POINTWISE TO GLOBAL GOOD REDUCTION PROBLEMS

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Contents

1. Introduction

1.1. **Motivation and main results.** Let $\mathcal{O}_F$ denote a complete DVR with fraction field F. Consider $\mathcal{X}$ a smooth, geometrically connected, separated scheme of finite type over $\mathcal{O}_F$, let X/F denote its generic fiber. Fix $Y \to X$ a smooth and proper family of varieties. We are concerned with the following problem: assuming (sufficiently many of) the closed fibers Y_x have good reduction, does Y extend to a smooth and proper scheme over $\mathcal{X}$? Does pointwise good reduction imply global good reduction?

This question is asked in the recent work of Cadoret and Tamagawa [CT26, §4.3], where it is answered positively in the case of pointed, smooth, proper curves of genus $g \geq 2$, of stable, proper curves of genus $g \geq 2$, and of abelian schemes if F has absolute ramification index $e \leq p-1$. The two main inputs of their argument are [CT26, Theorem 1] stating that a local system on X extends to $\mathcal{X}$ if and only if it is pointwise unramified for a relevant collection of closed points of X, as well as Serre-Tate-type criteria. It is also discussed in [CT26, §4.3, Example (2)] how *smooth integral canonical models* of Shimura varieties could be leveraged for a strategy when $\mathcal{O}_F$ is the ring of integers of a p-adic field. Indeed, by definition these models satisfy a property akin to the Néron extension property of Néron models, thus providing a Serre-Tate criterion over general bases.

The main goal of this document is to formalize the discussion in [CT26, §4.3, Example (2)]. We reduce the pointwise-to-global good reduction problem to the existence of a separated, Deligne-Mumford stack $\mathscr{S}/\mathcal{O}_F$ equipped with a local system $\mathbb{L}$ such that the pair $(\mathscr{S}, \mathbb{L})$ satisfies some "L-extension property", and show that moduli spaces of curves, as well as smooth integral canonical models of Shimura varieties, can fill this role. Consequently, we (re)establish:

Theorem 1.1: *Assume $p > 2$. Let $F/\mathbb{Q}_p$ be a finite field extension. Assume that $Y \to X$ is either:*

(a) *a smooth, proper family of genus g, n-pointed, geometrically connected curves with $2-2g < n$,*
(b) *($F/\mathbb{Q}_p$ unramified) a polarized abelian scheme,*
(c) *($F/\mathbb{Q}_p$ unramified) a primitively polarized K3 surface of degree $2d$ not divisible by p,*
(d) *($F/\mathbb{Q}_p$ unramified) a smooth, proper family of cubic fourfolds.*

If $Y \to X$ has pointwise good reduction, then it has global good reduction.

We give a precise definition of pointwise/global good reductions in Definition 3.3. Cases (c) and (d) are new, and are enabled by the work of Madapusi Pera in [MP16] and [MP15] on integral canonical models of orthogonal-type Shimura varieties. Case (a) was already known, and our proof is the same as in [CT26, §4.3, Example (1)], only reformulated to fit our framework. Case (b) is weaker than the statement in [CT26, §4.3, Example (3)]; however building upon [VZ10, Theorem 28] we also show the optimality of their assumption $e \leq p-1$, and relate it to the non-existence of a smooth integral

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canonical model for Siegel modular varieties if $e \geq p$. We have not investigated the possibility of a similar obstruction for orthogonal-type Shimura varieties involved in cases (c) and (d).

1.2. **Outline.** Section 2 introduces the formalism of local systems and torsors on stacks used in the rest of the document. Of particular importance to us are towers of stacks with finite étale transition maps inducing compatible systems of torsors: associated with this setup is a *canonical local system.* When the stacks carry specific moduli interpretations, we define a *universal local system* and argue that canonical and universal local systems agree, thus generalizing constructions and statements from [UY13] and [CM20].

We formulate in Section 3 the pointwise-to-global good reduction problem. We then show that even in the usually well-behaved case of abelian schemes the answer is negative: if $F/\mathbb{Q}_p$ has ramification index $e \geq p$, Corollary 3.8 provides a counterexample with arbitrary level structure. To obtain it, we recall the work of Vasiu-Zink in [VZ10] and descend their construction to finite type bases over $\mathcal{O}_F$.

Section 4 is the core of the proof toward Theorem 1.1. For the convenience of the reader, we motivate the technical results with case (c) in mind. With X/F and $\mathcal{X}/\mathcal{O}_F$ as above, let $Y \to X$ be a family of K3 surfaces (with extra structure) having pointwise good reduction, we wish to extend it to a family $\mathcal{Y} \to \mathcal{X}$ of K3 surfaces. Let $\mathscr{M}$ denote the moduli stack of K3 surfaces (with extra structure) over $\mathcal{O}_F$, the family defines a *PGR-morphism* $X \to \mathscr{M}$ (Definition 4.4) which we wish to extend to $\mathcal{X}$. If any PGR-morphism to $\mathscr{M}$ admits an extension over $\mathcal{O}_F$, we say that $\mathscr{M}$ *has PGR-extension* (Definition 4.6).

To argue that $\mathscr{M}$ has PGR-extension, we need to introduce two other extension properties of moduli stacks: *L-extension* and *smooth-extension* (Definition 4.2 and Definition 4.15). The majority of Section 4 is spent establishing the implications below:

(i) If $\mathscr{S}_\infty/\mathcal{O}_F$ is a stack with smooth-extension which can be expressed as a certain limit
$$\mathscr{S}_\infty = \varprojlim_K \mathscr{S}_K,$$
then each $\mathscr{S}_K$ has L-extension (Proposition 4.16). This is where the notion of canonical local system from Section 2 is used. The implication had already been observed in [BST24, §1.3] in the context of Shimura varieties.

(ii) If $\mathscr{S}/\mathcal{O}_F$ has L-extension, then it also has PGR-extension (Proposition 4.8).

(iii) If $\mathscr{S}/\mathcal{O}_F$ has PGR-extension and $\mathcal{U} \to \mathscr{S}$ is étale, then $\mathcal{U}$ has PGR-extension (Theorem 4.9). This fails for L-extension, as shown by the open inclusion of the generic fiber in $\mathscr{S}$.

(iv) If $\mathcal{U} \to \mathscr{M}$ is a finite torsor over $\mathcal{O}_F$ and if $\mathcal{U}$ has PGR-extension, then so does $\mathscr{M}$ (Proposition 4.13).

Summarizing, the extension properties imply each-other from top-right to bottom-left in the following diagram, using (i) to (iv) in order:

$$\begin{tikzcd}
 & & \mathscr{S}_\infty \arrow[d, "\text{projection}"] & \boxed{\text{smooth-extension}} \\
\boxed{\text{PGR-extension}} & \mathcal{U} \arrow[r, "\text{étale}"] \arrow[d, "\text{finite torsor}"'] & \mathscr{S} = \mathscr{S}_K & \boxed{\text{L-extension}} \\
\boxed{\text{PGR-extension}} & \mathscr{M} & &
\end{tikzcd}$$

The input allowing (ii) and (iv) is [CT26, Theorem 1]. We end Section 4 by introducing smooth integral canonical models of Shimura stacks (Definition 4.17) which have smooth-extension by definition.

In Section 5 we argue that the relevant moduli spaces $\mathscr{M}$ in Theorem 1.1 have PGR-extension. For case (a), it is known from [Sti05] that $\mathscr{M}$ has L-extension, hence PGR-extension. For case (b), the moduli space is the smooth integral canonical model of a Shimura stack of finite level, which also has L-extension; we then relate the counterexample from Section 3 to the non-existence of such models if F is too ramified (Proposition 5.7). Cases (c) and (d) require (iii) and (iv) additionally: the period morphism defines an étale map from a moduli space $\mathcal{U} = \mathscr{M}_K$ with K-level structure, which is a finite torsor over $\mathscr{M}$, to a smooth integral canonical model $\mathscr{S}_K$ of a Shimura stack of level K. Theorem 1.1 summarizes the contents of Theorem 5.2, Theorem 5.5, Theorem 5.15 and Theorem 5.19.

1.3. **Conventions and notations.** Some conventions for the rest of the document are:

- The ring $\mathcal{O}_F$ is a complete DVR with fraction field F. The integral closure of $\mathcal{O}_F$ in some fixed algebraic closure of F is denoted $\overline{\mathcal{O}_F}$.
- A variety over a field is a reduced separated scheme of finite type over the field.
- A *smooth model* of a variety X/F is a smooth algebraic space $\mathcal{X}/\mathcal{O}_F$ with generic fiber X/F.
- We fix $p > 0$ a prime number. We denote $\widehat{\mathbb{Z}}^p = \prod_{\ell \neq p} \mathbb{Z}_\ell$ the prime-to-p profinite integers, and $\mathbb{A}_f^p$ the prime-to-p adèles.

2. Canonical local systems on schemes and stacks

This section mostly serves as a toolbox of definitions: we advise the reader to skip it at first read, and come back to it whenever necessary.

We use the language of fibered categories and (algebraic) stacks over a site, and in particular the pro-étale site. The meaning of these terms varies accross references; for the convenience of the reader we spend the first paragraph expliciting our conventions.

The next two paragraphs define what is meant in the rest of the document by local systems and torsors on schemes and stacks. In particular, to certain projective systems of schemes/stacks called *fit systems* are attached *canonical* local systems which play a role in later applications; this construction is inspired from [UY13, §2] and [CM20, §3.1]. We show in Proposition 2.20 that there is a universal fit system of classifying stacks inducing all other fit systems. This construction is applied later in the proof of Proposition 4.16.

When the stacks of a fit system additionally carry a certain moduli interpretation, one can also define a *universal* local system: this is formalized in the third paragraph, where it is shown that it is canonically isomorphic to the canonical local system. This identification recovers [UY13, paragraph following Remark 2.8] (see also [CM20, Proposition 5.2]) by applying it to points on Siegel modular varieties, as well as [CM20, end of the proof of Theorem 6.6] for points on moduli spaces of K3 surfaces. This result is not essential for the rest of the document.

We fix S a base scheme and set $\mathscr{C} := (\mathrm{Sch}/S)_{\text{proét}}$ the big pro-étale site over S; in the rest of the document we will implicitely set $S = \operatorname{Spec} \mathcal{O}_F$. Unless stated otherwise, all fibered categories, stacks and morphisms between them are over $\mathscr{C}$. For all S-scheme T, we write $\mathscr{C}_T = (\mathrm{Sch}/T)_{\text{proét}}$ the restriction of $\mathscr{C}$ to T. We fix R a topological ring and V a free R-module of finite rank, we let $\mathrm{GL}(V)$ denote the topological group of R-linear automorphisms of V. We also fix Q a topological group.

2.1. **(Algebraic) stacks on the big pro-étale site.** In the rest of this document, we will need the notion of (algebraic and Deligne-Mumford) stacks over the big pro-étale site over S. In this paragraph we make our conventions precise.

The pro-étale site over S has three properties of interest to us, making it a better candidate than étale, fppf or fpqc topologies. First, sheafification and stackification exist (they are harder to define, and more choice-dependent, in the fpqc topology; see the paragraph following [Sta26, Tag 022C]); we need them to define induction functors in Definition 2.8 and Remark 2.9. Second, projective limits of finite étale covers are still covers, so the notion of, for instance, $\mathbb{Z}_\ell$-local system falls under the same formalism as that of $\mathbb{Z}/\ell^n\mathbb{Z}$ local systems for the étale topology, without the need for a special treatment via projective systems and lisse sheaves; this is not permitted by the étale/fppf topologies. Lastly, we briefly use in Lemma 2.23 that the pro-étale topos is replete.

2.1.1. *"The" pro-étale site over S.* The reader willing to ignore set-theoretic issues is encouraged to skip this paragraph and proceed with the intuitive understanding that the big pro-étale site over S is the category of all schemes on S with coverings refining the étale site, in that projective limits of finite étale covers are again covers.

We define a site as in [Sta26, Tag 00VG]. In particular, we require that objects, morphisms between objects and the collection of all coverings within the site are all sets. This is to ensure the existence of sheafification of presheaves (resp. stackification of prestacks) over a site. The big category Sch/S does not satisfy these conditions for any topology: the collection of objects needs to be reduced to a set. When shrinking, one has to ensure to keep all S-schemes which are essential in context[1], as well

[1]For instance, we will be studying applications to certain Shimura varieties over $S = \operatorname{Spec} F$ and $S = \operatorname{Spec} \mathcal{O}_F$, so whichever set we replace Sch/S with, it should contain those varieties.

as making sure the resulting set is still stable under useful operations. The question thus becomes: is it possible to find a *set* $\mathcal{S}$ within the collection of all S-schemes such that

(i) the set $\mathcal{S}$ contains a prescribed set $\mathcal{S}_0$ of S-schemes, and
(ii) for any $T \in \mathcal{S}$ and any scheme U obtained from T by some "reasonable scheme-theoretic construction", there exists $U' \in \mathcal{S}$ such that $U' \cong U$ as S-schemes.

The answer is positive, and is made precise in [Sta26, Tag 000H]. For convenience, we still denote this shrunk down category as Sch/S. To equip it with a topology τ, namely fppf or pro-étale, the collection of all coverings needs also to be reduced with similar care; this is done in [Sta26, Tag 000W]. It is shown in [Sta26, Tag 00VY] that the resulting category of sheaves on Sch/S is independent of the set of τ-coverings chosen in this manner.

The upshot is that there are many possible "big fppf/pro-étale" sites over S, but for all intents and purposes in this document any choice is virtually equivalent.

Definition 2.1: For the remainder of these notes, we fix $(\mathrm{Sch}/S)_{\mathrm{fppf}}$ and $\mathscr{C} := (\mathrm{Sch}/S)_{\mathrm{pro\acute{e}t}}$ big fppf and pro-étale sites over S, as in [Sta26, Tag 021R] and [Sta26, Tag 098G] respectively. We can and will arrange so that the underlying categories are the same, and so that our choice of fppf coverings is a subset of our choice of pro-étale coverings. We call them *the* big sites over S.

2.1.2. *Algebraic stacks on a site.* We use the language of fibered categories and stacks over a site as in [Vis], namely, a stack over a site is a category fibered in groupoids for which every descent datum is effective. In particular, we can make sense of stacks over the pro-étale site $\mathscr{C}$ over S.

Algebraic stacks, on the other hand, are only defined over $(\mathrm{Sch}/S)_{\mathrm{fppf}}$ (or even $(\mathrm{Sch}/S)_{\mathrm{\acute{e}t}}$) in most references. Our conventions are inspired from [LMB18, 4].

Definition 2.2: A stack over $(\mathrm{Sch}/S)_{\mathrm{fppf}}$ (resp. $\mathscr{C} = (\mathrm{Sch}/S)_{\mathrm{pro\acute{e}t}}$) is *algebraic* if its diagonal is representable, quasi-compact and separated, and if there exists a surjective smooth morphism (called an *atlas*) from an algebraic space. It is *Deligne-Mumford* if the atlas can be chosen étale. A Deligne-Mumford stack is *separated* if its diagonal is proper (equiv. finite).

We will work with standard fppf algebraic or Deligne-Mumford stacks (classifying stacks, moduli stacks of curves, abelian varieties,...), and view them as pro-étale stacks. When we do so, it is allowed by the following fact, without further mention.

Fact 2.3 ([Sta26],Tag 0GRH)**:** *An algebraic (resp. Deligne-Mumford) stack over* $(\mathrm{Sch}/S)_{fppf}$ *with ind-quasi-affine diagonal has all fpqc descent data effective. In particular, it is automatically an algebraic (resp. Deligne-Mumford) stack over* $\mathscr{C}$.

In this document, stacks are not required to be algebraic until Proposition 3.11.

2.2. **Local systems and torsors on schemes.** We fix an S-scheme T.

Definition 2.4: Let A be a topological space. Let $\mathcal{F}_A$ denote the presheaf on $\mathscr{C}_T$ which associates to any $U \in \mathscr{C}_T$ the set of continuous functions $U \to A$. This is a sheaf by [BS13, Lemma 4.2.12].

If A is discrete and T is qcqs then $\mathcal{F}_A$ is the usual constant sheaf $\underline{A}$. The sheaf $\mathcal{F}_R$ is a sheaf of rings, and $\mathcal{F}_V$ is an $\mathcal{F}_R$-module.

