dx = \psi(0) e^{i\pi \text{sgn} Q/4} \left(\frac{\pi}{\lambda}\right)^{d/2} + O(\lambda^{-d/2-1}).$$

□

Now we can do the general case with the help of the Morse lemma. In the d-dimensionnal setting, it states the following : if $\nabla\phi(x_0) = 0$ and if $\nabla^2\phi(x_0)$ is invertible, then there exists a diffeomorphism Φ around x_0 such that :

$$\phi(\Phi(y)) = \phi(x_0) + Q(y)/2$$

where Q is of the preceding form, with $\text{sgn}Q = \text{sgn}\nabla^2\phi(x_0)$. Moreover, we have $|\det(d\Phi)_0| = |\det \nabla^2\phi(x_0)|^{-1/2}$. With this, we can prove the general case.

Proof. First of all, by the non stationnary phase, we can localize our integral around x_0 like this :

$$I(\lambda) = \int_{B(x_0,r)} e^{i\lambda\phi} \psi\chi dx + O(\lambda^{-N})$$

for any N, with $\chi = 1$ around x_0. Then, the Morse lemma allow us to write :

$$\int e^{i\lambda\phi} \psi\chi dx = \int e^{i\lambda\phi(x_0) + i\lambda Q(y)/2} (\psi\chi)(\Phi(y)) \ |\det \nabla^2\phi(x_0)|^{1/2} dy$$

From which we can conclude with the preceding part of the proof.

□

Notice that our estimate doesn't cover the case of worst type of singularity (as soon as the Hessian is degenerate). For example, it doesn't cover the case where $\nabla\phi = 0$ on a whole submanifold of $\mathbb{R}^d$ that is not reduced to a point. Finding precise estimates in this kind of setting is really hard in general: one reason for this is the difficulty to apply a generalized Morse lemma if the Hessian is not invertible.

3.1.3 Smooth measures on surfaces

With the previous estimates, we are finally able to study the Fourier transform of smooth measures supported on surfaces. The possible decay of the Fourier transform will heavily depend on the geometry of the surface. The best decay happen when the Gauss curvature of the surface doesn't vanish, but in the other case we may still have some decay, thanks to the Van der Corput type of results we obtained. Once again, we follow Stein for this part. ([SM93])

In the first theorem, we consider a hypersurface $S \subset \mathbb{R}^d$ with nonvanishing Gaussian curvature. In our setting, it will mean the following. Around any point p in S, there exists a change of coordinates with translation and rotations from which we can see S as a graph $x_d = \phi(x_1, \dots, x_{d-1})$ for some smooth function ϕ satisfying $\phi(0) = 0$ and $\nabla\phi(0) = 0$. The eigenvalues of $\nabla^2\phi(0)$ are called the principal curvature of S at p, and $\det \nabla^2\phi(0)$ is called the Gaussian curvature of S at p.

We have the following result.

Theorem 58. *Let S be a smooth hypersurface with nonvanishing gaussian curvature. Let σ be a smooth measure on S with compact support. Then*

$$\widehat{\sigma}(\xi) \lesssim |\xi|^{-(d-1)/2}.$$

Notice that this proves the decay of the Fourier transform of the surface measure for the sphere that we already used. It also prove the decay of the Fourier transform of the measure on the parabola that we used to state the strichartz estimates for the Schrodinger equation.

Proof. Since the support of our measure is supposed to be compact, we can cover it by a finite number of small balls on which we can see S as a graph. Each of these integrals are of the form :

$$\int e^{-2i\pi\xi\cdot x}\chi(x)d\sigma(x) = \int e^{-2i\pi\left(\tilde{\xi}\cdot\tilde{x}+\xi_d\phi(\tilde{x})\right)}\psi(\tilde{x})d\tilde{x}$$

Where ψ is smooth and have a small compact support. The gradient of the phase is $\tilde{\xi}$. Hence, we have two case depending on ξ : if $\tilde{\xi}$ is away from zero, then the integral is quickly decreasing in $|\xi|$ by the nonstationnary phase. In the other case, we have $\xi_d \sim |\xi|$, and so we look at

$$\int e^{\xi_d\phi(\tilde{x})}\left(\psi(\tilde{x})e^{-2i\pi\tilde{\xi}\cdot\tilde{x}}\right)d\tilde{x}$$

Since the hessian of ϕ is non zero by hypothesis, the estimate of the theorem 55 apply, and we get a bound of the form

$$\left|\int e^{\xi_d\phi(\tilde{x})}\left(\psi(\tilde{x})e^{-2i\pi\tilde{\xi}\cdot\tilde{x}}\right)d\tilde{x}\right| \lesssim |\xi_d|^{-(d-1)/2} \simeq |\xi|^{-(d-1)/2}.$$

□

Notice that we proved that any hypersurface with nonvanishing gaussian curvature is Salem. In the case where the Gaussian curvature vanish, we still may have some pointwise decay, depending of what we will call the *type* of our hypersurface.

Definition 15. Let $S \subset \mathbb{R}^d$ be a submanifold of dimension $1 \leq m \leq d-1$. Geometrically, we will say that S have finite type if, at each point, S has at most a finite order of contact with any affine hyperplane. More precisely, we will say that S is of type k if the following holds.

Let p be any point in S. Locally, we can parametrize our manifold with an immersion $\phi : U \subset \mathbb{R}^m \to \mathbb{R}^d$. Then, for any unit vector η and for any $x_0 \in U$, the quantity $(\phi(x) - \phi(x_0)) \cdot \eta$ should vanish at order at most k. In other words, there should exists a multi index $|\alpha| \leq k$ such that $\partial^\alpha(\phi \cdot \eta)(x_0) \neq 0$.

In this case, the smallest k possible is called the type of S.

Theorem 59. *Let S be a manifold of type k. Let σ be a smooth measure on S with compact support. Then :*

$$|\widehat{\sigma}(\xi)| \lesssim |\xi|^{-1/k}.$$

Proof. As before, we can use partition of unity to write our Fourier transform as a finite sum of integrals of the form

$$\int_S e^{-2i\pi\xi\cdot x}\chi(x)d\sigma(x) = \int_{\mathbb{R}^m} e^{-2i\pi\xi\cdot\phi(x)}\psi(x)dx$$

Where ψ is smooth and have a very localized support. Write $\xi = |\xi|\eta$ with η unitary. Then, since $\partial^\alpha(\eta\cdot\phi) \neq 0$ for some $|\alpha| \leq k$, the theorem 54 give us the desired bound.

□

Notice that, since the cone is of type one, we can hope to adapt the strichartz estimates in the case of the wave equation, as previously mentionned.

This conclude this first half of the third chapter : I wanted to show old techniques that are in the landscape of harmonic analysis to study the decay of some Fourier transform and more general oscillatory integrals. But, as we saw, those techniques relies heavily on the smoothness hypothesis of the measures, and are not adaptable to fractal measures. We need new techniques.

Hopefully for us, such techniques exists, and we will explain one of them in the next part.

I like to think of it like this. Consider a fractal measure μ on $\mathbb{R}$ for example. In the smooth case, the proof of the fourier decay, when it is doable, is just an integration by parts (where we derivate the measure). This is not doable, so let's think a bit of how we could mimic the techniques anyway. What is an integration by parts really ? It is just a consequence of the fondamental theorem of analysis, $\int f' = 0$ if f have compact support. This, in turn, is a consequence of the fact that the lebesgue measure is invariant by translation, namely : $\int f(x+h)dx = \int f(x)dx$. Finally, this is just a rescaling of the following dynamical property : $\int \mathcal{L}f = \int f$ where $\mathcal{L}f(x) = f(x+1)$. The rescaling is doable because $\mathbb{R}$ is smooth (it's a Lie group).

So it turns out that our methods of the stationary phase rely on the fact that the translation leaves invariant $\mathbb{R}$ and its natural measure. Now, this we can mimic.

