POINTWISE TO GLOBAL GOOD REDUCTION PROBLEMS

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Contents

1. Introduction

1.1. **Motivation and main results.** Let $\mathcal{O}_F$ denote a complete DVR with fraction field F. Consider $\mathcal{X}$ a smooth, separated scheme of finite type over $\mathcal{O}_F$, let X/F denote its generic fiber. Fix $Y \to X$ a smooth and proper family of varieties. We are concerned with the following problem: assuming (sufficiently many of) the closed fibers Y_x have good reduction, does Y extend to a smooth and proper scheme over $\mathcal{X}$? Does pointwise good reduction imply global good reduction?

This question is asked in the recent work of Cadoret and Tamagawa [CT26, §4.3], where it is answered positively in the case of pointed, smooth, proper curves of genus $g \geq 2$, of stable, proper curves of genus $g \geq 2$, and of abelian schemes if F has absolute ramification index $e \leq p-1$. The two main inputs of their argument are [CT26, Theorem 1] stating that a local system on X extends to $\mathcal{X}$ if and only if it is pointwise unramified for a relevant collection of closed points of X, as well as Serre-Tate-type criteria. It is also discussed in [CT26, §4.3, Example (2)] how *integral canonical models* of Shimura varieties could be leveraged for a strategy when $\mathcal{O}_F$ is the ring of integers of a p-adic field. Indeed, by definition these models satisfy a property akin to the Néron extension property of Néron models, thus providing a Serre-Tate criterion over general bases.

The main goal of this document is to formalize the discussion in [CT26, §4.3, Example (2)]. We reduce the pointwise-to-global good reduction problem to the existence of a separated, Deligne-Mumford stack $\mathscr{S}/\mathcal{O}_F$ equipped with a local system $\mathbb{L}$ such that the pair $(\mathscr{S}, \mathbb{L})$ satisfies some "L-extension property", and show that moduli spaces of curves, as well as integral canonical models of Shimura varieties, can fill this role. Consequently, we (re)establish:

Theorem 1.1: *Assume $p > 2$. Let $F/\mathbb{Q}_p$ be a finite field extension of ramification index e. Assume that $Y \to X$ is either:*

(a) *a smooth, proper family of genus g, n-pointed, geometrically connected curves with $2-2g < n$,*
(b) *($e \leq p-1$) a polarized abelian scheme,*
(c) *($e \leq p-1$) a primitively polarized K3 surface,*
(d) *($e \leq p-1$) a smooth, proper family of cubic fourfolds.*

If $Y \to X$ has pointwise good reduction, then it has global good reduction.

Cases (a) and (b) were already known, and our proof is the same as in [CT26, §4.3, Examples (1), (3)], only reformulated to fit our framework. Building upon [VZ10, Theorem 28] we also show the optimality of their assumption $e \leq p-1$, and relate it to the non-existence of a smooth integral

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canonical model for Siegel modular varieties if $e \geq p$. We have not investigated the possibility of a similar obstruction for orthogonal-type Shimura varieties involved in cases (c) and (d).

1.2. **Outline and strategy of proof.** Section 2 introduces the formalism of local systems and torsors on stacks used in the rest of the document. Of particular importance to us are towers of stacks with finite étale transition maps inducing compatible systems of torsors called *fit systems*: associated with this setup is a *canonical local system*. When the stacks carry specific moduli interpretations, we define a *universal local system* and argue that canonical and universal local systems agree, thus generalizing constructions and statements from [UY13] and [CM20].

We formulate in Section 3 the pointwise-to-global good reduction problem. We then show that even in the usually well-behaved case of abelian schemes the answer is negative: if $F/\mathbb{Q}_p$ has ramification index $e \geq p$, Corollary 3.12 provides a principally polarized counterexample with arbitrary level structure. To obtain it, we recall the work of Vasiu-Zink in [VZ10] and descend their construction to finite type bases over $\mathcal{O}_F$.

We spend some time in Section 4 recalling the main inputs from [CT26] to be used further down. Our assumptions and conventions differ slightly from the reference, so for convenience we state their results, as well as certain proofs, with the terminology of Section 2.

Section 5 is the core of the proof toward Theorem 1.1. For the convenience of the reader, we motivate the technical results with case (c) in mind. With X/F and $\mathcal{X}/\mathcal{O}_F$ as above, let $Y \to X$ be a family of K3 surfaces (with extra structure) having pointwise good reduction, we wish to extend it to a family $\mathcal{Y} \to \mathcal{X}$ of K3 surfaces. Let $\mathscr{M}$ denote the moduli stack of K3 surfaces (with extra structure) over $\mathcal{O}_F$, the family defines a *PGR-morphism* $X \to \mathscr{M}$ (Definition 5.4) which we wish to extend to $\mathcal{X}$. If any PGR-morphism to $\mathscr{M}$ admits an extension over $\mathcal{O}_F$, we say that $\mathscr{M}$ has *PGR-extension* (Definition 5.6).

To argue that $\mathscr{M}$ has PGR-extension, we need to introduce two other extension properties of moduli stacks: *L-extension* and *smooth-extension* (Definition 5.2 and Definition 5.15). The majority of Section 5 is spent establishing the implications below:

(i) If $\mathscr{S}_\infty/\mathcal{O}_F$ is a stack with smooth-extension which can be expressed as the limit of a fit system
$$\mathscr{S}_\infty = \varprojlim_K \mathscr{S}_K,$$
then each $\mathscr{S}_K$ has L-extension (Proposition 5.18). This is where the notion of canonical local system from Section 2 is used. The implication had already been observed in [BST24, §1.3] in the context of Shimura varieties.

(ii) If $\mathscr{S}/\mathcal{O}_F$ has L-extension, then it also has PGR-extension (Proposition 5.8).

(iii) If $\mathscr{S}/\mathcal{O}_F$ has PGR-extension and $\mathcal{U} \to \mathscr{S}$ is étale, then $\mathcal{U}$ has PGR-extension (Theorem 5.9). This fails for L-extension, as shown by the open inclusion of the generic fiber in $\mathscr{S}$.

(iv) If $\mathcal{U} \to \mathscr{M}$ is a finite torsor over $\mathcal{O}_F$ and if $\mathcal{U}$ has PGR-extension, then so does $\mathscr{M}$ (Proposition 5.13).

The inputs allowing (ii) and (iv) are the results recalled in Section 4. Summarizing, the extension properties imply each-other from top-right to bottom-left in the following diagram, using (i) to (iv) in order:

$$\begin{tikzcd}
 & & \mathscr{S}_\infty \arrow[d, "\text{projection}"] & \boxed{\text{smooth-extension}} \\
\boxed{\text{PGR-extension}} & \mathcal{U} \arrow[r, "\text{étale}"] \arrow[d, "\text{finite torsor}"'] & \mathscr{S} = \mathscr{S}_K & \boxed{\text{L-extension}} \\
\boxed{\text{PGR-extension}} & \mathscr{M} & &
\end{tikzcd}$$

In Section 6 we investigate our primary source of stacks with smooth-extension: integral canonical models of Shimura varieties. These are schemes $\mathscr{S}_\infty$ usually defined over valuation rings of number fields. Unfortunately, none of the extension properties above are preserved under base-change in general: for $F/\mathbb{Q}_p$ an arbitrary finite extension, $\mathscr{S}_{\infty,\mathcal{O}_F}$ need not have smooth extension. Using an imput from Vasiu-Zink in [VZ10, Corollary 5], the majority of Section 6 is spent recalling the construction of certain integral canonical models from the work of Kisin and Madapusi Pera in [Kis09] and [MP16] respectively, to check that smooth extension is preserved whenever $F/\mathbb{Q}_p$ has ramification index $e \leq p-1$.

In Section 7 we argue that the relevant moduli spaces $\mathscr{M}$ in Theorem 1.1 have PGR-extension using the models from Section 6 and implications from Section 5. For case (a), it is known from [Sti05] that $\mathscr{M}$ has L-extension, hence PGR-extension. Case (b) follows directly from extension properties of moduli spaces of abelian schemes derived in Section 6; we then relate the counterexample from Section 3 to the possible non-existence of integral canonical models for Siegel Shimura varieties if F is too ramified (Remark 7.4). Cases (c) and (d) are similar, but require (iii) and (iv) additionally: the period morphism defines an étale map from a moduli space $\mathcal{U} = \mathscr{M}_K$ with K-level structure, which is a finite torsor over $\mathscr{M}$, to an integral canonical model $\mathscr{S}_K$ of a Shimura stack of level K. Theorem 1.1 summarizes the contents of Theorem 7.2, Theorem 7.5, Theorem 7.13 and Theorem 7.18.

1.3. **Conventions and notations.** Some conventions for the rest of the document are:

- The ring $\mathcal{O}_F$ is a complete DVR with fraction field F. The integral closure of $\mathcal{O}_F$ in some fixed algebraic closure of F is denoted $\overline{\mathcal{O}_F}$.
- A variety over a field is a reduced separated scheme of finite type over the field.
- A *smooth model* of a variety X/F is a smooth algebraic space $\mathcal{X}/\mathcal{O}_F$ with generic fiber X/F.
- We fix $p > 0$ a prime number. We denote $\widehat{\mathbb{Z}}^p = \prod_{\ell \neq p} \mathbb{Z}_\ell$ the prime-to-p profinite integers, and $\mathbb{A}_f^p$ the prime-to-p adèles.

2. Canonical local systems on schemes and stacks

This section mostly serves as a toolbox of definitions: we advise the reader to skip it at first read, and come back to it whenever necessary.

We use the language of fibered categories and (algebraic) stacks over a site, and in particular the pro-étale site. The meaning of these terms varies accross references; for the convenience of the reader we spend the first paragraph expliciting our conventions.

The next two paragraphs define what is meant in the rest of the document by local systems and torsors on schemes and stacks. In particular, to certain projective systems of schemes/stacks called *fit systems* are attached *canonical* local systems which play a role in later applications; this construction is inspired from [UY13, §2] and [CM20, §3.1]. We show in Proposition 2.20 that there is a universal fit system of classifying stacks inducing all other fit systems.

When the stacks of a fit system additionally carry a certain moduli interpretation, one can also define a *universal* local system: this is formalized in the third paragraph, where it is shown that it is canonically isomorphic to the canonical local system. This identification recovers [UY13, paragraph following Remark 2.8] (see also [CM20, Proposition 5.2]) by applying it to points on Siegel modular varieties, as well as [CM20, end of the proof of Theorem 6.6] for points on moduli spaces of K3 surfaces. This result is not essential for the rest of the document.

We fix S a base scheme and set $\mathscr{C} := (\mathrm{Sch}/S)_{\text{proét}}$ the big pro-étale site over S; in the rest of the document we will often implicitely be the case that $S = \operatorname{Spec} \mathcal{O}_F$, $S = \operatorname{Spec} \mathbb{Z}_{(p)}$ or closed points thereof. Unless stated otherwise, all fibered categories, stacks and morphisms between them are over $\mathscr{C}$. For all S-scheme T, we write $\mathscr{C}_T = (\mathrm{Sch}/T)_{\text{proét}}$ the restriction of $\mathscr{C}$ to T. We fix R a topological ring and V a free R-module of finite rank, we let $\mathrm{GL}(V)$ denote the topological group of R-linear automorphisms of V. We also fix Q a topological group.

2.1. **(Algebraic) stacks on the big pro-étale site.** In the rest of this document, we will need the notion of (algebraic and Deligne-Mumford) stacks over the big pro-étale site over S. In this paragraph we make our conventions precise.

The pro-étale site over S has three properties of interest to us, making it a better candidate than étale, fppf or fpqc topologies. First, sheafification and stackification exist (they are harder to define, and more choice-dependent, in the fpqc topology; see the paragraph following [Sta26, Tag 022C]); we need them to define induction functors in Definition 2.8 and Remark 2.9. Second, projective limits of finite étale covers are still covers, so the notion of, for instance, $\mathbb{Z}_\ell$-local system falls under the same formalism as that of $\mathbb{Z}/\ell^n\mathbb{Z}$ local systems for the étale topology, without the need for a special treatment via projective systems and lisse sheaves; this is not permitted by the étale/fppf topologies. Lastly, we briefly use in Lemma 2.23 that the pro-étale topos is replete.

2.1.1. *"The" pro-étale site over S.* The reader willing to ignore set-theoretic issues is encouraged to skip this paragraph and proceed with the intuitive understanding that the big pro-étale site over S is the category of all schemes on S with coverings refining the étale site, in that projective limits of finite étale covers are again covers.

We define a site as in [Sta26, Tag 00VG]. In particular, we require that objects, morphisms between objects and the collection of all coverings within the site are all sets. This is to ensure the existence of sheafification of presheaves (resp. stackification of prestacks) over a site. The big category Sch/S does not satisfy these conditions for any topology: the collection of objects needs to be reduced to a set. When shrinking, one has to ensure to keep all S-schemes which are essential in context[1], as well as making sure the resulting set is still stable under useful operations. The question thus becomes: is it possible to find a *set* $\mathcal{S}$ within the collection of all S-schemes such that

(i) the set $\mathcal{S}$ contains a prescribed set $\mathcal{S}_0$ of S-schemes, and
(ii) for any $T \in \mathcal{S}$ and any scheme U obtained from T by some "reasonable scheme-theoretic construction", there exists $U' \in \mathcal{S}$ such that $U' \cong U$ as S-schemes.

The answer is positive, and is made precise in [Sta26, Tag 000H]. For convenience, we still denote this shrunk down category as Sch/S. To equip it with a topology τ, namely fppf or pro-étale, the collection of all coverings needs also to be reduced with similar care; this is done in [Sta26, Tag 000W]. It is shown in [Sta26, Tag 00VY] that the resulting category of sheaves on Sch/S is independent of the set of τ-coverings chosen in this manner.

The upshot is that there are many possible "big fppf/pro-étale" sites over S, but for all intents and purposes in this document any choice is virtually equivalent.

Definition 2.1: For the remainder of these notes, we fix $(\mathrm{Sch}/S)_{\mathrm{fppf}}$ and $\mathscr{C} := (\mathrm{Sch}/S)_{\mathrm{pro\acute{e}t}}$ big fppf and pro-étale sites over S, as in [Sta26, Tag 021R] and [Sta26, Tag 098G] respectively. We can and will arrange so that the underlying categories are the same, and so that our choice of fppf coverings is a subset of our choice of pro-étale coverings. We call them *the* big sites over S.

2.1.2. *Algebraic stacks on a site.* We use the language of fibered categories and stacks over a site as in [Vis], namely, a stack over a site is a category fibered in groupoids for which every descent datum is effective. In particular, we can make sense of stacks over the pro-étale site $\mathscr{C}$ over S.

Algebraic stacks, on the other hand, are only defined over $(\mathrm{Sch}/S)_{\mathrm{fppf}}$ (or even $(\mathrm{Sch}/S)_{\mathrm{\acute{e}t}}$) in most references. Our conventions are inspired from [LMB18, 4].

Definition 2.2: A stack over $(\mathrm{Sch}/S)_{\mathrm{fppf}}$ (resp. $\mathscr{C} = (\mathrm{Sch}/S)_{\mathrm{pro\acute{e}t}}$) is *algebraic* if its diagonal is representable, quasi-compact and separated, and if there exists a surjective smooth morphism (called an *atlas*) from an algebraic space. It is *Deligne-Mumford* if the atlas can be chosen étale. A Deligne-Mumford stack is *separated* if its diagonal is proper (equiv. finite).

We will work with standard fppf algebraic or Deligne-Mumford stacks (classifying stacks, moduli stacks of curves, abelian varieties,...), and view them as pro-étale stacks. When we do so, it is allowed by the following fact, without further mention.

Fact 2.3 ([Sta26],Tag 0GRH)**:** *An algebraic (resp. Deligne-Mumford) stack over $(\mathrm{Sch}/S)_{fppf}$ with ind-quasi-affine diagonal has all fpqc descent data effective. In particular, it is automatically an algebraic (resp. Deligne-Mumford) stack over $\mathscr{C}$.*

In this document, stacks are not required to be algebraic until Proposition 3.13.

2.2. **Local systems and torsors on schemes.** We fix an S-scheme T.

Definition 2.4: Let A be a topological space. Let $\mathcal{F}_A$ denote the presheaf on $\mathscr{C}_T$ which associates to any $U \in \mathscr{C}_T$ the set of continuous functions $U \to A$. This is a sheaf by [BS13, Lemma 4.2.12].

If A is discrete and T is qcqs then $\mathcal{F}_A$ is the usual constant sheaf $\underline{A}$. The sheaf $\mathcal{F}_R$ is a sheaf of rings, and $\mathcal{F}_V$ is an $\mathcal{F}_R$-module.

[1] For instance, we will be studying applications to certain Shimura varieties over $S = \operatorname{Spec}\mathbb{Q}$ and $S = \operatorname{Spec}\mathbb{Z}_{(p)}$, so whichever set we replace Sch/S with, it should contain those varieties.

Definition 2.5: An $\mathcal{F}_R$-module locally isomorphic to the $\mathcal{F}_R$-module $\mathcal{F}_V$ is called an *R-local system with fiber V*. The groupoid of R-local systems with fiber V over $\mathscr{C}_T$ is denoted $\mathbf{Loc}_V(T)$. The functor

$$\mathbf{Loc}_V : T \in \mathscr{C} \longmapsto \mathbf{Loc}_V(T)$$

defines a stack over $\mathscr{C}$.

The sheaf $\mathcal{F}_Q$ is a sheaf of groups.

Definition 2.6: Let $\mathcal{G}$ be a sheaf of groups on $\mathscr{C}$. A *$\mathcal{G}$-sheaf* $\mathcal{P}$ on $\mathscr{C}_T$ is a sheaf of sets together with a morphism $\mathcal{G}_T \times \mathcal{P} \to \mathcal{P}$ inducing a group action for every object of $\mathscr{C}_T$. A *$\mathcal{G}$-torsor* on $\mathscr{C}_T$ is a $\mathcal{G}$-sheaf locally isomorphic to the $\mathcal{G}$-sheaf $\mathcal{G}$. The category of $\mathcal{G}$-sheaves (resp. the groupoid of $\mathcal{G}$-torsors) over $\mathscr{C}_T$ is denoted $\mathcal{G}\mathbf{Sh}(T)$ (resp. $\mathbf{B}\mathcal{G}(T)$). As in Definition 2.5 this defines a fibered category (resp. a stack) $\mathcal{G}\mathbf{Sh}$ (resp. $\mathbf{B}\mathcal{G}$) on $\mathscr{C}$. When $\mathcal{G} = \mathcal{F}_Q$, we write $Q\mathbf{Sh}$ (resp. $\mathbf{B}Q$) and call its objects Q-sheaves (resp. Q-torsors).

Fact 2.7 ([Gir20], III, Corollaire 2.2.6)**:** *If $Q = \mathrm{GL}(V)$ then there is an equivalence*

$$\mathrm{Triv} : \mathbf{Loc}_V \to \mathbf{B}\mathrm{GL}(V)$$

mapping a local system $\mathbb{L}$ over T to its sheaf of trivializations $\underline{\mathrm{Isom}}(\mathbb{L}, \mathcal{F}_V)$ over T.

Definition 2.8: Let $\phi : \mathcal{H} \to \mathcal{G}$ be a morphism of sheaves of groups over $\mathscr{C}$. The forgetful functor from $\mathcal{G}\mathbf{Sh}$ to $\mathcal{H}\mathbf{Sh}$ admits a left-adjoint

$$\mathrm{Ind}_{\mathcal{H}}^{\mathcal{G}} : \mathcal{H}\mathbf{Sh} \to \mathcal{G}\mathbf{Sh}.$$

mapping any $\mathcal{H}$-sheaf $\mathcal{P}$ over T to the $\mathcal{G}$-sheaf $\mathrm{Ind}_{\mathcal{H}}^{\mathcal{G}}(\mathcal{P})$ over T defined as the quotient of $\mathcal{G} \times \mathcal{P}$ under the $\mathcal{H}$-action:

$$\begin{aligned} \mathcal{H} \times (\mathcal{G} \times \mathcal{P}) &\to \mathcal{G} \times \mathcal{P} \\ (h, g, p) &\mapsto (g\phi(h)^{-1}, h \cdot p). \end{aligned}$$

This functor restricts to torsors and defines $\mathrm{Ind}_{\mathcal{H}}^{\mathcal{G}} : \mathbf{B}\mathcal{H} \to \mathbf{B}\mathcal{G}$. If $\mathcal{H} = \mathcal{F}_Q$, $\mathcal{G} = \mathcal{F}_{Q'}$ and ϕ comes from a continuous group homomorphism $Q \to Q'$, we denote it $\mathrm{Ind}_Q^{Q'} : \mathbf{B}Q \to \mathbf{B}Q'$.

Remark 2.9: There is an alternative description for $\mathrm{Ind}_{\mathcal{H}}^{\mathcal{G}}$. Recall that $\mathbf{B}\mathcal{G}$ is the stack associated to the prestack $\mathbf{B}\mathcal{G}^{\mathrm{pre}}$ such that for all T, $\mathbf{B}\mathcal{G}^{\mathrm{pre}}(T)$ has a single object $\{*\}$ and $\mathrm{Aut}(\{*\}) = \mathcal{G}(T)$. The morphism $\mathcal{H} \to \mathcal{G}$ induces a morphism of prestacks

$$\mathbf{B}\mathcal{H}^{\mathrm{pre}} \longrightarrow \mathbf{B}\mathcal{G}^{\mathrm{pre}},$$

the associated functor between stacks $\mathbf{B}\mathcal{H} \to \mathbf{B}\mathcal{G}$ is the induction $\mathrm{Ind}_{\mathcal{H}}^{\mathcal{G}}$. To check that both definitions agree, verify the result for the trivial torsor, then argue locally for an arbitrary torsor.

Using Fact 2.7 and Definition 2.8, any continuous representation of Q on V defines a functor

$$\mathbf{B}Q \to \mathbf{Loc}_V$$

sending P to the R-local system with fiber V associated with $\mathrm{Ind}_Q^{\mathrm{GL}(V)} P$.

We will be interested in torsors over $\mathscr{C}$ of geometric origin.

Definition 2.10: Let $T' \to T$ be an algebraic space over T. Fix a homomorphism $Q \to \mathrm{Aut}_T(T')$. We say that $T' \to T$ is a *Q-torsor* over T if the functor of points $h_{T'}$ defined by

$$(U \to T) \in \mathscr{C} \mapsto h_{T'}(U) = \mathrm{Hom}_T(U, T'),$$

is a Q-torsor on $\mathscr{C}_T$ as in Definition 2.6 for the induced action of $\mathcal{F}_Q$. If Q is finite then $T' \to T$ is finite étale and T' is a scheme.

If K_0 is a topological group, let $\mathcal{C}(K_0)$ denote the set of all open normal subgroups of K_0.

Definition 2.11: Let K_0 be a profinite group. Let $(T_K)_{K \in \mathcal{C}(K_0)}$ be a projective system of schemes. We say that it is *fit for K_0*, or a *K_0-fit system*, if it satisfies the following assumptions:

(i) for each $K \in \mathcal{C}(K_0)$, the morphism $T_K \to T_{K_0}$ is a finite K_0/K-torsor over T_{K_0} as in Definition 2.10, and
(ii) for every $K' \subseteq K$ in $\mathcal{C}(K_0)$, the morphism $T_{K'} \to T_K$ is K_0-equivariant for the K_0-action induced from (i).

Then for every $K \in \mathcal{C}(K_0)$ the map $T_K \to T_{K_0}$ is a finite étale cover, thus it makes sense to define the projective limit

$$T_\infty := \varprojlim_{K \in \mathcal{C}(K_0)} T_K$$

which is a pro-étale cover of T_{K_0}, and a K_0-torsor over T_{K_0} as in Definition 2.10 with $h_{T_\infty} = \varprojlim_K h_{T_K}$ in $K_0\mathbf{Sh}(T_{K_0})$.

The groupoid of K_0-fit systems over a fixed scheme T_{K_0} is denoted $K_0\mathbf{FS}(T_{K_0})$, and $K_0\mathbf{FS}$ is a stack over $\mathscr{C}$. We will see in Proposition 2.20 that it is equivalent to $\mathbf{B}K_0$ via $(T_K)_K \mapsto T_\infty$.