Definition 2.5: An $\mathcal{F}_R$-module locally isomorphic to the $\mathcal{F}_R$-module $\mathcal{F}_V$ is called an *R-local system with fiber V*. The groupoid of R-local systems with fiber V over $\mathscr{C}_T$ is denoted $\mathbf{Loc}_V(T)$. The functor

$$\mathbf{Loc}_V : T \in \mathscr{C} \longmapsto \mathbf{Loc}_V(T)$$

defines a stack over $\mathscr{C}$.

The sheaf $\mathcal{F}_Q$ is a sheaf of groups.

Definition 2.6: Let $\mathcal{G}$ be a sheaf of groups on $\mathscr{C}$. A *$\mathcal{G}$-sheaf* $\mathcal{P}$ on $\mathscr{C}_T$ is a sheaf of sets together with a morphism $\mathcal{G}_T \times \mathcal{P} \to \mathcal{P}$ inducing a group action for every object of $\mathscr{C}_T$. A *$\mathcal{G}$-torsor* on $\mathscr{C}_T$ is a $\mathcal{G}$-sheaf locally isomorphic to the $\mathcal{G}$-sheaf $\mathcal{G}$. The category of $\mathcal{G}$-sheaves (resp. the groupoid of $\mathcal{G}$-torsors) over

$\mathscr{C}_T$ is denoted $\mathcal{G}\mathbf{Sh}(T)$ (resp. $\mathbf{B}\mathcal{G}(T)$). As in Definition 2.5 this defines a fibered category (resp. a stack) $\mathcal{G}\mathbf{Sh}$ (resp. $\mathbf{B}\mathcal{G}$) on $\mathscr{C}$. When $\mathcal{G} = \mathcal{F}_Q$, we write $Q\mathbf{Sh}$ (resp. $\mathbf{B}Q$) and call its objects Q-sheaves (resp. Q-torsors).

Fact 2.7 ([Gir20], III, Corollaire 2.2.6)**:** *If $Q = \mathrm{GL}(V)$ then there is an equivalence*

$$\mathrm{Triv} : \mathbf{Loc}_V \to \mathbf{B}\mathrm{GL}(V)$$

mapping a local system $\mathbb{L}$ over T to its sheaf of trivializations $\underline{\mathrm{Isom}}(\mathbb{L}, \mathcal{F}_V)$ over T.

Definition 2.8: Let $\phi : \mathcal{H} \to \mathcal{G}$ be a morphism of sheaves of groups over $\mathscr{C}$. The forgetful functor from $\mathcal{G}\mathbf{Sh}$ to $\mathcal{H}\mathbf{Sh}$ admits a left-adjoint

$$\mathrm{Ind}_{\mathcal{H}}^{\mathcal{G}} : \mathcal{H}\mathbf{Sh} \to \mathcal{G}\mathbf{Sh}.$$

mapping any $\mathcal{H}$-sheaf $\mathcal{P}$ over T to the $\mathcal{G}$-sheaf $\mathrm{Ind}_{\mathcal{H}}^{\mathcal{G}}(\mathcal{P})$ over T defined as the quotientof $\mathcal{G} \times \mathcal{P}$ under the $\mathcal{H}$-action:

$$\begin{aligned} \mathcal{H} \times (\mathcal{G} \times \mathcal{P}) &\to \mathcal{G} \times \mathcal{P} \\ (h, g, p) &\mapsto (g\phi(h)^{-1}, h \cdot p). \end{aligned}$$

This functor restricts to torsors and defines $\mathrm{Ind}_{\mathcal{H}}^{\mathcal{G}} : \mathbf{B}\mathcal{H} \to \mathbf{B}\mathcal{G}$. If $\mathcal{H} = \mathcal{F}_Q$, $\mathcal{G} = \mathcal{F}_{Q'}$ and ϕ comes from a continuous group homomorphism $Q \to Q'$, we denote it $\mathrm{Ind}_Q^{Q'} : \mathbf{B}Q \to \mathbf{B}Q'$.

Remark 2.9: There is an alternative description for $\mathrm{Ind}_{\mathcal{H}}^{\mathcal{G}}$. Recall that $\mathbf{B}\mathcal{G}$ is the stack associated to the prestack $\mathbf{B}\mathcal{G}^{\mathrm{pre}}$ such that for all T, $\mathbf{B}\mathcal{G}^{\mathrm{pre}}(T)$ has a single object $\{*\}$ and $\mathrm{Aut}(\{*\}) = \mathcal{G}(T)$. The morphism $\mathcal{H} \to \mathcal{G}$ induces a morphism of prestacks

$$\mathbf{B}\mathcal{H}^{\mathrm{pre}} \longrightarrow \mathbf{B}\mathcal{G}^{\mathrm{pre}},$$

the associated functor between stacks $\mathbf{B}\mathcal{H} \to \mathbf{B}\mathcal{G}$ is the induction $\mathrm{Ind}_{\mathcal{H}}^{\mathcal{G}}$. To check that both definitions agree, verify the result for the trivial torsor, then argue locally for an arbitrary torsor.

Using Fact 2.7 and Definition 2.8, any continuous representation of Q on V defines a functor

$$\mathbf{B}Q \to \mathbf{Loc}_V$$

sending P to the R-local system with fiber V associated with $\mathrm{Ind}_Q^{\mathrm{GL}(V)} P$.

We will be interested in torsors over $\mathscr{C}$ of geometric origin.

Definition 2.10: Let $T' \to T$ be an algebraic space over T. Fix a homomorphism $Q \to \mathrm{Aut}_T(T')$. We say that $T' \to T$ is a *Q-torsor* over T if the functor of points $h_{T'}$ defined by

$$(U \to T) \in \mathscr{C} \mapsto h_{T'}(U) = \mathrm{Hom}_T(U, T'),$$

is a Q-torsor on $\mathscr{C}_T$ as in Definition 2.6 for the induced action of $\mathcal{F}_Q$. If Q is finite then $T' \to T$ is étale; if moreover $T' \to T$ is finite then T' is a scheme.

If K_0 is a topological group, let $\mathcal{C}(K_0)$ denote the set of all open normal subgroups of K_0.

Definition 2.11: Let K_0 be a profinite group. Let $(T_K)_{K \in \mathcal{C}(K_0)}$ be a projective system of schemes. We say that it is *fit for* K_0, or a *K_0-fit system*, if it satisfies the following assumptions:

(i) for each $K \in \mathcal{C}(K_0)$, the morphism $T_K \to T_{K_0}$ is a finite K_0/K-torsor over T_{K_0} as in Definition 2.10, and
(ii) for every $K' \subseteq K$ in $\mathcal{C}(K_0)$, the morphism $T_{K'} \to T_K$ is K_0-equivariant for the K_0-action induced from (i).

Then for every $K \in \mathcal{C}(K_0)$ the map $T_K \to T_{K_0}$ is a finite étale cover, thus it makes sense to define the projective limit

$$T_\infty \coloneqq \varprojlim_{K \in \mathcal{C}(K_0)} T_K$$

which is a pro-étale cover of T_{K_0}, and a K_0-torsor over T_{K_0} as in Definition 2.10 with $h_{T_\infty} = \varprojlim_K h_{T_K}$ in $K_0\mathbf{Sh}(T_{K_0})$.

The groupoid of K_0-fit systems over a fixed scheme T_{K_0} is denoted $K_0\mathbf{FS}(T_{K_0})$, and $K_0\mathbf{FS}$ is a stack over $\mathscr{C}$. We will see in Proposition 2.20 that it is equivalent to $\mathbf{B}K_0$ via $(T_K)_K \mapsto T_\infty$.

Definition 2.12: Let K_0 be a profinite group with a fixed continuous embedding $K_0 \hookrightarrow \mathrm{GL}(V)$. Let $(T_K)_{K\in\mathcal{C}(K_0)}$ be schemes fit for K_0, let h_{T_∞} be the K_0-torsor from Definition 2.11. The corresponding *canonical local system* $\mathbb{L}^{\mathrm{can}} \in \mathbf{Loc}_V(T_{K_0})$ is the R-local system associated with $\mathrm{Ind}_{K_0}^{\mathrm{GL}(V)}(h_{T_\infty})$.

Remark 2.13: Let us give a more down-to-earth description of the canonical local system in case T_{K_0} is a connected scheme with a K_0-fit system $(T_K)_K$. Fix $(\bar{t}_K)_K$ a compatible system of geometric points of $(T_K)_K$. For each $K \in \mathcal{C}(K_0)$, $T_K \to T_{K_0}$ is a K_0/K-torsor on T_{K_0} in the usual sense for the étale topology on S. Having fixed geometric points, this torsor is uniquely described by a continuous morphism

$$\pi_1(T_{K_0}, \bar{t}_{K_0}) \to K_0/K.$$

The compatibility conditions allow to take the projective limit over K so as to find a continuous morphism

$$\pi_1(T_{K_0}, \bar{t}_{K_0}) \to K_0.$$

The composition with the embeddings $K_0 \hookrightarrow \mathrm{GL}(V)$ yields the monodromy action on the fibers of $\mathbb{L}^{\mathrm{can}}$. If T_{K_0} is locally noetherian but no longer connected, we can argue component-wise.

Example 2.14: Let (G, Ω) be a Shimura datum with reflex field E, set $S = \operatorname{Spec} E$. Let K_0 be a neat compact open subgroup of $G(\mathbb{A}_f)$. If $K \in \mathcal{C}(K_0)$, let $T_K = \mathrm{Sh}_K(G, \Omega)$ denote the associated Shimura variety of level K. Then $(\mathrm{Sh}_K(G, \Omega))_K$ is fit for K_0, so $\mathrm{Sh}_{K_0}(G, \Omega)$ carries a canonical local system. The neatness assumption is equivalent to condition (i) of Definition 2.11. We can remove this assumption by replacing schemes with stacks.

2.3. **Local systems and torsors on stacks.** Let $\mathscr{X}$ be a stack on $\mathscr{C}$. We extend the notions of the previous paragraph to $\mathscr{X}$.

Definition 2.15: We denote by $Q\mathbf{Sh}(\mathscr{X})$ (resp. $\mathbf{Loc}_V(\mathscr{X})$, $\mathbf{B}Q(\mathscr{X})$, $K_0\mathbf{FS}(\mathscr{X})$) the category of morphisms of fibered categories $\mathscr{X} \to Q\mathbf{Sh}$ (resp. $\mathscr{X} \to \mathbf{Loc}_V$, $\mathscr{X} \to \mathbf{B}Q$, $\mathscr{X} \to K_0\mathbf{FS}$), the objects of which are called *Q-sheaves on $\mathscr{X}$* (resp. *R-local systems with fiber V on $\mathscr{X}$*, *Q-torsors on $\mathscr{X}$*, *K_0-fit systems over $\mathscr{X}$*).

Explicitely, an object of $Q\mathbf{Sh}(\mathscr{X})$ is the data, for each $\xi : T \to \mathscr{X}$, of a Q-sheaf $F_\xi \in Q\mathbf{Sh}(T)$, collectively satisfying compatibility conditions as ξ and T vary. Identical descriptions hold for $\mathbf{Loc}_V(\mathscr{X})$ and $\mathbf{B}Q(\mathscr{X})$.

The induction functor and the equivalence of Fact 2.7 automatically extend to Q-sheaves, local systems and torsors on $\mathscr{X}$. Consequently, any continuous representation of Q on V defines a functor

$$\mathbf{B}Q(\mathscr{X}) \to \mathbf{Loc}_V(\mathscr{X})$$

sending a Q-torsor P on $\mathscr{X}$ to the R-local system with fiber V on $\mathscr{X}$ associated with $\mathrm{Ind}_Q^{\mathrm{GL}(V)} P$.

We will be interested in a geometric description of objects of $\mathbf{B}Q(\mathscr{X})$ and $K_0\mathbf{FS}(\mathscr{X})$.

Definition 2.16: Let $f : \mathscr{X}' \to \mathscr{X}$ be a representable morphism of stacks over $\mathscr{C}$. Assume $\mathscr{X}'$ is equipped with a pseudo-Q-action[2] preserving f[3]. For any scheme $T \to \mathscr{X}$, the fiber product $T' := T \times_{\mathscr{X}} \mathscr{X}'$ is an algebraic space which inherits a Q-action $Q \to \mathrm{Aut}_T(T')$. We say that f is a *Q-torsor* if for all such T the morphism $T' \to T$ is a Q-torsor as in Definition 2.10. This defines a Q-torsor $h_{\mathscr{X}'}$ on $\mathscr{X}$ as in Definition 2.15.

Definition 2.17: Let K_0 be a profinite group. Let $(\mathscr{X}_K)_{K\in\mathcal{C}(K_0)}$ be a projective system of stacks. We say that it is *fit for K_0*, or a *K_0-fit system*, if it satisfies the following assumptions:

[2]Formally, a pseudo-Q-action is a 2-morphism $\mathbf{B}Q \to \mathbf{Cat}$ with essential image $\mathscr{X}'$. Explicitely, this is the data for every $q \in Q$ of an equivalence $a_q : \mathscr{X}' \xrightarrow{\sim} \mathscr{X}'$ such that there are isomorphisms $a_q \circ a_{q'} \simeq a_{qq'}$ satisfying the associativity condition.

[3]Formally, for any $q \in Q$ inducing an equivalence $a_q : \mathscr{X}' \to \mathscr{X}'$, there is an isomorphism $f \circ a_q \simeq f$ satisfying omitted compatibility conditions.

(i) for each $K \in \mathcal{C}(K_0)$, the morphism $\mathscr{X}_K \to \mathscr{X}_{K_0}$ is a finite K_0/K-torsor as in Definition 2.16, and
(ii) for every $K' \subseteq K$ in $\mathcal{C}(K_0)$, the morphism $\mathscr{X}_{K'} \to \mathscr{X}_K$ is pseudo-K_0-equivariant[4] for the K_0-action induced from (i).

Then for every $K \in \mathcal{C}(K_0)$ the map $\mathscr{X}_K \to \mathscr{X}_{K_0}$ is finite étale. One can define the projective limit

$$\mathscr{X}_\infty := \varprojlim_{K \in \mathcal{C}(K_0)} \mathscr{X}_K$$

which is pro-étale over $\mathscr{X}_{K_0}$, and is a K_0-torsor as in Definition 2.16 with $h_{\mathscr{X}_\infty} = \varprojlim_K h_{\mathscr{X}_K}$ in $K_0\mathbf{Sh}(\mathscr{X}_{K_0})$.

We show that both notions of K_0-fit systems on a fixed stack $\mathscr{X}_{K_0}$ coincide. Provided a projective system $(\mathscr{X}_K)_{K \in \mathcal{C}(K_0)}$ as in Definition 2.17 and an S-scheme T_{K_0}, we define a functor $\mathscr{X}_{K_0} \to K_0\mathbf{FS}$ by sending a T-object $T \to \mathscr{X}$ to the K_0-fit system $(T \times_{\mathscr{X}} \mathscr{X}_K)_{K \in \mathcal{C}(K_0)}$. Conversely, provided a functor $\mathscr{X}_{K_0} \to K_0\mathbf{FS}$, for any $K \in \mathcal{C}(K_0)$ we define $\mathscr{X}_K(T_{K_0})$ as follows: for any S-scheme T_{K_0}, we consider the groupoid

$$\mathscr{X}_K(T_{K_0}) = \{(\xi, T_K) \mid \xi \in \mathscr{X}_{K_0}(T_{K_0}) \text{ mapping to } (T_{K'})_{K' \in \mathcal{C}(K_0)} \in K_0\mathbf{FS}(T_{K_0}) \text{ via } \mathscr{X}_{K_0} \to K_0\mathbf{FS}\}.$$

One then checks that $\mathscr{X}_K$ is a stack, and that $(\mathscr{X}_K)_{K \in \mathcal{C}(K_0)}$ is a K_0-fit system on $\mathscr{X}_{K_0}$ as in Definition 2.17. The constructions are quasi-inverses of each-other.

Definition 2.18: Let K_0 be a profinite group with a fixed continuous embedding $K_0 \hookrightarrow \mathrm{GL}(V)$. Let $(\mathscr{X}_K)_{K \in \mathcal{C}(K_0)}$ be stacks fit for K_0, let $h_{\mathscr{X}_\infty}$ be the K_0-torsor defined in Definition 2.17. The corresponding *canonical local system* $\mathbb{L}^{\mathrm{can}} \in \mathbf{Loc}_V(\mathscr{X}_{K_0})$ is the R-local system associated to $\mathrm{Ind}_{K_0}^{\mathrm{GL}(V)}(h_{\mathscr{X}_\infty})$.