A lot of measures μ that comes from a dynamical setting are fixed points for a dynamical operator that we call a *transfer operator* $\mathcal{L}$. Using the invariance $\mathcal{L}^*\mu = \mu$ will be of great help to study our Fourier transforms. In a sense, this relation can be though as a description of the structure of the measure, just like how the fact that the Lebesgue measure is invariant under translation characterize it. Understanding a transfer operator associated to a measure is a key step if we want be able to work with this measure.

The next paragraph will be devoted to give a quick introduction to the notion of transfer operator through what is called the thermodynamical formalism.

3.2 The fractal case

3.2.1 Transfer operators

In this section, we will give a very quick glimpse to the thermodynamical formalism and the topic of dynamical systems. Our main reference will be [Zi96].

First of all, what is a dynamical system ? A dynamical system is the data of a topological space X, and of a continuous operator $T : X \to X$ that will mix our space X. We will often be interested to equip X with some measures μ that are invariant under the action of T, that is :

$$\forall A \in \mathcal{T},\ \mu(T^{-1}(A)) = \mu(A).$$

Intuitively, it means that no mass is loss when we apply our operator. The goal of the topic is to study the trajectories $(T^n(x_0))_n$ for some $x_0 \in X$. One famous subtopic of dynamical system is ergodic theory.

Let's give an important example of dynamical system.

Definition 16 (Symbolic dynamics)**.**

Let $\mathcal{A}$ be a finite set that we will call an alphabet. We consider $X := \mathcal{A}^{\mathbb{N}}$ the topological space of all infinite words $\mathbf{a} = a_0 a_1 \dots$ with letters in $\mathcal{A}$. The *shift operator* $T : X \to X$ is defined by

$$T(a_0 a_1 a_2 \dots) := a_1 a_2 a_3 \dots .$$

We can construct some invariant measures for this operator as follow.
Choose any probability measure $\mu_{\mathcal{A}}$ on $\mathcal{A}$, and equip X with the following probability measure:

$$\forall \alpha \in X,\ \mathbb{P}(a_0 = \alpha_0, a_1 = \alpha_1, \dots, a_n = \alpha_n) = \prod_{k=0}^{n} \mu_{\mathcal{A}}(a_i = \alpha_i).$$

Then we see that $\mathbb{P}$ is T-invariant.

Symbolic dynamics are the model example of dynamical system in a lot of situations. Indeed, we will often be able to conjugate other dynamical system to this one, or to slight modifications of this one. For example, recall the case of the cantor set.

Example 7. Recall that the triadic Cantor set is the set $C = \left\{ \sum_{n=1}^{\infty} \frac{2\varepsilon_n}{3^n},\ \varepsilon \in \{0,1\}^{\mathbb{N}^*} \right\}$.

Under this form, the fact that it is invariant under the map $T(x) = 3x \bmod 1$ is obvious. It is also quite obvious with this definition of the Cantor set that C is homeomorphic to $\{0,1\}^{\mathbb{N}}$.

Through this homeomorphism, the operator T is identified to the shift. The map $x \mapsto x/3$ is identified to $f_0(\mathbf{a}) = 0a_0a_1\dots$, while the map $x \mapsto x/3 + 2/3$ is identified to $f_1(\mathbf{a}) = 1a_0a_1\dots$. Notice that they are the two branch of the inverse of T.

The intervals $I_{\alpha_0 \dots \alpha_n}$ are also identified to the cylinders $\{a_0 = \alpha_0, \dots, a_n = \alpha_n\}$.

Finally, the Cantor measure correspond to a probability measure like constructed just before, where we choose $\mu_{\mathcal{A}}$ to be uniform. It is invariant under T.

Example 8. Recall the case of (affine) self similar compact sets. Consider $g_1, \dots, g_k$ some similarity transformation with ratio $r_1, \dots, r_k \in]0,1[$, and suppose that K is a nonempty compact set such that

$$\bigsqcup_i g_i(K) = K.$$

Moreover, suppose that there exists $V_1, \dots, V_k$ some bounded open sets such that $\overline{V_i} \cap \overline{V_j} = \emptyset$ if $i \neq j$, and such that $K \subset g_i^{-1}(V_i)$. In this case, let's show that the dynamic on K can be identified to the symbolic dynamic.

The encoding of a point $x \in K$ into an infinite word can be made like this. Let $x \in K$. Since, by hypothesis, $K = \bigsqcup g_i(K) \subset \bigsqcup V_i$, there exists a unique $a_0 \in \{1, \dots, k\}$ such that $x \in V_{a_0}$. So our word that represent x will begin like this : $x \simeq a_0?? \dots$.

Now, to find the next letter (or digit), we apply what we want to think as the shift.
Define $T : \bigsqcup_i V_i \to \mathbb{C}$ as $T(x) := g_i^{-1}(x)$ if $x \in V_i$. Then, since $x \in K$, $T(x) \in K$, and so there exists some index a_1 such that $T(x) \in V_{a_1}$.
We continue like this forever : for any $n \geq 0$, define a_n as the only index such that $T^n(x) \in V_{a_n}$.

Then, the word that is associated to x will be $\mathbf{a} := a_0 a_1 a_2 \dots \in \{1, \dots, k\}^{\mathbb{N}} =: X$.
This define a map $\Phi : K \to X$.

Let's show that T is an homeomorphism.

First of all, let's show that Φ is a bijection by constructing the inverse map. Let $x \in K$ and let $\mathbf{a} = \Phi(x)$ be its associated word. We can recover x by the data of $\mathbf{a}$ like this. Take an arbitrary $x_0 \in K$. For each $N \geq 0$, define $x_N := g_{a_0} \dots g_{a_N}(x_0)$. Then the word associated to x_N is of the form $a_0 \dots a_N b_0 \dots$, and from this we easily see that $x_N \to x$. Indeed, by construction, we have $T^N(x_N) \in V_{a_N}$ and $T^N(x) \in V_{a_N}$. Since the sets V_i all have diameter bounded by some constant C, and since all the g_i are k-lipschitz for some $k < 1$, we then have :

$$|x_N - x| = |g_{a_0} \dots g_{a_{N-1}} T^N(x_N) - g_{a_0} \dots g_{a_{N-1}} T^N(x)| \leq C k^N \to 0.$$

And so we have found an inverse for Φ. Moreover, the previous computations show that T^{-1} is continuous, and since X and K are compact, it follow that T is an homeomorphism.

The preceding case was affine. We already know that those kind of set are bad choices to try to construct sets with strictly positive fourier dimension. One example where the fourier dimension will actualy be strictly positive is the case of non-linear cantor sets. Let's define one of those.

Definition 17 (Schottky and Fushian group, [BD17])**.**

This example is linked to the hyperbolic plane. For this example, we will use the disk model of the hyperbolic plane : recall that the positive isometries of $\mathbb{D}$ for the hyperbolic metric are :

$$\text{Iso}^+(\mathbb{D}) = \left\{ z \mapsto e^{i\theta} \frac{z + c}{\overline{c}z + 1} \mid c \in \mathbb{D},\ \theta \in \mathbb{R} \right\}.$$

Also recall that, for $\gamma \in \text{Iso}^+(\mathbb{D})$, we have $\gamma(\mathbb{S}^1) = \mathbb{S}^1$, $\gamma(\mathbb{D}) = \mathbb{D}$, and $\gamma(\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}) = \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$.

A discrete subgroup $\Gamma < \text{Iso}^+(\mathbb{D})$ is called a Fuschian group. Hyperbolic geometers are often interested in the study of the surface obtained by a quotient $\mathbb{D}/\Gamma$, we call them hyperbolic surfaces. The spectral properties of those surfaces are deeply linked to the properties of the group Γ and to what is called the limit set Λ_Γ of Γ.

It is defined like this : take any $z_0 \in \mathbb{D}$, and look at the orbit Γz_0. The orbit accumulates into the circle $\mathbb{S}^1$, often creating a fractal. We then define $\Lambda_\Gamma \subset \mathbb{S}^1$ to be this fractal, the set of accumulation point of our orbit. This doesn't depend on the starting point z_0.

