Exponential Periods - an example around Euler's constant

Valentin Carpintero Pérez

IMJ-PRG

19 octobre 2026

Plan

1. Exponential Periods
2. Comparison isomorphism and period matrix
3. How to compute (co)homogy relative to a SNCD
4. Euler's constant

Periods

Definition

A **period** is a complex number whose real and imaginary parts can be written as absolutely convergent integrals

$$\int_S g(x_1, \ldots, x_n) dx_1 \ldots dx_n$$

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$$\int_S g(x_1, \ldots, x_n) dx_1 \ldots dx_n$$

Example

$$\pi = \int_{\{x^2+y^2 \leq 1\}} dxdy \qquad \log 2 = \int_1^2 \frac{dx}{x}$$

$\sqrt{2}$ and all algebraic numbers

$\zeta(2)$ and all MZV

Exponential Periods

Definition

An **exponential period** is a complex number whose real and imaginary parts can be written as absolutely convergent integrals

$$\int_S e^{-f(x_1,\ldots,x_n)} g(x_1,\ldots,x_n)\,dx_1\ldots dx_n$$

Exponential Periods

Definition

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$$\int_S e^{-f(x_1,\ldots,x_n)} g(x_1,\ldots,x_n)\,dx_1\ldots dx_n$$

Remark

Taking $f = 0$, every "classical" period is an exponential period.

Exponential Periods

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$$\int_S e^{-f(x_1,\ldots,x_n)} g(x_1,\ldots,x_n)dx_1\ldots dx_n$$

Example

$e^{\alpha}, \quad \alpha \in \bar{\mathbb{Q}}$

$$\sqrt{\pi} = \int_{-\infty}^{+\infty} e^{-x^2} dx$$

Euler's constant γ

special values of Γ, Bessel, $E-$ *functions*

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Comparison isomorphism (classical case)

Setup :
$k \subseteq \mathbb{C}$
X smooth variety $/k$

Theorem (Grothendieck)

There is a functorial canonical isomorphism

$$H^n_{dR}(X) \otimes_k \mathbb{C} \xrightarrow{\cong} H^n_B(X) \otimes_{\mathbb{Q}} \mathbb{C}$$

given on affine X by

$$[\omega] \longmapsto \left([\sigma] \mapsto \int_\sigma \omega \right)$$

Period pairing

Setup :
$k \subseteq \mathbb{C}$
X affine smooth variety $/k$

Theorem (Grothendieck)

There is a perfect pairing

$$H^n_{dR}(X) \otimes H^B_n(X) \longrightarrow \mathbb{C}$$
$$[\omega] \otimes [\sigma] \longmapsto \int_\sigma \omega$$

Period pairing - Exponential case

Setup :
$k \subseteq \mathbb{C}$
X affine smooth variety $/k$
$f : X \to \mathbb{A}^1$ regular function

Theorem (Grothendieck, Hien-Roucairol, Bloch-Esnault)

There is a perfect pairing

$$H^n_{dR}(X, f) \otimes H_n^{Rd}(X, f) \longrightarrow \mathbb{C}$$
$$[\omega] \otimes [\sigma] \longmapsto \int_\sigma e^{-f}\omega$$

Period matrix

Theorem (Grothendieck, Hien-Roucairol, Bloch-Esnault)

There is a perfect pairing

$$H^n_{dR}(X, f) \otimes H_n^{Rd}(X, f) \longrightarrow \mathbb{C}$$
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Definition (Period matrix)

Take $\sigma_1, \ldots, \sigma_d$ a basis of $H_n^{Rd}(X, f)$
and $\omega_1, \ldots, \omega_d$ a basis of $H^n_{dR}(X, f)$

$\left(\int_{\sigma_i} e^{-f}\omega_j\right)_{i,j=1,\ldots,d}$

is a **period matrix**.