If $\mathscr{X}_{K_0}$ is a scheme, then so are all the $\mathscr{X}_K$, and Definitions 2.12 and 2.18 coincide.

Example 2.19: Let (G, Ω) be a Shimura datum with reflex field E. Let K_0 be a neat compact open subgroup of $G(\mathbb{A}_f)$. Let $\mathrm{Sh}_\infty(G, \Omega) = \varprojlim_{K \in \mathcal{C}(K_0)} \mathrm{Sh}_K(G, \Omega)$ on which $G(\mathbb{A}_f)$ acts. For K' *any* compact open subgroup of $G(\mathbb{A}_f)$, define $\mathrm{Sh}_{K'}[G, \Omega]$ as the quotient stack $[\mathrm{Sh}_\infty(G, \Omega)/K']$ (if K' is neat this agrees with the earlier definition of $\mathrm{Sh}_{K'}(G, \Omega)$). If we fix K_0' *any* compact open subgroup of $G(\mathbb{A}_f)$ then $(\mathrm{Sh}_{K'}[G, \Omega])_{K' \in \mathcal{C}(K_0')}$ is a K_0'-fit system of stacks; in particular $\mathrm{Sh}_{K_0'}[G, \Omega]$ carries a canonical local system.

2.4. The universal K_0-fit system.

Induction between classifying stacks provide a universal K_0-fit system.

Proposition 2.20: *Let K_0 be a second countable profinite group.*

(i) *If $K \in \mathcal{C}(K_0)$, the induction functor defines a finite K_0/K-torsor $\mathbf{B}K \to \mathbf{B}K_0$ on $\mathbf{B}K_0$ as in Definition 2.16, so that $(\mathbf{B}K)_{K \in \mathcal{C}(K_0)}$ is fit for K_0. The associated K_0-torsor recovers the universal torsor on $\mathbf{B}K_0$ defined by the identity $\mathbf{B}K_0 \to \mathbf{B}K_0$.*
(ii) *There is an equivalence of stacks $K_0\mathbf{FS} \longleftrightarrow \mathbf{B}K_0$ given by the functors:*

$$(T_K)_K \longmapsto T_\infty$$
$$(T_{K_0} \times_{\mathbf{B}K_0} \mathbf{B}K)_K \longleftarrow [T_{K_0} \longrightarrow \mathbf{B}K_0].$$

Remark 2.21: Proposition 2.20 (ii) immediately extends to stacks: any fit system $(\mathscr{X}_K)_K$ on $\mathscr{X}_{K_0}$ is obtained by pulling back the universal system $(\mathbf{B}K)_K$ along the morphism $\mathscr{X}_{K_0} \to \mathbf{B}K_0$ defining $\mathscr{X}_\infty \to \mathscr{X}_{K_0}$. For this reason we call $(\mathbf{B}K)_K$ the *universal K_0-fit system.*

We prove Proposition 2.20. The proof of (i) relies on the following lemma.

[4]This is the categorical version of an equivariant morphism, sending the pseudo-action on $\mathscr{X}_{K'}$ to the pseudo-action on $\mathscr{X}_K$ up to isomorphisms satisfying omitted compatibility conditions.

Lemma 2.22: *Let $\mathcal{H} \to \mathcal{G}$ be a morphism of sheaves of groups on $\mathscr{C}$. Then there is a 2-cartesian square*

$$\begin{tikzcd} \mathbf{B}\mathcal{H} \arrow[r] \arrow[d] & \mathbf{B}\mathcal{G} \arrow[d] \\ \mathrm{pt} \arrow[r] & \mathbf{B}(\mathcal{G}/\mathcal{H}) \end{tikzcd}$$

where all arrows are induction functors. Here pt *denotes $\mathscr{C}$ viewed as a stack over itself, or as the classifying stack of the trivial group.*

Proof. As stackification commutes with 2-fiber product (see [Sta26, Tag 04Y1]), it suffices to show the result at the level of prestacks. See Remark 2.9 for the definitions of the prestacks $\mathbf{B}\mathcal{H}^{\mathrm{pre}}$, $\mathbf{B}\mathcal{G}^{\mathrm{pre}}$ and $\mathbf{B}(\mathcal{G}/\mathcal{H})^{\mathrm{pre}}$, and of the induction functor between them. Let T be an S-scheme, a T-object of $\mathrm{pt} \times_{\mathbf{B}(\mathcal{G}/\mathcal{H})^{\mathrm{pre}}} \mathbf{B}\mathcal{G}^{\mathrm{pre}}$ is just the datum of an element $q \in (\mathcal{G}/\mathcal{H})(T)$, and one checks that two objects defined by q and q' are isomorphic if and only if $q'q^{-1}$ lies in the image of $\mathcal{G}(T)$. As $\mathcal{G} \to \mathcal{G}/\mathcal{H}$ is surjective, we find that any two objects of the fiber product are locally isomorphic, and so the same holds true of the stackified fiber product. Since this stack always admits a T-point induced by the neutral element of $(\mathcal{G}/\mathcal{H})(T)$, it is the classifying stack of the automorphism group sheaf of this element. This is easily computed to be $\mathcal{H}$. ☐

Let K_0 be profinite and second-countable. We apply Lemma 2.22 to $\mathcal{G} = \mathcal{F}_{K_0}$ and $\mathcal{H} = \mathcal{F}_K$. We need two more lemmas.

Lemma 2.23: *For K_0 profinite and second countable, and for any $K \in \mathcal{C}(K_0)$, there is a natural isomorphism $\mathcal{F}_{K_0}/\mathcal{F}_K \simeq \mathcal{F}_{K_0/K}$.*

Proof. As K_0 is second countable, we can find a countable basis of neighborhoods $(H_n)_{n\geq 0}$ of the neutral element, consisting of elements of $\mathcal{C}(K_0)$ such that $H_{n+1} \subseteq H_n$. Consider the surjections

$$\mathcal{F}_{K_0/H_n} \longrightarrow \mathcal{F}_{K_0/H_n}/\mathcal{F}_{K/(K\cap H_n)}.$$

Note that since K_0 is profinite, $\mathcal{F}_{K_0} = \varprojlim_n \mathcal{F}_{K/H_n}$. As the topos of sheaves on $\mathscr{C} = (Sch/S)_{\text{proét}}$ is replete (see [BS13, Lemma 2.4.9, Definition 3.1.1, Proposition 3.2.3]), it follows from [BS13, Lemma 3.1.8] that the morphism

$$\mathcal{F}_{K_0} \longrightarrow \varprojlim_n (\mathcal{F}_{K_0/H_n}/\mathcal{F}_{K/(K\cap H_n)})$$

is surjective. Its kernel is easily seen to be $\varprojlim_n \mathcal{F}_{K/(K\cap H_n)} = \mathcal{F}_K$. Moreover, there is a natural isomorphism

$$\mathcal{F}_{K_0/H_n}/\mathcal{F}_{K/(K\cap H_n)} \simeq \mathcal{F}_{K_0/KH_n}.$$

Le left-hand side equals $\mathcal{F}_{K_0/K}$ for n large enough. This provides the sought-after isomorphism. ☐

Lemma 2.24: *Let $\mathcal{G}$ be a sheaf of groups over $\mathscr{C}$, fix T an S-scheme, and $\mathcal{P}$ a $\mathcal{G}$-torsor over T. Let $T \to \mathbf{B}\mathcal{G}$ be the morphism defining $\mathcal{P}$. Then there is a 2-cartesian diagram*

$$\begin{tikzcd} \tilde{\mathcal{P}} \arrow[r] \arrow[d] & T \arrow[d] \\ \mathrm{pt} \arrow[r] & \mathbf{B}\mathcal{G}. \end{tikzcd}$$

Here $\tilde{\mathcal{P}}$ is the sheaf on $\mathscr{C}$ obtained as the extension from $\mathscr{C}_T$ to $\mathscr{C}$ of $\mathcal{P}$ by the empty set (see [Sta26, Tag 00Y0, (5)]*).*

If $\mathcal{G} = \mathcal{F}_Q$ for a profinite group Q, then both $\mathcal{P}$ and $\tilde{\mathcal{P}}$ are represented by the same scheme (viewed as a T-scheme and as an S-scheme respectively). In particular, $\mathrm{pt} \to \mathbf{B}Q$ *is a Q-torsor which is the universal family over $\mathbf{B}Q$.*

Proof. A direct computation shows that the 2-fiber product is the sheaf of sets described as:

$$(U \to S) \longmapsto \coprod_{u:U\to T} \mathcal{P}(u : U \to T).$$

From [Sta26, Tag 03CD] this coincides with $\tilde{\mathcal{P}}$. The second statement follows from [Sta26, Tag 03HU] along with the fact that (pro)finite torsors are representable by (pro)finite étale covers. $\square$

Remark 2.25: Lemma 2.24 immediately extends to stacks: if Q is profinite, any Q-torsor $\mathscr{X}' \to \mathscr{X}$ fits in the 2-cartesian square

$$\begin{tikzcd} \mathcal{X}' \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\ \mathrm{pt} \arrow[r] & \mathbf{B}Q. \end{tikzcd}$$

Back to the proof. From Lemma 2.22 and Lemma 2.23 we find the 2-cartesian diagram

$$\begin{tikzcd} \mathbf{B}K \arrow[r] \arrow[d] & \mathbf{B}K_0 \arrow[d] \\ \mathrm{pt} \arrow[r] & \mathbf{B}K_0/K. \end{tikzcd}$$

Lemma 2.24 ensures that the bottom arrow, and hence also the top arrow, are K_0/K-torsors. It is easy to check that the associated K_0-torsor

$$\mathbf{B}K_0 \longrightarrow \varprojlim_K \mathbf{B}K_0/K = \mathbf{B}K_0$$

is (isomorphic to) the identity functor. This proves (i).

To prove (ii), notice that for a fit system $(T_K)_K$ with K_0-torsor T_∞ over T_{K_0} corresponding to $T_{K_0} \to \mathbf{B}K_0$, Lemma 2.22 and Lemma 2.24 induce a 2-commutative diagram of 2-cartesian squares

$$\begin{tikzcd} T_\infty \arrow[r] \arrow[d] & \mathrm{pt} \arrow[d] & \\ T_K \arrow[r] \arrow[d] & \mathbf{B}K \arrow[r] \arrow[d] & \mathrm{pt} \arrow[d] \\ T_{K_0} \arrow[r] & \mathbf{B}K_0 \arrow[r] & \mathbf{B}K_0/K. \end{tikzcd}$$

It follows that there is an isomorphism $T_K \simeq T_{K_0} \times_{\mathbf{B}K_0} \mathbf{B}K$. Conversely, provided an S-scheme T_{K_0} and a K_0-torsor $T_{K_0} \to \mathbf{B}K_0$, let $T_\infty \coloneqq T_{K_0} \times_{\mathbf{B}K_0} \mathrm{pt}$ be the T_{K_0}-scheme representing it. For any $K \in \mathcal{C}(K_0)$ we set

$$T_K \coloneqq T_{K_0} \times_{\mathbf{B}K_0} \mathbf{B}K$$

so that the diagram above holds again. This diagram shows that T_∞ is (isomorphic to) the K_0-torsor associated with $(T_K)_K$. We are done.

2.5. **Canonical vs universal local systems.** Let us first motivate our goal and the formalism we introduce to achieve it. Fix some integer $g > 0$ and let $\mathscr{M}_{K_0}$ denote the moduli stack over $\mathbb{Q}$ of principally polarized abelian schemes of dimension g with K_0-level structure, where K_0 is a compact open subgroup of $\mathrm{GSp}_{2g}(\mathbb{A}_f)$. Replacing K_0 with finite index open subgroups K, we find a K_0-fit system $(\mathscr{M}_K)_K$ of stacks in the sense of Definition 2.17, so we can consider the corresponding *canonical* $\mathbb{A}_f$-local system $\mathbb{L}^{\mathrm{can}}$ on $\mathscr{M}_{K_0}$. There is a second local system on $\mathscr{M}_{K_0}$: the relative cohomology of the universal abelian scheme, which we call the *universal* local system $\mathbb{L}_{K_0}$. We aim to show that $\mathbb{L}^{\mathrm{can}}$ and $\mathbb{L}_{K_0}$ are canonically isomorphic, which is the content of Proposition 2.30 below.

To do so, we formalize the notion of a moduli stack with level structure in Definition 2.26 and Definition 2.28. We sketch how this is done in the explicit case of $\mathscr{M}_{K_0}$. For any scheme $T/\mathbb{Q}$, an object in $\mathscr{M}_{K_0}(T)$ is a triple $(f : A \to T, \lambda, [\eta]_{K_0})$ where

- $(f : A \to T, \lambda)$ is a principally polarized abelian scheme of dimension g, and
- $[\eta]_{K_0}$ is a section of the quotient by $\mathcal{F}_{K_0}$ of some subsheaf of $\underline{\mathrm{Isom}}(R^1 f_* \mathbb{A}_f, \mathcal{F}_V)$ where $V = \mathbb{A}_f^{2g}$ is acted-on by K_0; namely we restrict to those isomorphisms that are compatible with the usual symplectic structure on V, and the symplectic structure on $R^1 f_* \mathbb{A}_f$ induced from λ.

To define $\mathscr{M}_{K_0}$, we first consider the stack $\mathscr{X}$ of principally polarized abelian schemes of dimension g and view relative cohomology as a morphism from $\mathscr{X}$ to $\mathbf{Loc}_V$. From there, we could define a stack of triples $(f : A \to T, \lambda, [\eta]_{K_0})$ with the slight caveat that $[\eta]_{K_0}$ is a section of the whole sheaf

$\underline{\mathrm{Isom}}(R^1 f_* \mathbb{A}_f, \mathcal{F}_V)/\mathcal{F}_{K_0}$ (no compatibility condition with the symplectic structures). To formalize replacing every $\mathrm{GL}_d(\mathbb{A}_f)$-torsor $\underline{\mathrm{Isom}}(R^1 f_* \mathbb{A}_f, \mathcal{F}_V)$ with the appropriate $\mathrm{GSp}_{2g}(\mathbb{A}_f)$-subsheaf (which is a $\mathrm{GSp}_{2g}(\mathbb{A}_f)$-torsor), we use the language of natural transformations between fibered categories.

Fix an embedding $Q \hookrightarrow \mathrm{GL}(V)$. Recall the equivalence of stacks

$$\mathrm{Triv} : \mathbf{Loc}_V \to \mathbf{B}\mathrm{GL}(V)$$

assigning to any $T \in \mathscr{C}$ and any $\mathbb{L} \in \mathbf{Loc}_V(T)$ the $\mathrm{GL}(V)$-torsor $\underline{\mathrm{Isom}}(\mathbb{L}, \mathcal{F}_V)$ over T.

Definition 2.26: A *Q-moduli datum* is a quadruple $(\mathscr{X}, \mathbb{L}_{\mathscr{X}}, \mathrm{Lev}, \iota)$ consisting of

(i) a stack $\mathscr{X}$ over $\mathscr{C}$,
(ii) a local system $\mathbb{L}_{\mathscr{X}} : \mathscr{X} \to \mathbf{Loc}_V$ on $\mathscr{X}$; we let $\mathrm{Triv}_{\mathscr{X}} = \mathrm{Triv} \circ \mathbb{L}_{\mathscr{X}}$,
(iii) a Q-torsor $\mathrm{Lev} : \mathscr{X} \to \mathbf{B}Q$ on $\mathscr{X}$, and
(iv) a natural transformation ι between the morphisms of fibered categories:

$$\begin{tikzcd}
& \mathbf{B}\mathrm{GL}(V) \arrow[dr, bend left, "\text{forget}"] \arrow[dd, phantom, "\Uparrow\iota"] & \\
\mathscr{X} \arrow[ur, bend left, "\mathrm{Triv}_{\mathscr{X}}"] \arrow[dr, bend right, "\mathrm{Lev}"'] & & Q\mathbf{Sh}. \\
& \mathbf{B}Q \arrow[ur, bend right, "\text{forget}"'] &
\end{tikzcd}$$

A Q-moduli datum is to be understood as follows. The functor $\mathrm{Triv}_{\mathscr{X}} : \mathscr{X} \to \mathbf{B}\mathrm{GL}(V)$ assigns to any $T \in \mathscr{C}$ and any $A \in \mathscr{X}(T)$ the $\mathrm{GL}(V)$-torsor $\underline{\mathrm{Isom}}(\mathbb{L}_{\mathscr{X}}(A), \mathcal{F}_V)$ over T. The functor Lev assigns to the same data a Q-subsheaf of $\mathrm{Triv}_{\mathscr{X}}(A)$ which is a Q-torsor for the induced action, considering only isomorphisms with additional constraints. The condition of being a Q-subsheaf for each $T \in \mathscr{C}$ and $A \in \mathscr{X}(T)$ is formalized by ι, which is automatically a natural monomorphism.