Then our group Γ will act on Λ_Γ, thus creating an interesting dynamical system on a fractal. In general, looking at the accumulation points of subgroups of moebius transformations often allow us to generate interesting dynamical systems.

Here we will be interested to one particular kind of Fuschian groups : we call them Shottky groups. To define a shottky group, we proceed like this.

Choose nonintersecting closed disks $D_1, \dots D_{2r}$ orthogonal to $\mathbb{S}^1$, and choose an involution $a \in \{1, \dots, 2r\} \mapsto \overline{a} \in \{1, \dots, 2r\}$ to create couple of disks.
Then, choose for each disk D_a a map $\gamma_a \in \text{Iso}^+(\mathbb{D})$ such that

$$\gamma_a(\widehat{\mathbb{C}} \setminus \mathring{D}_{\overline{a}}) = D_a \ , \ \gamma_a^{-1} = \gamma_{\overline{a}}.$$

The group Γ generated by $\gamma_1, \ldots, \gamma_{2r}$ is called a Schottky group.

With the same kind of consideration as before, we can see that the limit set Λ_Γ of our Schottky group is Cantor-like. But in this case, the natural encoding of the dynamic into symbolic dynamics will be a bit subtler.

To represent some point $x \in \Lambda_\Gamma$ as a word, we proceed like before. Our first digit will be the only $a_0 \in \mathcal{A} := \{1, \ldots, 2r\}$ such that $x \in D_{a_0}$. To look at the next digit, we have to apply what we would like to think as the shift map. It would be defined like this :

$$T: \begin{array}{ccc} \bigsqcup_{a\in\mathcal{A}} \overset{\circ}{D}_a & \longrightarrow & \mathbb{C} \\ x \in D_a & \longmapsto & \gamma_a^{-1}(x) \end{array}$$

So the next digit would be the only $a_1 \in \mathcal{A}$ such that $T(x) \in D_{a_1}$. But we have to be careful: a_1 can not be anyone. Indeed, by the hypothesis made on our γ_a, we know that $\gamma_{a_0}^{-1}(x) \notin D_{\overline{a_0}}$. So $a_1 \neq \overline{a_0}$.

More generally, we see that the natural word associated to $x \in \Lambda_\Gamma$, namely $\mathbf{a} := a_0 a_1 \ldots$ with $T^n(x) \in D_{a_n}$, satisfy the following property : $\forall n, a_{n+1} \neq \overline{a_n}$. So, in this case, the symbolical dynamic will be a bit different : this time, Λ_Γ is identified to the space $\mathcal{W}$ defined by:

$$\mathcal{W} := \{\mathbf{a} \in \mathcal{A}^{\mathbb{N}} \mid \forall n, a_{n+1} \neq \overline{a_n}\}.$$

The map T is then identified to the usual symbolic shift, but restricted conveniently $T : \mathcal{W} \to \mathcal{W}$. This setting is a particular case of what we call a subshift of finite type: it is when T is corestricted to words that doesn't have some fixed finite subwords. (Here, the finite word that we don't want to see are $a_1\overline{a_1}$, $a_2\overline{a_2}$, and so on.)

A lot of dynamical systems can be reduced to study some symbolical dynamics, so now we will focus on that. Let's get to the heart of the section: the transfer operator and the notion of pressure.

So we fix some finite alphabet $\mathcal{A}$ and set $X = \mathcal{A}^{\mathbb{N}}$ with the product topology. Note T the usual shift operator.

Definition 18. Let $\varphi \in C^0(X, \mathbb{R})$, that we call a *potential*. The transfer operator $\mathcal{L}_\varphi$ acts on the space of all continuous and real valued functions $f : X \to \mathbb{R}$, and is defined by:

$$\mathcal{L}_\varphi f(x) := \sum_{y \in T^{-1}(x)} e^{\varphi(y)} f(y) = \sum_{a \in \mathcal{A}} e^{\varphi(ax)} f(ax).$$

This operator depends on a choice of a potential: it is just something that we choose ourselves, conveniently, so that it ponders all the branches of T^{-1} accordingly to our intuition. In our fractal dynamic case, we will make sure to choose φ such that it represent the variation of the size of the partitions represented by the $\{cx \mid x \in X\}$. In a sense, in T we encoded the dynamics of our fractal, but in φ we could say that we will encode a part of its geometry. We will see that soon.

Some remark on the transfer operator: first of all, if we denote $S_n\varphi := \sum_{k=0}^{n-1} \varphi \circ T^k$, then we see that

$$\mathcal{L}_\varphi f(x) = \sum_{y \in T^{-n}(x)} e^{S_n(\varphi)(y)} f(y).$$

Second, by duality, our transfer operator also acts on the space of finite signed measures on X. Indeed, for a finite signed measure μ on X, we can define $\mathcal{L}_\varphi^* \mu$ by duality as follow:

$$\forall f \in C(X, \mathbb{R}), \int_X f d\mathcal{L}_\varphi^* \mu := \int_X \mathcal{L}_\varphi f d\mu.$$

Now, the goal for us is to understand this operator. To do this, we first have to introduce an important quantity associated to φ : its pressure.

Definition 19. Let $\varphi \in C^0(X, \mathbb{R})$ be a potential.
The following limit exists and is called the pressure of φ:

$$P(\varphi) := \lim_{n\to\infty} \frac{1}{n} \log \left(\sum_{c\in\mathcal{A}^n} e^{(S_n\varphi)_c} \right)$$

Where we denoted, for $c \in \mathcal{A}^n$ a finite word, $\psi_c := \sup\{\psi(cx) \mid x \in X\} = \sup_{cX} \psi$.

Proof. Set $V_n := \sum_{c\in\mathcal{A}^n} e^{(S_n\varphi)_c}$. Then we see that

$$V_{n+m} = \sum_{cd\in\mathcal{A}^{n+m}} e^{(S_{n+m}\varphi)_{cd}}$$

And, for $x \in X$, we have:

$$S_{n+m}\varphi(cdx) = \sum_{k=0}^{n+m-1} \varphi \circ T^k(cdx)$$

$$= \sum_{k=0}^{n-1} \varphi \circ T^k(cdx) + \sum_{k=0}^{m-1} \varphi \circ T^k(dx) \leq (S_n\varphi)_c + (S_m\varphi)_d$$

Hence,

$$V_{n+m} \leq \sum_{cd\in\mathcal{A}^{n+m}} e^{(S_n\varphi)_c+(S_p\varphi)_d} = V_nV_m,$$

Which means that $\log(V_n)$ is subadditive. By Fekete's subadditive lemma, the sequence $\log(V_n)/n$ converges.

□

The pressure of the potential is related to the spectrum of transfer operators, as we will see in the next theorem. To state it, I just need to specify one last thing: we equip X with its usual ultrametric distance $d(x, y) := \exp(-\inf\{n,\ x_n \neq y_n\})$. Now the notion of $\varphi : X \to \mathbb{R}$ being Hölder make sense. Let's state the most important theorem of the section.

Theorem 60 (Perron-Frobenius-Ruelle)**.** *Let $\varphi : X \to \mathbb{R}$ be Hölder. Then :*

- *There exists a simple eigenvalue $\beta > 0$ of $\mathcal{L}_\varphi$ and an associated eigenfunction $h > 0$ in $C(X, \mathbb{R})$.*
- *There exists a unique probability measure $\mu \in \mathcal{P}(X)$ such that $\mathcal{L}^*_\varphi\mu = \beta\mu$. Moreover, for any function $v \in C(X)$, $\beta^{-n}\mathcal{L}^n_\varphi(v)$ converge uniformly on X to $h \int vd\mu/ \int hd\mu$.*

Finally, we have $\log \beta = P(\varphi)$.