Definition 2.12: Let K_0 be a profinite group with a fixed continuous embedding $K_0 \hookrightarrow \mathrm{GL}(V)$. Let $(T_K)_{K \in \mathcal{C}(K_0)}$ be schemes fit for K_0, let h_{T_∞} be the K_0-torsor from Definition 2.11. The corresponding *canonical local system* $\mathbb{L}^{\mathrm{can}} \in \mathbf{Loc}_V(T_{K_0})$ is the R-local system associated with $\mathrm{Ind}_{K_0}^{\mathrm{GL}(V)}(h_{T_\infty})$.

Remark 2.13: Let us give a more down-to-earth description of the canonical local system in case T_{K_0} is a connected scheme with a K_0-fit system $(T_K)_K$. Fix $(\bar{t}_K)_K$ a compatible system of geometric points of $(T_K)_K$. For each $K \in \mathcal{C}(K_0)$, $T_K \to T_{K_0}$ is a K_0/K-torsor on T_{K_0} in the usual sense for the étale topology on S. Having fixed geometric points, this torsor is uniquely described by a continuous morphism

$$\pi_1(T_{K_0}, \bar{t}_{K_0}) \to K_0/K.$$

The compatibility conditions allow to take the projective limit over K so as to find a continuous morphism

$$\pi_1(T_{K_0}, \bar{t}_{K_0}) \to K_0.$$

The composition with the embeddings $K_0 \hookrightarrow \mathrm{GL}(V)$ yields the monodromy action on the fibers of $\mathbb{L}^{\mathrm{can}}$. If T_{K_0} is locally noetherian but no longer connected, we can argue component-wise.

Example 2.14: Let (G, Ω) be a Shimura datum with reflex field E, set $S = \operatorname{Spec} E$. Let K_0 be a neat compact open subgroup of $G(\mathbb{A}_f)$. If $K \in \mathcal{C}(K_0)$, let $T_K = \mathrm{Sh}_K(G, \Omega)$ denote the associated Shimura variety of level K. Then $(\mathrm{Sh}_K(G, \Omega))_K$ is fit for K_0, so $\mathrm{Sh}_{K_0}(G, \Omega)$ carries a canonical local system. The neatness assumption is equivalent to condition (i) of Definition 2.11. We can remove this assumption by replacing schemes with stacks.

2.3. **Local systems and torsors on stacks.** Let $\mathscr{X}$ be a stack on $\mathscr{C}$. We extend the notions of the previous paragraph to $\mathscr{X}$.

Definition 2.15: We denote by $Q\mathbf{Sh}(\mathscr{X})$ (resp. $\mathbf{Loc}_V(\mathscr{X})$, $\mathbf{B}Q(\mathscr{X})$, $K_0\mathbf{FS}(\mathscr{X})$) the category of morphisms of fibered categories $\mathscr{X} \to Q\mathbf{Sh}$ (resp. $\mathscr{X} \to \mathbf{Loc}_V$, $\mathscr{X} \to \mathbf{B}Q$, $\mathscr{X} \to K_0\mathbf{FS}$), the objects of which are called *Q-sheaves on $\mathscr{X}$* (resp. *R-local systems with fiber V on $\mathscr{X}$*, *Q-torsors on $\mathscr{X}$*, *K_0-fit systems over $\mathscr{X}$*).

Explicitely, an object of $Q\mathbf{Sh}(\mathscr{X})$ is the data, for each $\xi : T \to \mathscr{X}$, of a Q-sheaf $F_\xi \in Q\mathbf{Sh}(T)$, collectively satisfying compatibility conditions as ξ and T vary. Identical descriptions hold for $\mathbf{Loc}_V(\mathscr{X})$ and $\mathbf{B}Q(\mathscr{X})$.

The induction functor and the equivalence of Fact 2.7 automatically extend to Q-sheaves, local systems and torsors on $\mathscr{X}$. Consequently, any continuous representation of Q on V defines a functor

$$\mathbf{B}Q(\mathscr{X}) \to \mathbf{Loc}_V(\mathscr{X})$$

sending a Q-torsor P on $\mathscr{X}$ to the R-local system with fiber V on $\mathscr{X}$ associated with $\mathrm{Ind}_Q^{\mathrm{GL}(V)} P$.

We will be interested in a geometric description of objects of $\mathbf{B}Q(\mathscr{X})$ and $K_0\mathbf{FS}(\mathscr{X})$.

Definition 2.16: Let $f : \mathscr{X}' \to \mathscr{X}$ be a representable morphism of stacks over $\mathscr{C}$. Assume $\mathscr{X}'$ is equipped with a pseudo-Q-action[2] preserving f[3]. For any scheme $T \to \mathscr{X}$, the fiber product $T' \coloneqq T \times_{\mathscr{X}} \mathscr{X}'$ is an algebraic space which inherits a Q-action $Q \to \mathrm{Aut}_T(T')$. We say that f is a *Q-torsor* if for all such T the morphism $T' \to T$ is a Q-torsor as in Definition 2.10. This defines a Q-torsor $h_{\mathscr{X}'}$ on $\mathscr{X}$ as in Definition 2.15.

Definition 2.17: Let K_0 be a profinite group. Let $(\mathscr{X}_K)_{K \in \mathcal{C}(K_0)}$ be a projective system of stacks. We say that it is *fit for* K_0, or a *K_0-fit system*, if it satisfies the following assumptions:

(i) for each $K \in \mathcal{C}(K_0)$, the morphism $\mathscr{X}_K \to \mathscr{X}_{K_0}$ is a finite K_0/K-torsor as in Definition 2.16, and

(ii) for every $K' \subseteq K$ in $\mathcal{C}(K_0)$, the morphism $\mathscr{X}_{K'} \to \mathscr{X}_K$ is pseudo-K_0-equivariant[4] for the K_0-action induced from (i).

Then for every $K \in \mathcal{C}(K_0)$ the map $\mathscr{X}_K \to \mathscr{X}_{K_0}$ is finite étale. One can define the projective limit

$$\mathscr{X}_\infty \coloneqq \varprojlim_{K \in \mathcal{C}(K_0)} \mathscr{X}_K$$

which is pro-étale over $\mathscr{X}_{K_0}$, and is a K_0-torsor as in Definition 2.16 with $h_{\mathscr{X}_\infty} = \varprojlim_K h_{\mathscr{X}_K}$ in $K_0\mathbf{Sh}(\mathscr{X}_{K_0})$.

We show that both notions of K_0-fit systems on a fixed stack $\mathscr{X}_{K_0}$ coincide. Provided a projective system $(\mathscr{X}_K)_{K \in \mathcal{C}(K_0)}$ as in Definition 2.17 and an S-scheme T_{K_0}, we define a functor $\mathscr{X}_{K_0} \to K_0\mathbf{FS}$ by sending a T-object $T \to \mathscr{X}$ to the K_0-fit system $(T \times_{\mathscr{X}} \mathscr{X}_K)_{K \in \mathcal{C}(K_0)}$. Conversely, provided a functor $\mathscr{X}_{K_0} \to K_0\mathbf{FS}$, for any $K \in \mathcal{C}(K_0)$ we define $\mathscr{X}_K(T_{K_0})$ as follows: for any S-scheme T_{K_0}, we consider the groupoid

$$\mathscr{X}_K(T_{K_0}) = \{(\xi, T_K) \mid \xi \in \mathscr{X}_{K_0}(T_{K_0}) \text{ mapping to } (T_{K'})_{K' \in \mathcal{C}(K_0)} \in K_0\mathbf{FS}(T_{K_0}) \text{ via } \mathscr{X}_{K_0} \to K_0\mathbf{FS}\}.$$

One then checks that $\mathscr{X}_K$ is a stack, and that $(\mathscr{X}_K)_{K \in \mathcal{C}(K_0)}$ is a K_0-fit system on $\mathscr{X}_{K_0}$ as in Definition 2.17. The constructions are quasi-inverses of each-other.

Definition 2.18: Let K_0 be a profinite group with a fixed continuous embedding $K_0 \hookrightarrow \mathrm{GL}(V)$. Let $(\mathscr{X}_K)_{K \in \mathcal{C}(K_0)}$ be stacks fit for K_0, let $h_{\mathscr{X}_\infty}$ be the K_0-torsor defined in Definition 2.17. The corresponding *canonical local system* $\mathbb{L}^{\mathrm{can}} \in \mathbf{Loc}_V(\mathscr{X}_{K_0})$ is the R-local system associated to $\mathrm{Ind}_{K_0}^{\mathrm{GL}(V)}(h_{\mathscr{X}_\infty})$.

If $\mathscr{X}_{K_0}$ is a scheme, then so are all the $\mathscr{X}_K$, and Definitions 2.12 and 2.18 coincide.

Example 2.19: Let (G, Ω) be a Shimura datum with reflex field E. Let K_0 be a neat compact open subgroup of $G(\mathbb{A}_f)$. Let $\mathrm{Sh}_\infty(G, \Omega) = \varprojlim_{K \in \mathcal{C}(K_0)} \mathrm{Sh}_K(G, \Omega)$ on which $G(\mathbb{A}_f)$ acts. For K' *any* compact open subgroup of $G(\mathbb{A}_f)$, define $\mathrm{Sh}_{K'}[G, \Omega]$ as the quotient stack $[\mathrm{Sh}_\infty(G, \Omega)/K']$ (if K' is neat this agrees with the earlier definition of $\mathrm{Sh}_{K'}(G, \Omega)$). If we fix K'_0 *any* compact open subgroup of $G(\mathbb{A}_f)$ then $(\mathrm{Sh}_{K'}[G, \Omega])_{K' \in \mathcal{C}(K'_0)}$ is a K'_0-fit system of stacks; in particular $\mathrm{Sh}_{K'_0}[G, \Omega]$ carries a canonical local system.

2.4. The universal K_0-fit system.

Induction between classifying stacks provide a universal K_0-fit system.

Proposition 2.20: *Let K_0 be a second countable profinite group.*

(i) *If $K \in \mathcal{C}(K_0)$, the induction functor defines a finite K_0/K-torsor $\mathbf{B}K \to \mathbf{B}K_0$ on $\mathbf{B}K_0$ as in Definition 2.16, so that $(\mathbf{B}K)_{K \in \mathcal{C}(K_0)}$ is fit for K_0. The associated K_0-torsor recovers the universal torsor on $\mathbf{B}K_0$ defined by the identity $\mathbf{B}K_0 \to \mathbf{B}K_0$.*

[2] Formally, a pseudo-Q-action is a 2-morphism $\mathbf{B}Q \to \mathbf{Cat}$ with essential image $\mathscr{X}'$. Explicitely, this is the data for every $q \in Q$ of an equivalence $a_q : \mathscr{X}' \xrightarrow{\sim} \mathscr{X}'$ such that there are isomorphisms $a_q \circ a_{q'} \simeq a_{qq'}$ satisfying the associativity condition.

[3] Formally, for any $q \in Q$ inducing an equivalence $a_q : \mathscr{X}' \to \mathscr{X}'$, there is an isomorphism $f \circ a_q \simeq f$ satisfying omitted compatibility conditions.

[4] This is the categorical version of an equivariant morphism, sending the pseudo-action on $\mathscr{X}_{K'}$ to the pseudo-action on $\mathscr{X}_K$ up to isomorphisms satisfying omitted compatibility conditions.

(ii) *There is an equivalence of stacks $K_0\mathbf{FS} \longleftrightarrow \mathbf{B}K_0$ given by the functors:*

$$(T_K)_K \longmapsto T_\infty$$
$$(T_{K_0} \times_{\mathbf{B}K_0} \mathbf{B}K)_K \longleftarrow [T_{K_0} \longrightarrow \mathbf{B}K_0].$$

Remark 2.21: Proposition 2.20 (ii) immediately extends to stacks: any fit system $(\mathscr{X}_K)_K$ on $\mathscr{X}_{K_0}$ is obtained by pulling back the universal system $(\mathbf{B}K)_K$ along the morphism $\mathscr{X}_{K_0} \to \mathbf{B}K_0$ defining $\mathscr{X}_\infty \to \mathscr{X}_{K_0}$. For this reason we call $(\mathbf{B}K)_K$ the *universal K_0-fit system.*

We prove Proposition 2.20. The proof of (i) relies on the following lemma.

Lemma 2.22: *Let $\mathcal{H} \to \mathcal{G}$ be a morphism of sheaves of groups on $\mathscr{C}$. Then there is a 2-cartesian square*

$$\begin{tikzcd} \mathbf{B}\mathcal{H} \arrow[r] \arrow[d] & \mathbf{B}\mathcal{G} \arrow[d] \\ \mathrm{pt} \arrow[r] & \mathbf{B}(\mathcal{G}/\mathcal{H}) \end{tikzcd}$$

where all arrows are induction functors. Here pt *denotes $\mathscr{C}$ viewed as a stack over itself, or as the classifying stack of the trivial group.*

Proof. As stackification commutes with 2-fiber product (see [Sta26, Tag 04Y1]), it suffices to show the result at the level of prestacks. See Remark 2.9 for the definitions of the prestacks $\mathbf{B}\mathcal{H}^{\mathrm{pre}}$, $\mathbf{B}\mathcal{G}^{\mathrm{pre}}$ and $\mathbf{B}(\mathcal{G}/\mathcal{H})^{\mathrm{pre}}$, and of the induction functor between them. Let T be an S-scheme, a T-object of $\mathrm{pt} \times_{\mathbf{B}(\mathcal{G}/\mathcal{H})^{\mathrm{pre}}} \mathbf{B}\mathcal{G}^{\mathrm{pre}}$ is just the datum of an element $q \in (\mathcal{G}/\mathcal{H})(T)$, and one checks that two objects defined by q and q' are isomorphic if and only if $q'q^{-1}$ lies in the image of $\mathcal{G}(T)$. As $\mathcal{G} \to \mathcal{G}/\mathcal{H}$ is surjective, we find that any two objects of the fiber product are locally isomorphic, and so the same holds true of the stackified fiber product. Since this stack always admits a T-point induced by the neutral element of $(\mathcal{G}/\mathcal{H})(T)$, it is the classifying stack of the automorphism group sheaf of this element. This is easily computed to be $\mathcal{H}$. □

Let K_0 be profinite and second-countable. We apply Lemma 2.22 to $\mathcal{G} = \mathcal{F}_{K_0}$ and $\mathcal{H} = \mathcal{F}_K$. We need two more lemmas.

Lemma 2.23: *For K_0 profinite and second countable, and for any $K \in \mathcal{C}(K_0)$, there is a natural isomorphism $\mathcal{F}_{K_0}/\mathcal{F}_K \simeq \mathcal{F}_{K_0/K}$.*

Proof. As K_0 is second countable, we can find a countable basis of neighborhoods $(H_n)_{n\geq 0}$ of the neutral element, consisting of elements of $\mathcal{C}(K_0)$ such that $H_{n+1} \subseteq H_n$. Consider the surjections

$$\mathcal{F}_{K_0/H_n} \longrightarrow \mathcal{F}_{K_0/H_n}/\mathcal{F}_{K/(K\cap H_n)}.$$

Note that since K_0 is profinite, $\mathcal{F}_{K_0} = \varprojlim_n \mathcal{F}_{K/H_n}$. As the topos of sheaves on $\mathscr{C} = (Sch/S)_{\text{proét}}$ is replete (see [BS13, Lemma 2.4.9, Definition 3.1.1, Proposition 3.2.3]), it follows from [BS13, Lemma 3.1.8] that the morphism

$$\mathcal{F}_{K_0} \longrightarrow \varprojlim_n (\mathcal{F}_{K_0/H_n}/\mathcal{F}_{K/(K\cap H_n)})$$

is surjective. Its kernel is easily seen to be $\varprojlim_n \mathcal{F}_{K/(K\cap H_n)} = \mathcal{F}_K$. Moreover, there is a natural isomorphism

$$\mathcal{F}_{K_0/H_n}/\mathcal{F}_{K/(K\cap H_n)} \simeq \mathcal{F}_{K_0/KH_n}.$$

Le left-hand side equals $\mathcal{F}_{K_0/K}$ for n large enough. This provides the sought-after isomorphism. □

Lemma 2.24: *Let $\mathcal{G}$ be a sheaf of groups over $\mathscr{C}$, fix T an S-scheme, and $\mathcal{P}$ a $\mathcal{G}$-torsor over T. Let $T \to \mathbf{B}\mathcal{G}$ be the morphism defining $\mathcal{P}$. Then there is a 2-cartesian diagram*

$$\begin{tikzcd} \tilde{\mathcal{P}} \arrow[r] \arrow[d] & T \arrow[d] \\ \mathrm{pt} \arrow[r] & \mathbf{B}\mathcal{G}. \end{tikzcd}$$

Here $\tilde{\mathcal{P}}$ is the sheaf on $\mathscr{C}$ obtained as the extension from $\mathscr{C}_T$ to $\mathscr{C}$ of $\mathcal{P}$ by the empty set (see [Sta26, Tag 00Y0, (5)]*).*

If $\mathcal{G} = \mathcal{F}_Q$ for a profinite group Q, then both $\mathcal{P}$ and $\tilde{\mathcal{P}}$ are represented by the same scheme (viewed as a T-scheme and as an S-scheme respectively). In particular, $\mathrm{pt} \to \mathbf{B}Q$ is a Q-torsor which is the universal family over $\mathbf{B}Q$.

Proof. A direct computation shows that the 2-fiber product is the sheaf of sets described as:

$$(U \to S) \longmapsto \coprod_{u:U\to T} \mathcal{P}(u : U \to T).$$

From [Sta26, Tag 03CD] this coincides with $\tilde{\mathcal{P}}$. The second statement follows from [Sta26, Tag 03HU] along with the fact that (pro)finite torsors are representable by (pro)finite étale covers. $\square$

Remark 2.25: Lemma 2.24 immediately extends to stacks: if Q is profinite, any Q-torsor $\mathscr{X}' \to \mathscr{X}$ fits in the 2-cartesian square

$$\begin{tikzcd} \mathcal{X}' \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\ \mathrm{pt} \arrow[r] & \mathbf{B}Q. \end{tikzcd}$$

Back to the proof. From Lemma 2.22 and Lemma 2.23 we find the 2-cartesian diagram

$$\begin{tikzcd} \mathbf{B}K \arrow[r] \arrow[d] & \mathbf{B}K_0 \arrow[d] \\ \mathrm{pt} \arrow[r] & \mathbf{B}K_0/K. \end{tikzcd}$$

Lemma 2.24 ensures that the bottom arrow, and hence also the top arrow, are K_0/K-torsors. It is easy to check that the associated K_0-torsor

$$\mathbf{B}K_0 \longrightarrow \varprojlim_K \mathbf{B}K_0/K = \mathbf{B}K_0$$

is (isomorphic to) the identity functor. This proves (i).

To prove (ii), notice that for a fit system $(T_K)_K$ with K_0-torsor T_∞ over T_{K_0} corresponding to $T_{K_0} \to \mathbf{B}K_0$, Lemma 2.22 and Lemma 2.24 induce a 2-commutative diagram of 2-cartesian squares

$$\begin{tikzcd} T_\infty \arrow[r] \arrow[d] & \mathrm{pt} \arrow[d] & \\ T_K \arrow[r] \arrow[d] & \mathbf{B}K \arrow[r] \arrow[d] & \mathrm{pt} \arrow[d] \\ T_{K_0} \arrow[r] & \mathbf{B}K_0 \arrow[r] & \mathbf{B}K_0/K. \end{tikzcd}$$

It follows that there is an isomorphism $T_K \simeq T_{K_0} \times_{\mathbf{B}K_0} \mathbf{B}K$. Conversely, provided an S-scheme T_{K_0} and a K_0-torsor $T_{K_0} \to \mathbf{B}K_0$, let $T_\infty := T_{K_0} \times_{\mathbf{B}K_0} \mathrm{pt}$ be the T_{K_0}-scheme representing it. For any $K \in \mathcal{C}(K_0)$ we set

$$T_K := T_{K_0} \times_{\mathbf{B}K_0} \mathbf{B}K$$

so that the diagram above holds again. This diagram shows that T_∞ is (isomorphic to) the K_0-torsor associated with $(T_K)_K$. We are done.

2.5. **Canonical vs universal local systems.** Let us first motivate our goal and the formalism we introduce to achieve it. Fix some integer $g > 0$ and let $\mathscr{M}_{K_0}$ denote the moduli stack over $\mathbb{Q}$ of principally polarized abelian schemes of dimension g with K_0-level structure, where K_0 is a compact open subgroup of $\mathrm{GSp}_{2g}(\mathbb{A}_f)$. Replacing K_0 with finite index open subgroups K, we find a K_0-fit system $(\mathscr{M}_K)_K$ of stacks in the sense of Definition 2.17, so we can consider the corresponding *canonical* $\mathbb{A}_f$-local system $\mathbb{L}^{\mathrm{can}}$ on $\mathscr{M}_{K_0}$. There is a second local system on $\mathscr{M}_{K_0}$: the relative cohomology of the universal abelian scheme, which we call the *universal* local system $\mathbb{L}_{K_0}$. We aim to show that $\mathbb{L}^{\mathrm{can}}$ and $\mathbb{L}_{K_0}$ are canonically isomorphic, which is the content of Proposition 2.30 below.

To do so, we formalize the notion of a moduli stack with level structure in Definition 2.26 and Definition 2.28. We sketch how this is done in the explicit case of $\mathscr{M}_{K_0}$. For any scheme $T/\mathbb{Q}$, an object in $\mathscr{M}_{K_0}(T)$ is a triple $(f : A \to T, \lambda, [\eta]_{K_0})$ where

- $(f : A \to T, \lambda)$ is a principally polarized abelian scheme of dimension g, and
- $[\eta]_{K_0}$ is a section of the quotient by $\mathcal{F}_{K_0}$ of some subsheaf of $\underline{\mathrm{Isom}}(R^1 f_* \mathbb{A}_f, \mathcal{F}_V)$ where $V = \mathbb{A}_f^{2g}$ is acted-on by K_0; namely we restrict to those isomorphisms that are compatible with the usual symplectic structure on V, and the symplectic structure on $R^1 f_* \mathbb{A}_f$ induced from λ.

To define $\mathscr{M}_{K_0}$, we first consider the stack $\mathscr{X}$ of principally polarized abelian schemes of dimension g and view relative cohomology as a morphism from $\mathscr{X}$ to $\mathbf{Loc}_V$. From there, we could define a stack of triples $(f : A \to T, \lambda, [\eta]_{K_0})$ with the slight caveat that $[\eta]_{K_0}$ is a section of the whole sheaf $\underline{\mathrm{Isom}}(R^1 f_* \mathbb{A}_f, \mathcal{F}_V)/\mathcal{F}_{K_0}$ (no compatibility condition with the symplectic structures). To formalize replacing every $\mathrm{GL}_d(\mathbb{A}_f)$-torsor $\underline{\mathrm{Isom}}(R^1 f_* \mathbb{A}_f, \mathcal{F}_V)$ with the appropriate $\mathrm{GSp}_{2g}(\mathbb{A}_f)$-subsheaf (which is a $\mathrm{GSp}_{2g}(\mathbb{A}_f)$-torsor), we use the language of natural transformations between fibered categories.

Fix an embedding $Q \hookrightarrow \mathrm{GL}(V)$. Recall the equivalence of stacks

$$\mathrm{Triv} : \mathbf{Loc}_V \to \mathbf{B}\mathrm{GL}(V)$$

assigning to any $T \in \mathscr{C}$ and any $\mathbb{L} \in \mathbf{Loc}_V(T)$ the $\mathrm{GL}(V)$-torsor $\underline{\mathrm{Isom}}(\mathbb{L}, \mathcal{F}_V)$ over T.

