DE JONG - OORT'S "ON EXTENDING FAMILIES OF CURVES" DETAILED

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Contents

1. Introduction

I wrote these (messy) notes in an effort to fill the gaps in my understanding of the arguments of [dJO97], excluding §6. The paper establishes that any family of stable curves defined over a dense open subset U of a regular scheme S extends to a family over S, provided it has *locally constant topological type*. (we refer the reader to [dJO97, §3] for what this means) Any argument I ultimately failed to understand is indicated by a footnote.

Section 2 states the main result, and proves it assuming certain indermediate lemmas. The proofs of the latter are postponed until Section 3. I recommend reading the original article, and only switching to the present notes when in need of additional details.

I did not try to stay concise, and my proofs are likely suboptimal. Any mistake is my own. I thank Livia Grammatica, Pierre Godfard and Maxime Cazaux for helpful discussions.

2. Proof of main result

Let S be a scheme and D a normal crossing divisor in S: this means that S is regular along D, and étale-locally D is the zero set of a product $t_1 \dots t_r$, where $t_1, \dots, t_r$ are part of a regular system of parameters. Let $U \coloneqq S \setminus D$.

Theorem 1: *Let V be an open subscheme of S containing U and the generic points of D. Let $\mathcal{C}_U \to U$ be a stable curve of locally constant topological type, and assume it extends to a stable curve $\mathcal{C}_V \to V$. Then it extends to a stable curve over S.*

In this section we follow the proof of Theorem 1 from [dJO97], assuming intermediate results which we now state.

2.1. **Key intermediate results.** We collect intermediate results to be used in the argument of the main result. Proofs are deferred to Section 3.

Lemma 2: *Let S ba a locally Noetherian scheme, U a dense open subscheme, and $f : \mathcal{C}_U \to U$ a stable curve over U.*

(i) *Suppose V is an open subscheme of S containing U, such that for any $s \in V \setminus U$, the local ring $\mathcal{O}_{S,s}$ either has depth ≥ 2, or has depth 1 and is geometrically unibranch. Then there is at most one extension of $\mathcal{C}_U$ to a stable curve over V.*

(ii) *Let $s \in S$, consider $\operatorname{Spec} \mathcal{O}_{S,s} \to S$. If $\mathcal{C}_U$ extends to a stable curve over $\operatorname{Spec} \mathcal{O}_{S,s}$, then it extends to a stable curve over an open subscheme V of S containing s and U.*

As an immediate corollary, we find:

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Corollary 3: *Let S ba a locally Noetherian scheme, U a dense open subscheme, and $f : \mathcal{C}_U \to U$ a stable curve over U. Then there exists a unique maximal open subscheme V of S, containing U, over which $\mathcal{C}_U$ extends to a stable curve. In particular, V contains all $s \in S$ such that $\mathcal{C}_U$ extends to a stable curve over* $\operatorname{Spec} \mathcal{O}_{S,s}$.

Theorem 4: *Let S be a quasi-compact, excellent, reduced, normal scheme of dimension 2, with function field $K(S)$. Let U be a dense open subscheme, and $f : \mathcal{C}_U \to U$ a stable curve over U. Assume the following holds:*

For all discrete valuation ring R in $K(S)$ and $\phi : \operatorname{Spec} R \to S$ with $\phi(\eta) \in U$ (here $\eta \in \operatorname{Spec} R$ denotes the generic point), the pullback curve $\mathcal{C}_\eta$ under ϕ extends to a stable curve over $\operatorname{Spec} R$.

Then there exists a normalized blowing up $\tilde{S} \to S$ with center in $S \setminus U$ such that $\mathcal{C}_U$ extends to a stable curve over $\tilde{S}$.

Lemma 5: *Let k be a field, let $(\mathcal{C}, \Sigma) \to \mathbb{P}^1_k$ be a stable r-pointed curve of genus g. If the topological type of $(\mathcal{C}, \Sigma)$ is locally constant over $\mathbb{P}^1_k \setminus \{0, \infty\}$, then $(\mathcal{C}, \Sigma)$ is constant, i.e. there exists a stable pair (C_0, Σ_0) over k such that*

$$(C, \Sigma) \simeq (C_0 \times_k \mathbb{P}^1_k, \Sigma_0 \times_k \mathbb{P}^1_k).$$

2.2. **Reduction steps.** We argue that in Theorem 1, we can assume $S = \operatorname{Spec} R$ where R is a regular, complete local ring with separably closed residue field.

Using Lemma 2 we are reduced to the case where R is a Noetherian, regular local ring; let s be the closed point of S. Let $j : V \hookrightarrow S$ denote the largest open subscheme over which $\mathcal{C}_U$ extends, provided by Corollary 3, write $f : \mathcal{C}_V \to V$ the corresponding extension. For each n we put

$$\mathcal{F}_n = j_* f_* (\omega_{\mathcal{C}_V/V}^{\otimes 3n})^1.$$

Since $\operatorname{codim}_S(S \setminus V) \geq 2$ by assumption, these are coherent sheaves on S. The direct sum is a graded algebra $\mathcal{F} \coloneqq \bigoplus_{n \geq 0} \mathcal{F}_n$, notice that $\operatorname{Proj}(\mathcal{F}_{|V})$ recovers $\mathcal{C}_V$ since $\omega_{\mathcal{C}_V/V}^{\otimes 3}$ is relatively very ample. Notice that

(i) $\mathcal{C}_V \to V$ extends to a projective morphism over S if and only if $\mathcal{F}$ is generated by $\mathcal{F}_1$, in which case the extension is $\mathcal{C} \coloneqq \operatorname{Proj}(\mathcal{F}) \to S$,
(ii) in this case, $\mathcal{C} \to S$ is a stable curve if and only if (see [DM69, Definition 1.1]) all $\mathcal{F}_i$ are flat (i.e. free, because finitely generated over a Noetherian local ring) and $\mathcal{C}_s$ is a stable curve.