This theorem is a really powerful tool to easily construct invariant measures. I won't prove it as the proof is quite technical and long, but let's state some ideas.

First of all, let's consider λ another eigenvalue of $\mathcal{L}_\varphi$. Then, for its associated eigenfunction, we see that $\beta^{-n}\mathcal{L}^n_\varphi = (\lambda/\beta)^n v \to h \int vd\mu/ \int hd\mu$. So the sequence $(\lambda/\beta)^n$ is bounded, which means that $|\lambda| \leq \beta$.

In fact, we can show that when $\mathcal{L}_\varphi$ acts on Hölder functions, there exists a radius $r < \beta$ such that $\mathrm{Sp}(\mathcal{L}_\varphi) \subset B(0, r) \cup \{\beta\}$.

The idea of the proof is the following: first of all, by modifying φ, we can rescale our operator so that $\beta = 1$. Then, the existence of the eigenfunction and of the eigenmeasure can be obtained by the use of some sophisticated fixed point theorem, as the Shauder-Tychonov theorem. A detailled proof can be found in [Zi96].

Before going on, let's state an important fact about those invariant measures: they satisfy some good estimate properties.

Theorem 61. *Suppose that φ is α-Hölder. Then, with the same notations as before, we have:*

$$C^{-1}e^{-nP(\varphi)+S_n\varphi(x)} \leq \mu(x_1 \dots x_nX) \leq Ce^{-nP(\varphi)+S_n\varphi(x)}$$

For some constant $C > 0$. We say that μ is a Gibbs measure with parameter $-P(\varphi)$.

Proof. We have:

$$\int_X e^{-\varphi} \mathbb{1}_{x_1 \dots x_n X} d\mu = e^{-P(\varphi)} \int_X \mathcal{L}_\varphi \left(e^{-\varphi} \mathbb{1}_{x_1 \dots x_n X} \right) d\mu$$

And

$$\mathcal{L}_\varphi \left(e^{-\varphi} \mathbb{1}_{x_1 \dots x_n X} \right)(z) = \sum_{a \in \mathcal{A}} e^{\varphi(az)} e^{-\varphi(az)} \mathbb{1}_{x_1 \dots x_n X}(az) = \mathbb{1}_{x_1 \dots x_n X}(x_1 z).$$

Hence :

$$\int_X e^{-\varphi} \mathbb{1}_{x_1 \dots x_n X} d\mu = e^{-P(\varphi)} \mu(x_2 \dots x_n X).$$

Moreover, since φ is α-Hölder, we can write:

$$e^{-\exp(-\alpha n)} \mu(x_1 \dots x_n X) \leq e^{\varphi(x)} \int_X e^{-\varphi} \mathbb{1}_{x_1 \dots x_n X} d\mu \leq e^{\exp(-\alpha n)} \mu(x_1 \dots x_n X).$$

And so

$$e^{-P(\varphi) - \exp(\alpha n)} \leq \frac{\mu(x_1 \dots x_n X)}{\mu(x_2 \dots x_n X)} e^{-\varphi(x)} \leq e^{-P(\varphi) + \exp(\alpha n)}$$

Then, multiplying those inequalities, we get

$$C^{-1} e^{-nP(\varphi) + S_n(\varphi)(x)} \leq \mu(x_1 \dots x_n X) \leq C e^{-nP(\varphi) + S_n(\varphi)(x)}$$

Where $C := \exp\left(\sum_{n=0}^{\infty} \exp(-\alpha n)\right)$

□

Now this is very interesting too. Those estimates that satisfy Gibbs measures are similar to the kind of estimates satisfies by Ahlfors-David regular measures, and indeed, measures that comes from the PFR theorem will often behave nicely and satisfy some regularity properties.

Now let's manipulate those notions in the special case of linear self-similar sets.

Example 9. We come back to the setting of the example 8: we consider $g_1, \dots, g_k$ some similarity transformation with ratio $r_1, \dots, r_k \in]0,1[$, and we suppose that K is a nonempty compact set such that

$$\bigsqcup_i g_i(K) = K.$$

Moreover, we suppose that there exists $V_1, \dots, V_k$ some bounded open sets such that $\overline{V_i} \cap \overline{V_j} = \emptyset$ if $i \neq j$, and such that $K \subset g_i^{-1}(V_i)$. In this case, we can naturally identify the dynamic on K to the symbolic dynamic, where what takes the role of the shift is

$$\begin{array}{rccc} T: & \bigsqcup_{a \in \mathcal{A}} V_a & \longrightarrow & \mathbb{C} \\ & x \in V_a & \longmapsto & g_a^{-1}(x) \end{array}$$

Now, we have to think to a potential for our transfer operator. One natural choice would be just to encode the various contraction rate associated to our g_a. In other words, the following choice for φ is natural:

$$\varphi(x) := \log(r_a) \text{ if } x \in V_a.$$

In this case, we see that $S_n\varphi(x) = \log\left(r_{a_0} r_{a_1} \dots r_{a_{n-1}}\right)$, where $a_0 a_1 \dots$ is the word associated to x. The number $r_{a_0} r_{a_1} \dots r_{a_{n-1}}$ then have the order of magnitude of $\operatorname{diam}(V_{a_0 a_1 \dots a_n})$, where $V_{a_0 a_1 \dots a_n} := g_{a_0} \dots g_{a_{n-1}}(V_{a_n})$.

In particular, the Gibbs estimate gives us, for the measure given by the PFT :

$$\mu(V_{a_0 a_1 \dots a_n}) \simeq \operatorname{diam}(V_{a_0 a_1 \dots a_n}) e^{-nP(\varphi)}$$

Unfortunately, this isn't really what we would like to see. To get something that look likes an AD-regular measure, we would need to get rid of that pressure term. So we need $P(\varphi) = 0$.

So let's modify our choice of potential. Let's introduce a parameter $t \in \mathbb{R}$, and look at what happen when we consider the potential $t\varphi$. We have

$$S_n(t\varphi)(x) = t \log(r_{a_0} r_{a_1} \dots r_{a_{n-1}}).$$

Let's compute the pressure:

$$P(t\varphi) = \lim_n \frac{1}{n} \log \left(\sum_{c \in \mathcal{A}^n} e^{S_n(t\varphi)_c} \right)$$

$$= \lim_n \frac{1}{n} \log \left(\sum_{a_0 \dots a_{n-1} \in \mathcal{A}^n} r_{a_0}^t \dots r_{a_{n-1}}^t \right)$$

$$= \log \left(\sum_{a \in \mathcal{A}} r_a^t \right).$$

We then see that $P(\delta\varphi) = 0$ as soon as

$$\sum_{a \in \mathcal{A}} r_a^\delta = 1,$$

and we recognize the dimension of our set K. But now we are actually able to nearly prove that this is its dimension. Indeed, in this case, the invariant measure given by the PFR theorem will satisfy the following estimate, since it is a Gibbs measure:

$$\mu(V_{a_0 \dots a_n}) \simeq \operatorname{diam}(V_{a_0 \dots a_n})^\delta$$

Which nearly means that μ is Ahlfors-David regular of dimension δ !
This basically implies that $\dim_H K = \delta$.

Now that we have found a really good candidate for our potential, let's look at who really is our transfer operator. We have:

$$\mathcal{L}_\varphi f(x) = \sum_{a \in \mathcal{A}} r_a^\delta \, f(\gamma_a(x)).$$

Since we have defined δ especially for it, we have $\beta = 1$, and so our measure μ is invariant under the transfer operator. It means that we have the following property :

$$\sum_{a \in \mathcal{A}} \int_K r_a^\delta f(\gamma_a(x)) d\mu(x) = \int_K f d\mu.$$

Which means that our measure satisfies some autosimilarity properties.
We can rewrite it as:

$$d\mu(x) = \sum_{a \in \mathcal{A}} |r_a|^\delta d\mu(g_a^{-1}(x))$$

And once again we recover a knowned relation: if we look at what happen in the case of the triadic cantor set, we have $r_a = 1/3$, $\delta = \log(2)/\log(3)$, $g_0(x) = x/3$ and $g_2(x) = x/3 + 2/3$, and therefore

$$d\mu(x) = \frac{d\mu(3x) + d\mu(3x - 2)}{2}.$$

Finally, let's conclude this part by explaining what happens in the case of Schottky groups. Recall the setting : we have non-intersecting closed disks $D_1, \dots D_{2r}$ orthogonal to $\mathbb{S}^1$, and an involution $a \in \{1, \dots, 2r\} \mapsto \overline{a} \in \{1, \dots, 2r\}$.