Definition 2.26: A *Q-moduli datum* is a quadruple $(\mathscr{X}, \mathbb{L}_{\mathscr{X}}, \mathrm{Lev}, \iota)$ consisting of

(i) a stack $\mathscr{X}$ over $\mathscr{C}$,
(ii) a local system $\mathbb{L}_{\mathscr{X}} : \mathscr{X} \to \mathbf{Loc}_V$ on $\mathscr{X}$; we let $\mathrm{Triv}_{\mathscr{X}} = \mathrm{Triv} \circ \mathbb{L}_{\mathscr{X}}$,
(iii) a Q-torsor $\mathrm{Lev} : \mathscr{X} \to \mathbf{B}Q$ on $\mathscr{X}$, and
(iv) a natural transformation ι between the morphisms of fibered categories:

$$\begin{tikzcd} & \mathbf{B}\mathrm{GL}(V) \arrow[dr, bend left, "\text{forget}"] \arrow[d, phantom, "\Uparrow\iota"] & \\ \mathscr{X} \arrow[ur, bend left, "\mathrm{Triv}_{\mathscr{X}}"] \arrow[dr, bend right, "\mathrm{Lev}"'] & {} & Q\mathbf{Sh}. \\ & \mathbf{B}Q \arrow[ur, bend right, "\text{forget}"'] & \end{tikzcd}$$

A Q-moduli datum is to be understood as follows. The functor $\mathrm{Triv}_{\mathscr{X}} : \mathscr{X} \to \mathbf{B}\mathrm{GL}(V)$ assigns to any $T \in \mathscr{C}$ and any $A \in \mathscr{X}(T)$ the $\mathrm{GL}(V)$-torsor $\underline{\mathrm{Isom}}(\mathbb{L}_{\mathscr{X}}(A), \mathcal{F}_V)$ over T. The functor Lev assigns to the same data a Q-subsheaf of $\mathrm{Triv}_{\mathscr{X}}(A)$ which is a Q-torsor for the induced action, considering only isomorphisms with additional constraints. The condition of being a Q-subsheaf for each $T \in \mathscr{C}$ and $A \in \mathscr{X}(T)$ is formalized by ι, which is automatically a natural monomorphism.

Lemma 2.27: *Let $(\mathscr{X}, \mathbb{L}_{\mathscr{X}}, \mathrm{Lev}, \iota)$ be a Q-moduli datum. Then $\mathbb{L}_{\mathscr{X}}$ is the R-local system with fiber V associated with the $\mathrm{GL}(V)$-torsor $\mathrm{Ind}_Q^{\mathrm{GL}(V)}(\mathrm{Lev})$.*

Proof. Condition (iv) is equivalent to a natural isomorphism

$$\begin{tikzcd} \mathscr{X} \arrow[rr, "\mathrm{Triv}_L"] \arrow[dr, bend right, "\mathrm{Lev}"'] & \Uparrow & \mathbf{B}\mathrm{GL}(V) \\ & \mathbf{B}Q \arrow[ur, bend right, "\mathrm{Ind}_Q^{\mathrm{GL}(V)}"'] & \end{tikzcd}.$$

The equivalence follows formally from the fact that $\mathrm{Ind}_Q^{\mathrm{GL}(V)} : Q\mathbf{Sh} \to \mathrm{GL}(V)\mathbf{Sh}$ is left adjoint to the forgetful functor $\mathrm{GL}(V)\mathbf{Sh} \to Q\mathbf{Sh}$. □

It follows that a Q-moduli datum is determined up to isomorphism by the Q-torsor Lev on $\mathscr{X}$.

We are interested in the following moduli stacks:

Definition 2.28: Let K_0 be a compact subgroup of Q which is profinite, and K an open subgroup of K_0. Let $(\mathscr{X}, \mathbb{L}_{\mathscr{X}}, \mathrm{Lev}, \iota)$ be a Q-moduli datum. The associated *moduli stack of level K* is the stack $\mathscr{M}_K$ over $\mathscr{C}$ assigning to each $T \in \mathscr{C}$ a pair $(A, [\eta]_K)$, where

(i) A is an object in $\mathscr{X}(T)$, and

(ii) $[\eta]_K \in H^0(T, \mathrm{Lev}(A)/\mathcal{F}_K)$ is a global section of the quotient of $\mathrm{Lev}(A)$ by the action of $\mathcal{F}_K$.

If K is normal in K_0, the obvious morphism $\mathscr{M}_K \to \mathscr{M}_{K_0}$ is a K_0/K-torsor as in Definition 2.16.

Fix K_0 and $(\mathscr{X}, \mathbb{L}_{\mathscr{X}}, \mathrm{Lev}, \iota)$ as in Definition 2.28. The projective system $(\mathscr{M}_K)_{K \in \mathcal{C}(K_0)}$ is fit for K_0. By Definition 2.18 we can consider the canonical local system $\mathbb{L}^{\mathrm{can}}$ on $\mathscr{M}_{K_0}$. We now define the universal local system on $\mathscr{M}_{K_0}$ and show that it is canonically isomorphic to $\mathbb{L}^{\mathrm{can}}$. Observe that the projection onto the first component defines a morphism of stacks $p : \mathscr{M}_{K_0} \to \mathscr{X}$.

Definition 2.29: The *universal local system* $\mathbb{L}_{K_0} \in \mathbf{Loc}_V(\mathscr{M}_{K_0})$ on $\mathscr{M}_{K_0}$ is the pullback of $\mathbb{L}_{\mathscr{X}}$ by p, that is:

$$\mathscr{M}_{K_0} \xrightarrow{p} \mathscr{X} \xrightarrow{\mathbb{L}_{\mathscr{X}}} \mathbf{Loc}_V.$$

The *universal torsor* $\mathrm{Lev}_{K_0} \in \mathbf{B}Q(X_{K_0})$ on $\mathscr{M}_{K_0}$ is the pullback of Lev by p, that is:

$$\mathscr{M}_{K_0} \xrightarrow{p} \mathscr{X} \xrightarrow{\mathrm{Lev}} \mathbf{B}Q.$$

By Lemma 2.27, $\mathbb{L}_{K_0}$ is the R-local system with fiber V associated with $\mathrm{Ind}_Q^{\mathrm{GL}(V)}(\mathrm{Lev}_{K_0})$.

Proposition 2.30: *There is a canonical isomorphism* $\mathbb{L}^{\mathrm{can}} \simeq \mathbb{L}_{K_0}$.

Proof. Recall that $\mathbb{L}^{\mathrm{can}}$ is a local system defined from a $\mathrm{GL}(V)$-torsor, itself induced from the K_0-torsor $h_{\mathscr{M}_\infty}$ as in Definition 2.17. We will define a canonical morphism of K_0-sheaves between $h_{\mathscr{M}_\infty}$ and Lev_{K_0} (Q-torsor viewed as a K_0-sheaf). By adjunction, as in the proof of Lemma 2.27, this defines a canonical isomorphism between the Q-torsors $\mathrm{Ind}_{K_0}^{Q}(h_{\mathscr{M}_\infty})$ and Lev_{K_0}. Applying $\mathrm{Ind}_Q^{\mathrm{GL}(V)}$, we find the desired isomorphism.

Defining a morphism $h_{\mathscr{M}_\infty} \to \mathbf{Loc}_{K_0}$ of K_0-sheaves on $\mathscr{M}_{K_0}$ amounts to find, for every $\xi : T \to \mathscr{M}_{K_0}$, a morphism of K_0-sheaves $h_{\mathscr{M}_\infty}(\xi) \to \mathbf{Loc}_{K_0}(\xi)$ on T which is natural in T. Fixing $T \to \mathscr{M}_{K_0}$, let $(A, [\eta]_{K_0}) \in \mathscr{M}_{K_0}(T)$ be the corresponding object. A section of $h_{\mathscr{M}_\infty}(A, [\eta]_{K_0})$ is a compatible collection of sections $T \to \mathscr{M}_K$ over $\mathscr{M}_{K_0}$ as K varies, i.e a compatible system of pairs $(A, [\eta]_K)$ as K varies, which is determined by the projective system of level structures $([\eta]_K)_K \in \varprojlim_K H^0(T, \mathrm{Lev}(A)/\mathcal{F}_K)$. Now there are K_0 equivariant identifications

$$\varprojlim_K H^0(T, \mathrm{Lev}(A)/\mathcal{F}_K) = H^0(T, \varprojlim_K \mathrm{Lev}(A)/\mathcal{F}_K) \simeq H^0(T, \mathrm{Lev}(A)) = H^0(T, \mathrm{Lev}_{K_0}(A, [\eta]_{K_0})).$$

For the middle isomorphism we used that K_0 is profinite. This defines the desired morphism

$$h_{\mathscr{M}_\infty}(A, [\eta]_{K_0}) \longrightarrow \mathrm{Lev}_{K_0}(A, [\eta]_{K_0})$$

which is indeed natural in $(A, [\eta]_{K_0})$. □

Example 2.31: Let (G, Ω) be a Shimura datum with reflex field E. Let K_0 be a compact open subgroup of $G(\mathbb{A}_f)$. If $\mathrm{Sh}_{K_0}[G, \Omega]/E$ carries a modular interpretation (e.g. is Hodge-type), then the local system on $\mathrm{Sh}_{K_0}[G, \Omega]$ coming from the relative cohomology of the universal family coincides with the canonical local system.

3. Pointwise and global good reduction problems

3.1. **Formulation of the question.** We define the notions of pointwise and global good reduction.

Definition 3.1: Let Y/F be a smooth proper variety. We say that it has *good reduction* if it is the generic fiber of some smooth proper model $\mathcal{Y}/\mathcal{O}_F$.

We define the integral locus of a stack over $\mathcal{O}_F$. To this end, we introduce some notation. If X/F is a stack, its set of closed points is denoted by $|X|$. This is the set of F-morphisms $\operatorname{Spec} E \to X$, with E/F a finite extension, where two morphisms $\operatorname{Spec} E \to X$ and $\operatorname{Spec} E' \to X$ are identified if there exists a field E'', embeddings $E, E' \hookrightarrow E''$ and a 2-commutative diagram

$$\begin{tikzcd} \operatorname{Spec} E'' \arrow[r] \arrow[d] & \operatorname{Spec} E' \arrow[d] \\ \operatorname{Spec} E \arrow[r] & X. \end{tikzcd}$$

If X is a scheme, this coincides with the closed points of its underlying topological space in the usual sense.

Definition 3.2: Let $\mathcal{X}/\mathcal{O}_F$ be a stack with generic fiber X/F. We define the *integral locus* of $\mathcal{X}$ as

$$|\mathcal{X}|_\eta^{\text{int}} = \operatorname{im}(\mathcal{X}(\overline{\mathcal{O}_F}) \to |X|).$$

It is the set of closed points of X which admit a specialization in the special fiber of $\mathcal{X}$. In other words, $x_0 : \operatorname{Spec} E \to X$ lies in $|\mathcal{X}|_\eta^{\text{int}}$ if and only if, for some finite extension E'/E there exists $\operatorname{Spec} \mathcal{O}_{E'} \to \mathcal{X}$ and a 2-commutative diagram

$$\begin{tikzcd} \operatorname{Spec} E' \arrow[r] \arrow[d, hook] & \operatorname{Spec} E \arrow[r, "x_0"] & X \arrow[d, hook] \\ \operatorname{Spec} \mathcal{O}_{E'} \arrow[rr, "\exists"] & & \mathcal{X}. \end{tikzcd}$$

If $\mathcal{X} = X$, the integral locus is empty and if $\mathcal{X}/\mathcal{O}_F$ is a proper scheme, then $|\mathcal{X}|_\eta^{\text{int}} = |X|$.

Definition 3.3: Let X/F be a smooth variety with smooth model $\mathcal{X}/\mathcal{O}_F$. Consider a smooth proper morphism of schemes $Y \to X$.

We say that this family has *pointwise good reduction* (relative to $\mathcal{X}$) if for every $x \in |\mathcal{X}|_\eta^{\text{int}}$, the fiber $Y_x/F(x)$ has good reduction.

We say that this family has *global good reduction* (relative to $\mathcal{X}$) if there exists a smooth proper map $\mathcal{Y} \to \mathcal{X}$ of algebraic spaces over $\mathcal{O}_F$ with generic fiber $Y \to X$.

We are interested in whether pointwise good reduction implies global good reduction, and if not, what may some obstructions be.

Remark 3.4: By default, we require that the model $\mathcal{Y} \to \mathcal{X}$ extends as much of the structure of $Y \to X$ as possible. Typically if $Y \to X$ is a polarized abelian scheme or K3 surface with some level structure, we require $\mathcal{Y} \to \mathcal{X}$ to be so too.

3.2. **Abelian schemes that do not extend.** For abelian schemes over a sufficiently ramified p-adic field F, pointwise good reduction does not imply global good reduction. We construct a counterexample in the next paragraph. Prior to this, we need some recollections on how abelian schemes over some dense open subscheme can fail to extend to an abelian scheme over the whole base.

We recall the context behind this question. In [FC13, Chapter V, Corollary 6.8] it is stated that given a base regular scheme S with maximal points of characteristic zero, and a closed subscheme Z of codimension at least 2, any abelian scheme over $S \setminus Z$ uniquely extends to an abelian scheme over S[5]. A counterexample due to Raynaud-Ogus-Gabber (see [dJO97, §6]) showed that the claim is wrong as it stands. Vasiu-Zink later generalized this construction in [VZ10]; the goal of this paragraph is to recall their work so as to prove Proposition 3.5 below. From now on we consider a finite extension $F/\mathbb{Q}_p$ of ramification index at least p.

Proposition 3.5: *There exists a local étale $\mathcal{O}_F[[T]]$-algebra R with maximal ideal $\mathfrak{m}$ and an abelian scheme $A_U \to U := \operatorname{Spec} R \backslash \{\mathfrak{m}\}$, together with a degree-p^2 (resp. principal) polarization $\lambda : A_U \to A_U^\vee$, such that A_U does not extend to an abelian scheme over* $\operatorname{Spec} R$.

Moreover, let $\overline{\eta} \to \operatorname{Spec} R$ be a geometric point over the generic point and N a prime-to-p positive integer. We can arrange for $A_{\overline{\eta}}[N]$ to have trivial monodromy.

[5]In the language of Definition 6.4, the statement claims that any such S is healthy.

Remark 3.6: In the language of the Siegel Shimura variety, the second part of Proposition 3.5 states that we can arrange for $A_U \to U$ to have any prime-to-p level structure. From now on we use this terminology for conciseness.

The rest of the paragraph is dedicated to the proof of Proposition 3.5. The main tool for constructing abelian schemes that do not extend is the following lemma.

Lemma 3.7 ([VZ10], Lemma 27)**:** *Let R denote a regular local ring of dimension ≥ 2 and mixed characteristic $(0,p)$. Let $\mathfrak{m}$ denote its maximal ideal, and set $U = \operatorname{Spec} R \setminus \{\mathfrak{m}\}$.*

Let $D \to H$ be a homomorphism of finite flat group schemes over $\operatorname{Spec} R$ which is not a monomorphism, but whose restriction over U is a monomorphism[6]*. Let B be an abelian scheme over $\operatorname{Spec} R$ in which H embeds. Define the quotient abelian scheme $A_U = B_U/D_U$ over U.*

If R' is a faithfully flat local R-algebra of relative dimension 0, with maximal ideal $\mathfrak{m}'$, let $U' := U \times_R \operatorname{Spec} R' = \operatorname{Spec} R' \setminus \{\mathfrak{m}'\}$. Then $A_U \times_U \operatorname{Spec} R' \to U'$ is an abelian scheme which does not extend into an abelian scheme over $\operatorname{Spec} R'$.

Proof. The morphism $\operatorname{Spec} R' \to \operatorname{Spec} R$ is fpqc, so the assumptions on monomorphisms still hold after base-change to R'. Let D', H', A'_U, B' denote the group schemes base-changed to R', then $A'_U = B'_U/D'_U$. Hence we reduce to the case $R' = R$.

Suppose that A_U extends to an abelian scheme $A \to \operatorname{Spec} R$. By the Néron extension property (see [BLR12, 8.4, Corollary 6]) the isogeny $B_U \to A_U$ extends to a homomorphism $B \to A$ which is automatically an isogeny. Its kernel K is a finite flat group scheme which coincides with D_U over U; we will argue that there is an identification $D \simeq K$ as B-schemes, which then implies that $D \to H$ is a monomorphism over the whole base, contradiction.

Over U, the isogeny $B \to B/H$ factors through A_U; there is thus a factorization between Néron models

$$\begin{tikzcd} & A = B/K \arrow[dr] & \\ B \arrow[ur] \arrow[rr] & & B/H \end{tikzcd}$$

over $\operatorname{Spec} R$ which induces a closed embedding of kernels $K \hookrightarrow H$. We use that quotients of finite flat commutative group schemes exist and are again finite flat (see [Tat, §3.5]). The kernel N of the composition $D \to H \to H/K$ is a finite flat subgroup of D, so D/N is finite flat and trivial over U, hence trivial everywhere so that $N = D$, and $D \to H/K$ is trivial. Thus $D \to H \hookrightarrow B$ factors through $K \hookrightarrow B$. The same argument with the quotient K/D shows that this factorization is an isomorphism over B as desired. ☐

Disregarding polarizations for now, the proof of Proposition 3.5 is the following. According to [VZ10, Theorem 28 (i)] and its proof we can take the power series ring $R = \mathcal{O}_F[[T]]$ and find a homomorphism of group schemes $D \to H$ over R satisfying the assumptions of Lemma 3.7, hence an abelian scheme $A_U \to U$ which does not extend to an abelian scheme over $\operatorname{Spec} R$. By Lemma 3.7 again we can replace R with an appropriate étale extension of local rings to satisfy the monodromy condition on $A_{\overline{\eta}}[N]$.

To find an example with degree-p^2 polarization, we have to take a deeper dive into the proof of [VZ10, Theorem 28 (i)] and argue in three steps. First we will see that D can be arranged to be of order p. Second, we shall observe that the abelian scheme $B \to U$ in which H embeds can be chosen to be principally polarized. Lastly, we will show that under these conditions, the dual $A_U^\vee \to U$ has degree-p^2 polarization and does not extend to $\operatorname{Spec} R$.

Step 1. We argue that D can be chosen to have order p. Set $R := \mathcal{O}_F[[T_1]]$. Let k denote the residue field of F, so that $\mathcal{O}_F/W(k)$ is totally ramified of index $\geq p$. In particular, we can write $\mathcal{O}_F \simeq W(k)[T_2]/(E(T_2))$ where $E(T_2)$ is an Eisenstein polynomial. Multiplying by a unit, we may assume its constant coefficient is p, so we can write $E(T_2) = h - p$ where $h \in (T_2) \subseteq W(k)[T_2]$. Now, if we write $\mathfrak{S} := W(k)[[T_1, T_2]]$, we have shown that

$$R \simeq \mathfrak{S}/(h-p)$$

[6] [VZ10, Lemma 27] is stated with $H \to D$ which is an epimorphism over U but not globally. However the proof only uses the Cartier dual morphism $D^t \to H^t$: here we consider the dual statement to skip taking duals.

where $h \in (T_1, T_2) \setminus p\mathfrak{S}$ is such that the reduction $\overline{h}$ of h modulo p is divisible by T_2^p. This shows that [VZ10, Theorem 28 (i)] applies, the proof of which finds a homomorphism of group schemes $D \to H$ over R satisfying the assumptions of Lemma 3.7. Let us sketch this construction to argue that D can be chosen of order p. The key ingredient is this.

Fact 3.8 ([Lau10], Theorem 10.7)**:** *Let R be a complete regular local ring with perfect residue field k of characteristic $p > 0$. Thus we can write $R = \mathfrak{S}/E\mathfrak{S}$ where*

$$\mathfrak{S} = W(k)[[T_1, \dots, T_r]]$$

and $E \in \mathfrak{S}$ is a power series with constant term p.

Then the category of connected finite flat group schemes over R of p-power order is equivalent to the category of nilpotent Breuil modules relative to $\mathfrak{S}$[7]*.*

Following [Lau10, Definition 8.4], a *Breuil module relative to $\mathfrak{S} \to R$* is, roughly, a triple (M, φ, ψ) consisting of

- a finitely generated $\mathfrak{S}$-module M anihilated by a power of p, and of projective dimension ≤ 1,
- maps $\varphi : M \to M$ and $\psi : M \to M$ which are σ^{-1} and σ-linear respectively, where $\sigma : \mathfrak{S} \to \mathfrak{S}$ is some appropriate lift of the Frobenius, satisfying

$$\psi \circ \varphi = \varphi \circ \psi = E.$$

A Breuil module is *nilpotent* if φ or ψ is nilpotent modulo the maximal ideal of $\mathfrak{S}$. When $R = \operatorname{Spec} k$, Breuil modules relative to $W(k) \to k$ recover the more familiar notion of Dieudonné modules. In our setting we consider the case $r = 2$ and the map

$$\mathfrak{S} \to R \simeq \mathfrak{S}/(h - p)$$

is the quotient map. In the proof of [VZ10, Theorem 28 (i)], the authors define an epimorphism $H^t \to D^t$ by constructing nilpotent Breuil modules and an appropriate morphism between them. In particular, the Breuil module for D has underlying module $\mathfrak{S}/p\mathfrak{S}$. This implies that D has order p. To see this, observe that it suffices to prove it for the fiber of D above the closed point $s : \operatorname{Spec} k \to \operatorname{Spec} R$. The Dieudonné module of D_s is the base change

$$\mathfrak{S}/p\mathfrak{S} \otimes_{\mathfrak{S}} W(k) \simeq k$$

which is a $W(k)$-module of length 1. Thus D_s has order p. The same argument shows that H has order p^3, although we make no use of it.

Step 2. We argue that H embeds in a principally polarized abelian scheme. This follows from the following theorem of Raynaud.

Fact 3.9 ([BBM], Théorème 3.1.1)**:** *Let S be a noetherian local scheme, G a finite flat group scheme over S. Then there exists a principally polarized abelian scheme $B \to S$ in which G embeds as a closed subgroup scheme.*

In the reference $B \to S$ is only claimed to be projective with no mention of polarization. In the proof, B is constructed as the Jacobian of a relative smooth curve over S, hence it is principally polarized.

Step 3. Let $A_U \coloneqq B_U/D_U$ as in Lemma 3.7; we argue that the dual $A_U^\vee \to U$ is a degree-p^2 polarized abelian scheme that does not extend to $\operatorname{Spec} R$. Consider the dual isogeny

$$q^\vee : A_U^\vee \to B_U^\vee$$

to the quotient $q : B_U \to A_U$. A priori this is a morphism of algebraic spaces over $\operatorname{Spec} R$, but B_U is principally polarized, hence $B_U^\vee$ is a scheme, and so is its cover $A_U^\vee$. As q and $q^\vee$ have same degree, which is the order of D_U, we find that the pullback of the principal polarization on $B_U^\vee$ through $q^\vee$ defines a degree-p^2 polarization on $A_U^\vee$. If $A_U^\vee$ were to extend to $\operatorname{Spec} R$, its dual would extend A_U, which is impossible according to Lemma 3.7.

[7]For $p \neq 2$, [Lau10, Theorem 1.2] states an equivalence between finite flat group schemes and Breuil modules. The adjectives *connected* and *nilpotent* allow the result to extend to characteristic 2.

It remains to deduce a principally polarized example. Let A_U denote the abelian scheme from above, the Zarhin trick states that $(A_U \times A_U^\vee)^4 \to U$ is principally polarized. The following lemma shows that it does not extends over $\operatorname{Spec} R$. This concludes the proof of Proposition 3.5.

Lemma 3.10: *Let W be a scheme, and $V \subseteq W$ a dense open subscheme. Let A_V, B_V be abelian schemes over V. If $C_V \coloneqq A_V \times B_V$ extends to an abelian scheme C over W, then so do A_V and B_V.*

Proof. Let p_U, q_u be the idempotent morphisms from $C_V \to C_V$ with images A_V and B_V respectively. By the Néron extension property (see [BLR12, 8.4, Corollary 6]), they extend to idempotent morphisms on C. Then the images (or kernels) of p and q define the sought-after abelian schemes. □

3.3. **Abelian schemes with pointwise but not global good reduction.** Consider F and N as in the previous paragraph. We first argue that Proposition 3.5 also holds when R is a smooth local $\mathcal{O}_F$-algebra.

Proposition 3.11: *There exists a smooth local $\mathcal{O}_F$-algebra R with maximal ideal $\mathfrak{m}$ and an abelian scheme $A_U \to U \coloneqq \operatorname{Spec} R \backslash \{\mathfrak{m}\}$, together with a degree-p^2 (resp. principal) polarization $\lambda : A_U \to A_U^\vee$, such that A_U does not extend to an abelian scheme over $\operatorname{Spec} R$.*

Moreover, we can arrange for $A_U \to U$ to have any prime-to-p level structure.