Lemma 2.27: *Let $(\mathscr{X}, \mathbb{L}_{\mathscr{X}}, \mathrm{Lev}, \iota)$ be a Q-moduli datum. Then $\mathbb{L}_{\mathscr{X}}$ is the R-local system with fiber V associated with the $\mathrm{GL}(V)$-torsor $\mathrm{Ind}_Q^{\mathrm{GL}(V)}(\mathrm{Lev})$.*

Proof. Condition (iv) is equivalent to a natural isomorphism

$$\begin{tikzcd}
\mathscr{X} \arrow[rr, "\mathrm{Triv}_L"] \arrow[dr, bend right, "\mathrm{Lev}"'] & \arrow[d, phantom, "\Uparrow"] & \mathbf{B}\mathrm{GL}(V) \\
& \mathbf{B}Q \arrow[ur, bend right, "\mathrm{Ind}_Q^{\mathrm{GL}(V)}"'] &
\end{tikzcd}$$

The equivalence follows formally from the fact that $\mathrm{Ind}_Q^{\mathrm{GL}(V)} : Q\mathbf{Sh} \to \mathrm{GL}(V)\mathbf{Sh}$ is left adjoint to the forgetful functor $\mathrm{GL}(V)\mathbf{Sh} \to Q\mathbf{Sh}$. □

It follows that a Q-moduli datum is determined up to isomorphism by the Q-torsor Lev on $\mathscr{X}$.
We are interested in the following moduli stacks:

Definition 2.28: Let K_0 be a compact subgroup of Q which is profinite, and K an open subgroup of K_0. Let $(\mathscr{X}, \mathbb{L}_{\mathscr{X}}, \mathrm{Lev}, \iota)$ be a Q-moduli datum. The associated *moduli stack of level K* is the stack $\mathscr{M}_K$ over $\mathscr{C}$ assigning to each $T \in \mathscr{C}$ a pair $(A, [\eta]_K)$, where

(i) A is an object in $\mathscr{X}(T)$, and
(ii) $[\eta]_K \in H^0(T, \mathrm{Lev}(A)/\mathcal{F}_K)$ is a global section of the quotient of $\mathrm{Lev}(A)$ by the action of $\mathcal{F}_K$.

If K is normal in K_0, the obvious morphism $\mathscr{M}_K \to \mathscr{M}_{K_0}$ is a K_0/K-torsor as in Definition 2.16.

Fix K_0 and $(\mathscr{X}, \mathbb{L}_{\mathscr{X}}, \mathrm{Lev}, \iota)$ as in Definition 2.28. The projective system $(\mathscr{M}_K)_{K \in \mathcal{C}(K_0)}$ is fit for K_0. By Definition 2.18 we can consider the canonical local system $\mathbb{L}^{\mathrm{can}}$ on $\mathscr{M}_{K_0}$. We now define the universal local system on $\mathscr{M}_{K_0}$ and show that it is canonically isomorphic to $\mathbb{L}^{\mathrm{can}}$. Observe that the projection onto the first component defines a morphism of stacks $p : \mathscr{M}_{K_0} \to \mathscr{X}$.

Definition 2.29: The *universal local system* $\mathbb{L}_{K_0} \in \mathbf{Loc}_V(\mathscr{M}_{K_0})$ on $\mathscr{M}_{K_0}$ is the pullback of $\mathbb{L}_{\mathscr{X}}$ by p, that is:

$$\mathscr{M}_{K_0} \xrightarrow{p} \mathscr{X} \xrightarrow{\mathbb{L}_{\mathscr{X}}} \mathbf{Loc}_V.$$

The *universal torsor* $\mathrm{Lev}_{K_0} \in \mathbf{B}Q(X_{K_0})$ on $\mathscr{M}_{K_0}$ is the pullback of Lev by p, that is:

$$\mathscr{M}_{K_0} \xrightarrow{p} \mathscr{X} \xrightarrow{\mathrm{Lev}} \mathbf{B}Q.$$

By Lemma 2.27, $\mathbb{L}_{K_0}$ is the R-local system with fiber V associated with $\mathrm{Ind}_Q^{\mathrm{GL}(V)}(\mathrm{Lev}_{K_0})$.

Proposition 2.30: *There is a canonical isomorphism* $\mathbb{L}^{\mathrm{can}} \simeq \mathbb{L}_{K_0}$.

Proof. Recall that $\mathbb{L}^{\mathrm{can}}$ is a local system defined from a $\mathrm{GL}(V)$-torsor, itself induced from the K_0-torsor $h_{\mathscr{M}_\infty}$ as in Definition 2.17. We will define a canonical morphism of K_0-sheaves between $h_{\mathscr{M}_\infty}$ and Lev_{K_0} (Q-torsor viewed as a K_0-sheaf). By adjunction, as in the proof of Lemma 2.27, this defines a canonical isomorphism between the Q-torsors $\mathrm{Ind}_{K_0}^{Q}(h_{\mathscr{M}_\infty})$ and Lev_{K_0}. Applying $\mathrm{Ind}_Q^{\mathrm{GL}(V)}$, we find the desired isomorphism.

Defining a morphism $h_{\mathscr{M}_\infty} \to \mathbf{Loc}_{K_0}$ of K_0-sheaves on $\mathscr{M}_{K_0}$ amounts to find, for every $\xi : T \to \mathscr{M}_{K_0}$, a morphism of K_0-sheaves $h_{\mathscr{M}_\infty}(\xi) \to \mathbf{Loc}_{K_0}(\xi)$ on T which is natural in T. Fixing $T \to \mathscr{M}_{K_0}$, let $(A, [\eta]_{K_0}) \in \mathscr{M}_{K_0}(T)$ be the corresponding object. A section of $h_{\mathscr{M}_\infty}(A, [\eta]_{K_0})$ is a compatible collection of sections $T \to \mathscr{M}_K$ over $\mathscr{M}_{K_0}$ as K varies, i.e a compatible system of pairs $(A, [\eta]_K)$ as K varies, which is determined by the projective system of level structures $([\eta]_K)_K \in \varprojlim_K H^0(T, \mathrm{Lev}(A)/\mathcal{F}_K)$. Now there are K_0 equivariant identifications

$$\varprojlim_K H^0(T, \mathrm{Lev}(A)/\mathcal{F}_K) = H^0(T, \varprojlim_K \mathrm{Lev}(A)/\mathcal{F}_K) \simeq H^0(T, \mathrm{Lev}(A)) = H^0(T, \mathrm{Lev}_{K_0}(A, [\eta]_{K_0})).$$

For the middle isomorphism we used that K_0 is profinite. This defines the desired morphism

$$h_{\mathscr{M}_\infty}(A, [\eta]_{K_0}) \longrightarrow \mathrm{Lev}_{K_0}(A, [\eta]_{K_0})$$

which is indeed natural in $(A, [\eta]_{K_0})$. □

Example 2.31: Let (G, Ω) be a Shimura datum with reflex field E. Let K_0 be a compact open subgroup of $G(\mathbb{A}_f)$. If $\mathrm{Sh}_{K_0}[G, \Omega]/E$ carries a modular interpretation (e.g. is Hodge-type), then the local system on $\mathrm{Sh}_{K_0}[G, \Omega]$ coming from the relative cohomology of the universal family coincides with the canonical local system.

3. Pointwise and global good reduction problems

3.1. Formulation of the question.

We define the notions of pointwise and global good reduction.

Definition 3.1: Let Y/F be a smooth proper variety. We say that it has *good reduction* if it is the generic fiber of some smooth proper model $\mathcal{Y}/\mathcal{O}_F$.

We define the integral locus of a stack over $\mathcal{O}_F$. To this end, we introduce some notation. If X/F is a stack, its set of closed points is denoted by $|X|$. This is the set of F-morphisms $\operatorname{Spec} E \to X$, with E/F a finite extension, where two morphisms $\operatorname{Spec} E \to X$ and $\operatorname{Spec} E' \to X$ are identified if there exists a field E'', embeddings $E, E' \hookrightarrow E''$ and a 2-commutative diagram

$$\begin{array}{ccc} \operatorname{Spec} E'' & \longrightarrow & \operatorname{Spec} E' \\ \downarrow & & \downarrow \\ \operatorname{Spec} E & \longrightarrow & X. \end{array}$$

If X is a scheme, this coincides with the closed points of its underlying topological space in the usual sense.

Definition 3.2: Let $\mathcal{X}/\mathcal{O}_F$ be a stack with generic fiber X/F. We define the *integral locus* of $\mathcal{X}$ as

$$|\mathcal{X}|^{\mathrm{int}}_{\eta} = \mathrm{im}(\mathcal{X}(\overline{\mathcal{O}_F}) \to |X|).$$

It is the set of closed points of X which admit a specialization in the special fiber of $\mathcal{X}$. In other words, $x_0 : \operatorname{Spec} E \to X$ lies in $|\mathcal{X}|_\eta^{\mathrm{int}}$ if and only if, for some finite extension E'/E there exists $\operatorname{Spec} \mathcal{O}_{E'} \to \mathcal{X}$ and a 2-commutative diagram

$$\begin{tikzcd} \operatorname{Spec} E' \arrow[r] \arrow[d, hook] & \operatorname{Spec} E \arrow[r, "x_0"] & X \arrow[d, hook] \\ \operatorname{Spec} \mathcal{O}_{E'} \arrow[rr, "\exists"] & & \mathcal{X}. \end{tikzcd}$$

If $\mathcal{X} = X$, the integral locus is empty and if $\mathcal{X}/\mathcal{O}_F$ is a proper scheme, then $|\mathcal{X}|_\eta^{\mathrm{int}} = |X|$.

Definition 3.3: Let X/F be a smooth variety with smooth model $\mathcal{X}/\mathcal{O}_F$. Consider a smooth proper morphism of schemes $Y \to X$.

We say that this family has *pointwise good reduction* (relative to $\mathcal{X}$) if for every $x \in |\mathcal{X}|_\eta^{\mathrm{int}}$, the fiber $Y_x/F(x)$ has good reduction.

We say that this family has *global good reduction* (relative to $\mathcal{X}$) if there exists a smooth proper map $\mathcal{Y} \to \mathcal{X}$ of algebraic spaces over $\mathcal{O}_F$ with generic fiber $Y \to X$.

We are interested in whether pointwise good reduction implies global good reduction, and if not, what may some obstructions be.

Remark 3.4: By default, we require that the model $\mathcal{Y} \to \mathcal{X}$ extends as much of the structure of $Y \to X$ as possible. Typically if $Y \to X$ is a polarized abelian scheme or K3 surface with some level structure, we require $\mathcal{Y} \to \mathcal{X}$ to be so too.

3.2. **The ramification obstruction.** For abelian schemes over a sufficiently ramified p-adic field F, pointwise good reduction does not imply global good reduction. We build a counterexample upon the work of Vasiu-Zink in [VZ10][5], of which we recall some steps.

Lemma 3.5 ([VZ10], Lemma 27)**:** *Let R denote a regular local ring of dimension 2 and mixed characteristic $(0,p)$. Let $\mathfrak{m}$ denote its maximal ideal, and set $U = \operatorname{Spec} R \setminus \{\mathfrak{m}\}$.*

Let $D \to H$ be a homomorphism of finite flat group schemes over $\operatorname{Spec} R$ which is not a monomorphism, but whose restriction over U is a monomorphism[6]. Let B be an abelian scheme over $\operatorname{Spec} R$ in which H embeds. Define the quotient abelian scheme $A_U = B_U/D_U$ over U.

If R' is a faithfully flat local R-algebra of relative dimension 0, with maximal ideal $\mathfrak{m}'$, let $U' := U \times_R \operatorname{Spec} R' = \operatorname{Spec} R' \setminus \{\mathfrak{m}'\}$. Then $A_U \times_U \operatorname{Spec} R' \to U'$ is an abelian scheme which does not extend into an abelian scheme over $\operatorname{Spec} R'$.

Proof. The morphism $\operatorname{Spec} R' \to \operatorname{Spec} R$ is fpqc, so the assumptions on monomorphisms still hold after base-change to R'. Let D', H', A'_U, B' denote the group schemes base-changed to R', then $A'_U = B'_U/D'_U$. Hence we reduce to the case $R' = R$.

Suppose that A_U extends to an abelian scheme $A \to \operatorname{Spec} R$. By the Néron extension property (see [BLR12, 8.4, Corollary 6]) the isogeny $B_U \to A_U$ extends to a homomorphism $B \to A$ which is automatically an isogeny. Its kernel K is a finite flat group scheme which coincides with D_U over U; we will argue that there is an identification $D \simeq K$ as B-schemes, which then implies that $D \to H$ is a monomorphism over the whole base, contradiction.

Over U, the isogeny $B \to B/H$ factors through A_U; there is thus a factorization between Néron models

$$\begin{tikzcd} & A = B/K \arrow[dr] & \\ B \arrow[ur] \arrow[rr] & & B/H \end{tikzcd}$$

[5]Vasiu-Zink generalize a counterexample to [FC13, Chapter V, Corollary 6.8] due to Raynaud-Ogus-Gabber (see [dJO97, §6]).

[6][VZ10, Lemma 27] is stated with $H \to D$ which is an epimorphism over U but not globally. However the proof only uses the Cartier dual morphism $D^t \to H^t$: here we consider the dual statement to skip taking duals.

over $\operatorname{Spec} R$ which induces a closed embedding of kernels $K \hookrightarrow H$. We use that quotients of finite flat commutative group schemes exist and are again finite flat (see [Tat, §3.5]). The kernel N of the composition $D \to H \to H/K$ is a finite flat subgroup of D, so D/N is finite flat and trivial over U, hence trivial everywhere so that $N = D$, and $D \to H/K$ is trivial. Thus $D \to H \hookrightarrow B$ factors through $K \hookrightarrow B$. The same argument with the quotient K/D shows that this factorization is an isomorphism over B as desired. □

Proposition 3.6: *Suppose $F/\mathbb{Q}_p$ has ramification index at least p. Then for any positive prime-to-p integer N there exists an étale local $\mathcal{O}_F[[T]]$-algebra R with maximal ideal $\mathfrak{m}$ and an abelian scheme $A_U \to U := \operatorname{Spec} R \setminus \{\mathfrak{m}\}$ which does not extend to an abelian scheme over $\operatorname{Spec} R$, and such that $A_U[N] \to U$ is a trivial étale cover.*

Proof. According to [VZ10, Theorem 28 (i)] and its proof we can take the power series ring $R = \mathcal{O}_F[[T]]$ and find a homomorphism of group schemes $D \to H$ over R satisfying the assumptions of Lemma 3.5, hence an abelian scheme $A_U \to U$ which does not extend to an abelian scheme over $\operatorname{Spec} R$. By Lemma 3.5 we can replace R with an appropriate étale local ring extension to ensure that $A_U[N] \to U$ is a trivial cover. □

Next we argue that we can replace R with a finite type $\mathcal{O}_F$-algebra.

Proposition 3.7: *Suppose $F/\mathbb{Q}_p$ has ramification index at least p. Then for any positive prime-to-p integer N there exists a smooth local $\mathcal{O}_F$-algebra R with maximal ideal $\mathfrak{m}$ and an abelian scheme $A_U \to U := \operatorname{Spec} R \setminus \{\mathfrak{m}\}$ which does not extend to an abelian scheme over $\operatorname{Spec} R$, and such that $A_U[N] \to U$ is a trivial étale cover.*

Proof. We denote by $\tilde{R}$ the finite étale local $\mathcal{O}_F[[T]]$-algebra with maximal ideal $\tilde{\mathfrak{m}}$, by $\tilde{U} := \operatorname{Spec} \tilde{R} \setminus \{\tilde{\mathfrak{m}}\}$ the open subset and by $f : \tilde{A}_U \to \tilde{U}$ an abelian scheme provided by Proposition 3.6. We wish to descend all of this data over a smooth $\mathcal{O}_F$-algebra.