Moreover,for each disk D_a, we have a map $\gamma_a \in \mathrm{Iso}^+(\mathbb{D})$ such that

$$\gamma_a(\widehat{\mathbb{C}} \setminus \overset{\circ}{D}_{\overline{a}}) = D_a \ , \ \gamma_a^{-1} = \gamma_{\overline{a}}.$$

The group Γ generated by $\gamma_1, \dots, \gamma_{2r}$ is our Schottky group. The orbits under the action of this group accumulates onto a set $\Lambda_\Gamma \subset \mathbb{S}^1$ that we wish to study.

In this case, the shift map is

$$T : \begin{array}{ccc} \bigsqcup_{a \in \mathcal{A}} \overset{\circ}{D}_a & \longrightarrow & \mathbb{C} \\ x \in D_a & \longmapsto & \gamma_a^{-1}(x) \end{array}$$

and can be identified to the corestriction of the symbolic shift map to the set $\mathcal{W} := \{\mathbf{a} \in \mathcal{A}^{\mathbb{N}} \mid \forall n, a_{n+1} \neq \overline{a_n}\}$.. It can be showed that, in this case, the PFR theorem is still true.

So now, we need to choose a potential. Like before, we need something that encodes the geometry of our fractal. The analog of our previous choice, in this case, will be to choose

$$\varphi := -\log(|T'|).$$

The main interest of that choice is that we have

$$S_n(\varphi)(x) = -\log(|(T^n)'|) \simeq \log(\mathrm{diam}\gamma_{a_0} \dots \gamma_{a_{n-1}}(D_{a_n}))$$

Where $a_0 a_1 \dots$ is the word associated to x.
With this choice of function, the following theorem holds.

Theorem 62 (Bowen formula)**.**

The Hausdorff dimension of Λ_Γ is the only number $\delta \geq 0$ such that $P(\delta\varphi) = 0$.

So we fix this δ. Let's investigate who is the transfer operator. This case is a bit special, as we have to make sure to stay in the set $\mathcal{W}$. We have :

$$\mathcal{L}_{\delta\varphi} f(x) = \sum_{y \in T^{-1}(x)} e^{\delta\varphi(y)} f(y),$$

and $T^{-1}(x) = \{\gamma_b(x) \mid b \neq \overline{a} \text{ where } x \in D_a\}$. So :

$$\forall x \in D_a, \ \mathcal{L}_\varphi f(x) = \sum_{b \neq \overline{a}} e^{\delta\varphi(\gamma_b(x))} f(\gamma_b(x))$$

And $\varphi(\gamma_b(x)) = -\log|T'(\gamma_b(x))| = -\log|(\gamma_b^{-1})'(\gamma_b(x))| = \log|\gamma_b'(x)|$. Hence :

$$\mathcal{L}_{\delta\varphi} f(x) = \sum_{b \neq \overline{a}} f(\gamma_b(x))|\gamma_b'(x)|^\delta.$$

The Perron-Frobenius-Ruelle theorem then tells us that there exists a unique probability measure μ on Λ_Γ that satisfy $\mathcal{L}_{\delta\varphi}^* \mu = \mu$. In other words, we have for this measure :

$$\sum_{a \in \mathcal{A}} \sum_{b \neq \overline{a}} \int_{D_a} f(\gamma_b(x))|\gamma_b'(x)|^\delta d\mu(x) = \int_{\Lambda_\Gamma} f(x) d\mu(x).$$

for any continuous function $f : \Lambda_\Gamma \to \mathbb{C}$.

The measure that we obtain is known since the work of Patterson [Pa76] and Sullivan [Su79]. At the time, this is not how they constructed it: the standard way for the construction of a Patterson-Sullivan measure is by taking the weak limit of a sequence of measures on a fixed orbit $z_0\Gamma$, whose support accumulate on the limit set. The measure obtained by this construction depend on the chosen point z_0, but they are all conformally related to each other. Our measure fixed by the transfer operator is the case where $z_0 = 0$.

This measure satisfy some strong properties that we will need in the next section. First of all, μ is not only invariant by the transfer operator: it behaves nicely under the action of any element of our group Γ. In fact, we have:

$$\forall \gamma \in \Gamma, \int_{\Lambda_\Gamma} f(x) d\mu(x) = \int_{\Lambda_\Gamma} f(\gamma(x)) |\gamma'(x)|^\delta d\mu(x).$$

Or in other words:

$$\gamma_* \mu = |\gamma'|^{-\delta} d\mu.$$

Notice that this property imply the invariance by our transfer operator. Indeed, we see that:

$$\sum_{a \in \mathcal{A}} \sum_{b \neq \overline{a}} \int_{D_a} f(\gamma_b(x)) |\gamma_b'(x)|^\delta d\mu(x) = \sum_{a \in \mathcal{A}} \sum_{b \neq \overline{a}} \int_{\gamma_b(D_a)} f d\mu = \int_{\Lambda_\Gamma} f d\mu.$$

Finally, let's state that our Patterson-Sullivan measure is Ahlfors-David regular of dimension δ on Λ_Γ.

Notice that our measure is self similar, but in a nonlinear way, and so it have good chances to exhibit some Fourier decay. The next and final section of this thesis will explain how we can prove that $c_n(\mu)$ have some polynomial decay, using the previously mentioned results of Bourgain and some transfer operator.

3.2.2 Fourier transform of a Patterson-Sullivan measure.

This goal of this section is to explain the techniques used in [BD17] by Bourgain and Dyatlov, in 2017, to show that the limit set Λ_Γ of a Fuschian Schottky group Γ has strictly positive Fourier dimension (in the sense of Fourier coefficients of measures on the circle). This result was motivated by the study of the spectral properties of some hyperbolic surfaces: the main result of the paper was the proof of a Fractal Uncertainty Principle on Λ_Γ via the techniques showed in the chapter 2, section 3. To get this FUP, they showed that "the" associated Patterson-Sullivan measure μ satisfy some good Fourier decay properties.

Let's get into the details. First of all, in the setting of this paper, they do explicit computations: but they are significantly easier if we don't use the ball model for the hyperbolic plane, but instead use the half-plane model. So here's what we do:

Consider :

$$\Phi : \begin{array}{ccc} \widehat{\mathbb{C}} & \longrightarrow & \widehat{\mathbb{C}} \\ z & \longmapsto & \frac{z-i}{z+i} \end{array} .$$

This map, the Cayley transform, is known to conjugate the disk model to the half plane model, like so :

$$\Phi(\mathbb{H}) = \mathbb{D} \ , \ \Phi(\overline{\mathbb{R}}) = \mathbb{S}^1.$$

Where $\mathbb{H} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$.

Through this transformation, the Schottky group that we previously considered become a subgroup Γ of $\mathrm{Iso}^+(\mathbb{H})$, where

$$\mathrm{Iso}^+(\mathbb{H}) = \left\{ z \mapsto \frac{az+b}{cz+d} \mid ad - bc = 1, \ a,b,c,d \in \mathbb{R} \right\}.$$

Our disks orthogonal to $\mathbb{S}^1$ becomes disks orthogonal to $\overline{\mathbb{R}}$, and so, they are also centered at points in $\overline{\mathbb{R}}$. Without loss of generality, we may suppose that they are centered at points in $\mathbb{R}$. We note $I_a := D_a \cap \mathbb{R}$. Our limit set becomes a compact, fractal set $\Lambda_\Gamma \subset \mathbb{R}$ with same Hausdorff dimension δ than before.