Proof. We denote by $\tilde{R}$ the finite local étale $\mathcal{O}_F[[T]]$-algebra with maximal ideal $\tilde{\mathfrak{m}}$, by $\tilde{U} \coloneqq \operatorname{Spec} \tilde{R} \backslash \{\tilde{\mathfrak{m}}\}$ the open subset and by $f : \tilde{A}_U \to \tilde{U}$ an abelian scheme provided by Proposition 3.5. We wish to descend all of this data over a smooth $\mathcal{O}_F$-algebra.

We first argue that $\mathcal{O}_F \to \tilde{R}$ is a regular ring homomorphism, to do so we may assume $\tilde{R} = \mathcal{O}_F[[T]]$. Flatness is easy. It is clear that $k[[T]]$ is geometrically regular, where k is the residue field of $\mathcal{O}_F$. It remains to show that $\mathcal{O}_F[[T]] \otimes_{\mathcal{O}_F} F = \mathcal{O}_F[[T]][1/p]$ is regular. By [Sta26, Tag 07EL] we reduce to showing that $F \to \mathcal{O}_F[[T]][1/p]$ is formally smooth for the (T)-adic topology, which is easily checked.

We may now apply Popescu's theorem, see [Sta26, Tag 07GC]: there is a cofiltered system (R_i) of smooth $\mathcal{O}_F$-algebras such that $\tilde{R} = \varinjlim R_i$. All maps between the R_i and $\tilde{R}$ factor through local and integral rings, so we may assume that the R_i are local integral with maximal ideal $\mathfrak{m}_i$, and set $U_i \coloneqq \operatorname{Spec} R_i \setminus \{\mathfrak{m}_i\}$. We let $p_i : \tilde{U} \to U_i$ be the maps induced from the projective system. We wish to descend f to an abelian scheme over some U_i, along with the extra data.

Since the U_i are Noetherian, by [Sta26, Tag 0CNL] for large enough i there is a smooth and proper morphism $f_i : A_i \to \operatorname{Spec} U_i$ descending f. Using [Sta26, Tag 0CNR] we can assume that the group scheme structure, polarization and level structure descend to f_i, thus f_i defines an abelian scheme. Let $A_U = A_i$ and $U = U_i$ for this appropriate choice of i.

If $A_U \to U$ were to extend to an abelian scheme over $\operatorname{Spec} R$, its pullback to $\operatorname{Spec} \tilde{R}$ would extend $\tilde{A}_U \to \tilde{U}$, contradicting the definition of $\tilde{A}_U$. □

Corollary 3.12: *With notations from Proposition 3.11, set $\mathcal{X} = \operatorname{Spec} R$ which is a smooth model over $\mathcal{O}_F$ of its generic fiber $\iota \colon X \hookrightarrow \mathcal{X}$, and set $A_X \coloneqq \iota^* A_U$. Then $A_X \to X$ is a degree-p^2 (resp. principally) polarized abelian scheme having pointwise good reduction but not global good reduction as an abelian scheme.*

Moreover, we can arrange for $A_X \to X$ to have any prime-to-p level structure.

Proof. We argue that the polarized abelian scheme $A_X \to X$ does not have global good reduction. Assume $A_X \to X$ extends to an abelian scheme $A \to \mathcal{X}$, we argue that this also extends $A_U \to U$, contradicting Proposition 3.11. The polarization on A_X extends to a polarization on A due to the Néron extension property (see Fact 6.13). Let $j \colon U \to \mathcal{X}$ be the inclusion of open set, we find that the polarized abelian schemes

$$A_U \to U, \qquad j^* A \to U$$

have isomorphic generic fibers. Separatedness of the stack of polarized abelian schemes implies that they are isomorphic (Proposition 3.13).

It remains to check that $A_X \to X$ has pointwise good reduction. Let $\ell \neq p$ be a prime number, the local system $\mathbb{L} = R^1 f_* \mathbb{Q}_\ell$ over U corresponds to an ℓ-adic representation of the étale fundamental group $\pi_1(U)$. As $\mathcal{X} \setminus U$ has codimension at least 2 in $\mathcal{X}$, Zariski-Nagata purity implies that $\mathbb{L}$ extends to a local system on $\mathcal{X}$. In particular, for any $x \in |\mathcal{X}|_\eta^{\mathrm{int}}$ the local system $\mathbb{L}_x$ is unramified, which by [ST71] means that $(A_X)_x$ has good reduction. □

For this next proposition we deal with Deligne-Mumford stacks over the pro-étale site over a base scheme S. The reader is refered to Section 2.1, and more precisely Definition 2.2, for the definition of these objects.

Proposition 3.13: *Let $\mathscr{X}$ be a separated Deligne-Mumford stack over some base scheme S. Let Y be a locally Noetherian, normal S-scheme and U a dense open subscheme of Y. Let $f, g : Y \to \mathscr{X}$ be two morphisms such that $f_{|U} \simeq g_{|U}$. Then $f \simeq g$.*

Proof. Let $I = \underline{\mathrm{Isom}}(f, g)$ the algebraic stack obtained by the pullback of the diagonal Δ

$$\begin{tikzcd} I \arrow[r, "\theta"] \arrow[d] & Y \arrow[d, "f\times g"] \\ \mathscr{X} \arrow[r, "\Delta"] & \mathscr{X} \times_S \mathscr{X}. \end{tikzcd}$$

By assumption, Δ, and hence also θ, is finite, so by [LMB18, Théorème A.2] I is a scheme. The assumption $f_{|U} \simeq g_{|U}$ means that θ has a section over U $s : U \to I_{|U}$ which is an open immersion by [Sta26, Tag 024T]. Let Z denote the scheme-theoretic closure of $s(U)$ in I. We want to show that $\theta_{|Z} : Z \to Y$ is an isomorphism. Working componentwise, we can assume that Y, and so U and Z, are irreducible. Since $\theta(Z)$ contains U, $\theta_{|Z}$ is dominant. By [Sta26, Tag 0356] Z is reduced. Thus $\theta_{|Z} : Z \to Y$ is a finite birational morphism between integral schemes, with Y normal: by [Sta26, Tag 0AB1] it is an isomorphism. □

Remark 3.14: The normality assumption on Y in Proposition 3.13 cannot be dropped. Over some base field k, consider Y to be $\mathbb{P}^1$ with 0 and ∞ glued together, let U be Y minus the singular point, and set $\mathscr{X} = \mathbf{B}\mathbb{Z}/2\mathbb{Z}$. Let Y_1 be two disjoint copies of Y and Y_2 be two copies of $\mathbb{P}^1$ glued at zero and glued at infinity. Then there are obvious projection maps $Y_i \to Y$, $i \in \{1, 2\}$, which are double étale covers, and which both restrict to the trivial double cover over U. In this way, we find two non-isomorphic maps $Y \to \mathscr{X}$ which are isomorphic over the dense open subset U.

In Section 5 we will apply Proposition 3.13 repeatedly as follows: for $\mathcal{X}/\mathcal{O}_F$ a regular scheme, and $\mathscr{S}/\mathcal{O}_F$ a separated Deligne-Mumford stack, two $\mathcal{O}_F$ morphisms $f, g : \mathcal{X} \to \mathscr{S}$ that are isomorphic over $\operatorname{Spec} F$ are isomorphic.

4. A pointwise criterion for extending local systems

In this section, we recall a theorem of Cadoret-Tamagawa in [CT26] to be used in Section 5, giving a pointwise criterion to check whether a local system over a smooth variety X/F extends to a smooth model $\mathcal{X}/\mathcal{O}_F$.

Fix $\mathcal{X}/\mathcal{O}_F$ a smooth, separated scheme of finite type, let X denote its generic fiber. Consider a finite Galois[8] cover $\psi : Y \to X$, set $\mathcal{Y}$ the relative normalization of $\mathcal{X}$ in $Y \to X \to \mathcal{X}$. We begin with a pointwise criterion to check whether $\mathcal{Y} \to \mathcal{X}$ is again étale. For any closed point $x \in |X|$ we say that ψ is *unramified at* x if for any (equiv. all) $y \in |Y|$ mapping to x, the induced morphism $\operatorname{Spec} \mathcal{O}_{F(y)} \to \operatorname{Spec} \mathcal{O}_{F(x)}$ is étale (i.e. $F(y)/F(x)$ is unramified). For a subset $\Sigma \subseteq |X|$ we say that ψ is *unramified at* Σ if it is unramified at every $x \in \Sigma$.

Theorem 4.1 ([CT26], Theorem 14 (U))**:** *The following are equivalent:*

(i) *The map $\mathcal{Y} \to \mathcal{X}$ is étale, and*
(ii) *The map ψ is unramified at $|\mathcal{X}|_\eta^{\mathrm{int}}$ (recall Definition 3.2).*

[8]By this, we mean that the pre-image of any connected component of X defines a finite Galois cover in the usual sense (see [Sta26, Tag 03SF]).

The reference has the added assumption that X be geometrically connected. Let us show that it can be removed. We only need to show (ii) $\Longrightarrow$ (i). First, as $\mathcal{X}/\mathcal{O}_F$ is smooth, we can write

$$\mathcal{X} = \bigsqcup_i \mathcal{X}_i$$

where each $\mathcal{X}_i$ is the closure in $\mathcal{X}$ of a component of X. Arguing with each $\mathcal{X}_i$ individually, we may assume that X is connected. Furthermore we may assume that the special fiber of $\mathcal{X}$ is nonempty, otherwise the theorem is vacuously true.

According to Lemma 4.2 below, for some unramified extension F'/F the components of $X_{F'}$ are geometrically connected. As relative normalization commutes with étale base-change (see [Sta26, Tag 0ABP]) and étaleness is étale-local, it suffices to show that $\mathcal{Y}_{\mathcal{O}_{F'}} \to \mathcal{X}_{\mathcal{O}_{F'}}$ is étale. The reference ensures that this is true above every component of $\mathcal{X}_{\mathcal{O}_{F'}}$, and we are done.

Lemma 4.2: *Let $\mathcal{X}$ be a smooth scheme of finite type over $\mathcal{O}_F$, with connected generic fiber X/F and nonempty special fiber $\mathcal{X}_s$. Then for some finite unramified field extension F'/F, the components of the base-change $X_{F'}$ are geometrically connected.*

Proof. It suffices to show that if $\mathcal{X}_s$ is geometrically connected, then so is X; in this case, it suffices to pick an unramified extension F'/F, with residue field extension k'/k, so that $\mathcal{X}s, k'$ has its components geometrically connected.

Assume $\mathcal{X}_s$ geometrically connected. Suppose $X_{F'} = X_1 \sqcup X_2$ for some finite extension F'/F. Then $\mathcal{X}_{\mathcal{O}_{F'}}$ has at least two irreducible components intersecting at its special fiber, contradicting regularity. $\square$

Let $\mathcal{G}$ be a sheaf of groups group, consider $\mathcal{X}, X$ as above. Let P be a $\mathcal{G}$-torsor on X (see Definition 2.6). For any closed point $x \in |X|$, we say that P *is unramified at* x if the $\mathcal{G}$-torsor P_x over $\operatorname{Spec} F(x)$ extends to a $\mathcal{G}$-torsor over $\operatorname{Spec} \mathcal{O}_{F(x)}$. For a subset $\Sigma \subseteq |X|$ we say that P is *unramified at* Σ if it is unramified at every $x \in \Sigma$.

Corollary 4.3: *Let Q be a profinite group, suppose $\mathcal{G} = \mathcal{F}_Q$. The following are equivalent:*

(i) *P extends to a G-torsor over $\mathcal{X}$, and*
(ii) *P is unramified at $|\mathcal{X}|_\eta^{\mathrm{int}}$.*

Proof. We only need to show (ii) $\Longrightarrow$ (i). Let us first assume that Q is finite. The torsor P is represented by a scheme whose components are finite Galois covers of X. Assumption (*ii*) implies that the morphism $P \to X$ is unramified at $|\mathcal{X}|_\eta^{\mathrm{int}}$, hence by Theorem 4.1 the relative normalization $\mathcal{P}$ of $\mathcal{X}$ in $P \to X \to \mathcal{X}$ defines an étale map $\mathcal{P} \to \mathcal{X}$. It remains to show that this is a Q-torsor. Normality of $\mathcal{X}$ implies that the restriction from finite étale covers of $\mathcal{X}$ to those of X is fully faithful, so that the isomorphism

$$\mathcal{F}_Q \times_X P \longrightarrow P \times_X P$$

extends to an isomorphism $\mathcal{F}_Q \times_{\mathcal{X}} \mathcal{P} \to \mathcal{P} \times_{\mathcal{X}} \mathcal{P}$, and we are done.

For the general case, write $Q = \varprojlim_i Q/K_i$ where each K_i is an open subgroup of Q. For each i write $Q_i := Q/K_i$, and $P_i := P/K_i$, so that P_i is a Q_i-torsor over X, unramified at $|\mathcal{X}|_\eta^{\mathrm{int}}$. From the finite group case, P_i extends to a Q_i-torsor $\mathcal{P}_i$ over $\mathcal{X}$, and the quotient morphisms $P_i \to P_j$ extend to quotient morphisms $\mathcal{P}_i \to \mathcal{P}_j$. Now, set

$$\mathcal{P} := \varprojlim_i \mathcal{P}_i$$

which is a Q-torsor over $\mathcal{X}$ extending P. $\square$

Let R be a topological ring, V a free R-module of finite rank. Let $\mathbb{L}$ be an R-local system with fiber V over X (see Definition 2.5). For any closed point $x \in |X|$, we say that $\mathbb{L}$ *is unramified at* x if the local system $\mathbb{L}_x$ over $\operatorname{Spec} F(x)$ extends to a local system over $\operatorname{Spec} \mathcal{O}_{F(x)}$. For a subset $\Sigma \subseteq |X|$ we say that $\mathbb{L}$ is *unramified at* Σ if it is unramified at every $x \in \Sigma$.

Corollary 4.4 ([CT26], Theorem 1 (U-1))**:** *Suppose R is profinite. The following are equivalent:*

(i) *$\mathbb{L}$ extends to a local system over $\mathcal{X}$, and*
(ii) *$\mathbb{L}$ is unramified at $|\mathcal{X}|_\eta^{\mathrm{int}}$.*

Proof. By Fact 2.7 the statement is equivalent to that of Corollary 4.3 with $Q = \mathrm{GL}(V)$ profinite. $\square$

Remark 4.5: In [CT26], *local systems* are defined over a *connected* scheme T as morphism $\pi_1(T) \to Q$ where Q is some profinite group. With the terminology of Section 2 such morphisms define Q-torsors on T, which induce a local system only after a choice of embedding $Q \hookrightarrow \mathrm{GL}(V)$.

5. Extension properties for good reduction

In this section we will deal with schemes and stacks over $\mathcal{O}_F$: we use the subscripts η and s to denote the generic and special fibers of these objects respectively.

Definition 5.1: Let $\mathcal{X}, \mathscr{S}/\mathcal{O}_F$ be stacks. A *generic morphism* $f : \mathcal{X} \rightsquigarrow \mathscr{S}$ is a morphism of stacks between the generic fibers $f_\eta : \mathcal{X}_\eta \to \mathscr{S}_\eta$.

We study the question of extending generic morphisms to actual morphisms between the stacks.

5.1. L-extension and PRG-extension.

Definition 5.2: Let $\mathscr{S}/\mathcal{O}_F$ be a separated Deligne-Mumford stack, and R a profinite ring. Let $\mathbb{L}$ be an R-local system on $\mathscr{S}$ (see Definition 2.5, Definition 2.15). We say that the pair $(\mathscr{S}, \mathbb{L})$ has *L-extension* if any generic morphism $f : \mathcal{X} \rightsquigarrow \mathscr{S}$ from a smooth scheme $\mathcal{X}/\mathcal{O}_F$, satisfying that the pullback $f_\eta^{-1}\mathbb{L}$ to $\mathcal{X}_\eta$ extends to a local system over $\mathcal{X}$, automatically and uniquely extends to an $\mathcal{O}_F$-morphism $\mathcal{X} \to \mathscr{S}$.

The uniqueness in Definition 5.2 is automatic by Proposition 3.13. This will also be the case for all other extension properties considered.

Lemma 5.3: *Let $\mathscr{S}/\mathcal{O}_F$ be a stack, $\mathbb{L}$ a local system on $\mathscr{S}$ such that $(\mathscr{S}, \mathbb{L})$ has $\mathbb{L}$-extension. Consider $\pi : \mathscr{S}' \to \mathscr{S}$ a surjective étale morphism over $\mathscr{S}/\mathcal{O}_F$. Then $(\mathscr{S}', \pi^*\mathbb{L})$ has $\mathbb{L}$-extension.*

Proof. Let $\mathcal{X}/\mathcal{O}_F$ be a smooth scheme and $f : \mathcal{X} \rightsquigarrow \mathscr{S}'$ a generic morphism. Using $\mathbb{L}$-extension, the composition $\mathcal{X} \rightsquigarrow \mathscr{S}' \to \mathscr{S}$ extends uniquely to $h : \mathcal{X} \to \mathscr{S}$. The situation is summarized in the following commutative diagram

$$\begin{tikzcd}
\mathcal{X}_\eta \arrow[r, "f_\eta"] \arrow[d, hook] & \mathscr{S}'_\eta \arrow[r] \arrow[d, hook] & \mathscr{S}_\eta \arrow[d, hook] \\
\mathcal{X} \arrow[r, dashed, "?"] \arrow[rr, bend right, "h"'] & \mathscr{S}' \arrow[r] & \mathscr{S}.
\end{tikzcd}$$

Let $\mathcal{X}' \coloneqq \mathcal{X} \times_{\mathscr{S}} \mathscr{S}'$ which is an étale cover of $\mathcal{X}$. We wish to show that $h' : \mathcal{X}' \to \mathscr{S}'$ descends to a morphism $\mathcal{X} \to \mathscr{S}'$, in other words we need to show that both composition arrows

$$\mathcal{X}' \times_{\mathcal{X}} \mathcal{X}' \rightrightarrows \mathcal{X}' \to \mathscr{S}'$$

coincide. Since this holds over the generic fibers, the result follows from Proposition 3.13. $\square$

In our applications however, we will be considering non-surjective étale morphisms $\mathcal{U} \to \mathscr{S}$: it is no longer true that $\mathcal{U}$ inherits L-extension. Fortunately, L-extension implies a weaker property called PGR-extension, which we define below. In the next paragraph we show that PGR-extension pulls back under étale maps.

Definition 5.4: Let $\mathcal{X}, \mathscr{M}/\mathcal{O}_F$ be stacks. A *PGR-morphism* (as in Pointwise Good Reduction) is a generic morphism $f : \mathcal{X} \rightsquigarrow \mathscr{M}$ such that $f_\eta(|\mathcal{X}|_\eta^{\mathrm{int}}) \subseteq |\mathscr{M}|_\eta^{\mathrm{int}}$. In other words, for any finite extension E/F and any $\mathcal{O}_E$-point of $\mathcal{X}$, the image by f_η of the induced E-point x of $\mathcal{X}_\eta$ extends to an $\mathcal{O}_{E'}$-point of $\mathscr{S}$ for some finite extension E'/E:

$$\begin{tikzcd}
& \operatorname{Spec} E \arrow[r, "x"] \arrow[d] & \mathcal{X}_\eta \arrow[r, "f_\eta"] \arrow[d] & \mathscr{S}_\eta \arrow[d] \\
\operatorname{Spec} \mathcal{O}_{E'} \arrow[r, "\exists"] \arrow[rrr, dashed, bend right, "\exists"] & \operatorname{Spec} \mathcal{O}_E \arrow[r] & \mathcal{X} & \mathscr{S}.
\end{tikzcd}$$

The restriction of an $\mathcal{O}_F$-morphism $\mathcal{X} \to \mathscr{M}$ to the special fibers is a PGR-morphism.

Remark 5.5: The acronym PGR is motivated by the case, of interest to us, where $\mathscr{M}/\mathcal{O}_F$ is a moduli stack for some families of smooth and proper algebraic spaces over $\mathcal{O}_F$. In this case, a PGR-morphism $f : \mathcal{X}_\eta \rightsquigarrow \mathscr{M}$ corresponds to a smooth proper family $Y \to \mathcal{X}_\eta$ with pointwise good reduction as in Definition 3.3.

Definition 5.6: Let $\mathscr{M}/\mathcal{O}_F$ be a separated Deligne-Mumford stack. We say that it has *PGR-extension* if every PGR-morphism $f : \mathcal{X} \rightsquigarrow \mathscr{M}$ from a smooth scheme $\mathcal{X}/\mathcal{O}_F$ automatically and uniquely extends to an $\mathcal{O}_F$-morphism $\mathcal{X} \to \mathscr{M}$.

Remark 5.7: Assuming $\mathscr{M}/\mathcal{O}_F$ is a moduli stack for some smooth and proper families of schemes over $\mathcal{O}_F$, having PGR-extension exactly states that such a family with pointwise good reduction has global good reduction.

Proposition 5.8: *Let $\mathscr{S}/\mathcal{O}_F$ be a separated Deligne-Mumford stack with a local system $\mathbb{L}$. Assume the pair $(\mathscr{S}, \mathbb{L})$ has L-extension as in Definition 5.2. Then $\mathscr{S}$ has PGR-extension.*

Proof. Let $\mathcal{X}/\mathcal{O}_F$ be as in Definition 5.6, consider $f : \mathcal{X} \rightsquigarrow \mathscr{S}$ a PGR-morphism. We want to show that f extends to a morphism $\mathcal{X} \to \mathscr{S}$ over $\mathcal{O}_F$. It suffices to show that $f^{-1}\mathbb{L}$ extends to a local system on $\mathcal{X}$ and apply L-extension.

Let $x \in |\mathcal{X}|_\eta^{\text{int}}$, then $f(x) \in |\mathscr{S}|_\eta^{\text{int}}$ is in the image of some $\overline{\mathcal{O}_F}$-point of $\mathscr{S}$. In particular, the Galois representation defined by the pullback of $\mathbb{L}$ to $f(x)$ is unramified, hence so is its pullback to x. Thus $f^{-1}\mathbb{L}$ is unramified at every point of $|\mathcal{X}|_\eta^{\text{int}}$; by Corollary 4.4 it extends to a local system over $\mathcal{X}$. $\square$

5.2. **Pulling back PGR-extension along étale maps.** We aim to show the following statement.

Theorem 5.9: *Let $\mathscr{S}/\mathcal{O}_F$ be a separated Deligne-Mumford stack with PGR-extension. Let $\mathcal{U}/\mathcal{O}_F$ be an algebraic stack étale over $\mathscr{S}$. Then $\mathcal{U}$ has PGR-extension.*

We first consider the case when $\mathscr{S}$ is a scheme.