We first argue that $\mathcal{O}_F \to \tilde{R}$ is a regular ring homomorphism, to do so we may assume $\tilde{R} = \mathcal{O}_F[[T]]$. It is clear that $k[[T]]$ is geometrically regular, where k is the residue field of $\mathcal{O}_F$. It remains to show that $\mathcal{O}_F[[T]] \otimes_{\mathcal{O}_F} F = \mathcal{O}_F[[T]][1/p]$ is regular. By [Sta26, Tag 07EL] we reduce to showing that $F \to \mathcal{O}_F[[T]][1/p]$ is formally smooth for the (T)-adic topology, which is easily checked.

We may now apply Popescu's theorem, see [Sta26, Tag 07GC]: there are smooth $\mathcal{O}_F$-algebras R_i such that $\tilde{R} = \varinjlim R_i$. All maps between the R_i and $\tilde{R}$ factor through local and integral rings, so we may assume that the R_i are local integral with maximal ideal $\mathfrak{m}_i$, and set $U_i := \operatorname{Spec} R_i \setminus \{\mathfrak{m}_i\}$. We let $p_i : \tilde{U} \to U_i$ be the maps induced from the projective system. We wish to descend f to an abelian scheme over some U_i.

Since the U_i are Noetherian, by [Sta26, Tag 0CNL] for large enough i there is a smooth and proper morphism $f_i : A_i \to \operatorname{Spec} U_i$ descending f. Using [Sta26, Tag 0CNR] we can assume that the group scheme structure and N-torsion structure descend to f_i, thus f_i defines an abelian scheme. Let $A_U = A_i$ and $U = U_i$ for this appropriate choice of i.

If $A_U \to U$ were to extend to an abelian scheme over $\operatorname{Spec} R$, its pullback to $\operatorname{Spec} \tilde{R}$ would extend $\tilde{A}_U \to \tilde{U}$, contradicting the definition of $\tilde{A}_U$. □

Corollary 3.8: *With notations from Proposition 3.7, set $\mathcal{X} = \operatorname{Spec} R$ which is a smooth model over $\mathcal{O}_F$ of its generic fiber $\iota\colon X \hookrightarrow \mathcal{X}$, and set $A_X := \iota^* A_U$. Then $A_X \to X$ has pointwise good reduction but not global good reduction as an abelian scheme, and $A_X[N] \to X$ is a trivial étale cover.*

Proof. We argue that the abelian scheme $A_X \to X$ does not have global good reduction. Let η denote the generic point of X, hence also of U and $\mathcal{X}$, fix a symmetric ample line bundle $\mathscr{L}_\eta$ on the generic abelian variety A_η. Assume $A_X \to X$ extends to an abelian scheme $A \to \mathcal{X}$, we argue that this also extends $A_U \to U$, contradicting Proposition 3.7. According to Fact 3.10 there are positive integers n, m such that $\mathscr{L}_\eta^{\otimes n}$ and $\mathscr{L}_\eta^{\otimes m}$ extend to relatively ample line bundles on $A_U \to U$ and $A \to \mathcal{X}$ respectively. Let $j\colon U \to \mathcal{X}$ be the inclusion of open set; replacing n and m with nm, we find that the polarized abelian schemes

$$A_U \to U, \qquad j^* A \to U$$

have isomorphic generic fibers. Separatedness of the stack of polarized abelian schemes implies that they are isomorphic (Proposition 3.11).

It remains to check that $A_X \to X$ has pointwise good reduction. Let $\ell \neq p$ be a prime number, the local system $\mathbb{L} = R^1 f_* \mathbb{Q}_\ell$ over U corresponds to an ℓ-adic representation of the étale fundamental group $\pi_1(U)$. As $\mathcal{X} \setminus U$ has codimension at least 2 in $\mathcal{X}$, Zariski-Nagata purity implies that $\mathbb{L}$ extends to a local system on $\mathcal{X}$. In particular, for any $x \in |\mathcal{X}|_\eta^{\mathrm{int}}$ the local system $\mathbb{L}_x$ is unramified, which by [ST71] means that $(A_X)_x$ has good reduction. □

Remark 3.9: In Proposition 3.6, Proposition 3.7 and Corollary 3.8 we could have replaced "$A_U[N] \to U$ is a trivial étale cover" with other restrictions on the monodromy of the N-torsion as long as they are satisfied up to finite étale cover of the base. In the language of Shimura varieties, we could have asked for $A_U \to U$ to have a prime-to-p level structure (as in Section 4.4 below).

Fact 3.10 ([Ray06], Remarque XI 1.3 (e))**:** *Let S be a locally Noetherian, normal scheme with generic point η, consider an abelian scheme $A \to S$, and a symmetric ample line bundle $\mathscr{L}_\eta$ on the generic fiber A_η. Then there exists a positive integer n such that $\mathscr{L}_\eta^{\otimes n}$ extends to a symmetric relatively ample line bundle on $A \to S$.*

For this next proposition we deal with Deligne-Mumford stacks over the pro-étale site over a base scheme S. The reader is refered to Section 2.1, and more precisely Definition 2.2, for the definition of these objects.

Proposition 3.11: *Let $\mathscr{X}$ be a separated Deligne-Mumford stack over some base scheme S. Let Y be a locally Noetherian, normal S-scheme and U a dense open subscheme of Y. Let $f, g : Y \to \mathscr{X}$ be two morphisms such that $f_{|U} \simeq g_{|U}$. Then $f \simeq g$.*

Proof. Let $I = \underline{\mathrm{Isom}}(f, g)$ the algebraic stack obtained by the pullback of the diagonal Δ

$$\begin{array}{ccc} I & \xrightarrow{\theta} & Y \\ \downarrow & & \downarrow{\scriptstyle f \times g} \\ \mathscr{X} & \xrightarrow{\Delta} & \mathscr{X} \times_S \mathscr{X}. \end{array}$$

By assumption, Δ, and hence also θ, is finite, so by [LMB18, Théorème A.2] I is a scheme. The assumption $f_{|U} \simeq g_{|U}$ means that θ has a section over U $s : U \to I_{|U}$ which is an open immersion by [Sta26, Tag 024T]. Let Z denote the scheme-theoretic closure of $s(U)$ in I. We want to show that $\theta_{|Z} : Z \to Y$ is an isomorphism. Working componentwise, we can assume that Y, and so U and Z, are irreducible. Since $\theta(Z)$ contains U, $\theta_{|Z}$ is dominant. By [Sta26, Tag 0356] Z is reduced. Thus $\theta_{|Z} : Z \to Y$ is a finite birational morphism between integral schemes, with Y normal: by [Sta26, Tag 0AB1] it is an isomorphism. □

Remark 3.12: The normality assumption on Y in Proposition 3.11 cannot be dropped. Over some base field k, consider Y to be $\mathbb{P}^1$ with 0 and ∞ glued together, let U be Y minus the singular point, and set $\mathscr{X} = \mathbf{B}\mathbb{Z}/2\mathbb{Z}$. Let Y_1 be two disjoint copies of Y and Y_2 be two copies of $\mathbb{P}^1$ glued at zero and glued at infinity. Then there are obvious projection maps $Y_i \to Y$, $i \in \{1, 2\}$, which are double étale covers, and which both restrict to the trivial double cover over U. In this way, we find two non-isomorphic maps $Y \to \mathscr{X}$ which are isomorphic over the dense open subset U.

In Section 4 we will apply Proposition 3.11 repeatedly as follows: for $\mathcal{X}/\mathcal{O}_F$ a regular scheme, and $\mathscr{S}/\mathcal{O}_F$ a separated Deligne-Mumford stack, two $\mathcal{O}_F$ morphisms $f, g : \mathcal{X} \to \mathscr{S}$ that are isomorphic over $\operatorname{Spec} F$ are isomorphic.

4. Extension properties for good reduction

In this section we will deal with schemes and stacks over $\mathcal{O}_F$: we use the subscripts η and s to denote the generic and special fibers of these objects respectively.

Definition 4.1: Let $\mathcal{X}, \mathscr{S}/\mathcal{O}_F$ be stacks. A *generic morphism* $f : \mathcal{X} \rightsquigarrow \mathscr{S}$ is a morphism of stacks between the generic fibers $f_\eta : \mathcal{X}_\eta \to \mathscr{S}_\eta$.

We study the question of extending generic morphisms to actual morphisms between the stacks.

4.1. **L-extension and PRG-extension.**

Definition 4.2: Let $\mathscr{S}/\mathcal{O}_F$ be a separated Deligne-Mumford stack. Let $\mathbb{L}$ be a local system on $\mathscr{S}$ (see Definition 2.5, Definition 2.15). We say that the pair $(\mathscr{S}, \mathbb{L})$ *has L-extension* if any generic morphism $f : \mathcal{X} \to \mathscr{S}$ from a smooth scheme $\mathcal{X}/\mathcal{O}_F$, satisfying that the pullback $f_\eta^{-1}\mathbb{L}$ to $\mathcal{X}_\eta$ extends to a local system over $\mathcal{X}$, automatically and uniquely extends to an $\mathcal{O}_F$-morphism $\mathcal{X} \to \mathscr{S}$.

The uniqueness in Definition 4.2 is automatic by Proposition 3.11. This will also be the case for all other extension properties considered.

Lemma 4.3: *Let $\mathscr{S}/\mathcal{O}_F$ be a stack, $\mathbb{L}$ a local system on $\mathscr{S}$ such that $(\mathscr{S}, \mathbb{L})$ has $\mathbb{L}$-extension. Consider $\pi : \mathscr{S}' \to \mathscr{S}$ a surjective étale morphism over $\mathscr{S}/\mathcal{O}_F$. Then $(\mathscr{S}', \pi^*\mathbb{L})$ has $\mathbb{L}$-extension.*

Proof. Let $\mathcal{X}/\mathcal{O}_F$ be a smooth scheme and $f : \mathcal{X} \rightsquigarrow \mathscr{S}'$ a generic morphism. Using $\mathbb{L}$-extension, the composition $\mathcal{X} \rightsquigarrow \mathscr{S}' \to \mathscr{S}$ extends uniquely to $h : \mathcal{X} \to \mathscr{S}$. The situation is summarized in the following commutative diagram

$$\begin{tikzcd}
\mathcal{X}_\eta \arrow[r, "f_\eta"] \arrow[d, hook] & \mathscr{S}'_\eta \arrow[r] \arrow[d, hook] & \mathscr{S}_\eta \arrow[d, hook] \\
\mathcal{X} \arrow[r, dashed, "?"] \arrow[rr, bend right, "h"'] & \mathscr{S}' \arrow[r] & \mathscr{S}.
\end{tikzcd}$$

Let $\mathcal{X}' := \mathcal{X} \times_{\mathscr{S}} \mathscr{S}'$ which is an étale cover of $\mathcal{X}$. We wish to show that $h' : \mathcal{X}' \to \mathscr{S}'$ descends to a morphism $\mathcal{X} \to \mathscr{S}'$, in other words we need to show that both composition arrows

$$\mathcal{X}' \times_{\mathcal{X}} \mathcal{X}' \rightrightarrows \mathcal{X}' \to \mathscr{S}'$$

coincide. Since this holds over the generic fibers, the result follows from Proposition 3.11. $\square$

In our applications however, we will be considering non-surjective étale morphisms $\mathcal{U} \to \mathscr{S}$: it is no longer true that $\mathcal{U}$ inherits L-extension. Fortunately, L-extension implies a weaker property called PGR-extension, which we define below. In the next paragraph we show that PGR-extension pulls back under étale maps.

Definition 4.4: Let $\mathcal{X}, \mathscr{M}/\mathcal{O}_F$ be stacks. A *PGR-morphism* (as in Pointwise Good Reduction) is a generic morphism $f : \mathcal{X} \rightsquigarrow \mathscr{M}$ such that $f_\eta(|\mathcal{X}|_\eta^{\text{int}}) \subseteq |\mathscr{M}|_\eta^{\text{int}}$. In other words, for any finite extension E/F and any $\mathcal{O}_E$-point of $\mathcal{X}$, the image by f_η of the induced E-point x of $\mathcal{X}_\eta$ extends to an $\mathcal{O}_{E'}$-point of $\mathscr{S}$ for some finite extension E'/E:

$$\begin{tikzcd}
 & \operatorname{Spec} E \arrow[r, "x"] \arrow[d] & \mathcal{X}_\eta \arrow[r, "f_\eta"] \arrow[d] & \mathscr{S}_\eta \arrow[d] \\
\operatorname{Spec} \mathcal{O}_{E'} \arrow[r, "\exists"] \arrow[rrr, dashed, bend right, "\exists"'] & \operatorname{Spec} \mathcal{O}_E \arrow[r] & \mathcal{X} & \mathscr{S}.
\end{tikzcd}$$

The restriction of an $\mathcal{O}_F$-morphism $\mathcal{X} \to \mathscr{M}$ to the special fibers is a PGR-morphism.

Remark 4.5: The acronym PGR is motivated by the case, of interest to us, where $\mathscr{M}/\mathcal{O}_F$ is a moduli stack for some families of smooth and proper algebraic spaces over $\mathcal{O}_F$. In this case, a PGR-morphism $f : \mathcal{X}_\eta \rightsquigarrow \mathscr{M}$ corresponds to a smooth proper family $Y \to \mathcal{X}_\eta$ with pointwise good reduction as in Definition 3.3.

Definition 4.6: Let $\mathscr{M}/\mathcal{O}_F$ be a separated Deligne-Mumford stack. We say that it *has PGR-extension* if every PGR-morphism $f : \mathcal{X} \rightsquigarrow \mathscr{M}$ from a smooth scheme $\mathcal{X}/\mathcal{O}_F$, with geometrically connected irreducible components, automatically and uniquely extends to an $\mathcal{O}_F$-morphism $\mathcal{X} \to \mathscr{M}$.

Remark 4.7: Assuming $\mathscr{M}/\mathcal{O}_F$ is a moduli stack for some smooth and proper families of schemes over $\mathcal{O}_F$, having PGR-extension exactly states that such a family with pointwise good reduction has global good reduction.

Proposition 4.8: *Let $\mathscr{S}/\mathcal{O}_F$ be a separated Deligne-Mumford stack with a local system $\mathbb{L}$. Assume the pair $(\mathscr{S}, \mathbb{L})$ has L-extension as in Definition 4.2. Then $\mathscr{S}$ has PGR-extension.*

Proof. Let $\mathcal{X}/\mathcal{O}_F$ be as in Definition 4.6, consider $f : \mathcal{X} \rightsquigarrow \mathscr{S}$ a PGR-morphism. We want to show that f extends to a morphism $\mathcal{X} \to \mathscr{S}$ over $\mathcal{O}_F$. It suffices to show that $f^{-1}\mathbb{L}$ extends to a local system on $\mathcal{X}$ and apply L-extension.

Let $x \in |\mathcal{X}|_\eta^{\mathrm{int}}$, then $f(x) \in |\mathscr{S}|_\eta^{\mathrm{int}}$ is in the image of some $\overline{\mathcal{O}_F}$-point of $\mathscr{S}$. In particular, the Galois representation defined by the pullback of $\mathbb{L}$ to $f(x)$ is unramified, hence so is its pullback to x. Thus $f^{-1}\mathbb{L}$ is unramified at every point of $|\mathcal{X}|_\eta^{\mathrm{int}}$; by [CT26, Theorem 1 (i)] it extends to a local system over $\mathcal{X}$. □

4.2. **Pulling back PGR-extension along étale maps.** We aim to show the following statement.

Theorem 4.9: *Let $\mathscr{S}/\mathcal{O}_F$ be a separated Deligne-Mumford stack with PGR-extension. Let $\mathcal{U}/\mathcal{O}_F$ be an algebraic stack étale over $\mathscr{S}$. Then $\mathcal{U}$ has PGR-extension.*

We first consider the case when $\mathscr{S}$ is a scheme.