Twisted de Rham cohomology

Motivation :

Twisted de Rham cohomology

$$\mathcal{O}_X \longrightarrow \Omega^1_X \longrightarrow \Omega^2_X \longrightarrow \dots$$

$$\omega \longmapsto d\omega - df \wedge \omega$$

Definition (Twisted de Rham cohomology)

$$H^n_{dR}(X, f) := \mathbb{H}^n_{Zar}(X, (\Omega^\bullet_X, d_f))$$

Twisted de Rham cohomology - basic examples

Example

$X = \mathbb{A}^1, \quad f = x$

Twisted de Rham cohomology - basic examples

Example

$X = \mathbb{A}^1, \quad f = x^2$

Rapid decay homology

Motivation :

Rapid decay homology

Definition (Rapid decay homology)

$$H_n^{sing}(X(\mathbb{C}), \{\Re(f) \geq R\})$$

Rapid decay homology

Definition (Rapid decay homology)

$$H_n^{Rd}(X, f) := \lim_{R \to \infty} H_n^{sing}(X(\mathbb{C}), \{Re(f) \geq R\})$$

Rapid decay homology - a simple example

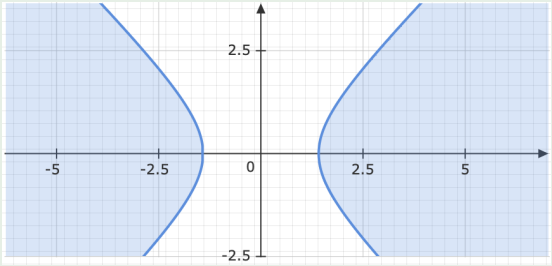
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Rapid decay homology - a simple example

Example

$X = \mathbb{A}^1, \quad f = x^2$



Period matrix of $\sqrt{\pi}$

Example :

$$H^1_{dR}(\mathbb{A}^1, x^2) = k\ [dx]$$
$$H^{Rd}_1(\mathbb{A}^1, x^2) = \mathbb{Q}[\mathbb{R}]$$

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Relative cohomology

Motivation :

Euler's constant integral representation

Euler's constant

$$\gamma := \lim_{n\to\infty} \Big(\sum_{k=1}^{n} \frac{1}{k} - \ln n \Big)$$

Euler's constant integral representation

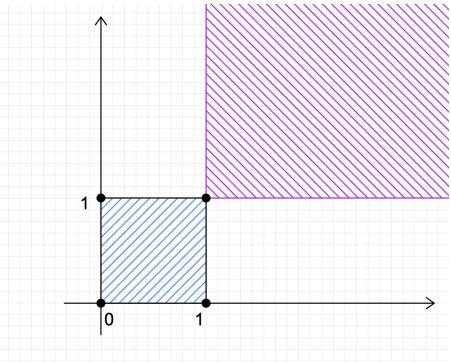
Euler's constant

$$\gamma := \lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \ln n \right) = \int_0^1 \int_0^1 e^{-xy}\, dxdy - \int_1^\infty \int_1^\infty e^{-xy}\, dxdy$$

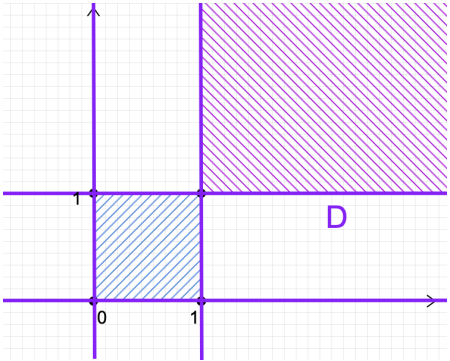
Euler's constant integral representation

Euler's constant

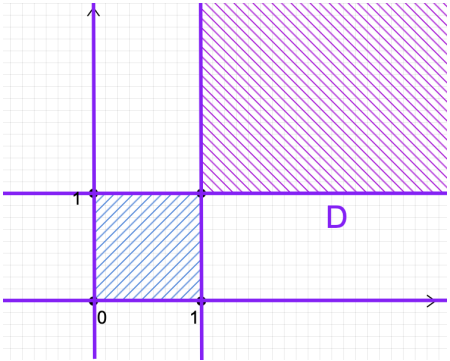
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Geometrical setup for Euler's constant



Geometrical setup for Euler's constant



Setup

$$X = \mathbb{A}^2$$
$$f = xy$$
$$D = \# = \{x(x-1)y(y-1) = 0\}$$

de Rham cohomology relative to a SNCD

Setup

X smooth affine variety/k

$D \subset X$smooth normal crossing divisor

with irreducible components $D_1, \ldots, D_r$

de Rham cohomology relative to a SNCD

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$$D^p := \bigsqcup_{|I|=p}^{r} D_I$$