Proposition 5.10: *Let $\mathscr{S}/\mathcal{O}_F$ be a separated scheme with PGR-extension. Let $\mathcal{U}/\mathcal{O}_F$ be a scheme étale over $\mathscr{S}$. Then $\mathcal{U}$ has PGR-extension.*

Proof. Let $\mathcal{X}/\mathcal{O}_F$ be as in Definition 5.4, consider $f : \mathcal{X} \rightsquigarrow \mathcal{U}$ a PGR-morphism. We want to show that f extends to a morphism $\mathcal{X} \to \mathcal{U}$ over $\mathcal{O}_F$. The composition $\mathcal{X}_\eta \to \mathcal{U}_\eta \to \mathscr{S}_\eta$ is again PGR, hence extends to $h : \mathcal{X} \to \mathscr{S}$. The situation is summarized in the following commutative diagram:

$$\begin{tikzcd}
\mathcal{X}_\eta \arrow[r, "f_\eta"] \arrow[d, hook] & \mathcal{U}_\eta \arrow[r, "g_\eta"] \arrow[d, hook] & \mathscr{S}_\eta \arrow[d, hook] \\
\mathcal{X} \arrow[r, dashed, "?"] \arrow[rr, bend right, "h"'] & \mathcal{U} \arrow[r, "g"] & \mathscr{S}.
\end{tikzcd}$$

Let $\Gamma_{f_\eta} \hookrightarrow \mathcal{X}_\eta \times_F \mathcal{U}_\eta$ be the graph of f_η, let Z denote its closure in $\mathcal{X} \times_{\mathcal{O}_F} \mathcal{U}$. It suffices to show that the projection $Z \to \mathcal{X}$ is an isomorphism, so that the extension of f_η is provided by

$$\tilde{f} : \mathcal{X} \to Z \hookrightarrow \mathcal{X} \times_{\mathcal{O}_F} \mathcal{U} \to \mathcal{U}.$$

Let $\Gamma_h \hookrightarrow \mathcal{X} \times_{\mathcal{O}_F} \mathcal{U}$ denote the graph of h. Let $\tilde{Z}$ denote the pullback of $\Gamma_h \to \mathscr{S}$ along g:

$$\begin{tikzcd} \tilde{Z} \arrow[r, hook] \arrow[d] & \mathcal{X} \times_{\mathcal{O}_F} \mathcal{U} \arrow[r, "p_2"] \arrow[d, "\mathrm{id}\times g"] & \mathcal{U} \arrow[d, "g"] \\ \Gamma_h \arrow[r, hook] & \mathcal{X} \times_{\mathcal{O}_F} \mathscr{S} \arrow[r, "p_1"] & \mathscr{S}. \end{tikzcd}$$

By definition $\tilde{Z}$ is a closed subscheme of $\mathcal{X} \times_{\mathcal{O}_F} \mathcal{U}$ containing Γ_{f_η}, hence Z is a closed subscheme of $\tilde{Z}$. By Lemma 5.11 below, the projection $Z \to \mathcal{X}$ is surjective (here we use again that f is PGR). The commutative diagram

$$\begin{tikzcd} Z \arrow[r, hook] \arrow[d] & \mathcal{X} \times_{\mathcal{O}_F} \mathcal{U} \arrow[r, "p_1"] \arrow[d, "\mathrm{id}\times \tilde{g}"] & \mathcal{X} \arrow[d, "\mathrm{id}"] \\ \Gamma_h \arrow[r, hook] \arrow[rr, bend right, "\sim"'] & \mathcal{X} \times_{\mathcal{O}_F} \mathscr{S} \arrow[r, "p_1"] & \mathcal{X} \end{tikzcd}$$

shows that this projection factors through $Z \to \Gamma_h \xrightarrow{\sim} \mathcal{X}$, thus $\theta : Z \to \Gamma_h$ is surjective. By Lemma 5.12 below, $Z \to \Gamma_h$ is étale. By [Gro66, Proposition 18.2.8] the number of points in the geometric fibers of θ is lower semicontinuous, bounded below by 1 by surjectivity, and equals 1 on the dense open subscheme Γ_{f_η} (as $\Gamma_{f_\eta} \to \Gamma_{g_\eta \circ f_\eta}$ is an isomorphism); it is thus constant equal to 1. By *loc.cit.*, θ is finite, and so by [Sta26, Tag 0AB1] it is an isomorphism. □

Lemma 5.11: *Let $\mathcal{X},\mathcal{U}/\mathcal{O}_F$ be schemes of finite type, with $\mathcal{X}/\mathcal{O}_E$ smooth, and $f : \mathcal{X} \rightsquigarrow \mathcal{U}$ a PGR-morphism with graph $\Gamma \hookrightarrow \mathcal{X}_\eta \times_F \mathcal{U}_\eta$. Let Z be the closure of Γ in $\mathcal{X} \times_{\mathcal{O}_F} \mathcal{U}$. Then the projection $Z \to \mathcal{X}$ is surjective.*

Proof. We show that the set-theoretic image of $p : Z \to \mathcal{X}$ contains all closed points. Then by Chevalley's theorem the image is a locally constructible subset of $\mathcal{X}$ containing all closed points, thus it is the whole of $\mathcal{X}$. Since $\Gamma \subseteq Z$, $\mathcal{X}_\eta$ is contained in $p(Z)$, so it remains to show that the closed points of the special fiber $\mathcal{X}_s$ of $\mathcal{X}$ are attained.

Let $x_s \in \mathcal{X}_s$ be a closed point. We use the smoothness of $\mathcal{X}$: by the infinitesimal lifting criterion, x_s lifts to an $\mathcal{O}_E$-point $x : \operatorname{Spec} \mathcal{O}_E \to \mathcal{X}$ for some finite extension E/F. Let $x_0 : \operatorname{Spec} E \to \mathcal{X}_\eta$ denote the special fiber of x. Let $y_0 = f_\eta(x_0)$, $y_0 \in |\mathcal{U}|_\eta^{\mathrm{int}}$ by the PGR assumption, hence y_0 is the special fiber of some $\mathcal{O}_{E'}$-point $y : \operatorname{Spec} \mathcal{O}_{E'} \to \mathcal{U}$ for E'/E a finite extension. We may assume $E = E'$, so that the pair (x, y) defines an $\mathcal{O}_E$-point of the product $\mathcal{X} \times_{\mathcal{O}_F} \mathcal{U}$. The situation is summarized in the following commutative diagram:

$$\begin{tikzcd} \operatorname{Spec} E \arrow[r, "(x_0,y_0)"] \arrow[d, hook] & \Gamma_{f_\eta} \arrow[r, hook] \arrow[d, hook] & \mathcal{X}_\eta \times_F \mathcal{U}_\eta \arrow[d, hook] \\ \operatorname{Spec} \mathcal{O}_E \arrow[r, dashed, "?"] \arrow[rr, bend right, "(x,y)"'] & Z \arrow[r, hook] & \mathcal{X} \times_{\mathcal{O}_F} \mathcal{U}. \end{tikzcd}$$

The factorization via the dashed arrow exists because all vertical arrows are inclusions of generic fibers in their closures. Now the composition morphism

$$\operatorname{Spec} \mathcal{O}_E \longrightarrow Z \longrightarrow \mathcal{X} \times_{\mathcal{O}_F} \mathcal{U} \longrightarrow \mathcal{X}$$

gives back the $\mathcal{O}_E$-point x, so x_s lies in the image of $Z \to \mathcal{X}$, as desired. □

Lemma 5.12: *Let $\phi : \tilde{Z} \to \Gamma$ be an étale morphism of schemes with Γ regular irreducible. Let $Z \hookrightarrow \tilde{Z}$ be an irreducible closed subscheme such that $\theta : Z \to \Gamma$ is again surjective. Then θ is étale.*

Proof. The surjectivity of θ ensures $\dim \tilde{Z} \geq \dim Z \geq \dim \Gamma = \dim \tilde{Z}$, so Z is an irreducible component of $\tilde{Z}$. As ϕ is étale, $\tilde{Z}$ is also regular, hence so are its irreducible components: $Z \hookrightarrow \tilde{Z}$ is an open immersion. Hence $\theta : Z \hookrightarrow \tilde{Z} \to \Gamma$ is étale. □

Proof of Theorem 5.9. Let $\mathcal{X}/\mathcal{O}_F$ be as in Definition 5.6, and $f : \mathcal{X} \rightsquigarrow \mathcal{U}$ a PGR-morphism. We want to show that f extends to a morphism $\mathcal{X} \to \mathcal{U}$ over $\mathcal{O}_F$.

Consider $\mathscr{S}'/\mathcal{O}_F$ a scheme which is an étale atlas over $\mathscr{S}/\mathcal{O}_F$. By Lemma 5.3 (rather, its PGR analogue), $\mathscr{S}'$ has PGR-extension. Let $\mathcal{X}'/\mathcal{O}_F$ and $\mathcal{U}'/\mathcal{O}_F$ be the algebraic spaces obtained by base change by $\mathscr{S}' \to \mathscr{S}$. Since $\mathscr{S}' \to \mathscr{S}$ is an étale presentation of a Deligne-Mumford stack with separated and quasi-compact diagonal, $\mathcal{X}'$ and $\mathcal{U}'$ are schemes (see [Alp24, Remark 3.1.5]). By Proposition 5.10, $\mathcal{U}'$ has PGR-extension, and since $f' : \mathcal{X}' \rightsquigarrow \mathcal{U}'$ is again PGR, thus we find a unique morphism

$$\tilde{f}' : \mathcal{X}' \to \mathcal{U}'$$

extending f'. Now it suffices to show that the composition $\mathcal{X}' \to \mathcal{U}' \to \mathcal{U}$ descends to a morphism $\mathcal{X} \to \mathcal{U}$, in other words we need to show that both composition arrows

$$\mathcal{X}' \times_{\mathcal{X}} \mathcal{X}' \rightrightarrows \mathcal{X}' \to \mathcal{U}' \to \mathcal{U}$$

are isomorphic. This follows from Proposition 3.13. □

We end this paragraph by showing that PGR-extension descends along finite torsors.

Proposition 5.13: *Let H be a finite group. Let $\mathscr{M}/\mathcal{O}_F$ be a separated Deligne-Mumford stack. Let $P : \mathscr{M} \to \mathbf{B}H$ denote an H-torsor on $\mathscr{M}$, consider $\mathcal{U} \to \mathscr{M}$ the corresponding finite étale cover. If $\mathcal{U}$ has PGR-extension, then so does $\mathscr{M}$.*

Proof. Let $\mathcal{X}/\mathcal{O}_F$ be as in Definition 5.6, and $f : \mathcal{X} \rightsquigarrow \mathscr{M}$ a PGR-morphism. We want to show that f extends to a morphism $\mathcal{X} \to \mathscr{M}$ over $\mathcal{O}_F$.

Let $\mathcal{U}'_\eta := \mathcal{X}_\eta \times_{\mathscr{M}_\eta} \mathcal{U}_\eta$. We will argue that $\mathcal{U}'_\eta/F$ has a smooth model $\mathcal{U}'/\mathcal{O}_F$, and that there is an H-torsor $\mathcal{U}' \to \mathcal{X}$ extending $\mathcal{U}'_\eta \to \mathcal{X}_\eta$. The following commutative diagram summarizes the situation

$$\begin{tikzcd}
\mathcal{U}'_\eta \arrow[r] \arrow[d] \arrow[drr] & \mathcal{X}_\eta \arrow[d, "f_\eta"'] \arrow[drr] & & \\
\mathcal{U}_\eta \arrow[r] \arrow[drr] & \mathscr{M}_\eta \arrow[drr] & \mathcal{U}' \arrow[r] \arrow[d, dashed, "?"] & \mathcal{X} \arrow[d, dashed, "?"] \\
& & \mathcal{U} \arrow[r] & \mathscr{M}
\end{tikzcd}$$

It then follows from Lemma 5.14 below that the projection $\mathcal{U}' \rightsquigarrow \mathcal{U}$ is a PGR-morphism, hence extends to $\mathcal{U}' \to \mathcal{U}$[9]. It remains to show that the composition $\mathcal{U}' \to \mathcal{U} \to \mathscr{M}$ descends to a morphism $\mathcal{X} \to \mathscr{M}$, this is done exactly as in the proof of Theorem 5.9 using Proposition 3.13.

We construct $\mathcal{U}'$ as follows. From Remark 2.25 we have the 2-cartesian diagram

$$\begin{tikzcd}
\mathcal{U}'_\eta \arrow[r] \arrow[d] & \mathcal{X}_\eta \arrow[d] \\
\mathrm{pt} \arrow[r] & \mathbf{B}H.
\end{tikzcd}$$

If we can show that $\mathcal{X}_\eta \to \mathbf{B}H$ extends to $\mathcal{X} \to \mathbf{B}H$, we then define $\mathcal{U}$ as the fiber product

$$\begin{tikzcd}
\mathcal{U}' \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\
\mathrm{pt} \arrow[r] & \mathbf{B}H.
\end{tikzcd}$$

(i.e. the total space of the H-torsor on $\mathcal{X}$, which is a scheme) Then $\mathcal{U}' \to \mathcal{X}$ has all required properties. To extend $\mathcal{X}_\eta \to \mathbf{B}H$, we use Corollary 4.3. Let $x \in |\mathcal{X}|_\eta^{\mathrm{int}}$ with residue field $F(x)$, consider the H-torsor $P = \mathcal{U} \times_{\mathscr{M}} \operatorname{Spec} F(x)$ over $\operatorname{Spec} F(x)$. By Definition, $\operatorname{Spec} F(x) \to \mathscr{M}$ extends to $\operatorname{Spec} \mathcal{O}_{F(x)} \to \mathscr{M}$, so $\mathcal{P} := \mathcal{U} \times_{\mathscr{M}} \operatorname{Spec} \mathcal{O}_{F(x)}$ is an H-torsor over $\mathcal{O}_{F(x)}$ extending P. □

Lemma 5.14: *Let $\mathscr{M}/\mathcal{O}_F$ be an algebraic stack, $\mathcal{U} \to \mathscr{M}$ a proper affine morphism of stacks, and let $\mathcal{X}/\mathcal{O}_F$ be as in Definition 5.6, together with a PGR-morphism $f : \mathcal{X} \rightsquigarrow \mathscr{M}$. Let $\mathcal{U}'_\eta := \mathcal{U}_\eta \times_{\mathscr{M}_\eta} \mathcal{X}_\eta$, suppose it admits a model $\mathcal{U}'/\mathcal{O}_F$ such that the generic morphism $\mathcal{U}' \rightsquigarrow \mathcal{X}$ extends to $\mathcal{U}' \to \mathcal{X}$ over $\mathcal{O}_F$. Then the generic morphism $\mathcal{U}' \rightsquigarrow \mathcal{U}$ is PGR.*

[9]While the existence of $\mathcal{U}$ holds even assuming H profinite, here we need H to be finite because $\mathcal{U}'/\mathcal{O}_F$ is required to be smooth (in particular, finite type) in Definition 5.6.

Proof. Consider the 2-cartesian diagram

$$\begin{tikzcd} \mathcal{U}'_\eta \arrow[r] \arrow[d] & \mathcal{X}_\eta \arrow[d] \\ \mathcal{U}_\eta \arrow[r] & \mathscr{M}_\eta. \end{tikzcd}$$

Since $\mathcal{U}' \rightsquigarrow \mathcal{X}$ extends, it is a PGR morphism, hence so is the composition $\mathcal{U}' \rightsquigarrow \mathscr{M}$. It is enough to show that the pre-image of $|\mathscr{M}|_\eta^{\mathrm{int}}$ via $\mathcal{U} \to \mathscr{M}$ is $|\mathcal{U}|_\eta^{\mathrm{int}}$. Let $x : \operatorname{Spec} E \to \mathcal{U}$, for some finite E/F, mapping to $|\mathscr{M}|_\eta^{\mathrm{int}}$; up to extending E, we may assume that there is a 2-commutative diagram of solid arrows

$$\begin{tikzcd} \operatorname{Spec} E \arrow[r] \arrow[d] & \mathcal{U} \arrow[d] \\ \operatorname{Spec} \mathcal{O}_E \arrow[r] \arrow[ur, dashed] & \mathscr{M}. \end{tikzcd}$$

We argue that there exists a dashed arrow making the diagram 2-commutative, so that x lies in $|\mathcal{U}|_\eta^{\mathrm{int}}$. First, we base-change $\mathcal{U} \to \mathscr{M}$ along the bottom horizontal arrow so as to reduce to the case:

$$\begin{tikzcd} \operatorname{Spec} E \arrow[r] \arrow[d] & \mathcal{U}_{\mathcal{O}_E} \arrow[d] \\ \operatorname{Spec} \mathcal{O}_E \arrow[r, "\mathrm{id}"] \arrow[ur, dashed] & \operatorname{Spec} \mathcal{O}_E \end{tikzcd}$$

where the vertical arrow on the right is proper and affine. We conclude with the valuative criterion for properness of a morphism of schemes[10]. □

5.3. Smooth-extension.

Definition 5.15: Let $\mathscr{S}/\mathcal{O}_F$ be a separated Deligne-Mumford stack. We say that it has *smooth-extension* if every generic morphism $f : \mathcal{X} \rightsquigarrow \mathscr{S}$ from a regular, formally smooth scheme $\mathcal{X}/\mathcal{O}_F$, automatically and uniquely extends to an $\mathcal{O}_F$-morphism $\mathcal{X} \to \mathscr{S}$.

Restricting Definition 5.15 to smooth separated schemes $\mathcal{X}/\mathcal{O}_F$ would recover the definition of Néron models. As we will be dealing with pro-schemes, we need to remove the finite type assumption.

We state two lemmas to be used in the following section.

Lemma 5.16: *Let $\mathscr{S}/\mathcal{O}_F$ be a separated Deligne-Mumford stack with smooth-extension.*

(i) *Let $\mathscr{Z} \hookrightarrow \mathscr{S}$ a closed substack. Then $\mathscr{Z}$ has smooth-extension.*
(ii) *Suppose that there exists a surjection $U \to \mathscr{S}$ from a scheme U such that every quasi-compact open of U has finitely many irreducible components. Let $\mathscr{S}^\nu \to \mathscr{S}$ denote the normalization[11] of $\mathscr{S}$. Suppose moreover that the generic fiber $\mathscr{S}_F$ is normal, i.e. that $(\mathscr{S}^\nu)_F \to \mathscr{S}_F$ is an isomorphism. Then $\mathscr{S}^\nu$ has smooth-extension.*

Proof. (i) This follows formally from the fact that any morphism $\mathscr{X} \to \mathscr{S}$ such that $\mathscr{X}_\eta$ factors through $\mathscr{Z}_\eta$, factors through $\mathscr{Z}$.

(ii) We may assume that the generic fiber $\mathscr{S}_F$ is nonempty, otherwise so is $(\mathscr{S}^\nu)_F$, and $\mathscr{S}^\nu$ satisfies smooth extension vacuously.

To show smooth-extension for $\mathscr{S}^\nu$, it suffices to argue that any $\mathcal{O}_F$-morphism $\mathcal{X} \to \mathscr{S}$ from a flat, regular $\mathcal{O}_F$-scheme $\mathcal{X}$ factors through $\mathscr{S}^\nu$. Let $V \to \mathscr{S}$ be an étale atlas (in particular V_F is nonempty), set $V^\nu := V \times_{\mathscr{S}} \mathscr{S}^\nu$, which by definition of $\mathscr{S}^\nu$ is the normalization of V. Let $\mathcal{X}_V := \mathcal{X} \times_{\mathscr{S}} V$ and

[10] Dropping the affineness assumption, the valuative criterion for properness of a morphism of stacks only gives the dashed arrow after an extension of valuation ring, which need not be finite in general, so we cannot conclude. More generally, we could have replaced "affine" with "representable by a scheme".

[11] See [Sta26, Tag 0GMH] for the notion of normalization of an algebraic stack; the surjection $U \to \mathscr{S}$ is a sufficient condition for the existence of $\mathscr{S}^\nu$

$\mathcal{X}_{V^\nu} := \mathcal{X} \times_{\mathscr{S}} V^\nu$. Consider the cartesian diagram of schemes

$$\begin{tikzcd} \mathcal{X}_{V^\nu} \arrow[r] \arrow[d, "h"] & V^\nu \arrow[d, "g"] \\ \mathcal{X}_V \arrow[r] & V. \end{tikzcd}$$

We aim to show that h is an isomorphism. Flatness of $\mathcal{X}/\mathcal{O}_F$ ensures that the generic fiber of $\mathcal{X}_V$ is nonempty, a fortiori dense open. Notice that g is integral and is an isomorphism between the generic fibers (as $\mathscr{S}_F$, hence also V_F, is normal), thus the same can be said for h. Furthermore, $\mathcal{X}_V$ is regular as an étale cover of a regular scheme. In particular it is locally Noetherian, hence a disjoint union of (open) connected components; replacing $\mathcal{X}_V$ with such a component, we can assume that it is connected regular, hence integral. It follows that the generic fiber $\mathcal{X}_{V^\nu,F}$ is an integral dense open subscheme of $\mathcal{X}_{V^\nu}$, so the latter is integral. The fact that h is an isomorphism now follows from [Sta26, Tag 0AB1]. Thus we have found a lift of $\mathcal{X}_V \to V$ to $\mathcal{X}_V \to V^\nu$.

It remains to show that the composition $\mathcal{X}_V \to V^\nu \to \mathscr{S}^\nu$ descends to a morphism $\mathcal{X} \to \mathscr{S}^\nu$, in other words we need to show that both composition arrows

$$\mathcal{X}_V \times_{\mathcal{X}} \mathcal{X}_V \rightrightarrows \mathcal{X}_V \to \mathscr{S}^\nu$$

are isomorphic. This follows from Proposition 3.13. □

Lemma 5.17: *Let $\mathcal{O}'$ be a faithfully flat, formally smooth $\mathcal{O}_F$-algebra with the following property: for any regular, formally smooth $\mathcal{O}_F$-scheme $\mathcal{X}$, the scheme $\mathcal{X}_{\mathcal{O}'}$ is again regular.*

Let $\mathscr{S}/\mathcal{O}_F$ be a separated Deligne-Mumford stack such that $\mathscr{S}_{\mathcal{O}'}/\mathcal{O}$ has smooth extension. Then $\mathscr{S}/\mathcal{O}_F$ has smooth extension.

Proof. Consider a generic morphism $f : \mathcal{X} \rightsquigarrow \mathscr{S}$ from a regular, formally smooth scheme $\mathcal{X}/\mathcal{O}_F$. By assumption $\mathcal{X}_{\mathcal{O}'}$ is a regular, formally smooth $\mathcal{O}'$-scheme. By smooth-extension we find $\mathcal{X}_{\mathcal{O}'} \to \mathscr{S}_{\mathcal{O}'}$. Now it suffices to show that the composition $\mathcal{X}_{\mathcal{O}'} \to \mathscr{S}_{\mathcal{O}'} \to \mathscr{S}$ descends to a morphism $\mathcal{X} \to \mathscr{S}$, in other words we need to show by fpqc descent that both composition arrows

$$\mathcal{X}_{\mathcal{O}'} \times_{\mathcal{X}} \mathcal{X}_{\mathcal{O}'} \rightrightarrows \mathcal{X}_{\mathcal{O}'} \to \mathscr{S}_{\mathcal{O}'} \to \mathscr{S}$$

are isomorphic. This follows from Proposition 3.13 once we show that $\mathcal{X}_{\mathcal{O}'} \times_{\mathcal{X}} \mathcal{X}_{\mathcal{O}'}$ is regular. Observe that $\mathcal{X}_{\mathcal{O}'}$ is regular, formally smooth over $\mathcal{O}_F$ (here we use that $\mathcal{O}'$ is formally smooth over $\mathcal{O}_F$), so its base-change along $\operatorname{Spec}\mathcal{O}' \to \operatorname{Spec}\mathcal{O}_F$ is regular by assumption. This base-change recovers $\mathcal{X}_{\mathcal{O}'} \times_{\mathcal{X}} \mathcal{X}_{\mathcal{O}'}$. □

L-extension can be inferred from smooth-extension. The following statement is a reformulation of [BST24, §1.3].

Proposition 5.18: *Let R be a topological ring, let V be a free R-module of finite rank, let K_0 be a profinite subgroup of $\mathrm{GL}(V)$. Recall that $\mathcal{C}(K_0)$ denotes the set of all open normal subgroups of K_0.*

Let $(\mathscr{S}_K)_{K \in \mathcal{C}(K_0)}$ be a projective system of separated Deligne-Mumford stacks over $\mathcal{O}_F$ which is fit for K_0 as in Definition 2.17. Let

$$\mathscr{S}_\infty := \varprojlim_{K \in \mathcal{C}(K_0)} \mathscr{S}_K.$$

Let $\mathbb{L}^{\mathrm{can}}$ be the canonical local system on $\mathscr{S}_{K_0}$ as in Definition 2.18. If $\mathscr{S}_\infty$ has smooth-extension, then $(\mathscr{S}_{K_0}, \mathbb{L}^{\mathrm{can}})$ has L-extension.

Proof. Let $\mathcal{X}/\mathcal{O}_F$ be a smooth scheme, let $f : \mathcal{X} \rightsquigarrow \mathscr{S}_{K_0}$ such that $f_\eta^{-1}\mathbb{L}^{\mathrm{can}}$ extends to a local system $\mathbb{L}$ on $\mathcal{X}$. We want to show that f extends to $\tilde{f} : \mathcal{X} \to \mathscr{S}_{K_0}$.