Proposition 4.10: *Let $\mathscr{S}/\mathcal{O}_F$ be a separated scheme with PGR-extension. Let $\mathcal{U}/\mathcal{O}_F$ be a scheme étale over $\mathscr{S}$. Then $\mathcal{U}$ has PGR-extension.*

Proof. Let $\mathcal{X}/\mathcal{O}_F$ be as in Definition 4.4, consider $f : \mathcal{X} \rightsquigarrow \mathcal{U}$ a PGR-morphism. We want to show that f extends to a morphism $\mathcal{X} \to \mathcal{U}$ over $\mathcal{O}_F$. The composition $\mathcal{X}_\eta \to \mathcal{U}_\eta \to \mathscr{S}_\eta$ is again PGR, hence extends to $h : \mathcal{X} \to \mathscr{S}$. The situation is summarized in the following commutative diagram:

$$\begin{tikzcd}
\mathcal{X}_\eta \arrow[r, "f_\eta"] \arrow[d, hook] & \mathcal{U}_\eta \arrow[r, "g_\eta"] \arrow[d, hook] & \mathscr{S}_\eta \arrow[d, hook] \\
\mathcal{X} \arrow[r, dashed, "?"] \arrow[rr, bend right, "h"'] & \mathcal{U} \arrow[r, "g"] & \mathscr{S}.
\end{tikzcd}$$

Let $\Gamma_{f_\eta} \hookrightarrow \mathcal{X}_\eta \times_F \mathcal{U}_\eta$ be the graph of f_η, let Z denote its closure in $\mathcal{X} \times_{\mathcal{O}_F} \mathcal{U}$. It suffices to show that the projection $Z \to \mathcal{X}$ is an isomorphism, so that the extension of f_η is provided by

$$\tilde{f} : \mathcal{X} \to Z \hookrightarrow \mathcal{X} \times_{\mathcal{O}_F} \mathcal{U} \to \mathcal{U}.$$

Let $\Gamma_h \hookrightarrow \mathcal{X} \times_{\mathcal{O}_F} \mathcal{U}$ denote the graph of h. Let $\tilde{Z}$ denote the pullback of $\Gamma_h \to \mathscr{S}$ along g:

$$\begin{tikzcd}
\tilde{Z} \arrow[r, hook] \arrow[d] & \mathcal{X} \times_{\mathcal{O}_F} \mathcal{U} \arrow[r, "p_2"] \arrow[d, "\mathrm{id} \times g"] & \mathcal{U} \arrow[d, "g"] \\
\Gamma_h \arrow[r, hook] & \mathcal{X} \times_{\mathcal{O}_F} \mathscr{S} \arrow[r, "p_1"] & \mathscr{S}.
\end{tikzcd}$$

By definition $\tilde{Z}$ is a closed subscheme of $\mathcal{X} \times_{\mathcal{O}_F} \mathcal{U}$ containing Γ_{f_η}, hence Z is a closed subscheme of $\tilde{Z}$. By Lemma 4.11 below, the projection $Z \to \mathcal{X}$ is surjective (here we use again that f is PGR). The commutative diagram

$$\begin{tikzcd}
Z \arrow[r, hook] \arrow[d] & \mathcal{X} \times_{\mathcal{O}_F} \mathcal{U} \arrow[r, "p_1"] \arrow[d, "\mathrm{id} \times \tilde{g}"] & \mathcal{X} \arrow[d, "\mathrm{id}"] \\
\Gamma_h \arrow[r, hook] \arrow[rr, bend right, "\sim"'] & \mathcal{X} \times_{\mathcal{O}_F} \mathscr{S} \arrow[r, "p_1"] & \mathcal{X}
\end{tikzcd}$$

shows that this projection factors through $Z \to \Gamma_h \xrightarrow{\sim} \mathcal{X}$, thus $\theta : Z \to \Gamma_h$ is surjective. By Lemma 4.12 below, $Z \to \Gamma_h$ is étale. By [Gro66, Proposition 18.2.8] the number of points in the geometric fibers of θ is lower semicontinuous, bounded below by 1 by surjectivity, and equals 1 on the dense open subscheme Γ_{f_η} (as $\Gamma_{f_\eta} \to \Gamma_{g_\eta \circ f_\eta}$ is an isomorphism); it is thus constant equal to 1. By *loc.cit.*, θ is finite, and so by [Sta26, Tag 0AB1] it is an isomorphism. $\square$

Lemma 4.11: *Let $\mathcal{X}$,$\mathcal{U}/\mathcal{O}_F$ be schemes of finite type, with $\mathcal{X}/\mathcal{O}_E$ smooth, and $f : \mathcal{X} \rightsquigarrow \mathcal{U}$ a PGR-morphism with graph $\Gamma \hookrightarrow \mathcal{X}_\eta \times_F \mathcal{U}_\eta$. Let Z be the closure of Γ in $\mathcal{X} \times_{\mathcal{O}_F} \mathcal{U}$. Then the projection $Z \to \mathcal{X}$ is surjective.*

Proof. We show that the set-theoretic image of $p : Z \to \mathcal{X}$ contains all closed points. Then by Chevalley's theorem the image is a locally constructible subset of $\mathcal{X}$ containing all closed points, thus it is the whole of $\mathcal{X}$. Since $\Gamma \subseteq Z$, $\mathcal{X}_\eta$ is contained in $p(Z)$, so it remains to show that the closed points of the special fiber $\mathcal{X}_s$ of $\mathcal{X}$ are attained.

Let $x_s \in \mathcal{X}_s$ be a closed point. We use the smoothness of $\mathcal{X}$: by the infinitesimal lifting criterion, x_s lifts to an $\mathcal{O}_E$-point $x : \operatorname{Spec} \mathcal{O}_E \to \mathcal{X}$ for some finite extension E/F. Let $x_0 : \operatorname{Spec} E \to \mathcal{X}_\eta$ denote the special fiber of x. Let $y_0 = f_\eta(x_0)$, $y_0 \in |\mathcal{U}|_\eta^{\mathrm{int}}$ by the PGR assumption, hence y_0 is the special fiber of some $\mathcal{O}_{E'}$-point $y : \operatorname{Spec} \mathcal{O}_{E'} \to \mathcal{U}$ for E'/E a finite extension. We may assume $E = E'$, so that the pair (x, y) defines an $\mathcal{O}_E$-point of the product $\mathcal{X} \times_{\mathcal{O}_F} \mathcal{U}$. The situation is summarized in the following commutative diagram:

$$\begin{tikzcd}
\operatorname{Spec} E \arrow[r, "(x_0,y_0)"] \arrow[d, hook] & \Gamma_{f_\eta} \arrow[r, hook] \arrow[d, hook] & \mathcal{X}_\eta \times_F \mathcal{U}_\eta \arrow[d, hook] \\
\operatorname{Spec} \mathcal{O}_E \arrow[r, dashed, "?"] \arrow[rr, bend right, "(x,y)"'] & Z \arrow[r, hook] & \mathcal{X} \times_{\mathcal{O}_F} \mathcal{U}.
\end{tikzcd}$$

The factorization via the dashed arrow exists because all vertical arrows are inclusions of generic fibers in their closures. Now the composition morphism

$$\operatorname{Spec} \mathcal{O}_E \longrightarrow Z \longrightarrow \mathcal{X} \times_{\mathcal{O}_F} \mathcal{U} \longrightarrow \mathcal{X}$$

gives back the $\mathcal{O}_E$-point x, so x_s lies in the image of $Z \to \mathcal{X}$, as desired. $\square$

Lemma 4.12: *Let $\phi : \tilde{Z} \to \Gamma$ be an étale morphism of schemes with Γ regular irreducible. Let $Z \hookrightarrow \tilde{Z}$ be an irreducible closed subscheme such that $\theta : Z \to \Gamma$ is again surjective. Then θ is étale.*

Proof. The surjectivity of θ ensures $\dim \tilde{Z} \geq \dim Z \geq \dim \Gamma = \dim \tilde{Z}$, so Z is an irreducible component of $\tilde{Z}$. As ϕ is étale, $\tilde{Z}$ is also regular, hence so are its irreducible components: $Z \hookrightarrow \tilde{Z}$ is an open immersion. Hence $\theta : Z \hookrightarrow \tilde{Z} \to \Gamma$ is étale. $\square$

Proof of Theorem 4.9. Let $\mathcal{X}/\mathcal{O}_F$ be as in Definition 4.6, and $f : \mathcal{X} \rightsquigarrow \mathcal{U}$ a PGR-morphism. We want to show that f extends to a morphism $\mathcal{X} \to \mathcal{U}$ over $\mathcal{O}_F$.

Consider $\mathscr{S}'/\mathcal{O}_F$ a scheme which is an étale atlas over $\mathscr{S}/\mathcal{O}_F$. By Lemma 4.3 (rather, its PGR analogue), $\mathscr{S}'$ has PGR-extension. Let $\mathcal{X}'/\mathcal{O}_F$ and $\mathcal{U}'/\mathcal{O}_F$ be the algebraic spaces obtained by base change by $\mathscr{S}' \to \mathscr{S}$. Since $\mathscr{S}' \to \mathscr{S}$ is an étale presentation of a Deligne-Mumford stack with separated and quasi-compact diagonal, $\mathcal{X}'$ and $\mathcal{U}'$ are schemes (see [Alp24, Remark 3.1.5]). By Proposition 4.10, $\mathcal{U}'$ has PGR-extension, and since $f' : \mathcal{X}' \rightsquigarrow \mathcal{U}'$ is again PGR, thus we find a unique morphism

$$\tilde{f}' : \mathcal{X}' \to \mathcal{U}'$$

extending f'. Now it suffices to show that the composition $\mathcal{X}' \to \mathcal{U}' \to \mathcal{U}$ descends to a morphism $\mathcal{X} \to \mathcal{U}$, in other words we need to show that both composition arrows

$$\mathcal{X}' \times_{\mathcal{X}} \mathcal{X}' \rightrightarrows \mathcal{X}' \to \mathcal{U}' \to \mathcal{U}$$

are isomorphic. This follows from Proposition 3.11. $\square$

We end this paragraph by showing that PGR-extension descends along finite torsors.

Proposition 4.13: *Let H be a finite group. Let $\mathscr{M}/\mathcal{O}_F$ be a separated Deligne-Mumford stack. Let $P : \mathscr{M} \to \mathbf{B}H$ denote an H-torsor on $\mathscr{M}$, consider $\mathcal{U} \to \mathscr{M}$ the corresponding finite étale cover. If $\mathcal{U}$ has PGR-extension, then so does $\mathscr{M}$.*

Proof. Let $\mathcal{X}/\mathcal{O}_F$ be as in Definition 4.6, and $f : \mathcal{X} \rightsquigarrow \mathscr{M}$ a PGR-morphism. We want to show that f extends to a morphism $\mathcal{X} \to \mathscr{M}$ over $\mathcal{O}_F$.

Let $\mathcal{U}'_\eta := \mathcal{X}_\eta \times_{\mathscr{M}_\eta} \mathcal{U}_\eta$. We will argue that $\mathcal{U}'_\eta/F$ has a smooth model $\mathcal{U}'/\mathcal{O}_F$, and that there is an H-torsor $\mathcal{U}' \to \mathcal{X}$ extending $\mathcal{U}'_\eta \to \mathcal{X}_\eta$. The following commutative diagram summarizes the situation

$$\begin{tikzcd}
\mathcal{U}'_\eta \arrow[r] \arrow[d] \arrow[drr] & \mathcal{X}_\eta \arrow[d, "f_\eta"'] \arrow[drr] & & \\
\mathcal{U}_\eta \arrow[r] \arrow[drr] & \mathscr{M}_\eta \arrow[drr] & \mathcal{U}' \arrow[r] \arrow[d, dashed, "?"] & \mathcal{X} \arrow[d, dashed, "?"] \\
& & \mathcal{U} \arrow[r] & \mathscr{M}
\end{tikzcd}$$

It then follows from Lemma 4.14 below that the projection $\mathcal{U}' \rightsquigarrow \mathcal{U}$ is a PGR-morphism, hence extends to $\mathcal{U}' \to \mathcal{U}$[7]. It remains to show that the composition $\mathcal{U}' \to \mathcal{U} \to \mathscr{M}$ descends to a morphism $\mathcal{X} \to \mathscr{M}$, this is done exactly as in the proof of Theorem 4.9 using Proposition 3.11.

We construct $\mathcal{U}'$ as follows. From Remark 2.25 we have the 2-cartesian diagram

$$\begin{tikzcd}
\mathcal{U}'_\eta \arrow[r] \arrow[d] & \mathcal{X}_\eta \arrow[d] \\
\mathrm{pt} \arrow[r] & \mathbf{B}H.
\end{tikzcd}$$

If we can show that $\mathcal{X}_\eta \to \mathbf{B}H$ extends to $\mathcal{X} \to \mathbf{B}H$, we then define $\mathcal{U}$ as the fiber product

$$\begin{tikzcd}
\mathcal{U}' \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\
\mathrm{pt} \arrow[r] & \mathbf{B}H.
\end{tikzcd}$$

(i.e. the total space of the H-torsor on $\mathcal{X}$, which is a scheme) Then $\mathcal{U}' \to \mathcal{X}$ has all required properties.

To extend $\mathcal{X}_\eta \to \mathbf{B}H$, we use [CT26, Theorem 1 (i)] as in Proposition 4.8. We can replace $\mathcal{X}$ with an irreducible component so as to assume it is geometrically connected. The H-torsor on $\mathcal{X}_\eta$ is defined by a morphism

$$\pi_1(\mathcal{X}_\eta) \to H;$$

(up to conjugation, so we suppress the choice of base point) if we show that for any $x \in |\mathcal{X}|^{\mathrm{int}}_\eta$ the Galois representation $\pi_1(x) \to H$ is unramified, then the cited result implies that $\pi_1(\mathcal{X}_\eta) \to H$ factors through some morphism

$$\pi_1(\mathcal{X}) \to H.$$

As in Proposition 4.8, this follows from the fact that $\mathcal{X} \rightsquigarrow \mathscr{M}$ is PGR. $\square$

Lemma 4.14: *Let $\mathscr{M}/\mathcal{O}_F$ be an algebraic stack, $\mathcal{U} \to \mathscr{M}$ a proper affine morphism of stacks, and let $\mathcal{X}/\mathcal{O}_F$ be as in Definition 4.6, together with a PGR-morphism $f : \mathcal{X} \rightsquigarrow \mathscr{M}$. Let $\mathcal{U}'_\eta := \mathcal{U}_\eta \times_{\mathscr{M}_\eta} \mathcal{X}_\eta$, suppose it admits a smooth model $\mathcal{U}/\mathcal{O}_F$ such that the generic morphism $\mathcal{U} \rightsquigarrow \mathcal{X}$ extends to $\mathcal{U} \to \mathcal{X}$ over $\mathcal{O}_F$. Then the generic morphism $\mathcal{U}' \rightsquigarrow \mathcal{U}$ is PGR.*

Proof. Consider the 2-cartesian diagram

$$\begin{tikzcd}
\mathcal{U}'_\eta \arrow[r] \arrow[d] & \mathcal{X}_\eta \arrow[d] \\
\mathcal{U}_\eta \arrow[r] & \mathscr{M}_\eta.
\end{tikzcd}$$

Since $\mathcal{U}' \rightsquigarrow \mathcal{X}$ extends, it is a PGR morphism, hence so is the composition $\mathcal{U}' \rightsquigarrow \mathscr{M}$. It is enough to show that the preimage of $|\mathscr{M}|^{\mathrm{int}}_\eta$ via $\mathcal{U} \to \mathscr{M}$ is $|\mathcal{U}|^{\mathrm{int}}_\eta$. Let $x : \operatorname{Spec} E \to \mathcal{U}$, for some finite E/F,

[7]While the existence of $\mathcal{U}$ holds even assuming H profinite, here we need H to be finite because $\mathcal{U}'/\mathcal{O}_F$ is required to be smooth (in particular, finite type) in Definition 4.6.

mapping to $|\mathscr{M}|^{\mathrm{int}}_\eta$; up to extending E, we may assume that there is a 2-commutative diagram of solid arrows

$$\begin{tikzcd} \operatorname{Spec} E \arrow[r] \arrow[d] & \mathcal{U} \arrow[d] \\ \operatorname{Spec} \mathcal{O}_E \arrow[r] \arrow[ur, dashed] & \mathscr{M}. \end{tikzcd}$$

We argue that there exists a dashed arrow making the diagram 2-commutative, so that x lies in $|\mathcal{U}|^{\mathrm{int}}_\eta$. First, we base-change $\mathcal{U} \to \mathscr{M}$ along the bottom horizontal arrow so as to reduce to the case:

$$\begin{tikzcd} \operatorname{Spec} E \arrow[r] \arrow[d] & \mathcal{U}_{\mathcal{O}_E} \arrow[d] \\ \operatorname{Spec} \mathcal{O}_E \arrow[r, "\mathrm{id}"] \arrow[ur, dashed] & \operatorname{Spec} \mathcal{O}_E \end{tikzcd}$$

where the vertical arrow on the right is proper and affine. We conclude with the valuative criterion for properness of a morphism of schemes[8]. □

4.3. **Smooth extension.**

Definition 4.15: Let $\mathscr{S}/\mathcal{O}_F$ be a separated Deligne-Mumford stack. We say that it *has smooth extension* if every generic morphism $f : \mathcal{X} \rightsquigarrow \mathscr{S}$ from a regular, formally smooth scheme $\mathcal{X}/\mathcal{O}_F$, automatically and uniquely extends to an $\mathcal{O}_F$-morphism $\mathcal{X} \to \mathscr{S}$.