Remember that our group is generated by some γ_a, $a \in \mathcal{A}$ such that, in this model:

$$\forall a \in \mathcal{A}, \gamma_a(\overline{\mathbb{R}} \setminus \overset{\circ}{I_{\overline{a}}}) = I_a.$$

The Patterson-Sullivan measure becomes a measure on Λ_Γ that satisfies the following invariance property:

$$\forall \gamma \in \Gamma, \ \forall f \in C^0(\mathbb{R}), \ \int_{\Lambda_\Gamma} f(x) d\mu(x) = \int_{\Lambda_\Gamma} f(\gamma(x)) |\gamma'(x)|_{\mathbb{D}}^\delta d\mu(x)$$

Where $|\gamma'|_{\mathbb{D}} := |(\Phi^{-1} \circ \gamma \circ \Phi)'(x)| = \frac{1+x^2}{1+\gamma^2(x)} \gamma'(x)$.

It is still Alhfors-David regular in this case. The main theorem of the paper is the following.

Theorem 63. *Let $\Gamma < Iso^+(\mathbb{H})$ be a Fuschian Shottky group, with a compact limit set $\Lambda_\Gamma \subset \mathbb{R}$ of dimension $\delta > 0$. Note μ the associated Patterson-Sullivan measure. Let $\varphi \in C^2(\mathbb{R},\mathbb{R})$ and $g \in C^1(\mathbb{R},\mathbb{C})$ such that:*

$$\|\varphi\|_{C^2} + \|g\|_{C^1} \leq C_0 \ , \ \inf_{\Lambda_\Gamma} |f'| \geq C_0^{-1}$$

for some $C_0 > 0$. Then, there exists $\varepsilon > 0$ depending only on δ, and $C > 0$ depending on Γ and C_0, such that

$$\left| \int_{\Lambda_\Gamma} e^{i\xi\varphi(x)} g(x) d\mu(x) \right| \leq C |\xi|^{-\varepsilon}.$$

We come back to our usual assumptions for the non-stationnary phase: it means that this theorem really isn't about f or g, but about the additive chaos of Λ_Γ and the nonlinear self similarity of μ.

Let's get into the details. First of all, we need some notations.

- For $n \geq 1$, define $\mathcal{W}_n$ the set of finite admissible words of length n by

$$\mathcal{W}_n := \{a_1 \dots a_n \mid a, \dots, a_n \in \mathcal{A},\ a_{j+1} \neq \overline{a_j}\ \forall j.\}$$

  Denote by $\mathcal{W} := \mathcal{W}$ the set of all finite words. Denote the empty word by $\emptyset$ and put $\mathcal{W}^0 = \mathcal{W} \backslash \{\emptyset\}$. For $\mathbf{a} = a_1 \dots a_n \in \mathcal{W}$, put $\overline{\mathbf{a}} := \overline{a_n} \dots \overline{a_1} \in \mathcal{W}$. If $\mathbf{a} \in \mathcal{W}^0$, put $\mathbf{a}' := a_1 \dots a_{n-1} \in \mathcal{W}$. Note that $\mathcal{W}$ forms a tree with root $\emptyset$ and each $\mathbf{a} \in \mathcal{W}^0$ having parents $\mathbf{a}'$.

- For $\mathbf{a} = a_1 \dots a_n$, $\mathbf{b} = b_1 \dots b_m \in \mathcal{W}$, we write $\mathbf{a} \to \mathbf{b}$ if either $\mathbf{a}$ or $\mathbf{b}$ is empty or $a_n \neq \overline{b_1}$. Under this condition the concatenation $\mathbf{ab}$ is a word.

- For $\mathbf{a}, \mathbf{b} \in \mathcal{W}$, we write $\mathbf{a} \prec \mathbf{b}$ if $\mathbf{a}$ is a prefix of $\mathbf{b}$, that is $\mathbf{b} = \mathbf{ac}$ for some $\mathbf{c} \in \mathcal{W}$.

- For $\mathbf{a} = a_1 \dots a_n$, $\mathbf{b} = b_1 \dots b_m \in \mathcal{W}^0$, we write $\mathbf{a} \rightsquigarrow \mathbf{b}$ if $a_n = b_1$. Note that when $\mathbf{a} \rightsquigarrow \mathbf{b}$, the concatenation **a'b** is a word of length $n + m - 1$.

- A finite set $Z \subset \mathcal{W}^0$ is called a partition if there exists N such that for each $\mathbf{a} \in \mathcal{W}$ with $|a| \geq N$, there exists a unique $\mathbf{b} \in Z$ such that $\mathbf{b} \prec \mathbf{a}$.

For each $\mathbf{a} \in \mathcal{W}$, define the group element $\gamma_\mathbf{a} \in \Gamma$ by

$$\gamma_\mathbf{a} := \gamma_{a_1} \cdots \gamma_{a_n}.$$

Note that each element of Γ is equal to $\gamma_\mathbf{a}$ for a unique choice of $\mathbf{a}$ and $\gamma_\mathbf{a}^{-1} = \gamma_{\overline{\mathbf{a}}}$. Also, $\gamma_{\mathbf{ab}} = \gamma_\mathbf{a}\gamma_\mathbf{b}$ when $\mathbf{a} \to \mathbf{b}$.

For each $\mathbf{a} \in \mathcal{W}^0$, define the intervals $I_\mathbf{a}$ as follow:

$$I_\mathbf{a} := \gamma_{\mathbf{a}'}(I_{a_n}).$$

By construction, we have $I_\mathbf{b} \subset I_\mathbf{a}$ when $\mathbf{a} \prec \mathbf{b}$, and $I_\mathbf{a} \cap I_\mathbf{b} = \emptyset$ when $|\mathbf{a}| = |\mathbf{b}|$, $\mathbf{a} \neq \mathbf{b}$. The limit set Λ_Γ is given by

$$\Lambda_\Gamma := \bigcap_n \bigsqcup_{\mathbf{a} \in \mathcal{W}_n} I_\mathbf{a}.$$

A finite set $Z \subset \mathcal{W}^0$ is a partition iff

$$\Lambda_\Gamma = \bigsqcup_{\mathbf{a} \in Z} (I_\mathbf{a} \cap \Lambda_\Gamma).$$

For an interval I, denote by $|I|$ its length. Finally, define, for any small parameter τ, the partition of resolution τ by :

$$Z(\tau) = \{\mathbf{a} \in \mathcal{W}^0 \mid |I_\mathbf{a}| \leq \tau < |I_{\mathbf{a}'}|\}$$

Where we put $|I_\emptyset| := \infty$. Then $Z(\tau)$ is a partition. Some computations allow the authors to see that $\# Z(\tau) \simeq \tau^{-\delta}$. Its no surprise that the Hausdorff dimension appear here.