By definition, $\mathbb{L}^{\mathrm{can}}$ is induced from a K_0-torsor on $\mathscr{S}_{K_0}$, thus $\mathbb{L}$ is induced from a K_0-torsor on $\mathcal{X}$. From Proposition 2.20, the corresponding morphism $\mathcal{X} \to \mathbf{B}K_0$ defines a K_0-fit system of schemes $(\mathcal{X}_K)_{K \in \mathcal{C}(K_0)}$, with $\mathcal{X}_{K_0} = \mathcal{X}$. Let

$$\mathcal{X}_\infty := \varprojlim_K \mathcal{X}_K$$

which is a regular formally smooth $\mathcal{O}_F$-scheme. The K_0-fit system of generic fibers $((\mathcal{X}_K)_\eta)_K$ over F coincides with the pullback along f_η of the fit system $((\mathscr{S}_K)_\eta)_K$. In particular, there is an induced

F-morphism

$$(\mathscr{X}_\infty)_\eta \to (\mathscr{S}_\infty)_\eta$$

By smooth-extension, there is an induced morphism

$$\mathscr{X}_\infty \to \mathscr{S}_\infty.$$

Now it suffices to show that the composition $\mathscr{X}_\infty \to \mathscr{S}_\infty \to \mathscr{S}_{K_0}$ descends to a morphism $\mathscr{X}_{K_0} \to \mathscr{S}_{K_0}$, in other words we need to show by fpqc descent that both composition arrows

$$\mathscr{X}_\infty \times_{\mathscr{X}_{K_0}} \mathscr{X}_\infty \rightrightarrows \mathscr{X}_\infty \to \mathscr{S}_\infty \to \mathscr{S}_{K_0}$$

are isomorphic. This follows from Proposition 3.13. $\square$

6. Smooth-extension for integral models of Shimura varieties

Producing schemes with smooth-extension will be our starting point for finding stacks with PGR-extension. One source such of schemes is the theory of *integral canonical models* of Shimura varieties, which we recall in the first paragraph below.

The remaining paragraphs deal with the following problem: having smooth-extension is not preserved under general base change. As integral canonical models of Shimura varieties are usually defined over a valuation ring $\mathcal{O}_{(v)}$ of their reflex field, it is not certain that their base-change to $\mathcal{O}_F$ still has smooth-extension if, say, $F/\mathbb{Q}_p$ is a finite, possibly ramified extension. We aim to show that there is no issue whenever the absolute ramification index e of $F/\mathbb{Q}_p$ is $\leq p-1$.

This issue leaves us no choice but to dive into the derivations. Namely, starting with a discrete valuation ring $\mathcal{O}$ faithfully flat over $\mathcal{O}_{(v)}$, we recall the constructions of integral canonical models $\mathscr{S}$ in the cases of Siegel, Hodge, abelian and orthogonal-type, and investigate the proofs that they have smooth-extensions to see that $\mathscr{S}_\mathcal{O}$ also does if $\mathcal{O}$ has absolute ramification index $e \leq p-1$. Some of the original derivations are technical and too long to fit faithfully in this section; we tried to find a balance between readable sketches of proofs and convincing arguments. This is in particular the case for Section 6.4 and Section 6.5.

The reader willing to take the results of this section for granted is advised to read only Section 6.1 and Section 6.2, and to go back to the statements of the remaining paragraphs while reading Section 7.

6.1. **Integral canonical models.** Let (G,Ω) be a Shimura datum with reflex field $E/\mathbb{Q}$, and $K_p \subseteq G(\mathbb{Q}_p)$ a hyperspecial subgroup. Let v be a place of E lying above p, $\mathcal{O}_{(v)}$ the local ring of $\mathcal{O}_E$ at v and $\mathcal{O}_v$ its v-adic completion. Consider a neat level $K = K^pK_p$ where K^p is a compact open subgroup of $G(\mathbb{A}_f^p)$, write $\mathrm{Sh}_K(G,\Omega)/E$ for the corresponding variety over E. Set

$$\mathrm{Sh}_{K_p}(G,\Omega) = \varprojlim_{K^p} \mathrm{Sh}_{K^pK_p}(G,\Omega)$$

which is an E-scheme with ind-continuous $G(\mathbb{A}_f^p)$-action (see Remark 6.2 below). If $K = K^pK_p$ need no longer be neat, we let $\mathrm{Sh}_K[G,\Omega] = [\mathrm{Sh}_{K_p}(G,\Omega)/K^p]$ denote the quotient stack; for K neat this is a scheme which coincides with $\mathrm{Sh}_K(G,\Omega)$.

Let $\mathcal{O}$ be a discrete valuation ring, faithfully flat over $\mathcal{O}_{(v)}$, write $\tilde{E}$ for its fraction field and $\mathfrak{m}_\mathcal{O}$ for its maximal ideal. We recall the notions of integral canonical models of Shimura varieties.

Definition 6.1: An *integral model* of $\mathrm{Sh}_{K_p}(G,\Omega)$ over $\mathcal{O}$ is a faithfully flat $\mathcal{O}$-scheme denoted $\mathscr{S}_{K_p}(G,\Omega)$ with an ind-continuous $G(\mathbb{A}_f^p)$-action (see Remark 6.2 below), such that

(i) there is a $G(\mathbb{A}_f^p)$-equivariant identification $\mathscr{S}_{K_p}(G,\Omega) \times_\mathcal{O} \tilde{E} \simeq \mathrm{Sh}_{K_p}(G,\Omega) \times_E \tilde{E}$, and

(ii) there exists a compact open subgroup $K_0^p \subseteq G(\mathbb{A}_f^p)$ such that for any $K_2^p \subseteq K_1^p \subseteq K_0^p$ open subgroups, the canonical quotient map

$$\mathscr{S}_{K_p}(G,\Omega)/K_1^p \to \mathscr{S}_{K_p}(G,\Omega)/K_2^p$$

is a finite étale morphism between separated schemes of finite type over $\mathcal{O}$.

Remark 6.2: We specify what is meant by the nonstandard terminology "ind-continuous action" on a scheme T over a base S. Recall from Definition 2.4 that to a topological group Q corresponds a scheaf of groups $\mathcal{F}_Q$ over the pro-étale site $(\mathrm{Sch}/S)_{\text{proét}}$, mapping an S-scheme U to the group of continuous functions $U \to Q$. A sheaf with an action of $\mathcal{F}_Q$ is called a Q-sheaf, as in Definition 2.6. A *continuous*

Q-action on an S-scheme T is defined to be a Q-sheaf structure on the functor of points h_T over $(\mathrm{Sch}/S)_{\text{proét}}$:

$$\mathcal{F}_Q \times h_T \longrightarrow h_T.$$

For Shimura varieties however, we need a weaker notion. Let $\mathcal{K}(Q)$ denote the set of all compact open subgroups of Q, and assume that

$$Q = \varinjlim_{K \in \mathcal{K}(Q)} K.$$

An *ind-continuous Q-action* on an S scheme T is defined to be an action of the sheaf of groups

$$\varinjlim_{K \in \mathcal{K}(Q)} \mathcal{F}_K,$$

which in general is a strict subsheaf of $\mathcal{F}_Q$, on the functor of points h_T over $(\mathrm{Sch}/S)_{\text{proét}}$. In this case, it makes sense to talk about the quotient stack $[T/K]$ for any $K \in \mathcal{K}(Q)$.

In the context of Shimura varieties, let K_0 be a compact open subgroup of $G(\mathbb{A}_f)$ and $K \in \mathcal{C}(K_0)$, it is known that there is an action of the finite constant group scheme K_0/K on $\mathrm{Sh}_K(G, \Omega)$ which preserves the projection map to $\mathrm{Sh}_{K_0}(G, \Omega)$. In terms of sheaves over the pro-étale site on $\operatorname{Spec} E$, there is an action

$$\mathcal{F}_{K_0/K} \times h_{\mathrm{Sh}_K(G,\Omega)} \longrightarrow h_{\mathrm{Sh}_K(G,\Omega)}.$$

If we let $\mathrm{Sh}_\infty(G, \Omega) = \varprojlim_{K \in \mathcal{C}(K_0)} \mathrm{Sh}_K(G, \Omega)$, we can pass to the projective limit to find

$$\mathcal{F}_{K_0} \times h_{\mathrm{Sh}_\infty(G,\Omega)} \longrightarrow h_{\mathrm{Sh}_\infty(G,\Omega)}.$$

We now take the inductive limit over K_0 to find the ind-continuous action. The arguement is unchanged when replacing $G(\mathbb{A}_f)$ with $G(\mathbb{A}_f^p)$.

Definition 6.1 is vacuous as it is: $\mathscr{S}_{K_p}(G, \Omega) \coloneqq \mathrm{Sh}_{K_p}(G, \Omega)$ qualifies as an integral model. We want our models to be canonical in some sense, which is achieved by requiring $\mathscr{S}_{K_p}(G, \Omega)$ to satisfy some extension property. Namely, let $\mathcal{P}$ be some property satisfied by $\mathcal{O}$-schemes, then one could ask for the following:

(i) the scheme $\mathscr{S}_{K_p}(G, \Omega)$ satisfies $\mathcal{P}$, and
(ii) every generic morphism $f : \mathcal{X} \rightsquigarrow \mathscr{S}_{K_p}(G, \Omega)$ from a scheme $\mathcal{X}/\mathcal{O}$ satisfying $\mathcal{P}$, automatically and uniquely extends to an $\mathcal{O}$-morphism $\mathcal{X} \to \mathscr{S}$.

These properties ensure the uniqueness of the integral model $\mathscr{S}_{K_p}(G, \Omega)$ up to unique isomorphism. Definition 5.15 motivates the case

$$\mathcal{P} = \text{being a regular, formally smooth } \mathcal{O}\text{-scheme.}$$

In fact, item (i) above is replaced with a slightly stronger condition.

Definition 6.3: Let $\mathscr{S}_{K_p}(G, \Omega)$ be an integral model as in Definition 6.1. It is a *smooth integral canonical model* if moreover:

(i) For small enough $K^p \subseteq K_0^p$, the scheme $\mathscr{S}_{K_p}(G, \Omega)/K^p$ is smooth, and
(ii) the scheme $\mathscr{S}_{K_p}(G, \Omega)$ has smooth extension as in Definition 5.15.

Unfortunately, useful integral models for orthogonal-type Shimura varieties constructed in [MP16] are not all shown to be formally smooth. Nevertheless, they satisfy an extension property which we now describe. The following notions originate from the work of Vasiu.

Definition 6.4 ([MP16], Definition 2.12)**:**

- A regular, local, faithfully flat $\mathbb{Z}_{(p)}$-algebra R with maximal ideal $\mathfrak{m}$ is *quasi-healthy* if every abelian scheme over $\operatorname{Spec}(R) \setminus \{\mathfrak{m}\}$ extends uniquely to an abelian scheme over $\operatorname{Spec} R$.
- A regular, faithfully flat $\mathbb{Z}_{(p)}$-scheme $\mathcal{X}$ is *healthy* is for every open subscheme U containing the generic fiber $\mathcal{X}_{\mathbb{Q}}$ and all generic points of the special fiber $\mathcal{X}_{\mathbb{F}_p}$, every abelian scheme over U extends uniquely to an abelian scheme over $\mathcal{X}$.
- A regular, faithfully flat $\mathbb{Z}_{(p)}$-scheme $\mathcal{X}$ is *locally healthy* if for every $x \in \mathcal{X}_{\mathbb{F}_p}$ of codimension at least 2, the complete local ring $\widehat{\mathcal{O}_{\mathcal{X},x}}$ is quasi-healthy.

Note that locally healthy implies healthy (see [MP16, Remark 2.13]). The following extension property will not be referenced outside of Section 6.

Definition 6.5: Let $\mathscr{S}/\mathcal{O}$ be a separated Deligne-Mumford stack. We say that it has *healthy-extension* if every generic morphism $f : \mathcal{X} \rightsquigarrow \mathscr{S}$ from a regular, locally healthy scheme $\mathcal{X}/\mathcal{O}_F$, automatically and uniquely extends to an $\mathcal{O}$-morphism $\mathcal{X} \to \mathscr{S}$.

We define integral canonical models in general by choosing $\mathcal{P}$ following Definition 6.5.

Definition 6.6 ([MP15], Definition 4.3)**:** Let $\mathscr{S}_{K_p}(G, \Omega)$ be an integral model as in Definition 6.1. It is an *integral canonical model* if moreover:

(i) the scheme $\mathscr{S}_{K_p}(G, \Omega)$ is regular, locally healthy, and
(ii) the scheme $\mathscr{S}_{K_p}(G, \Omega)$ has healthy-extension.

Although integral canonical models need not be formally smooth, the work of Vasiu-Zink still shows that they have smooth extension whenever $\mathcal{O}$ is not too ramified.

Proposition 6.7: *Suppose $\mathcal{O}$ has absolute ramification index $e \leq p-1$. Then any regular, formally smooth $\mathcal{O}$-scheme $\mathcal{X}$ is locally healthy, hence healthy.*

In particular, is $\mathscr{S}/\mathcal{O}$ is a separated Deligne-Mumford stack with healthy-extension, then it has smooth extension.

Proof. It is stated in [VZ10, Corollary 5] that such an $\mathcal{X}$ is healthy. Let us run through the argument to see that $\mathcal{X}$ is in fact locally healthy. Let $x \in \mathcal{X}_{\mathbb{F}_p}$ be of codimension ≥ 2, set $R = \mathcal{O}_{\mathcal{X},x}$. We want to show that the completion $\widehat{R}$ is quasi-healthy. Then, [VZ10, Theorem 3] provides a criterion on the strict completion $\widehat{A}^s$ of a ring A implying that A is quasi-healthy. Our assumptions ensure that the criterion is satisfied for $A = R$, so R is quasi-healthy. But R and $\widehat{R}$ share the same strict completion, so we find that $\widehat{R}$ is also quasi-healthy, as desired. The second statement follows. □

Corollary 6.8: *Suppose $\mathcal{O}$ has absolute ramification index $e \leq p-1$. Let $\mathscr{S}_{K_p}(G, \Omega)$ be an integral canonical model as in Definition 6.6. Then $\mathscr{S}_{K_p}(G, \Omega)$ has smooth extension.*

Let $\mathscr{S}_{K_p}(G, \Omega)$ be an integral model of $\mathrm{Sh}_{K_p}(G, \Omega)$ over $\mathcal{O}$. Suppose either that

- it is a smooth integral canonical model, or
- it is an integral canonical model, and $e \leq p-1$.

Then we have established that $\mathscr{S}_{K_p}(G, \Omega)/\mathcal{O}$ has smooth extension. For any level $K = K^pK_p$, we let $\mathscr{S}_K[G, \Omega] \coloneqq [\mathscr{S}_{K_p}(G, \Omega)/K^p]$ denote the quotient stack. Fix K_0^p any compact open subgroup of $G(\mathbb{A}_f^p)$, then the projective system $(\mathscr{S}_{K^pK_p}[G, \Omega])_{K^p \in \mathcal{C}(K_0^p)}$ is fit for K_0^p. From Definition 2.18 there is a canonical local system $\mathbb{L}^{\mathrm{can}}$ on $\mathscr{S}_{K_0^pK_p}[G, \Omega]$.

Proposition 6.9: *The pair $(\mathscr{S}_{K_0^pK_p}[G, \Omega], \mathbb{L}^{\mathrm{can}})$ has L-extension, hence also PGR-extension.*

Proof. This follows from Proposition 5.18 and Proposition 5.8. □

Integral canonical models have been shown to exist for Shimura varieties of Siegel, Hodge and abelian-type, as well as certain orthogonal-type[12] over a localization of the ring of integers of their reflex field. In the following paragraphs, we sketch their constructions. The important takeaway is that the abelian and orthogonal cases are bootstrapped from the Hodge case, itself bootstrapped from the Siegel case.

6.2. Smooth-extension for moduli spaces of polarized abelian schemes.

We begin with some recollections on moduli spaces of polarized abelian schemes. Set $g > 0$ an integer; all abelian schemes in this section are of relative dimension g. Let R be a ring, fix V a free R-module of rank $2g > 0$, and ψ a symplectic form on V, by which is meant an R-valued, alternating non-degenerate bilinear form.

[12] This is also true for PEL Shimura varieties, see [Kot], but we will not need them.

Then there exists a unique sequence (up to invertible factors) of elements $d_1|\dots|d_g$ of R such that ψ is represented in some basis by the matrix

$$J_\Delta := \begin{pmatrix} 0 & \Delta \\ -\Delta & 0 \end{pmatrix}$$

where $\Delta = \mathrm{diag}(d_1, \dots, d_g)$. We call a matrix of the form Δ a *symplectic type*.

We now assume that (V, ψ) is defined over $\mathbb{Z}$. Consider the $\mathbb{Z}$-group scheme $\mathcal{G} := \mathrm{GSp}(V, \psi)$, set $G := \mathcal{G}_\mathbb{Q}$ its generic fiber. After fixing an appropriate basis, the R-points of $\mathcal{G}$ for some algebra R are described by

$$\mathcal{G}(R) = \{M \in \mathrm{GL}_{2g}(R) \mid MJ_\Delta M^t = J_\Delta\}.$$

For a prime $p > 0$, $\mathcal{G}_{\mathbb{Z}_{(p)}}$ is reductive (equiv. smooth) if and only if none of the d_i are multiples of p, in which case we say that Δ *is prime to* p.

Let A be a polarized abelian variety over a field k of characteristic $p \geq 0$. The polarization induces a symplectic form on the $\widehat{\mathbb{Z}}^p$-Tate module of A, defined to be the product of all ℓ-adic Tate modules of A for $p \neq \ell > 0$. We say that A has *symplectic type* Δ if the type of the $\widehat{\mathbb{Z}}^p$-Tate module of A is Δ. For any base scheme T, a polarized abelian scheme A over T has *(symplectic) type* Δ if all its geometric fibers have symplectic type Δ.

Fix a prime $p > 0$. We study the moduli spaces of polarized abelian schemes of symplectic type Δ.

Definition 6.10: For any $\mathbb{Z}_{(p)}$-scheme T, we set $\mathscr{A}_{g,\Delta}(T)$ to be the groupoid of polarized abelian schemes $A \to T$ of symplectic type Δ. Then it is known that $\mathscr{A}_{g,\Delta}$ is a Deligne-Mumford stack over $\mathbb{Z}_{(p)}$, which is smooth if and only if Δ is prime to p.

We now add level structures to the mix using the formalism of Section 2.5. Set $R := \widehat{\mathbb{Z}}^p = \prod_{\ell \neq p} \mathbb{Z}_\ell$ and $V_R := V \otimes R$, as well as $Q = \mathcal{G}(R) \hookrightarrow \mathrm{GL}(V_R)$. Consider the following objects (see the first paragraph of Section 2.5 for the simpler case of principally polarized abelian schemes).

(i) Set $S := \operatorname{Spec} \mathbb{Z}_{(p)}$ and let $\mathscr{C}$ denote the big pro-étale site over S (see Section 2.1 for details), so that $\mathscr{A}_{g,\Delta}$ is a stack over $\mathscr{C}$.

(ii) For any $T \to \mathscr{A}_{g,\Delta}$ corresponding to a polarized abelian scheme $f : A \to T$, we set $\mathbb{L}(A)$ the R-local system defined as its (relative) $\widehat{\mathbb{Z}}^p$-Tate module. This functor defines a local system on $\mathscr{A}_{g,\Delta}$

$$\mathbb{L} : \mathscr{A}_{g,\Delta} \to \mathbf{Loc}_{V_R}$$

as defined in Definition 2.15.

(iii), (iv) Similarly, the functor sending $f : A \to T$ to the Q-subsheaf of $\underline{\mathrm{Isom}}(L(A), \mathcal{F}_{V_R})$ consisting of those isomorphisms compatible with the symplectic structures, defines a Q-torsor on $\mathscr{A}_{g,\Delta}$

$$\mathrm{Lev} : \mathscr{A}_{g,\Delta} \to \mathbf{B}Q$$

as defined in Definition 2.15.

This data specifies a Q-moduli datum as in Definition 2.26, we will now use Definition 2.28. Set $K_0 = \mathcal{G}(\widehat{\mathbb{Z}})$, so that $K_0^p = Q$ is a compact subgroup of Q[13], and $K_p := K_{0,p}$ is the stabilizer of $V_{\mathbb{Z}_p}$ in $G(\mathbb{Q}_p)$. For any compact open subgroup $K^p \subseteq K_0^p$, set $K = K_p K^p$.

Definition 6.11: We write $\mathscr{A}_{g,\Delta,K}$ for the moduli stack over $\mathbb{Z}_{(p)}$ of level K^p as in Definition 2.28. Its objects are pairs $(A, [\eta]_{K^p})$ where A is a polarized abelian scheme of type Δ, and $[\eta]_{K^p}$ is a global section of the quotient sheaf $\mathrm{Lev}(A)/\mathcal{F}_{K^p}$. For K^p sufficiently small, it is represented by a quasi-projective $\mathbb{Z}_{(p)}$-scheme. Additionally, we set

$$\mathscr{A}_{g,\Delta,K_p} := \varprojlim_{K^p \in \mathcal{C}(K_0^p)} \mathscr{A}_{g,\Delta,K}.$$

Its objects are pairs $(A, [\eta])$ where A is a polarized abelian scheme of type Δ, and $[\eta]$ is a global section of $\mathrm{Lev}(A)$. In particular, the $\widehat{\mathbb{Z}}^p$-Tate module of A has trivial monodromy.

Since K_p is fixed by our choice of symplectic structure, we also call $\mathscr{A}_{g,\Delta,K}$ the moduli stack of polarized abelian schemes of type Δ and of level K.

[13] We emphasize on this tautology to warn the reader that K_0^p (resp. K^p) here is what plays the role of K_0 (resp. K) in Definition 2.28.

Let $\mathcal{O}$ denote a discrete valuation ring that is a faithfully flat $\mathbb{Z}_{(p)}$-algebra. Our goal for the end of this paragraph is to show the following.

Fact 6.12: *The stack $\mathscr{A}_{g,\Delta,K_p,\mathcal{O}}/\mathcal{O}$ has healthy-extension as in Definition 6.5.*

The proof is adapted from [Mil92, Theorem 2.10]. The following ensures that we can ignore polarizations and focus on solely on the abelian schemes.

Fact 6.13 ([Mil92], Proposition 2.14)**:** *Let $A \to X$ be an abelian scheme on a normal scheme X, let $\lambda : A_U \to A_U^\vee$ be a polarization on A_U where $U \subseteq X$ is a dense open subscheme. Then λ extends to a polarization of A.*

Let $\mathcal{X}/\mathcal{O}$ be regular, locally healthy, together with a generic morphism

$$f : \mathcal{X} \rightsquigarrow \mathscr{A}_{g,\Delta,K_p,\mathcal{O}}.$$

Thus f corresponds to a polarized abelian scheme $A_{\mathcal{X}_\eta} \to \mathcal{X}_\eta$. It suffices to show that it extends to an abelian scheme over some open subscheme U containing $\mathcal{X}_\eta$ and each generic point of the special fiber $\mathcal{X}_s$. Let ξ be such a generic point; since $\mathcal{X}$ is regular we find that the local ring $\mathcal{O}_{\mathcal{X},\xi}$ is a DVR. Let η_ξ denote the generic point of $\operatorname{Spec} \mathcal{O}_{\mathcal{X},\xi}$.

By the Serre-Tate criterion, it suffices to show that the restriction of $A_{\mathcal{X}_\eta}$ to η_ξ has good reduction with respect to ξ, i.e. that its $\widehat{\mathbb{Z}}^p$-Tate module is unramified as a $\pi_1(\eta_\xi)$-module. In fact, the moduli interpretation of $\mathscr{A}_{g,\Delta,K_p,\mathcal{O}}$ (see Definition 6.11) ensures that the $\pi_1(\eta_\xi)$-action is trivial.

Corollary 6.14: *Suppose $\mathcal{O}$ has absolute ramification index $e \leq p-1$. Then $\mathscr{A}_{g,\Delta,K_p,\mathcal{O}}/\mathcal{O}$ has smooth-extension.*

Proof. This follows from Fact 6.12 and Proposition 6.7. □

6.3. **Smooth integral canonical models for Siegel and Hodge-type.** We follow the construction from [Kis09, §2.3]. Beginning with the Siegel case, let $V, \psi, \mathcal{G}, G, \Delta$ and $K_0 = \mathcal{G}(\widehat{\mathbb{Z}})$ as in Section 6.2. Let Ω be the space of complex structures on $V_\mathbb{R}$ for which $(V_\mathbb{R}, \psi_\mathbb{R})$ is a polarized Hodge structure. Then (G, Ω) is a Siegel Shimura datum with reflex field $\mathbb{Q}$, and for any compact open $K^p \subseteq K_0^p$ we let $\mathrm{Sh}_K[G, \Omega]$ denote the associated Shimura stack of level K.