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Definition/Proposition

$$\Omega^\bullet(X, D) := Tot^\bullet(K^{\bullet,\bullet}) \text{ where } K^{p,q} = \Omega^q(D^p)$$

$$H^n(X, D) = H^n(\Omega^\bullet(X, D))$$

de Rham cohomology relative to a SNCD

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$$H^n(X, D) = H^n(\Omega^\bullet(X, D))$$

$$\rightsquigarrow E^1_{p,q} = H^q_{dR}(\Omega^\bullet(D^p)) \Rightarrow H^{p+q}_{dR}(X, D)$$

de Rham cohomology relative to a SNCD

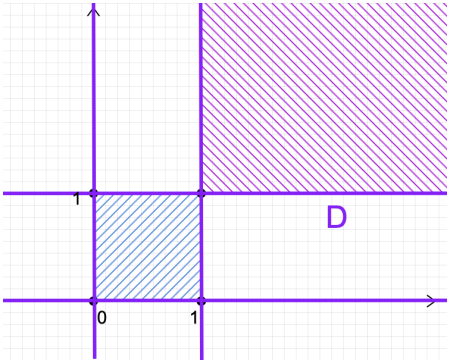
de Rham cohomology relative to a SNCD

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \vdots \\
\uparrow{\scriptstyle d} & & \uparrow & & \uparrow & & \\
\Omega^2(X) & \longrightarrow & \bigoplus_{i=1}^{r} \Omega^2(D_i) & \longrightarrow & \bigoplus_{1\leq i<j\leq r} \Omega^2(D_i \cap D_j) & \longrightarrow & \cdots \\
\uparrow{\scriptstyle d} & & \uparrow & & \uparrow & & \\
\Omega^1(X) & \longrightarrow & \bigoplus_{i=1}^{r} \Omega^1(D_i) & \longrightarrow & \bigoplus_{1\leq i<j\leq r} \Omega^1(D_i \cap D_j) & \longrightarrow & \cdots \\
\uparrow{\scriptstyle d} & & \uparrow & & \uparrow & & \\
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\end{array}$$

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Geometrical setup



Setup

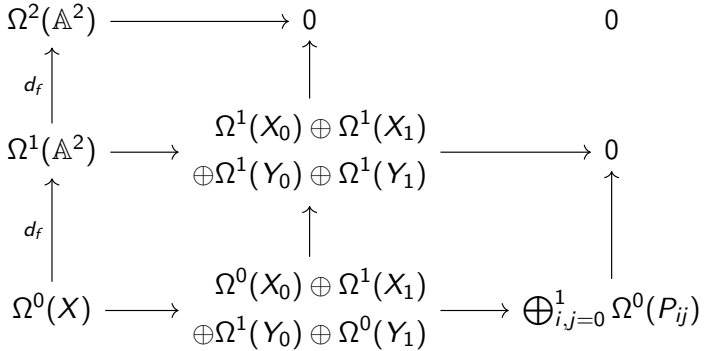
$$X = \mathbb{A}^2$$
$$f = xy$$
$$D = \# = \{x(x-1)y(y-1) = 0\}$$
$$D^{(0)} = X$$
$$D^{(1)} = X_0 \sqcup X_1 \sqcup Y_0 \sqcup Y_1$$
$$D^{(2)} = P_{00} \sqcup P_{01} \sqcup P_{10} \sqcup P_{11}$$

De Rham cohomology $H^n(\mathbb{A}^2, \#, xy)$

Page 0

De Rham cohomology $H^n(\mathbb{A}^2, \#, xy)$

Page 0



De Rham cohomology $H^n(\mathbb{A}^2, \#, xy)$

Page 1

$$H^2_{dR}(\mathbb{A}^2, xy) \longrightarrow 0 \longrightarrow 0$$

$$H^1_{dR}(\mathbb{A}^2, xy) \longrightarrow \begin{matrix} H^1_{dR}(X_0) \oplus H^1_{dR}(X_1, y) \\ \oplus H^1_{dR}(Y_0) \oplus H^1_{dR}(Y_1, x) \end{matrix} \longrightarrow 0$$

$$H^0_{dR}(\mathbb{A}^2, xy) \longrightarrow \begin{matrix} H^0_{dR}(X_0) \oplus H^0_{dR}(X_1, y) \\ \oplus H^0_{dR}(Y_0) \oplus H^0_{dR}(Y_1, x) \end{matrix} \longrightarrow \begin{matrix} H^0_{dR}(P_{00}) \oplus H^0_{dR}(P_{01}) \\ \oplus H^0_{dR}(P_{10}) \oplus H^0_{dR}(P_{11}) \end{matrix}$$