The main interest of the partition $Z(\tau)$ is that it gives us a tight control on the order of magnitude of the different quantities that will appear in the argument. It is associated to the following transfer operator:

$$\forall x \in I_b,\ \mathcal{L}_{Z(\tau)} f(x) := \sum_{\mathbf{a} \in Z(\tau), \mathbf{a} \rightsquigarrow b} f(\gamma_{\mathbf{a}'(x)}) w_{\mathbf{a}'}(x).$$

Where the weight $w_{\mathbf{a}'}(x)$ is defined by

$$w_{\mathbf{a}'}(x) := |\gamma_{\mathbf{a}'}(x)|_\mathbb{D}^\delta.$$

Then, our Patterson-Sullivan measure will satisfy the following invariance formula, for any continuous function f:

$$\int \mathcal{L}_{Z(\tau)} f d\mu = \int f d\mu.$$

And so, iterating the formula gives us:

$$\int f d\mu = \int \mathcal{L}_{Z(\tau)}^k f d\mu$$

Where

$$\forall x \in I_b,\ \mathcal{L}_{Z(\tau)}^k f(x) = \sum_{\substack{\mathbf{a}_1,\dots,\mathbf{a}_k \in Z(\tau) \\ \mathbf{a}_1 \rightsquigarrow \dots \rightsquigarrow \mathbf{a}_k \rightsquigarrow b}} f(\gamma_{\mathbf{a}_1' \dots \mathbf{a}_k'}(x)) w_{\mathbf{a}_1' \dots \mathbf{a}_k'}(x).$$

We will be interested to iterate even more, and for this we need even more notations. For some resolution $\tau > 0$, we note:

- $\mathbf{A} = (\mathbf{a}_0, \dots, \mathbf{a}_k) \in Z(\tau)^{k+1}$, $\mathbf{B} = (\mathbf{b}_0, \dots, \mathbf{b}_k) \in Z(\tau)^k$
- We write $\mathbf{A} \leftrightarrow \mathbf{B}$ iif $\mathbf{a}_{j-1} \rightsquigarrow \mathbf{b}_j \rightsquigarrow \mathbf{a}_j$ for all $j = 1, \dots k$
- If $\mathbf{A} \leftrightarrow \mathbf{B}$, then we define the words $\mathbf{A} * \mathbf{B} := \mathbf{a}_0'\mathbf{b}_1'\mathbf{a}_1'\mathbf{b}_2' \dots \mathbf{a}_{k-1}'\mathbf{b}_k'\mathbf{a}_k'$ and $\mathbf{A} \# \mathbf{B} := \mathbf{a}_0'\mathbf{b}_1'\mathbf{a}_1'\mathbf{b}_2' \dots \mathbf{a}_{k-1}'\mathbf{b}_k'$
- Denote by $b(\mathbf{A}) \in \mathcal{A}$ the last letter of $\mathbf{A}$.

Now we have all the necessary notations to be able to write correctly the following formula:

$$\int_{\Lambda_\Gamma} f d\mu = \sum_{\mathbf{A},\mathbf{B},\mathbf{A}\leftrightarrow\mathbf{B}} \int_{I_{b(\mathbf{A})}} f(\gamma_{\mathbf{A}*\mathbf{B}}(x)) w_{\mathbf{A}*\mathbf{B}}(x) d\mu(x).$$

And now the proof may begin. To keep it concise, I'll only outline the key arguments of the proof. Of course the interested reader can find the missing details in [BD17].

So, set $\varphi \in C^2(\mathbb{R}, \mathbb{R})$ and $g \in C^1(\mathbb{R}, \mathbb{C})$ be as in the theorem. We are interested in the following quantity:

$$\int_{\Lambda_\Gamma} e^{i\xi\varphi} g d\mu = \int_{\Lambda_\Gamma} f d\mu$$

Where $f := \exp(i\xi\varphi) g$. By the invariance property previously mentionned, we have

$$\int_{\Lambda_\Gamma} f d\mu = \sum_{\mathbf{A},\mathbf{B},\mathbf{A}\leftrightarrow\mathbf{B}} \int_{I_{b(A)}} e^{i\xi\varphi(\gamma_{\mathbf{A}*\mathbf{B}}(x))} g(\gamma_{\mathbf{A}*\mathbf{B}}(x)) w_{\mathbf{A}*\mathbf{B}}(x) d\mu(x).$$

The use of the regularity of g and its boundedness in the C^1 norm allow them to get rid of g. In addition, the weights have the order of magnitude of the associated diameter, which behaves accordingly to the intuition: we have $w_{\mathbf{A}} \simeq \tau^{k\delta}$. Those estimates allow them to write

$$\left| \int_{I_{b(A)}} e^{i\xi\varphi(\gamma_{\mathbf{A}*\mathbf{B}}(x))} g(\gamma_{\mathbf{A}*\mathbf{B}}(x)) w_{\mathbf{A}*\mathbf{B}}(x) d\mu(x) \right| \lesssim \tau^{2k\delta} \left| \int_{I_{b(A)}} e^{i\xi\varphi(\gamma_{\mathbf{A}*\mathbf{B}}(x))} w_{\mathbf{a}_k'}(x) d\mu(x) \right|$$

And then, by Cauchy-Schwartz:

$$\left| \int_{\Lambda_\Gamma} f d\mu \right|^2 \lesssim \tau^{(2k-1)\delta} \sum_{\mathbf{A},\mathbf{B},\mathbf{A}\leftrightarrow\mathbf{B}} \left| \int_{I_{b(A)}} e^{i\xi\varphi(\gamma_{\mathbf{A}*\mathbf{B}}(x))} w_{\mathbf{a}_k'}(x) d\mu(x) \right|^2$$

$$= \tau^{(2k-1)\delta} \sum_{\mathbf{A}} \int_{I_{b(A)}^2} w_{\mathbf{a}_k'}(x) w_{\mathbf{a}_k'}(y) \sum_{\mathbf{B},\mathbf{A}\leftrightarrow\mathbf{B}} e^{i\xi(\varphi(\gamma_{\mathbf{A}*\mathbf{B}}(x)) - \varphi(\gamma_{\mathbf{A}*\mathbf{B}}(y)))} d\mu(x) d\mu(y)$$

$$\lesssim \tau^{(2k+1)\delta} \sum_{\mathbf{A}} \int_{I_{b(\mathbf{A})}^2} \left| \sum_{\mathbf{B},\mathbf{A}\leftrightarrow\mathbf{B}} e^{i\xi(\varphi(\gamma_{\mathbf{A}*\mathbf{B}}(x)) - \varphi(\gamma_{\mathbf{A}*\mathbf{B}}(y)))} \right|^2 d\mu(x) d\mu(y).$$

From there, the goal is to linearize the phase to get a sum that theorem 22 can control (see the section 1.3.3). Again, the good regularity properties of φ allow us to see that

$$\begin{aligned}\varphi(\gamma_{\mathbf{A}*\mathbf{B}}(x)) - \varphi(\gamma_{\mathbf{A}*\mathbf{B}}(y)) &\simeq (\varphi\circ\gamma_{\mathbf{A}\#\mathbf{B}})'(t)(\gamma_{\mathbf{a}'_k}(x) - \gamma_{\mathbf{a}'_k}(y)) \\ &\simeq \varphi'(x_{\mathbf{a}_0})\gamma'_{\mathbf{a}'_0\mathbf{b}'_1}(s_1)\dots\gamma'_{\mathbf{a}'_{k-1}\mathbf{b}'_k}(s_k)(\gamma_{\mathbf{a}'_k}(x) - \gamma_{\mathbf{a}'_k}(y)) \\ &\simeq \tau^{2k}\varphi'(x_{\mathbf{a}_0})\zeta_{1,\mathbf{A}}(\mathbf{b}_1)\dots\zeta_{k,\mathbf{A}}(\mathbf{b}_k)(\gamma_{\mathbf{a}'_k}(x) - \gamma_{\mathbf{a}'_k}(y))\end{aligned}$$

Where

$$\zeta_{j,\mathbf{A}}(\mathbf{b}) := \tau^{-2}\gamma'_{\mathbf{a}'_{j-1}\mathbf{b}'}(x_{\mathbf{a}_j})$$

Notice that $\gamma'_{\mathbf{a}'}$ have the order of magnitude of a typical element of $Z(\tau)$: in particular, $\zeta_{j,\mathbf{A}}(\mathbf{b})$ doesn't scale with τ.

This gives us:

$$\left|\int f d\mu\right|^2 \lesssim \tau^{(2k+1)\delta}\sum_{\mathbf{A}}\int_{I^2_{b(\mathbf{A})}}\left|\sum_{\mathbf{B},\mathbf{A}\leftrightarrow\mathbf{B}} e^{2i\pi\eta\zeta_{1,\mathbf{A}}(\mathbf{b}_1)\dots\zeta_{k,\mathbf{A}}(\mathbf{b}_k)}\right|^2 d\mu(x)d\mu(y).$$

With $\eta := \xi\tau^{2k}\varphi'(x_{\mathbf{a}_0})(\gamma_{\mathbf{a}'_k}(x) - \gamma_{\mathbf{a}'_k}(y))$. At this point, it seems that choosing our resolution τ to satisfy $\tau = \xi^{-2k-3/2}$ may be a good idea: it will ensure that $\eta \sim \tau^{-1/2}$.