We define an integral model $\mathscr{S}_{K_p}(G, \Omega)$ of $\mathrm{Sh}_{K_p}[G, \Omega]$ with smooth-extension. It should be noted that $\mathscr{S}_{K_p}(G, \Omega)$ will not be smooth in general (and so will not be a smooth integral canonical model as in Definition 6.3) unless Δ is prime to p. Consider K^p sufficiently small so that $\mathscr{A}_{g,\Delta,K}$ is represented by a quasi-projective scheme; in this case we can identify

$$\mathrm{Sh}_K(G, \Omega) = \mathscr{A}_{g,\Delta,K,\mathbb{Q}}$$

and similarly replacing K with K_p.

Definition 6.15: Let $\mathscr{S}_K(G, \Omega)/\mathbb{Z}_{(p)}$ (resp. $\mathscr{S}_{K_p}(G, \Omega)/\mathbb{Z}_{(p)}$) denote the closure of $\mathrm{Sh}_K(G, \Omega)$ (resp. $\mathrm{Sh}_{K_p}(G, \Omega)$) in $\mathscr{A}_{g,\Delta,K}$ (resp. $\mathscr{A}_{g,\Delta,K_p}$).

We extend this definition to all $K^p \subseteq K_0^p$ as we did before, by setting $\mathscr{S}_K[G, \Omega] := [\mathscr{S}_{K_p}(G, \Omega)/K^p]$.

Proposition 6.16: *Let $\mathcal{O}$ denote a discrete valuation ring that is a faithfully flat $\mathbb{Z}_{(p)}$-algebra. Then $\mathscr{S}_{K_p}(G, \Omega)_\mathcal{O}/\mathcal{O}$ has healthy-extension.*

If $\mathcal{O}$ has absolute ramification index $e \leq p-1$, then $\mathscr{S}_{K_p}(G, \Omega)_\mathcal{O}/\mathcal{O}$ has smooth-extension.

Proof. This follows from Fact 6.12 and the healthy-extension counterpart of Lemma 5.16 (i). The second statement follows from Proposition 6.7. □

Corollary 6.17: *Let $\mathcal{O}$ denote a discrete valuation ring that is a faithfully flat $\mathbb{Z}_{(p)}$-algebra of absolute ramification index $e \leq p-1$. Denote F its field of fractions. If Δ is prime to p then K_p is hyperspecial, $\mathscr{A}_{g,\Delta,K}$ is smooth for all K and hence equals $\mathscr{S}_K[G, \Omega]$. In this case, $\mathscr{S}_{K_p}(G, \Omega)_\mathcal{O} = \mathscr{A}_{g,\Delta,K_p,\mathcal{O}}$ is a smooth integral canonical model of $\mathrm{Sh}_{K_p}(G, \Omega)_F$.*

Remark 6.18: It follows from Proposition 2.30 that the canonical local system on $\mathscr{S}_K[G,\Omega]$ coincides with the universal Tate module.

We move on to Hodge-type Shimura varieties. Let (G_1,Ω_1) be a Shimura datum of Hodge-type, meaning that there is an embedding

$$(G_1,\Omega_1)\hookrightarrow(G,\Omega).$$

Let E denote the reflex field of (G_1,Ω_1), fix v a place of E dividing p. Fix $\mathcal{K}_p\subseteq G_1(\mathbb{Q}_p)$ a hyperspecial level. We sketch the proof of the following.

Theorem 6.19 ([Kis09], Theorem 2.3.8)**:** *Suppose* $p>2$[14].

(1) *The scheme* $\mathrm{Sh}_{\mathcal{K}_p}(G_1,\Omega_1)$ *admits a smooth integral canonical model* $\mathscr{S}_{\mathcal{K}_p}(G_1,\Omega_1)$ *over* $\mathcal{O}_{E,(v)}$.
(2) *Moreover, for any discrete valuation ring* $\mathcal{O}$ *that is a faithfully flat* $\mathcal{O}_{E,(v)}$*-algebra of absolute ramification index* $e\leq p-1$*, with field of fractions* F*,* $\mathscr{S}_{\mathcal{K}_p}(G_1,\Omega_1)_{\mathcal{O}}$ *is again a smooth integral canonical model of* $\mathrm{Sh}_{\mathcal{K}_p}(G_1,\Omega_1)_F$.

Statement (1) is the theorem of Kisin. Statement (2) follows for free from the proof and Proposition 6.16.

For $\mathcal{K}^p$ a small enough compact open subgroup of $G_1(\mathbb{A}_f^p)$, it is shown in [Kis09, §2.3.2] that there exists a symplectic space (V,ψ) over $\mathbb{Z}$ (and hence, $\mathcal{G},G,\Delta$ as before; note that Δ need not be prime to p) and a subgroup $K^p\subseteq K_0^p$ such that the embedding of Shimura data induces an inclusion

$$\iota:\mathrm{Sh}_{\mathcal{K}}(G_1,\Omega_1)\hookrightarrow\mathrm{Sh}_K(G,\Omega)_E.$$

Then one defines $\mathscr{S}_{\mathcal{K}}(G_1,\Omega_1)$ as the normalization[15] of the closure of the image of ι in the model $\mathscr{S}_K(G,\Omega)$ from Definition 6.15. Then we set

$$\mathscr{S}_{\mathcal{K}_p}(G_1,\Omega_1):=\varprojlim_{\mathcal{K}^p}\mathscr{S}_{\mathcal{K}}(G_1,\Omega_1).$$

One argues that the pairs $(\mathcal{K}^p,K^p)$ can be chosen coherently so that there is an induced embedding

$$\mathrm{Sh}_{\mathcal{K}_p}(G_1,\Omega_1)\hookrightarrow\mathrm{Sh}_{K_p}(G,\Omega)_E,$$

hence taking the normalization of the closure in $\mathscr{S}_{K_p}(G,\Omega)$ of its image recovers $\mathscr{S}_{\mathcal{K}_p}(G_1,\Omega_1)$. Base-changing to $\mathcal{O}$, smooth-extension follows from Lemma 5.16 and Proposition 6.16. The proof of Theorem 6.19 is now complete.

6.4. **Smooth integral canonical models for abelian-type.** We sketch the construction from [Kis09, §3.4]. Let (G_2,Ω_2) be a Shimura datum of abelian type, meaning that there exists (G_1,Ω_1) a Shimura datum of Hodge type, and a central isogeny

$$G_1^{\mathrm{der}}\to G_2^{\mathrm{der}}$$

inducing an isomorphism $(G_1^{\mathrm{ad}},\Omega_1^{\mathrm{ad}})\simeq(G_2^{\mathrm{ad}},\Omega_2^{\mathrm{ad}})$ (notations from [Kis09, §3.3.6]). If E_i denotes the reflex field of (G_i,Ω_i), then $E_2\subseteq E_1$ and the extension is unramified at places dividing p; fix $v_1|v_2|p$ such places of E_1 and E_2. Fix $\tilde{\mathcal{K}}_p\subseteq G_2(\mathbb{Q}_p)$ hyperspecial, we wish to show:

Theorem 6.20 ([Kis09], Theorem 3.4.14)**:** *Suppose* $p>2$[16].

(1) *The scheme* $\mathrm{Sh}_{\tilde{\mathcal{K}}_p}(G_2,\Omega_2)$ *admits a smooth integral canonical model* $\mathscr{S}_{\tilde{\mathcal{K}}_p}(G_2,\Omega_2)$ *over* $\mathcal{O}_{E_2,(v_2)}$.
(2) *Moreover, for any discrete valuation ring* $\mathcal{O}$ *that is a faithfully flat* $\mathcal{O}_{E_2,(v_2)}$*-algebra of absolute ramification index* $e\leq p-1$*, with field of fractions* F*,* $\mathscr{S}_{\tilde{\mathcal{K}}_p}(G_2,\Omega_2)_{\mathcal{O}}$ *is again a smooth integral canonical model of* $\mathrm{Sh}_{\tilde{\mathcal{K}}_p}(G_2,\Omega_2)_F$.

Statement (1) is the theorem of Kisin. Statement (2) follows from the proof and Theorem 6.19.

Kisin shows that it is always possible to choose (G_1,Ω_1) as above, as well as a hyperspecial level $\mathcal{K}_p\in G_1(\mathbb{Q}_p)$, so that $\mathscr{S}_{\tilde{\mathcal{K}}_p}(G_2,\Omega_2)$ is obtained as follows. Let $\mathscr{S}_{\mathcal{K}_p}(G_1,\Omega_1)/\mathcal{O}_{E_1,(v_1)}$ be the smooth integral canonical model of $\mathrm{Sh}_{\mathcal{K}_p}(G_1,\Omega_1)/E_1$ previously constructed. Let $E_1^{v_1}/E_1$ be the maximal extension unramified at v_1; then it follows from [Kis09, Proposition 2.2.4] that the geometrically connected

[14] The statement holds true for $p=2$ under additional assumptions. We do not need this.

[15] It was shown in [Xu26] that the normalization step is unnecessary.

[16] The statement holds true for $p=2$ under additional assumptions. We do not need this.

components of $\mathscr{S}_{\mathcal{K}_p}(G_1,\Omega_1)$ are defined over $\mathcal{O}_{E_1^{v_1},(w)}$, where w divides v_1. Let $\mathscr{S}_{\mathcal{K}_p}(G_1,\Omega_1)^+/\mathcal{O}_{E_1^{v_1},(w)}$ denote one such component. As $\mathcal{O}_{E_1^{v_1},(w)}$ has ramification index less than p, $\mathscr{S}_{\mathcal{K}_p}(G_1,\Omega_1)_{\mathcal{O}_{E_1^{v_1},(w)}}$ has smooth extension, and so does its closed subscheme $\mathscr{S}_{\mathcal{K}_p}(G_1,\Omega_1)^+$ by Lemma 5.16. Then Kisin defines a group $\mathscr{A}_1$ acting on $\mathscr{S}_{\mathcal{K}_p}(G_1,\Omega_1)^+$, and a group homomorphism $\phi:\mathscr{A}_1\to\mathscr{A}_2$, so as to set (see [Kis09, §3.3.2, §3.4.11])

$$\mathscr{S}_{\tilde{\mathcal{K}}_p}(G_2,\Omega_2)_{\mathcal{O}_{E_1^{v_1},(w)}} := [\mathscr{A}_2\times\mathscr{S}_{\mathcal{K}_p}(G_1,\Omega_1)^+]/\mathscr{A}_1. \tag{1}$$

By [Kis09, Proposition 3.3.10] this is a model of $\mathrm{Sh}_{\tilde{\mathcal{K}}_p}(G_2,\Omega_2)_{E_1^{v_1}}$. To argue that $\mathscr{S}_{\tilde{\mathcal{K}}_p}(G_2,\Omega_2)_{\mathcal{O}_{E_1^{v_1},(w)}}$ has smooth-extension, one reduces to showing that the quotient $\mathscr{S}_{\mathcal{K}_p}(G_1,\Omega_1)^+/\Delta$ has smooth-extension, where $\Delta:=\ker(\phi)$ acts freely. The latter follows from the formal argument in [Moo98, Proposition 3.21.4]. Smooth extension can then be leveraged to extend descent data from $\mathrm{Sh}_{\tilde{\mathcal{K}}_p}(G_2,\Omega_2)_{E_1^{v_1}}$ to its model $\mathscr{S}_{\tilde{\mathcal{K}}_p}(G_2,\Omega_2)_{\mathcal{O}_{E_1^{v_1},(w)}}$, which is then defined over $\mathcal{O}_{E_2,(v_2)}$ as desired.

It remains to show that the base-change $\mathscr{S}_{\tilde{\mathcal{K}}_p}(G_2,\Omega_2)_{\mathcal{O}}$ also has smooth extension. If $\mathcal{O}_{E_1^{v_1},(w)}$ were a subring of $\mathcal{O}$, it would be enough to base change to $\mathcal{O}$ in Equation (1), state that $\mathscr{S}_{\mathcal{K}_p}(G_1,\Omega_1)^+_{\mathcal{O}}$ has smooth extension, and invoke [Moo98, Proposition 3.21.4] once again. As $\mathcal{O}$ need not contain $\mathcal{O}_{E_1^{v_1},(w)}$ in general, we take a small detour.

Let $\mathcal{O}^{\mathrm{sh}}$ denote the strict henselization of $\mathcal{O}$, so that $\mathcal{O}_{E_1^{v_1},(w)}\subseteq\mathcal{O}^{\mathrm{sh}}$. Thus we can write

$$\mathscr{S}_{\tilde{\mathcal{K}}_p}(G_2,\Omega_2)_{\mathcal{O}^{\mathrm{sh}}} = [\mathscr{A}_2\times\mathscr{S}_{\mathcal{K}_p}(G_1,\Omega_1)^+_{\mathcal{O}^{\mathrm{sh}}}]/\mathscr{A}_1.$$

Since $\mathcal{O}^{\mathrm{sh}}$ is a discrete DVR and a faithfully flat $\mathcal{O}_{E_1^{v_1},(w)}$-algebra, it follows from Theorem 6.19 that $\mathscr{S}_{\mathcal{K}_p}(G_1,\Omega_1)^+_{\mathcal{O}^{\mathrm{sh}}}$ has smooth extension. As before, applying [Moo98, Proposition 3.21.4] shows that $\mathscr{S}_{\tilde{\mathcal{K}}_p}(G_2,\Omega_2)_{\mathcal{O}^{\mathrm{sh}}}$ has smooth extension. As regularity is preserved under base-change along $\mathcal{O}\to\mathcal{O}^{\mathrm{sh}}$, we may apply Lemma 5.17 to deduce that $\mathscr{S}_{\tilde{\mathcal{K}}_p}(G_2,\Omega_2)_{\mathcal{O}}$ has smooth extension. The proof of Theorem 6.20 is now complete.

6.5. **Integral canonical model for certain orthogonal-type.** We sketch the construction from [MP16]. Let (L,Q) be a quadratic lattice over $\mathbb{Z}$ of signature $(n,2)$, $n\geq 1$. Let Ω denote the space of oriented negative definite planes in $L_{\mathbb{R}}$, set $\mathcal{G}_0:=\mathrm{SO}(L)$ and $G_0=(\mathcal{G}_0)_{\mathbb{Q}}$. Let K_L be the *discriminant kernel*, i.e. the largest subgroup of $\mathcal{G}_0(\widehat{\mathbb{Z}})$ acting trivially on the discriminant $\mathrm{disc}(L):=L^\vee/L$, where $L^\vee\subseteq L_{\mathbb{Q}}$ denotes the dual lattice; write $K_{0,p}$ for the p-part of K_L. Then (G_0,Ω) defines a shimura datum with reflex field $\mathbb{Q}$.

Integral canonical models for $\mathrm{Sh}_{K_{0,p}}(G_0,\Omega)$ have been constructed by Madapusi Pera under some conditions on L.

Definition 6.21: We say that $L_{\mathbb{Z}_{(p)}}$ is *maximal* if there is no bigger $\mathbb{Z}_{(p)}$-lattice in $L_{\mathbb{Q}}$ on which the quadratic form is $\mathbb{Z}_{(p)}$-valued. Otherwise it is *non-maximal*.

We say that L has *cyclic discriminant* if $\mathrm{disc}(L)$ is cyclic.

Theorem 6.22 ([MP16], Theorem 7.4 (1), Proposition 7.9)**:** *Assume $L_{\mathbb{Z}_{(p)}}$ is maximal (resp. is non-maximal with cyclic determinant).*

(1) *The scheme $\mathrm{Sh}_{K_{0,p}}(G_0,\Omega)$ admits an integral canonical model (resp. a smooth integral canonical model) over $\mathbb{Z}_{(p)}$, denoted $\mathscr{S}_{K_{0,p}}(G_0,\Omega)$.*
(2) *Moreover, let $\mathcal{O}$ be a discrete valuation ring $\mathcal{O}$ that is a faithfully flat $\mathbb{Z}_{(p)}$-algebra of absolute ramification index $e\leq p-1$, with field of fractions F. Assume that the strict completion $\widehat{\mathcal{O}}^s$ is finite over $W(\overline{\mathbb{F}_p})$.*
Then $\mathscr{S}_{K_{0,p}}(G_0,\Omega)_{\mathcal{O}}$ again has smooth-extension (resp. is again a smooth integral canonical model of $\mathrm{Sh}_{K_{0,p}}(G_0,\Omega)_F$).

Statement (1) are the results of Madapusi Pera. Statement (2) follows from the proof and Theorem 6.19. Assuming (1), it remains to show that the base-change $\mathscr{S}_{K_{0,p}}(G_0,\Omega)_{\mathcal{O}}$ has smooth-extension.

To do so, we need to introduce two additional Shimura data, for which we tried to stay as close as possible to the notations from [MP16][17]. We argue in three steps.

Step 1. Let $\mathcal{G} = \mathrm{GSpin}(L)$ and $G = \mathcal{G}_{\mathbb{Q}}$, so that (G, Ω) is a Shimura datum with reflex field Q. We claim that it suffices to show that Theorem 6.22 holds for $\mathrm{Sh}_{K_p}(G, \Omega)$ (here $K_p = \mathcal{G}(\mathbb{Z}_p)$ is hyperspecial). Assuming this, let $\mathscr{S}_{K_p}(G, \Omega)$ be the corresponding model. Then, Madapusi Pera defines a candidate integral model $\mathscr{S}_{K_{0,p}}(G_0, \Omega)$ of which $\mathscr{S}_{K_p}(G, \Omega)$ is a pro-étale cover. In particular, we have a pro-étale cover

$$\mathscr{S}_{K_p}(G, \Omega)_{\mathcal{O}} \longrightarrow \mathscr{S}_{K_{0,p}}(G_0, \Omega)_{\mathcal{O}}$$

where $\mathscr{S}_{K_p}(G, \Omega)_{\mathcal{O}}$ has smooth-extension by assumption. Descending smooth-extension to the scheme $\mathscr{S}_{K_{0,p}}(G_0, \Omega)_{\mathcal{O}}$ is done as in the case of abelian-type, by invoking the argument of [Moo98, Proposition 3.21.4] (see also the proof of [MP16, Theorem 7.4]).

Step 2. From now-on, the goal is to show that Theorem 6.22 holds for $\mathrm{Sh}_{K_p}(G, \Omega)$. In this step, we sketch the construction of the integral model $\mathscr{S}_{K_p}(G, \Omega)$. To this end, one embeds (L, Q) into an appropriate self-dual $\mathbb{Z}_{(p)}$-lattice $(\tilde{L}, \tilde{Q})$. We define $\tilde{\Omega}$, $\tilde{\mathcal{G}} = \mathrm{GSpin}(\tilde{L})$, $\tilde{G} = \tilde{\mathcal{G}}_{\mathbb{Q}}$ and $\tilde{K}_p = \tilde{\mathcal{G}}(\mathbb{Z}_p)$ as above; one shows that $(\tilde{G}, \tilde{\Omega})$ is a Shimura datum of Hodge-type and reflex field $\mathbb{Q}$. In particular:

- For each $\tilde{K} = \tilde{K}^p\tilde{K}_p \subseteq \tilde{G}(\mathbb{A}_f)$, $\mathrm{Sh}_K[\tilde{G}, \tilde{\Omega}]$ is a moduli space for certain polarized abelian schemes with level structure. We let $(\tilde{A}^{\mathrm{KS}}_{\tilde{K}}, \tilde{\lambda}^{\mathrm{KS}}, [\tilde{\eta}^{\mathrm{KS}}]_{\tilde{K}})$ be the universal *Kuga-Satake* object over $\mathrm{Sh}_K[\tilde{G}, \tilde{\Omega}]$.
- There is a smooth integral canonical model $\mathscr{S}_{\tilde{K}_p}(\tilde{G}, \tilde{\Omega})$, such that the base change to $\mathcal{O}$ is again a smooth integral canonical model. By construction $\mathscr{S}_{\tilde{K}}[\tilde{G}, \tilde{\Omega}]$ is also a moduli space, we write again $(\tilde{A}^{\mathrm{KS}}_{\tilde{K}}, \tilde{\lambda}^{\mathrm{KS}}, [\tilde{\eta}^{\mathrm{KS}}]_{\tilde{K}})$ for the universal object.

Fix $\tilde{K}^p \subseteq \tilde{G}(\mathbb{A}_f^p)$ small enough, and $K^p \subseteq G(\mathbb{A}_f^p)$ a subgroup of $\tilde{K}^p$. One then defines a finite unramified morphism $Z_{K^p} \to \mathscr{S}_{\tilde{K}}(\tilde{G}, \tilde{\Omega})$ via a description of the functor of points of Z_{K^p}. To describe it, Madapusi Pera introduces the notion of *special endomorphisms* corresponding to a subsheaf $[T \to \mathscr{S}_{\tilde{K}}(\tilde{G}, \tilde{\Omega})] \mapsto \tilde{L}(T)$ of the sheaf $\underline{\mathrm{End}}(\tilde{A}^{\mathrm{KS}}_{\tilde{K}})$ on $\mathscr{S}_{\tilde{K}}(\tilde{G}, \tilde{\Omega})$-schemes. The following lemma is key to establishing smooth-extension for Z_{K^p}.

Lemma 6.23: *Let T be a regular scheme faithfully flat over $\mathbb{Z}_{(p)}$, and consider a morphism $g : T \to \mathscr{S}_{\tilde{K}}(\tilde{G}, \tilde{\Omega})$. Then any section of $\tilde{L}(T_{\mathbb{Q}})$ uniquely extends to a section of $\tilde{L}(T)$.*

Proof. Let $A_T \to T$ denote the pullback of the Kuga-Satake abelian scheme via g. Let $f \in \tilde{L}(T_{\mathbb{Q}}) \subseteq \mathrm{End}(A_{T_{\mathbb{Q}}})$, then f extends to an endomorphism of A_T via the property of Néron models (see [BLR12, 8.4, Corollary 6]). Our assumptions on T show that any generic point of $T_{\mathbb{F}_p}$ is the specialization of a point in $T_{\mathbb{Q}}$, so by [MP16, Lemma 5.13] the extension of f is again special. □

Let $\Lambda := L^{\perp} \hookrightarrow \tilde{L}$ (here $\tilde{L}$ is the $\mathbb{Z}_{(p)}$-lattice, not the sheaf of special endomorphisms).To any $\mathscr{S}_{\tilde{K}}(\tilde{G}, \tilde{\Omega})$-scheme T, let $Z_{K^p}(T)$ denote the set of all $(A_T, \lambda_T, [\eta_T]_{\tilde{K}}, \iota, [\eta_\iota]_{K^p})$, where:

- The triple $(A_T, \lambda_T, [\eta_T]_{\tilde{K}^p})$ is the object associated with $T \to \mathscr{S}_{\tilde{K}}(\tilde{G}, \tilde{\Omega})$,
- The map $\iota : \underline{\Lambda}_T \to \tilde{L}(T)$ is an isometric embedding,
- The object $[\eta_\iota]_{K^p}$ is a section over T of a certain quotient sheaf I_ι^p/K^p (see [MP16, Definition 6.11] for details), and
- There are compatibility conditions between ι, $[\eta_T]_{\tilde{K}^p}$ and $[\eta_\iota]_{K^p}$ which we omit for simplicity. In particular, $[\eta_\iota]_{K^p}$ determines $[\eta_T]_{\tilde{K}^p}$, so that changing $\tilde{K}^p$ while leaving K^p fixed does not change the set $Z_{K^p}(T)$.