De Rham cohomology $H^n(\mathbb{A}^2, \#, xy)$

Page 1

$$
\begin{array}{ccc}
k\ dx \wedge dy & & \\
0 & 0 & 0 \\
0 & k\delta_{X_0} \oplus k\delta_{Y_0} \longrightarrow & \begin{array}{r} k\delta_{00} \oplus k\delta_{01} \\ \oplus k\delta_{10} \oplus k\delta_{11} \end{array}
\end{array}
$$

De Rham cohomology $H^n(\mathbb{A}^2, \#, xy)$

Page 2

$$k[dx \wedge dy]$$

$$0 \qquad 0$$

$$0 \qquad 0 \qquad k[\delta_{00}] \oplus k[\delta_{11}]$$

De Rham cohomology $H^n(\mathbb{A}^2, \#, xy)$

Lemma

$$H^n_{dR}(\mathbb{A}^2, \#, xy)$$

is concentrated in degree $n = 2$ *of dimension* 3. *A basis is given by triplets*

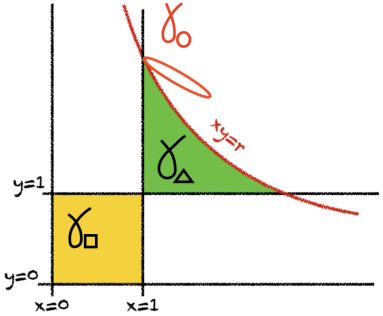
$$(dx \wedge dy, 0, 0)$$
$$(0, 0, \delta_{00}) \qquad \text{constant equal to } 1 \text{ on } P_{00}$$
$$(0, 0, \delta_{11}) \qquad \text{constant equal to } 1 \text{ on } P_{11}$$

and live in $\Omega^2(X) \oplus \Omega^1(D^{(1)}) \oplus \Omega^0(D^{(2)})$.

Rapid decay homology $H^n(\mathbb{A}^2, \#, xy)$

Rapid decay homology $H^n(\mathbb{A}^2, \#, xy)$

$$H_n^{Rd}(\mathbb{A}^2, \#, xy)$$
$$= \lim_R H_n(\mathbb{C}^2, \# \cup \{\Re(xy) \geq R\})$$



Rapid decay homology $H^n(\mathbb{A}^2, \#, xy)$

Lemma

$$H_n^{dR}(\mathbb{A}^2, \#, xy)$$

is concentrated in degree $n = 2$ of dimension 3. A basis is given by triplets

$$\gamma_\circ$$
$$\gamma_\square$$
$$\gamma_\triangle$$

and live in $C_2(X, f) \oplus C_1(D^1, f_{|D^1}) \oplus C_0(D^2)$.

Period matrix : $H^2(\mathbb{A}^2, \#, xy)$

Period matrix : $H^2(\mathbb{A}^2, \#, xy)$

$$\begin{pmatrix} 1 & e^{-1} & \iint_{[0,1]^2} e^{-xy}\,dxdy \\ 0 & e^{-1} & \iint_{[1,+\infty[^2} e^{-xy}\,dxdy \\ 0 & 0 & 2i\pi \end{pmatrix} \begin{matrix} \gamma_\square \\ \gamma_\triangle \\ \gamma_\circ \end{matrix}$$

$$\delta_{00} \quad \delta_{11} \quad dx \wedge dy$$

Period matrix : $H^2(\mathbb{A}^2, \#, xy)$

$$\begin{pmatrix} 1 & 0 & \gamma \\ 0 & e^{-1} & E_1(1) \\ 0 & 0 & 2i\pi \end{pmatrix} \begin{matrix} \gamma_\square - \gamma_\triangle \\ \gamma_\triangle \\ \gamma_\circ \end{matrix}$$

$$\delta_{00} \quad \delta_{11} \quad dx \wedge dy$$

Thank you for your attention ! :)