And now is the moment where we would like to use our theorem 22. Since our δ is known, we can consider the k and the ε given by the theorem. For this k, we may hope that our sum is controled, but we still have to verify that the theorem indeed applies. We need to verify the following non-concentration hypothesis:

$$\forall j, \forall\sigma \in \left[|\eta|^{-1}, |\eta|^{-\varepsilon}\right],\ \#\left\{(\mathbf{b},\mathbf{c}) \in Z(\tau)^2 \mid |\zeta_{j,\mathbf{A}}(\mathbf{b}) - \zeta_{j,\mathbf{A}}(\mathbf{c})| \leq \sigma\right\} \lesssim \tau^{2\delta}\sigma^\delta$$

And this is where the paper really becomes technical. Getting those non-concentration results is by far the most challenging part of the game. In the case of the paper, they are able to get it by combinatorial arguments in a fairly elementary way. This is only possible because, in the very particular case of Schottky groups, every computations can be made by hands, but the problem with this approach is that it doesn't generalize well into other fractals.

Anyway, they manage to prove that this non-concentration hypothesis is verified for a large enough members of $\mathbf{A}$ and $\mathbf{B}$, and this allow them to conclude more or less like this:

$$\left|\int f d\mu\right|^2 \lesssim \tau^{(2k+1)\delta}\sum_{\mathbf{A}}\int_{I^2_{b(\mathbf{A})}}\left(\tau^{-k\delta}|\eta|^{-\varepsilon}\right)^2 d\mu(x)d\mu(y) \lesssim |\xi|^\varepsilon.$$

Of course I oversimplified some arguments here, but this is the idea. Actually, the decay rate is not the same ε as the one given by the theorem 22.

This work of Bourgain and Dyatlov marked a turning point in the study of the fourier transform of fractal measures. In 2019, Li, Naud and Pan obtained a similar result for Shottky groups, where the centers of the disks are placed arbitrarily in $\mathbb{C}$, under an algebraic hypothesis on the group. The theorem is the following:

Theorem 64 ([LNP19]). *Assume that Γ is a Zariski dense Schottky group in $PSL_2(\mathbb{C})$, and let μ be a Patterson-Sullivan measure on Λ_Γ. Fix any neighborhood $\mathcal{U}$ of Λ_Γ. Let g be in $C^1(\mathcal{U},\mathbb{C})$ and φ be in $C^2(\mathcal{U},\mathbb{R})$, with*

$$M := \inf_{z\in\mathcal{U}}|\nabla_z\varphi| > 0.$$

Assume that $\|g\|_{C^1} + \|\varphi\|_{C^2} \leq M'$. Then there exists $C := C(M, M', \Gamma) > 0$ and $\varepsilon > 0$ depending only on μ, such that for all $t \in \mathbb{R}$ with $|t| \geq 1$,

$$\left|\int_{\Lambda_\Gamma} e^{it\varphi(z)}g(z)d\mu(z)\right| \leq C|t|^{-\varepsilon}$$

The main part of the proof is to prove the non-concentration property: they deal with it via results of representation theory and regularity estimates for stationary measures of certain random walks on linear groups.

Then recently, in a preprint posted in 2020, Sahlsten and Stevens generalized the theorem of Bourgain and Dyaltov in the case of any nonlinear real cantor sets. They make a thorough use of the thermodynamical formalism. Once again, the key part of the proof is to be able to deal with the non-concentration estimates that appear: they prove it, in their case, using large deviations techniques.

The theorem is the following.

Theorem 65. *[SS20]*
Let $I_1, \dots, I_N$, $N \geq 2$, be closed, disjoint and bounded intervals in $[0,1]$, and write $I := \cup_a I_a$. Let $T : I \to \mathbb{R}$ be a mapping such that each restriction $T_a := T_{|I_a}$ is a real analytic map and assume T is conjugated to the shift on $\mathcal{A}^{\mathbb{N}}$, where $\mathcal{A} = \{1, \dots, N\}$. Suppose that:

1. Uniform expansion: *There exists $\gamma > 1$ and $D > 0$ such that for all $n \in \mathbb{N}$ and all $x \in I$, we have*
$$|(T^n)'(x)| \geq D^{-1}\gamma^n.$$
2. Markov property: *For all $a, b \in \mathcal{A}$, if $T(I_b) \cap \overset{\circ}{I_a} \neq \emptyset$, then $T(I_b) \supset I_a$.*
3. Bounded distortions: *Define $\Phi := \log T'$. Then, there exists $B < \infty$ such that*
$$\|\Phi\|_\infty = \|T''/T'\|_\infty \leq B.$$
4. Total non-linearity: *There exists no $g \in C^1(I)$ such that*
$$\Phi = \psi_0 + g \circ T - g$$
where $\psi_0 : I \to R$ is constant on every $I_a \subset I$, $a \in \mathcal{A}$

Let $K := \bigcap_{n=0}^{\infty} T^{-n}(I)$ be the T-invariant cantor set. Let φ be a potential with exponentially vanishing variations (like in the theorem 58), and let μ be the equilibrium measure associated to $\mathcal{L}_\varphi$. Suppose moreover that μ is without atoms.

Then $\widehat{\mu}$ vanish with a polynomial rate. In particular, K has strictly positive Fourier dimension.

Chapter 4

Conclusion

Let's quickly summarize what we've learned.

First of all, it may seems strange at first, but Fourier transform is deeply linked to the notion of Hausdorff dimension. For a set $E \subset \mathbb{R}^d$, having Hausdorff dimension less than α is equivalent to the existence of a probability measure on E such that

$$\int_E |\widehat{\mu}(\xi)|^2 |\xi|^{d-\alpha} d\xi < \infty.$$

If the link between the geometry of a set and the possible existence of measures that have a Fourier decay *in average* is well known, the study of sets that may exhibit measures that have a pointwise Fourier decay is much more subtle.

Not much was really known about the topic until the breakthrough of Bourgain in ~ 2010, who adapted additive combinatorial arguments to the study of Fourier transform. The key idea is the sum-product phenomenon: the Fourier transform decay if the phase is additively chaotic, and it happens that multiplicative subgroups are additively pseudorandom. Hence, to improve our chances to get some Fourier decay, one may be tempted to consider multiplicative convolutions of measures like so:

$$\int_{\mathbb{R}} f(z) d\mu \odot \nu(z) := \int_E \int_F f(xy) d\mu(x) d\nu(y).$$

And it works. Under some natural non concentration hypothesis on the measure, there always exists some $k \geq 0$ such that $\widehat{\mu^{\odot k}}(\xi) \to 0$ at a polynomial rate (at some scale).

The next step is to be able to get back to informations on our measure μ. For this, the thermodynamics formalism comes to help: a lot of measures that arise in dynamical systems can be seen as equilibrium measures for a well chosen transfer operators. The invariance by this transfer operator allow us to relate our Fourier transform to quantities that look like $\widehat{\mu^{\odot k}(\xi)}$.

For this to have a chance to work, our set E must exhibit non-linear self similarity: this is the total non-linearity hypothesis made on our last theorem.

Let's conclude by saying that we virtually know nothing past the dimension 1.

Writing this Master thesis was a good exercise for me. Of course, I don't claim to have handled all the subject with this little paper, I just wanted to write down some of the things I learned during the last 2 months. The interested reader should definitively read more.

Finally, I would like to thank Frederic Naud for giving me most of the material that I needed to write this, for his recurrent availability and for the many discussions that we had the past two months.

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