It is shown in [MP16, Proposition 6.13] that Z_{K^p} is represented by a scheme finite and unramified over $\mathscr{S}_{\tilde{K}}(\tilde{G}, \tilde{\Omega})$. Next, Madapusi Pera argues that there is a canonical morphism

$$\mathrm{Sh}_K(G, \Omega) \to Z_{K^p, \mathbb{Q}}$$

which is an isomorphism onto an open and closed subscheme of the target (see [MP16, Lemma 7.1]). Before taking the Zariski closure, we first replace Z_{K^p} with an open subscheme $Z^{\mathrm{pr}}_{K^p}$ which contains the

[17]There are a total of four Shimura data introduced in [MP16]. We only mention three of them here: the remaining datum $(\tilde{G}_0, \tilde{\Omega})$ is used for the construction of $\mathscr{S}_{K_{0,p}}(G_0, \Omega)$, which we omit.

generic fiber $Z_{K^p,\mathbb{Q}}$ (see [MP16, Lemma 6.16] for details). We define $\mathscr{S}_K(G,\Omega)$ as the Zariski closure of the image of $\mathrm{Sh}_K(G,\Omega)$ in $Z^{\mathrm{pr}}_{K^p}$. Finally, we set

$$\mathscr{S}_{K_p}(G,\Omega) := \varprojlim_{K^p\subseteq \tilde{K}^p} \mathscr{S}_{K^pK_p}(G,\Omega).$$

Step 3. It remains to show that $\mathscr{S}_{K_p}(G,\Omega)_{\mathcal{O}}$ has smooth extension. It follows from the moduli description of Z_{K^p}, from Theorem 6.19 and from Lemma 6.23 that the projective limit

$$Z_{\mathcal{O}} := \varprojlim_{K^p} Z_{K^p,\mathcal{O}}$$

has smooth-extension. Let T be a regular, formally smooth $\mathcal{O}$-scheme, together with a generic morphism $T \rightsquigarrow \mathscr{S}_{K_p}(G,\Omega)_{\mathcal{O}}$. Then there is an extension

$$T \to Z_{\mathcal{O}}$$

and our goal is to show that this extension factors through $Z^{\mathrm{pr}}_{\mathcal{O}} := \varprojlim_{K^p} Z^{\mathrm{pr}}_{K^p,\mathcal{O}}$. We claim that T can be expressed as a filtered limit

$$T = \varprojlim_i T_i$$

where each T_i is smooth over $\mathcal{O}$. Indeed, $T \to \operatorname{Spec}\mathcal{O}$ is flat by [Gro66, Théorème 19.7.1], and each fiber is geometrically regular by [Sta26, Tag 07EL] and [Sta26, Tag 07EC]. The conclusion follows from Popescu's theorem, see [Sta26, Tag 07GC]. We are now reduced to the following statement, where we use the assumptions on $\widehat{\mathcal{O}}^s$ in Theorem 6.22.

Lemma 6.24: *Let T_i be a smooth $\mathcal{O}$-scheme, consider a morphism*

$$f : T_i \to Z_{K^p,\mathcal{O}}$$

such that the generic fiber f_F factors through $\mathrm{Sh}_{K^pK_p}(G,\Omega)_{\mathcal{O}}$. Then f factors through $Z^{\mathrm{pr}}_{K^p}$.

Sketch of proof. The original statement in [MP16, Lemma 6.16 (4)] deals with the case $\mathcal{O} = \mathbb{Z}_{(p)}$. The only point in the argument which needs some care after base-change to $\mathcal{O}$ is the use of the comparison isomorphism between crystalline and de Rham realization functors with $\mathcal{O}$-coefficients (in the proofs of [MP16, Lemma 6.14] and [MP16, Lemma 6.16 (3)]). Fortunately, the isomorphism still holds under the assumption $e \leq p-1$.

Additionally, the reference to [MP16, Lemma 6.14] in the proof is where our assumption on $\widehat{\mathcal{O}}^s$ originates from. □

Thus the extension $T \to Z_{\mathcal{O}}$ factors through $Z^{\mathrm{pr}}_{\mathcal{O}}$, hence through $\mathscr{S}_{K_p}(G,\Omega)_{\mathcal{O}}$. The proof of Theorem 6.22 is now complete.

7. Applications

Assume $p \neq 2$. Let $F/\mathbb{Q}_p$ be a finite field extension of ramification index e.

7.1. **Smooth curves.** Let $g \geq 2$, $n \geq 0$ be integers satisfying $2-2g < n$. Let $\mathscr{M}_{g,n}$ denote the separated Deligne-Mumford stack of genus-g, n-pointed, geometrically connected, smooth, proper curves. Let ℓ_1, ℓ_2 be prime numbers different from p, set $\mathscr{L} = \{\ell_1, \ell_2\}$, and $\mathbb{Z}_{\mathscr{L}} = \mathbb{Z}_{\ell_1} \times \mathbb{Z}_{\ell_2}$. In [Sti05, Remark 2.8 (a)] Stix defines a $\mathbb{Z}_{\mathscr{L}}$-local system $\mathbb{L}$[18] over $\mathscr{M}_{g,n} \times_{\mathbb{Z}} \mathbb{Z}\left[\frac{1}{\mathscr{L}}\right]$.

Let $\mathscr{S} := \mathscr{M}_{g,n} \times_{\mathbb{Z}} \mathcal{O}_F$, the pullback of $\mathbb{L}$ to $\mathscr{S}$ is again denoted $\mathbb{L}$.

Proposition 7.1: *The pair $(\mathscr{S}, \mathbb{L})$ has L-extension, hence $\mathscr{S}$ has PGR-extension.*

Proof. This is a corollary of [Sti05, Theorem 2.10 (B)] and Proposition 5.8. □

Consequently, we recover [CT26, §4.3, Example (1)].

Theorem 7.2: *Let $\mathscr{X}/\mathcal{O}_F$ be a smooth scheme with generic fiber X/F. Let $f : Y \to X$ be a smooth proper family of genus-g, n-pointed, geometrically connected, smooth, proper curves. If it has pointwise good reduction, then it has global good reduction.*

[18] We warn the reader that in [Sti05] the notation $\mathbb{L}$ refers to the set denoted $\mathscr{L}$ here.

7.2. **Abelian varieties.** We recall the setup from Section 6.2. Namely, let $g \geq 1$ be an integer and fix a symplectic abelian group (V, ψ) of rank $2g$. Let Δ denote its symplectic type. Consider the $\mathbb{Z}$-group scheme $\mathcal{G} := \mathrm{GSp}(V, \psi)$, set $G := \mathcal{G}_{\mathbb{Q}}$ its generic fiber. Let $K_0 = K_p K_0^p = \mathcal{G}(\widehat{\mathbb{Z}})$, and fix $K = K_p K^p \subseteq K_0$ a level structure; note that K_p is fixed and K^p varies. The stacks $\mathscr{A}_{g,\Delta}$, $\mathscr{A}_{g,\Delta,K}$ and $\mathscr{A}_{g,\Delta,K_p}$ have been defined in Definition 6.10 and Definition 6.11; in particular for $K = K_0$, $\mathscr{A}_{g,\Delta} = \mathscr{A}_{g,\Delta,K_0}$.

Proposition 7.3: *Assume $e \leq p-1$. Then $\mathscr{A}_{g,\Delta,K,\mathcal{O}_F}$ has L-extension, hence PGR-extension.*

Proof. The stack $\mathscr{A}_{g,\Delta,K_p,\mathcal{O}_F}$ has smooth extension by Theorem 6.20. By definition it is the projective limit of the K_0-fit system $(\mathscr{A}_{g,\Delta,K,\mathcal{O}_F})_K$; it follows from Proposition 5.18 that each $\mathscr{A}_{g,\Delta,K,\mathcal{O}_F}$ has smooth extension, hence PGR-extension by Proposition 5.8. □

Remark 7.4: The bound on e is optimal. Indeed, otherwise Corollary 3.12 shows that for $\Delta = (1, \ldots, 1)$ or $(1, \ldots, 1, p)$, there are some values of g such that $\mathscr{A}_{g,\Delta,K,\mathcal{O}_F}$ does not have PGR-extension for any level structure $K = K^p K_p$. This implies in particular that for $e \geq p$ and $\Delta = (1, \ldots, 1)$, the base-change $\mathscr{S}_{K_p}(G, \Omega)_{\mathcal{O}_F}$ of the Siegel modular variety (see Section 6.3) does not have smooth-extension, hence is no longer a smooth integral canonical model. This answers the question of [Vas99, Remark 3.2.12.1] negatively

As a corollary, we recover [CT26, §4.3.2, Example (3)]. Our proof up to this point has been a reformulation and generalization of theirs; no new idea were introduced.

Theorem 7.5: *Assume $e \leq p-1$. Let X/F be a smooth variety, and $\mathcal{X}/\mathcal{O}_F$ a smooth model. Let $A \to X$ be a polarized abelian scheme with level structure K. If it has pointwise good reduction, then it has global good reduction.*

7.3. **K3 surfaces.** Fix $d \geq 1$ an integer such that $p \nmid 2d$. The K3 lattice is the rank 22 quadratic lattice $\Lambda = E_8^{\oplus 2} \oplus U^{\oplus 3}$, where E_8 is the E8 lattice of rank 8 and U is the hyperbolic lattice of rank 2. Let $e, f \in U \subseteq \Lambda$ be isotropic elements satisfying $[e, f] = 1$ in some copy of the hyperbolic lattice, where $[\cdot, \cdot]$ is the bilinear symmetric form on U. Set $L_d = \langle e - df \rangle^{\perp} \subseteq \Lambda$ a sublattice of discriminant $2d$ and rank 21 and write $W := L_{d,\mathbb{Q}}$. We let $L_d^{\vee} \subseteq W$ denote the dual lattice. To this data is associated a Shimura stack of orthogonal type, as in Section 6.5 (see also [MP15, §3.1]).

Definition 7.6: Let $G = \mathrm{SO}(W)$ and Ω the space of oriented negative planes in $L_{d,\mathbb{R}}$. Let $K_0 = K_0^p K_p \subseteq G(\mathbb{A}_f)$ denote the largest subgroup of $\mathrm{SO}(L_d)(\widehat{\mathbb{Z}})$ acting trivially on $L_d^{\vee}/L_d$. For any level $K = K^p K_p \subseteq K_0$, let $\mathrm{Sh}_K[G, \Omega]$ be the Shimura stack of level K associated to the Shimura datum (G, Ω), over its reflex field $\mathbb{Q}$.

Proposition 7.7: *Assume $e \leq p-1$. The Shimura variety $\mathrm{Sh}_{K_p}(G, \Omega)$ admits a integral model $\mathscr{S}_{K_p}(G, \Omega)$ over $\mathbb{Q}$ such that*

1. *If $p^2 \nmid d$, then $\mathscr{S}_{K_p}(G, \Omega)$ is an integral canonical model over $\mathbb{Z}_{(p)}$ as in Definition 6.6, and $\mathscr{S}_{K_p}(G, \Omega)_{\mathcal{O}_F}$ has smooth extension.*
2. *If $p^2 \mid d$, then $\mathscr{S}_{K_p}(G, \Omega)$ is a smooth integral canonical model over $\mathbb{Z}_{(p)}$ as in Definition 6.3, and so is its base-change $\mathscr{S}_{K_p}(G, \Omega)_{\mathcal{O}_F}$.*

In any case, $\mathscr{S}_{K_p}(G, \Omega)_{\mathcal{O}_F}$ has smooth extension

Proof. This is Theorem 6.22. From [MP15, §4.1] we have that L_d has cyclic determinant, and is maximal at p if and only if $p^2 \nmid d$. □

We assume from now-on that $e \leq p-1$ and that $\mathscr{S}_{K_p}(G, \Omega)$ is as in Proposition 7.7. From Proposition 6.9 we find that the quotient stack $\mathscr{S}_K[G, \Omega]$ has PGR-extension.

Next we introduce the moduli space of primitively polarized K3 surfaces with level structure, following [MP15, §2.1, §2.10].

Definition 7.8: Let $\mathscr{M}_{2d}^{\circ}$ be the moduli problem that assigns to every $\mathbb{Z}_{(p)}$-scheme T the groupoid of tuples $(f : Y \to T, \xi)$ where

- $f : Y \to T$ is a K3 surface[19], and
- $\xi \in \underline{\mathrm{Pic}}(X/T)(T)$ is a primitive polarization of degree $2d$.

It is a separated Deligne-Mumford stack of finite type over $\mathbb{Z}_{(p)}$ ([Riz05] Theorem 4.3.3).

If $(f : Y \to T, \xi) \in \mathscr{M}^{\circ}_{2d}(T)$, let $\mathbf{H}^2_{\widehat{\mathbb{Z}}^p}(f) = \prod_{\ell \neq p} \mathbf{H}^2_{\ell}(f)$ be the $\widehat{\mathbb{Z}}^p$-local system on T defined as the product of the relative ℓ-adic cohomology sheaves. The polarization defines a relative Chern class $\mathrm{ch}_{\widehat{\mathbb{Z}}^p}(\xi) \in \mathbf{H}^2_{\widehat{\mathbb{Z}}^p}(f)(1)$, the sub-local-system orthogonal to this class is the relative primitive cohomology sheaf $\mathbf{P}^2_{\widehat{\mathbb{Z}}^p}(f)(1)$. Let $V := L_d \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}^p$. With notations from Section 2 there is a local system

$$\begin{aligned} \mathbb{L} : \mathscr{M}^{\circ}_{2d} &\longrightarrow \mathbf{Loc}_V \\ (f : Y \to T, \xi) &\longmapsto \mathbf{P}^2_{\widehat{\mathbb{Z}}^p}(f)(1). \end{aligned}$$

The cup-product induces a quadratic form on (primitive) cohomology. We define

$$\begin{aligned} \mathrm{Lev} : \mathscr{M}^{\circ}_{2d} &\longrightarrow \mathbf{B}\mathrm{SO}(L_d)(\widehat{\mathbb{Z}}^p) \\ (f : Y \to T, \xi) &\longmapsto [(U \to T) \mapsto \{\text{Isometries } \mathbf{P}^2_{\widehat{\mathbb{Z}}^p}(f)(1)_U \xrightarrow{\sim} \mathcal{F}_V\}]. \end{aligned}$$

Let $R = \widehat{\mathbb{Z}}^p$ and $Q = \mathrm{SO}(L_d)(R)$, now $(\mathscr{M}^{\circ}_{2d}, \mathbb{L}, \mathrm{Lev}, \iota)$ is a Q-moduli datum as in Definition 2.26, where ι is the obvious sheaf inclusion of $\mathrm{Lev}(f : Y \to T, \xi)$ inside $\underline{\mathrm{Isom}}(\mathbf{P}^2_{\widehat{\mathbb{Z}}^p}(f)(1), \mathcal{F}_V)$. For any open subgroup $K = K^p K_p \subseteq K_0$ we may consider the associated moduli stack of level K denoted $\mathscr{M}^{\circ}_{2d,K}$. As K_0 is a finite index open normal subgroup of Q, we find that the projection $\mathscr{M}^{\circ}_{2d,K_0} \to \mathscr{M}^{\circ}_{2d}$ is a Q/K_0-torsor.

Ideally, $\mathrm{Sh}_K[G, \Omega]$ would serve as a period domain for the cohomology of K3 surfaces, so as to induce a morphism $\mathscr{M}^{\circ}_{2d,K,\mathbb{C}} \to \mathrm{Sh}_K[G, \Omega]_{\mathbb{C}}$ which would descend over $\mathbb{Q}$, and finally extend to $\mathscr{M}^{\circ}_{2d,K,\mathcal{O}_F} \to \mathscr{S}_K[G, \Omega]$ by invoking some extension property. To make this argument work however, $\mathscr{M}^{\circ}_{2d}$ needs to be replaced with a double cover.

Definition 7.9: Let $\tilde{\mathscr{M}}^{\circ}_{2d} \to \mathscr{M}^{\circ}_{2d}$ be the double finite étale cover parametrizing isometric trivializations $\det(\mathbf{P}^2_2) \xrightarrow{\sim} \mathcal{F}_{\det(L_d) \otimes_{\mathbb{Z}} \mathbb{Z}_2}$ (notation from Definition 2.4). Its objects are called *oriented, primitively polarized K3-surfaces of degree* $2d$.

From [MP15, Proposition 4.2, Corollary 4.4 and Proposition 4.7] there is a natural period map $\tilde{\mathscr{M}}^{\circ}_{2d,\mathbb{C}} \to \mathrm{Sh}_{K_0}(G, \Omega)_{\mathbb{C}}$ which extends to a $\mathbb{Z}_{(p)}$-morphism $\tilde{\mathscr{M}}^{\circ}_{2d} \to \mathscr{S}_{K_0}(G, \Omega)$.

Fact 7.10 ([MP15], Proposition 4.8)**:** *The map* $\tilde{\mathscr{M}}^{\circ}_{2d} \to \mathscr{S}_{K_0}[G, \Omega]$ *is étale.*

Let $\tilde{\mathscr{M}}^{\circ}_{2d,K}$ denote the pullback of $\mathscr{M}^{\circ}_{2d,K} \to \mathscr{M}^{\circ}_{2d}$ along $\tilde{\mathscr{M}}^{\circ}_{2d} \to \mathscr{M}^{\circ}_{2d}$. The morphism in Fact 7.10 lifts to an étale morphism at level $K = K^p K_p \subseteq K_0$:

$$\tilde{\mathscr{M}}^{\circ}_{2d,K} \to \mathscr{S}_K[G, \Omega].$$

Remark 7.11: It follows from Proposition 2.30 that the pullback of $\mathbb{L}_{\mathrm{can}}$ on $\mathscr{S}_K[G, \Omega]$ to $\tilde{\mathscr{M}}^{\circ}_{2d,K}$ coincides with the universal relative primitive cohomology $\mathbf{P}^2_{\widehat{\mathbb{Z}}^p}(1)$.

Corollary 7.12: *The stacks* $\tilde{\mathscr{M}}^{\circ}_{2d,\mathcal{O}_F}$, $\mathscr{M}^{\circ}_{2d,\mathcal{O}_F}$, $\tilde{\mathscr{M}}^{\circ}_{2d,K,\mathcal{O}_F}$ *and* $\mathscr{M}^{\circ}_{2d,K,\mathcal{O}_F}$ *all have PGR-extension.*

Proof. For $\tilde{\mathscr{M}}^{\circ}_{2d,K,\mathcal{O}_F}$ and $\tilde{\mathscr{M}}^{\circ}_{2d,\mathcal{O}_F}$ this follows from $\mathscr{S}_K[G, \Omega]_{\mathcal{O}_F}$ having PGR-extension and Theorem 5.9. As $\tilde{\mathscr{M}}^{\circ}_{2d} \to \mathscr{M}^{\circ}_{2d}$ is a $\mathbb{Z}/2\mathbb{Z}$-torsor, PGR-extension for $\mathscr{M}^{\circ}_{2d,\mathcal{O}_F}$ and $\mathscr{M}^{\circ}_{2d,K,\mathcal{O}_F}$ is then deduced from Proposition 5.13. $\square$

Thus, (oriented) primitively polarized K3-surfaces with K-level structure having pointwise good reduction also have global good reduction. In the particular case of $\mathscr{M}^{\circ}_{2d,\mathcal{O}_F}$, we find:

[19] That is, Y is a proper and smooth algebraic space over T, with geometric fibers that are K3 surfaces in the usual sense.

Theorem 7.13: *Assume $e \leq p-1$. Let X/F be a smooth variety, and $\mathcal{X}/\mathcal{O}_F$ a smooth model. Consider $Y \to X$ a primitively polarized K3-surface of degree $2d$. If it has pointwise good reduction, then it has global good reduction.*

Remark 7.14: All statements of this paragraph still hold true if $\mathscr{M}^\circ_{2d}$ is replaced with $\mathscr{M}_{2d}$ the moduli stack defined as in Definition 7.8, except ξ is now a primitive quasi-polarization (as in [MP15, §2.1]).

Remark 7.15: In trying to generalize the argument from K3 surfaces to general hyperkähler varieties, only the analogue Fact 7.10 is still missing. One can attach a Shimura datum $(G^{\mathrm{HK}}, \Omega^{\mathrm{HK}})$ to the quadratic lattice defined by the second cohomology group of a fixed hyperkähler variety as in [Bin21, §4.5], as well as introduce the moduli stack $\mathscr{M}_{2d}$ (resp. $\tilde{\mathscr{M}}_{2d}$) of degree-$2d$ polarized, (resp. oriented) hyperkähler varieties over $\mathbb{Z}_{(p)}$, $p \nmid d$, as in [Bin21, Definition 3.3.1, Definition 4.3.3][20]. By Theorem 6.22 the variety $\mathrm{Sh}_{K_p}(G^{\mathrm{HK}}, \Omega^{\mathrm{HK}})/\mathbb{Q}$ has an integral canonical model $\mathscr{S}_{K_p}(G^{\mathrm{HK}}, \Omega^{\mathrm{HK}})/\mathbb{Z}_{(p)}$, where $K_p \subseteq G^{\mathrm{HK}}(\mathbb{A}_p)$ suitably chosen. By [Bin21, Theorem 4.5.2] there is an étale period map $\tilde{\mathscr{M}}_{2d,K,\mathbb{Q}} \to \mathrm{Sh}_K(G^{\mathrm{HK}}, \Omega^{\mathrm{HK}})$ over $\mathbb{Q}$, and the extension property yields

$$\tilde{\mathscr{M}}_{2d,K} \to \mathscr{S}_K(G^{\mathrm{HK}}, \Omega^{\mathrm{HK}})$$

over $\mathbb{Z}_{(p)}$. If this map were shown to be étale, it would follow as in Corollary 7.12 that $\tilde{\mathscr{M}}_{2d,K}$, $\mathscr{M}_{2d,K}$ and $\mathscr{M}_{2d}$ have PGR-extension, and so Theorem 7.13 would extend to hyperkähler varieties.

7.4. **Cubic fourfolds.** The argument for K3 surfaces applies *mutatis mutandis* to cubic fourfolds; we give references to the main ingredients following [MP15, §5.13].

Let M_0 be the even rank 2 quadratic lattice with bilinear form represented by $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ and set

$$M = E_8^{\oplus 2} \oplus U^{\oplus 2} \oplus M_0$$

which is has signature $(20, 2)$ and is maximal at all primes.

Definition 7.16: Let $G = \mathrm{SO}(V)$ and Ω the space of oriented negative planes in $M_{\mathbb{R}}$. Let $K_0 = K_0^p K_p \subseteq G(\mathbb{A}_f)$ denote the largest subgroup of $\mathrm{SO}(M)(\widehat{\mathbb{Z}})$ acting trivially on $M^\vee/M$. For any level $K = K^p K_p \subseteq K_0$, let $\mathrm{Sh}_K[G, \Omega]$ be the Shimura stack of level K associated to the Shimura datum (G, Ω), over its reflex field $\mathbb{Q}$.

From Theorem 6.22 we find:

Fact 7.17: *The scheme $\mathrm{Sh}_{K_p}(G, \Omega)$ admits an integral canonical model $\mathscr{S}_{K_p}(G, \Omega)$ over $\mathbb{Z}_{(p)}$. If $e \leq p-1$ then $\mathscr{S}_{K_p}(G, \Omega)_{\mathcal{O}_F}$ has smooth-extension.*

Consider $\mathscr{M}$ the moduli stack of cubic fourfolds over $\mathbb{Z}_{(p)}$, and $\tilde{\mathscr{M}}$ its double cover trivializing the determinant of primitive cohomology. For K as in Definition 7.16, we let $\mathscr{M}_K$ (resp. $\tilde{\mathscr{M}}_K$) be the moduli stack with K-level structure on degree-4 primitive cohomology which is a finite étale cover of $\mathscr{M}$ (resp. $\tilde{\mathscr{M}}$). In particular, $\mathscr{M}_{K_0} \to \mathscr{M}$ is a finite torsor.

From the modular description of $\mathrm{Sh}_{K_0}(G, \Omega)_{\mathbb{C}}$ (see [MP15, Proposition 3.3]) we find a period map:

$$\tilde{\mathscr{M}}_{\mathbb{C}} \to \mathrm{Sh}_{K_0}(G, \Omega)_{\mathbb{C}}$$

which descends over $\mathbb{Q}$ as in [MP15, Corollary 4.4]. By smooth-extension we find a period map $\tilde{\mathscr{M}} \to \mathscr{S}_{K_0}[G, \Omega]$ which is étale by [MP15, Theorem 5.14]. Thus for K as in Definition 7.16 we find étale maps $\tilde{\mathscr{M}}_K \to \mathscr{S}_K(G, \Omega)$. It follows that $\tilde{\mathscr{M}}_{K,\mathcal{O}_F}$, hence also $\mathscr{M}_{K,\mathcal{O}_F}$ and $\mathscr{M}_{\mathcal{O}_F}$, have PGR-extension. This proves the pointwise to global good reduction for (oriented) cubic fourfolds with K-level structure, and in the particular case of $\mathscr{M}$:

Theorem 7.18: *Assume $e \leq p-1$. Let X/F be a smooth variety, and $\mathcal{X}/\mathcal{O}_F$ a smooth model. Consider $Y \to X$ a smooth, proper family of cubic fourfolds. If it has pointwise good reduction, then it has global good reduction.*

[20] The stack introduced in [Bin21] is the disjoint union over d of the $\tilde{\mathscr{M}}_{2d,\mathbb{Q}}$.

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