Restricting Definition 4.15 to smooth separated schemes $\mathcal{X}/\mathcal{O}_F$ would recover the definition of Néron models. As we will be dealing with pro-schemes, we need to remove the finite type assumption. The following statement is a reformulation of [BST24, §1.3].

Proposition 4.16: *Let R be a topological ring, let V be a free R-module of finite rank, let K_0 be a profinite subgroup of $\mathrm{GL}(V)$. Recall that $\mathcal{C}(K_0)$ denotes the set of all open normal subgroups of K_0.*

Let $(\mathscr{S}_K)_{K\in\mathcal{C}(K_0)}$ be a projective system of separated Deligne-Mumford stacks over $\mathcal{O}_F$ which is fit for K_0 as in Definition 2.17. Let

$$\mathscr{S}_\infty := \varprojlim_{K\in\mathcal{C}(K_0)} \mathscr{S}_K.$$

Let $\mathbb{L}^{\mathrm{can}}$ be the canonical local system on $\mathscr{S}_{K_0}$ as in Definition 2.18. If $\mathscr{S}_\infty$ has smooth-extension, then $(\mathscr{S}_{K_0}, \mathbb{L}^{\mathrm{can}})$ has L-extension.

Proof. Let $\mathcal{X}/\mathcal{O}_F$ be a smooth scheme, let $f : \mathcal{X} \rightsquigarrow \mathscr{S}_{K_0}$ such that $f_\eta^{-1}\mathbb{L}^{\mathrm{can}}$ extends to a local system $\mathbb{L}$ on $\mathcal{X}$. We want to show that f extends to $\tilde{f} : \mathcal{X} \to \mathscr{S}_{K_0}$.

By definition, $\mathbb{L}^{\mathrm{can}}$ is induced from a K_0-torsor on $\mathscr{S}_{K_0}$, thus $\mathbb{L}$ is induced from a K_0-torsor on $\mathcal{X}$. From Proposition 2.20, the corresponding morphism $\mathcal{X} \to \mathbf{B}K_0$ defines a K_0-fit system of schemes $(\mathcal{X}_K)_{K\in\mathcal{C}(K_0)}$, with $\mathcal{X}_{K_0} = \mathcal{X}$. Let

$$\mathcal{X}_\infty := \varprojlim_K \mathcal{X}_K$$

which is a regular formally smooth $\mathcal{O}_F$-scheme. The K_0-fit system of generic fibers $((\mathcal{X}_K)_\eta)_K$ over F coincides with the pullback along f_η of the fit system $((\mathscr{S}_K)_\eta)_K$. In particular, there is an induced F-morphism

$$(\mathcal{X}_\infty)_\eta \to (\mathscr{S}_\infty)_\eta$$

By smooth extension, there is an induced morphism

$$\mathcal{X}_\infty \to \mathscr{S}_\infty.$$

Now it suffices to show that the composition $\mathcal{X}_\infty \to \mathscr{S}_\infty \to \mathscr{S}_{K_0}$ descends to a morphism $\mathcal{X}_{K_0} \to \mathscr{S}_{K_0}$, in other words we need to show by fpqc descent that both composition arrows

$$\mathcal{X}_\infty \times_{\mathcal{X}_{K_0}} \mathcal{X}_\infty \rightrightarrows \mathcal{X}_\infty \to \mathscr{S}_\infty \to \mathscr{S}_{K_0}$$

are isomorphic. This follows from Proposition 3.11. □

[8] Dropping the affineness assumption, the valuative criterion for properness of a morphism of stacks only gives the dashed arrow after an extension of valuation ring, which need not be finite in general, so we cannot conclude. More generally, we could have replaced "affine" with "representable by a scheme".

4.4. **Applications to Shimura varieties.** Let (G,Ω) be a Shimura datum with reflex field $E/\mathbb{Q}$, and $K_p \subseteq G(\mathbb{Q}_p)$ a hyperspecial subgroup. Let v be a place of E lying above p, $\mathcal{O}_{(v)}$ the local ring of $\mathcal{O}_E$ at v and $\mathcal{O}_v$ its v-adic completion. Consider a neat level $K = K^pK_p$ where K^p is a compact open subgroup of $G(\mathbb{A}_f^p)$, write $\mathrm{Sh}_K(G,\Omega)/E$ for the corresponding variety over E. Set

$$\mathrm{Sh}_{K_p}(G,\Omega) = \varprojlim_{K^p} \mathrm{Sh}_{K^pK_p}(G,\Omega)$$

which is an E-scheme with ind-continuous $G(\mathbb{A}_f^p)$-action (see Remark 4.18 below). If $K = K^pK_p$ need no longer be neat, we let $\mathrm{Sh}_K[G,\Omega] = [\mathrm{Sh}_{K_p}(G,\Omega)/K^p]$ denote the quotient stack; for K neat this is a scheme which coincides with $\mathrm{Sh}_K(G,\Omega)$.

Let $\mathcal{O}$ be a discrete valuation ring, faithfully flat over $\mathcal{O}_{(v)}$, write $\tilde{E}$ for its fraction field and $\mathfrak{m}_{\mathcal{O}}$ for its maximal ideal. We recall from [Vas99], [Moo98] and [Kis09] the notion of smooth integral canonical models of Shimura varieties.

Definition 4.17: A *smooth integral model* of $\mathrm{Sh}_{K_p}(G,\Omega)$ over $\mathcal{O}$ is a faithfully flat $\mathcal{O}$-scheme denoted $\mathscr{S}_{K_p}(G,\Omega)$ with an ind-continuous $G(\mathbb{A}_f^p)$-action (see Remark 4.18 below), such that

(i) there is a $G(\mathbb{A}_f^p)$-equivariant identification $\mathscr{S}_{K_p}(G,\Omega) \times_{\mathcal{O}} \tilde{E} \simeq \mathrm{Sh}_{K_p}(G,\Omega) \times_E \tilde{E}$, and

(ii) there exists a compact open subgroup $K_0^p \subseteq G(\mathbb{A}_f^p)$ such that for any $K_2^p \subseteq K_1^p \subseteq K_0^p$ open subgroups, the canonical quotient map

$$\mathscr{S}_{K_p}(G,\Omega)/K_1^p \to \mathscr{S}_{K_p}(G,\Omega)/K_2^p$$

is a finite étale morphism between smooth separated schemes of finite type over $\mathcal{O}$.

Moreover, $\mathscr{S}_{K_p}(G,\Omega)$ is a *smooth integral canonical model* if it also has smooth-extension as in Definition 4.15.

Remark 4.18: We specify what is meant by an "ind-continuous action" on a scheme T over a base S. Recall from Definition 2.4 that to a topological group Q corresponds a scheaf of groups $\mathcal{F}_Q$ over the pro-étale site $(\mathrm{Sch}/S)_{\text{proét}}$, mapping an S-scheme U to the group of continuous functions $U \to Q$. A sheaf with an action of $\mathcal{F}_Q$ is called a Q-sheaf, as in Definition 2.6. A *continuous Q-action* on an S-scheme T is defined to be a Q-sheaf structure on the functor of points h_T over $(\mathrm{Sch}/S)_{\text{proét}}$:

$$\mathcal{F}_Q \times h_T \longrightarrow h_T.$$

For Shimura varieties however, we need a weaker notion. Let $\mathcal{K}(Q)$ denote the set of all compact open subgroups of Q, and assume that

$$Q = \varinjlim_{K\in\mathcal{K}(Q)} K.$$

An *ind-continuous Q-action* on an S scheme T is defined to be an action of the sheaf of groups

$$\varinjlim_{K\in\mathcal{K}(Q)} \mathcal{F}_K,$$

which in general is a strict subsheaf of $\mathcal{F}_Q$, on the functor of points h_T over $(\mathrm{Sch}/S)_{\text{proét}}$. In this case, it makes sense to talk about the quotient stack $[T/K]$ for any $K \in \mathcal{K}(Q)$.

In the context of Shimura varieties, let K_0 be a compact open subgroup of $G(\mathbb{A}_f)$ and $K \in \mathcal{C}(K_0)$, it is known that there is an action of the finite constant group scheme K_0/K on $\mathrm{Sh}_K(G,\Omega)$ which preserves the projection map to $\mathrm{Sh}_{K_0}(G,\Omega)$. In terms of sheaves over the pro-étale site on $\operatorname{Spec} E$, there is an action

$$\mathcal{F}_{K_0/K} \times h_{\mathrm{Sh}_K(G,\Omega)} \longrightarrow h_{\mathrm{Sh}_K(G,\Omega)}.$$

If we let $\mathrm{Sh}_\infty(G,\Omega) = \varprojlim_{K\in\mathcal{C}(K_0)} \mathrm{Sh}_K(G,\Omega)$, we can pass to the projective limit to find

$$\mathcal{F}_{K_0} \times h_{\mathrm{Sh}_\infty(G,\Omega)} \longrightarrow h_{\mathrm{Sh}_\infty(G,\Omega)}.$$

We now take the inductive limit over K_0 to find the ind-continuous action. The arguement is unchanged when replacing $G(\mathbb{A}_f)$ with $G(\mathbb{A}_f^p)$.

Fix $\mathscr{S}_{K_p}(G,\Omega)$ a smooth integral canonical model of $\mathrm{Sh}_{K_p}(G,\Omega)$ over $\mathcal{O}$. For any level $K = K^pK_p$, we let $\mathscr{S}_K[G,\Omega] \coloneqq [\mathscr{S}_{K_p}(G,\Omega)/K^p]$ denote the quotient stack. Fix K_0^p any compact open subgroup of $G(\mathbb{A}_f^p)$, then the projective system $(\mathscr{S}_{K^pK_p}[G,\Omega])_{K^p\in\mathcal{C}(K_0^p)}$ is fit for K_0^p. From Definition 2.18 there is a canonical local system $\mathbb{L}^{\mathrm{can}}$ on $\mathscr{S}_{K_0^pK_p}[G,\Omega]$.

Proposition 4.19: *The pair $(\mathscr{S}_{K_0^pK_p}[G,\Omega],\mathbb{L}^{\mathrm{can}})$ has L-extension, hence also PGR-extension.*

Proof. This follows from Proposition 4.16 and Proposition 4.8. □

Let $\mathcal{O}\to\mathcal{O}'$ be a faithfully flat extension of discrete valuation rings. It is not a priori clear whether the base change $\mathscr{S}_{K_p}(G,\Omega)_{\mathcal{O}'}/\mathcal{O}'$ also has smooth-extension, as schemes $\mathcal{X}/\mathcal{O}'$ need not be formally smooth over $\mathcal{O}$. This holds under additional assumptions.

Fact 4.20 ([Vas99], Proposition 3.2.3.3)**:** *Let $\mathcal{O}\hookrightarrow\mathcal{O}'$ be a faithfully flat extension of discrete valuation rings with separable residue field extension, and such that $\mathfrak{m}_{\mathcal{O}}\mathcal{O}'$ is maximal in $\mathcal{O}'$. Let $\tilde{E}'$ be the fraction field of $\mathcal{O}'$. Then the base-change $\mathscr{S}_{K_p}(G,\Omega)_{\mathcal{O}'}$ is a smooth integral canonical model of $\mathrm{Sh}_{K_p}(G,\Omega)_{\tilde{E}'}$.*

Remark 4.21: The assumptions of Fact 4.20 are also required for the analogous statement in the theory of Néron models (see [BLR12, Theorem 7.2.1 (ii)]).

5. Applications

Assume $p\neq 2$. Let $F/\mathbb{Q}_p$ be a finite field extension.

5.1. **Smooth curves.** Let $g\geq 2$, $n\geq 0$ be integers satisfying $2-2g<n$. Let $\mathscr{M}_{g,n}$ denote the separated Deligne-Mumford stack of genus-g, n-pointed, geometrically connected, smooth, proper curves. Let ℓ_1,ℓ_2 be prime numbers different from p, set $\mathscr{L}=\{\ell_1,\ell_2\}$, and $\mathbb{Z}_{\mathscr{L}}=\mathbb{Z}_{\ell_1}\times\mathbb{Z}_{\ell_2}$. In [Sti05, Remark 2.8 (a)] Stix defines a $\mathbb{Z}_{\mathscr{L}}$-local system $\mathbb{L}$[9] over $\mathscr{M}_{g,n}\times_{\mathbb{Z}}\mathbb{Z}\left[\frac{1}{\mathscr{L}}\right]$.

Let $\mathscr{S}:=\mathscr{M}_{g,n}\times_{\mathbb{Z}}\mathcal{O}_F$, the pullback of $\mathbb{L}$ to $\mathscr{S}$ is again denoted $\mathbb{L}$.

Proposition 5.1: *The pair $(\mathscr{S},\mathbb{L})$ has L-extension, hence $\mathscr{S}$ has PGR-extension.*

Proof. This is a corollary of [Sti05, Theorem 2.10 (B)] and Proposition 4.8. □

Consequently, we recover [CT26, §4.3, Example (1)].

Theorem 5.2: *Let $\mathscr{X}/\mathcal{O}_F$ be a smooth, geometrically connected scheme with generic fiber X/F. Let $f:Y\to X$ be a smooth proper family of genus-g, n-pointed, geometrically connected, smooth, proper curves. If it has pointwise good reduction, then it has global good reduction.*

5.2. **Abelian varieties.** Let $g\geq 1$ be an integer, let V be a free abelian group of rank $2g$, let ψ be a nondegenerate symplectic form on V, set $G=\mathrm{GSp}(V_{\mathbb{Q}},\psi_{\mathbb{Q}})$. Let Ω be the space of complex structures on $V_{\mathbb{R}}$ for which (V,ψ) is a polarized Hodge structure. Fix $K_p\subseteq G(\mathbb{Q}_p)$ a hyperspecial level at p, $K=K^pK_p\subseteq G(\mathbb{A}_f)$ a compact open subgroup. The Siegel modular stack $\mathrm{Sh}_K[G,\Omega]$ over $\mathbb{Q}$ has been defined in Section 4.4, along with its canonical local system.

Remark 5.3: It follows from Example 2.31 that the canonical local system on $\mathscr{S}_K[G,\Omega]$ coincides with the universal Tate module.

Fact 5.4: *Assume $F/\mathbb{Q}_p$ is unramified. The Shimura variety $\mathrm{Sh}_{K_p}(G,\Omega)$ admits a smooth integral canonical model $\mathscr{S}_{K_p}(G,\Omega)$ over $\mathcal{O}_F$ as in Definition 4.17.*

Proof. The existence of $\mathscr{S}_{K_p}(G,\Omega)$ over $\mathbb{Z}_{(p)}$ is shown in [Mil92, Theorem 2.10], except the last part of the proof has to be modified following [Moo98, Corollary 3.8]. By Fact 4.20, its base-change to $\mathcal{O}_F$ is again a smooth integral canonical model. □

From Proposition 4.19 we find that the quotient stack $\mathscr{S}_K[G,\Omega]$ has PGR-extension. Thus, for $F/\mathbb{Q}_p$ unramified, polarized abelian schemes with K-level structure having pointwise good reduction also have global good reduction. By choosing $K^p=G(\widehat{\mathbb{Z}}^p)$, we find:

[9] We warn the reader that in [Sti05] the notation $\mathbb{L}$ refers to the set denoted $\mathscr{L}$ here.

Theorem 5.5: *Assume $F/\mathbb{Q}_p$ is unramified. Let X/F be a smooth, geometrically connected variety, and $\mathcal{X}/\mathcal{O}_F$ a smooth model. Consider $A \to X$ a polarized abelian scheme. If it has pointwise good reduction, then it has global good reduction.*

Proof. We may choose the symplectic module (V, ψ) so that the abelian scheme $A \to X$ defines a PGR-morphism $X \to \mathrm{Sh}_K[G, \Omega]$ with $K^p = G(\widehat{\mathbb{Z}}^p)$, which extends to $\mathcal{X} \to \mathscr{S}_K[G, \Omega]$ by PGR-extension. □

Theorem 5.5 is a weaker version of the following result of Cadoret-Tamagawa, which relies on [VZ10, Corollary 5] with no need for Shimura varieties.

Fact 5.6 ([CT26], §4.3.2, Example (3))**:** *Assume $F/\mathbb{Q}_p$ has ramification index $e \leq p-1$. Let X/F be a smooth, geometrically connected variety, and $\mathcal{X}/\mathcal{O}_F$ a smooth model. Let $A \to X$ be an abelian scheme. If it has pointwise good reduction, then it has global good reduction.*

The bound on e is optimal, as shown by Corollary 3.8. The theory of Shimura varieties provides a second interpretation for this obstruction. It was asked by Vasiu in [Vas99, Remark 3.2.12.1] if Fact 4.20 still holds for ramified extensions. For the Siegel modular variety, the answer is negative in general.

Proposition 5.7: *Assume $F/\mathbb{Q}_p$ has ramification index $e \geq p$. There are symplectic modules (V, ψ) such that smooth integral canonical models for $\mathrm{Sh}_{K_p}(G, \Omega)_F$ do not exist.*

Proof. If such models were to exist, Fact 5.4 and Theorem 5.5 would hold for F, but this is not the case as shown by Corollary 3.8. □

5.3. **K3 surfaces.** Fix $d \geq 1$ an integer such that $p \nmid 2d$. The K3 lattice is the rank 22 quadratic lattice $\Lambda = E_8^{\oplus 2} \oplus U^{\oplus 3}$, where E_8 is the E8 lattice of rank 8 and U is the hyperbolic lattice of rank 2. Let $e, f \in U \subseteq \Lambda$ be isotropic elements satisfying $[e, f] = 1$ in some copy of the hyperbolic lattice, where $[\cdot, \cdot]$ is the bilinear symmetric form on U. Set $L_d = \langle e - df \rangle^{\perp} \subseteq \Lambda$ a sublattice of discriminant $2d$ and rank 21 and write $W \coloneqq L_{d,\mathbb{Q}}$. We let $L_d^{\vee} \subseteq W$ denote the dual lattice. To this data is associated a Shimura stack of orthogonal type, following [MP15, §3.1].

Definition 5.8: Let $G = \mathrm{SO}(W)$ and Ω the space of oriented negative planes in $L_{d,\mathbb{R}}$. Let $K_0 = K_0^p K_p \subseteq G(\mathbb{A}_f)$ denote the largest subgroup of $\mathrm{SO}(L_d)(\widehat{\mathbb{Z}})$ acting trivially on $L_d^{\vee}/L_d$. For any level $K = K^p K_p \subseteq K_0$, let $\mathrm{Sh}_K[G, \Omega]$ be the Shimura stack of level K associated to the Shimura datum (G, Ω), over its reflex field $\mathbb{Q}$.

Fact 5.9: *Assume $F/\mathbb{Q}_p$ is unramified. The Shimura variety $\mathrm{Sh}_{K_p}(G, \Omega)$ admits a smooth integral canonical model $\mathscr{S}_{K_p}(G, \Omega)$ over $\mathcal{O}_F$.*

Proof. The existence of $\mathscr{S}_{K_p}(G, \Omega)$ over $\mathbb{Z}_{(p)}$ is established in [MP16, Proposition 7.9]. By Fact 4.20, its base-change to $\mathcal{O}_F$ is again a smooth integral canonical model. □

We assume from now-on that $F/\mathbb{Q}_p$ is unramified and that $\mathscr{S}_{K_p}(G, \Omega)$ is as in Fact 5.9. From Proposition 4.19 we find that the quotient stack $\mathscr{S}_K[G, \Omega]$ has PGR-extension.

Next we introduce the moduli space of primitively polarized K3 surfaces with level structure, following [MP15, §2.1, §2.10].

Definition 5.10: Let $\mathscr{M}_{2d}^{\circ}$ be the moduli problem that assigns to every $\mathbb{Z}_{(p)}$-scheme T the groupoid of tuples $(f : Y \to T, \xi)$ where

- $f : Y \to T$ is a K3 surface[10], and
- $\xi \in \underline{\mathrm{Pic}}(X/T)(T)$ is a primitive polarization of degree $2d$.

It is a separated Deligne-Mumford stack of finite type over $\mathbb{Z}_{(p)}$ ([Riz05] Theorem 4.3.3).

[10] That is, Y is a proper and smooth algebraic space over T, with geometric fibers that are K3 surfaces in the usual sense.

If $(f : Y \to T, \xi) \in \mathscr{M}^\circ_{2d}(T)$, let $\mathbf{H}^2_{\widehat{\mathbb{Z}}^p}(f) = \prod_{\ell \neq p} \mathbf{H}^2_\ell(f)$ be the $\widehat{\mathbb{Z}}^p$-local system on T defined as the product of the relative ℓ-adic cohomology sheaves. The polarization defines a relative Chern class $\mathrm{ch}_{\widehat{\mathbb{Z}}^p}(\xi) \in \mathbf{H}^2_{\widehat{\mathbb{Z}}^p}(f)(1)$, the sub-local-system orthogonal to this class is the relative primitive cohomology sheaf $\mathbf{P}^2_{\widehat{\mathbb{Z}}^p}(f)(1)$. Let $V := L_d \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}^p$. With notations from Section 2 there is a local system

$$\begin{aligned} \mathbb{L} : \mathscr{M}^\circ_{2d} &\longrightarrow \mathbf{Loc}_V \\ (f : Y \to T, \xi) &\longmapsto \mathbf{P}^2_{\widehat{\mathbb{Z}}^p}(f)(1). \end{aligned}$$

The cup-product induces a quadratic form on (primitive) cohomology. We define

$$\begin{aligned} \mathrm{Lev} : \mathscr{M}^\circ_{2d} &\longrightarrow \mathbf{B}\mathrm{SO}(L_d)(\widehat{\mathbb{Z}}^p) \\ (f : Y \to T, \xi) &\longmapsto [(U \to T) \mapsto \{\text{Isometries } \mathbf{P}^2_{\widehat{\mathbb{Z}}^p}(f)(1)_U \xrightarrow{\sim} \mathcal{F}_V\}]. \end{aligned}$$

Let $R = \widehat{\mathbb{Z}}^p$ and $Q = \mathrm{SO}(L_d)(R)$, now $(\mathscr{M}^\circ_{2d}, \mathbb{L}, \mathrm{Lev}, \iota)$ is a Q-moduli datum as in Definition 2.26, where ι is the obvious sheaf inclusion of $\mathrm{Lev}(f : Y \to T, \xi)$ inside $\underline{\mathrm{Isom}}(\mathbf{P}^2_{\widehat{\mathbb{Z}}^p}(f)(1), \mathcal{F}_V)$. For any open subgroup $K = K^pK_p \subseteq K_0$ we may consider the associated moduli stack of level K denoted $\mathscr{M}^\circ_{2d,K}$. As K_0 is a finite index open normal subgroup of Q, we find that the projection $\mathscr{M}^\circ_{2d,K_0} \to \mathscr{M}^\circ_{2d}$ is a Q/K_0-torsor.

Ideally, $\mathrm{Sh}_K[G, \Omega]$ would serve as a period domain for the cohomology of K3 surfaces, so as to induce a morphism $\mathscr{M}^\circ_{2d,K,\mathbb{C}} \to \mathrm{Sh}_K[G, \Omega]_{\mathbb{C}}$ which would descend over $\mathbb{Q}$, and finally extend to $\mathscr{M}^\circ_{2d,K,\mathcal{O}_F} \to \mathscr{S}_K[G, \Omega]$ by invoking some extension property. To make this argument work however, $\mathscr{M}^\circ_{2d}$ needs to be replaced with a double cover.

Definition 5.11: Let $\tilde{\mathscr{M}}^\circ_{2d} \to \mathscr{M}^\circ_{2d}$ be the double finite étale cover parametrizing isometric trivializations $\det(\mathbf{P}^2_2) \xrightarrow{\sim} \mathcal{F}_{\det(L_d)\otimes_{\mathbb{Z}}\mathbb{Z}_2}$ (notation from Definition 2.4). Its objects are called *oriented, primitively polarized K3-surfaces of degree* $2d$.

From [MP15, Proposition 4.2, Corollary 4.4 and Proposition 4.7] there is a natural period map $\tilde{\mathscr{M}}^\circ_{2d,\mathbb{C}} \to \mathrm{Sh}_{K_0}(G, \Omega)_{\mathbb{C}}$ which extends to an $\mathcal{O}_F$-morphism $\tilde{\mathscr{M}}^\circ_{2d,\mathcal{O}_F} \to \mathscr{S}_{K_0}(G, \Omega)$.

Fact 5.12 ([MP15], Proposition 4.8)**:** *The map $\tilde{\mathscr{M}}^\circ_{2d,\mathcal{O}_F} \to \mathscr{S}_{K_0}[G, \Omega]$ is étale.*

Let $\tilde{\mathscr{M}}^\circ_{2d,K}$ denote the pullback of $\mathscr{M}^\circ_{2d,K} \to \mathscr{M}^\circ_{2d}$ along $\tilde{\mathscr{M}}^\circ_{2d} \to \mathscr{M}^\circ_{2d}$. The morphism in Fact 5.12 lifts to an étale morphism at level $K = K^pK_p \subseteq K_0$:

$$\tilde{\mathscr{M}}^\circ_{2d,K,\mathcal{O}_F} \to \mathscr{S}_K[G, \Omega].$$

Remark 5.13: It follows from Proposition 2.30 that the pullback of $\mathbb{L}_{\mathrm{can}}$ on $\mathscr{S}_K[G, \Omega]$ to $\tilde{\mathscr{M}}^\circ_{2d,K}$ coincides with the universal relative primitive cohomology $\mathbf{P}^2_{\widehat{\mathbb{Z}}^p}(1)$.

Corollary 5.14: *The stacks $\tilde{\mathscr{M}}^\circ_{2d,\mathcal{O}_F}$, $\mathscr{M}^\circ_{2d,\mathcal{O}_F}$, $\tilde{\mathscr{M}}^\circ_{2d,K,\mathcal{O}_F}$ and $\mathscr{M}^\circ_{2d,K,\mathcal{O}_F}$ all have PGR-extension.*

Proof. For $\tilde{\mathscr{M}}^\circ_{2d,K,\mathcal{O}_F}$ and $\tilde{\mathscr{M}}^\circ_{2d,\mathcal{O}_F}$ this follows from $\mathscr{S}_K[G, \Omega]$ having PGR-extension and Theorem 4.9. As $\tilde{\mathscr{M}}^\circ_{2d} \to \mathscr{M}^\circ_{2d}$ is a $\mathbb{Z}/2\mathbb{Z}$-torsor, PGR-extension for $\mathscr{M}^\circ_{2d,\mathcal{O}_F}$ and $\mathscr{M}^\circ_{2d,K,\mathcal{O}_F}$ is then deduced from Proposition 4.13. □

Thus, for $F/\mathbb{Q}_p$ unramified, (oriented) primitively polarized K3-surfaces with K-level structure having pointwise good reduction also have global good reduction. In the particular case of $\mathscr{M}^\circ_{2d,\mathcal{O}_F}$, we find:

Theorem 5.15: *Assume $F/\mathbb{Q}_p$ is unramified. Let X/F be a smooth, geometrically connected variety, and $\mathcal{X}/\mathcal{O}_F$ a smooth model. Consider $Y \to X$ a primitively polarized K3-surface of degree $2d$.If it has pointwise good reduction, then it has global good reduction.*

Remark 5.16: All statements of this paragraph still hold true if $\mathscr{M}^\circ_{2d}$ is replaced with $\mathscr{M}_{2d}$ the moduli stack defined as in Definition 5.10, except ξ is now a primitive quasi-polarization (as in [MP15, §2.1]).

Remark 5.17: In trying to generalize the argument from K3 surfaces to general hyperkähler varieties, only the analogue Fact 5.12 is still missing. One can attach a Shimura datum $(G^{\mathrm{HK}}, \Omega^{\mathrm{HK}})$ to the quadratic lattice defined by the second cohomology group of a fixed hyperkähler variety as in [Bin21, §4.5], as well as introduce the moduli stack $\mathscr{M}_{2d}$ (resp. $\tilde{\mathscr{M}}_{2d}$) of degree-$2d$ polarized, (resp. oriented) hyperkähler varieties over $\mathbb{Z}_{(p)}$, $p \nmid d$, as in [Bin21, Definition 3.3.1, Definition 4.3.3][11]. By [MP16, Proposition 7.9] the variety $\mathrm{Sh}_{K_p}(G^{\mathrm{HK}}, \Omega^{\mathrm{HK}})/\mathbb{Q}$ has a smooth integral canonical model $\mathscr{S}_{K_p}(G^{\mathrm{HK}}, \Omega^{\mathrm{HK}})/\mathbb{Z}_{(p)}$, where $K_p \subseteq G^{\mathrm{HK}}(\mathbb{A}_p)$ suitably chosen. By [Bin21, Theorem 4.5.2] there is an étale period map $\tilde{\mathscr{M}}_{2d,K,\mathbb{Q}} \to \mathrm{Sh}_K(G^{\mathrm{HK}}, \Omega^{\mathrm{HK}})$ over $\mathbb{Q}$, and the extension property yields

$$\tilde{\mathscr{M}}_{2d,K} \to \mathscr{S}_K(G^{\mathrm{HK}}, \Omega^{\mathrm{HK}})$$

over $\mathbb{Z}_{(p)}$. If this map were shown to be étale, it would follow as in Corollary 5.14 that $\tilde{\mathscr{M}}_{2d,K}$, $\mathscr{M}_{2d,K}$ and $\mathscr{M}_{2d}$ have PGR-extension, and so Theorem 5.15 would extend to hyperkähler varieties.

5.4. **Cubic fourfolds.** The argument for K3 surfaces applies *mutatis mutandis* to cubic fourfolds; we give references to the main ingredients following [MP15, §5.13].

Let M_0 be the even rank 2 quadratic lattice with bilinear form represented by $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ and set

$$M = E_8^{\oplus 2} \oplus U^{\oplus 2} \oplus M_0$$

which is has signature $(20, 2)$ and is self dual away from 2 and 3.

Definition 5.18: Let $G = \mathrm{SO}(V)$ and Ω the space of oriented negative planes in $M_{\mathbb{R}}$. Let $K_0 = K_0^p K_p \subseteq G(\mathbb{A}_f)$ denote the largest subgroup of $\mathrm{SO}(M)(\widehat{\mathbb{Z}})$ acting trivially on $M^\vee/M$. For any level $K = K^p K_p \subseteq K_0$, let $\mathrm{Sh}_K[G, \Omega]$ be the Shimura stack of level K associated to the Shimura datum (G, Ω), over its reflex field $\mathbb{Q}$.

Consider $\mathscr{M}$ the moduli stack of cubic fourfolds over $\mathbb{Z}_{(p)}$, and $\tilde{\mathscr{M}}$ its double cover trivializing the determinant of primitive cohomology. For K as in Definition 5.18, we let $\mathscr{M}_K$ (resp. $\tilde{\mathscr{M}}_K$) be the moduli stack with K-level structure on degree-4 primitive cohomology which is a finite étale cover of $\mathscr{M}$ (resp. $\tilde{\mathscr{M}}$). In particular, $\mathscr{M}_{K_0} \to \mathscr{M}$ is a finite torsor.

From the modular description of $\mathrm{Sh}_{K_0}(G, \Omega)_{\mathbb{C}}$ (see [MP15, Proposition 3.3]) we find a period map:

$$\tilde{\mathscr{M}}_{\mathbb{C}} \to \mathrm{Sh}_{K_0}(G, \Omega)_{\mathbb{C}}$$

which descends over $\mathbb{Q}$ as in [MP15, Corollary 4.4]. By smooth-extension we find a period map $\tilde{\mathscr{M}} \to \mathscr{S}_{K_0}[G, \Omega]$ which is étale by [MP15, Theorem 5.14]. Thus for K as in Definition 5.18 we find étale maps $\tilde{\mathscr{M}}_K \to \mathscr{S}_K(G, \Omega)$. It follows that $\tilde{\mathscr{M}}_K$, hence also $\mathscr{M}_K$ and $\mathscr{M}$, have PGR-extension. This proves the pointwise to global good reduction for (oriented) cubic fourfolds with K-level structure, and in the particular case of $\mathscr{M}$:

Theorem 5.19: *Assume $F/\mathbb{Q}_p$ is unramified. Let X/F be a smooth, geometrically connected variety, and $\mathcal{X}/\mathcal{O}_F$ a smooth model. Consider $Y \to X$ a smooth, proper family of cubic fourfolds. If it has pointwise good reduction, then it has global good reduction.*

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