Let $R \to R'$ be a faithfully flat local ring extension, set $S' = \operatorname{Spec} R'$, $D' = D \times_S S'$ and U', V', $\mathcal{C}_{U'}$ and $\mathcal{C}_{V'}$ defined similarly. These objects also satisfy the assumptions of Theorem 1. Moreover, setting $\mathcal{F}'_n$ as above, it follows from flat base change that $\mathcal{F}'_n = \mathcal{F}_n \otimes \mathcal{O}_{S'}$. Assuming conditions (i) and (ii) hold for S', faithfully flat descent shows that they also hold for S. In particular, letting R' be the strict Henselization of the completion of R with respect to its maximal ideal, we find that it suffices to show the result for R'. Thus from now on we assume R is a regular complete local ring with separately closed residue field.

2.3. **Dimension 2 case.** We prove Theorem 1 assuming $S = \operatorname{Spec} R$ as in Section 2.2, with $\dim R = 2$. In this case, $V = S \setminus \{s\}$ where s denotes the closed point of S. Let us first summarize the argument into steps.

- Step 1: Let $G \to U$ denote the relative Pic° of $\mathcal{C}_U$, which is a semi-abelian scheme over U. Fixing $n \geq 3$ prime to the residue characteristic at s, we argue that we may assume $G[n]$ has trivial monodromy.
- Step 2: We deduce from Step 1 that $\mathcal{C}_U$ satisfies the assumptions of Theorem 4.
- Step 3: It follows from Step 2 that $\mathcal{C}_U$ extends to a stable curve $\tilde{\mathcal{C}}$ over a normalized blowing-up $\tilde{S} \to S$ of the base with center in $\{s\}$. We argue that we may assume $\tilde{S} \to S$ is a simple blowing up of the closed point $\{s\}$.
- Step 4: Let $E \subseteq \tilde{S}$ be the exceptional fiber, isomorphic to the projective line. We argue that there is $W \subset E$ consisting of at most two rational points, such that $\tilde{\mathcal{C}}_{E \setminus W}$ has locally constant topological type.
- Step 5: It follows from Step 4 and Lemma 5 that $\tilde{\mathcal{C}}_E$ is constant. We argue that $\tilde{\mathcal{C}}$ thus comes from a stable curve $\mathcal{C}$ over S, which concludes the proof.

[1] Our definition defers slightly from [dJO97]

We work out the details of each step.

Step 1. Let G and n as above. By [BLR90, §7.3, Lemma 2] the n-torsion $G[n] \to U$ is étale. Moreover we know that G extends to a semi-abelian scheme $\tilde{G} \to V$, and since n and $\operatorname{char} k(s)$ are coprime this implies that $G[n]$ is tamely ramified along D. In other words, the action of wild inertia (relative to the components of D) on the generic fiber $G_\eta[n]$ is trivial, $\eta \in S$ the generic point. We justify this claim. As G is semi-abelian, there is a decomposition over η

$$0 \longrightarrow T \longrightarrow G_\eta \longrightarrow A \longrightarrow 0$$

where T is a torus and A is an abelian variety, both over η. The closure of T in $\tilde{G}$ is again a torus (see [FC90, Proposition 2.9]), the quotient $\tilde{G}/T$ shows that A has semi-abelian reduction at (each irreducible components of) D. It follows from [GRR72, Proposition 3.5] that inertia acts unipotently on the n-torsion $A[n]$, hence that wild inertia acts trivially because unipotent subgroups of $\operatorname{Aut}(A[n])$ have prime-to-p order. The monodromy of $T[n]$ is provided by a direct sum of the $\operatorname{Gal}(\overline{\eta}/\eta)$-action on n-th roots of unity in an algebraic closure of the residue field $k(\eta)$; as R is strictly Henselian it follows that these roots lie in R, so the Galois action is trivial, and in particular wild inertia acts trivially. Thus we find a short exact sequence of $\operatorname{Gal}(\overline{\eta}/\eta)$-modules

$$0 \longrightarrow T[n] \longrightarrow G_\eta[n] \longrightarrow A[n] \longrightarrow 0$$

with wild inertia acting trivially on the right and the left. Thus it acts unipotently on $G_\eta[n]$, so wild inertia acts trivially as in the case of $A[n]$.

Now, assuming $G[n] \to U$ is finite (we show this in Lemma 6 below) we can apply Abhyankar's Lemma (see [Gro71, XIII, Proposition 5.2]) to find a finite flat covering $\pi : S' \to S$ such that:

- S' is regular,
- $U' := \pi^{-1}(U) \to U$ is finite étale,
- $D' := \pi^{-1}(D)_{red}$ is a divisor with normal crossings in S', and
- $G[n] \times_U U' \to U'$ is a trivial étale cover.

Arguing as in Section 2.2 it suffices to show the result replacing S with S'. Thus we can assume $G[n]$ is trivial. It remains to argue that $G[n] \to U$ is finite: this is not obvious, since the n-torsion of semi-abelian schemes (even over henselian local rings) is not finite in general. We will use that the topological type of $\mathcal{C}_U \to U$ is locally constant.

Lemma 6: *The quasi-finite étale morphism $G[n] \to U$ is finite.*

Proof. It suffices to show the result holds at the local rings of closed points of U, so we can assume U is the spectrum of a DVR. By faithfully flat descent, we may assume that the DVR is henselian. It follows from [BLR90, Paragraph following §7.3, Lemma 2] that $G[n] = H' \sqcup H''$ where $H' \to U$ is finite, and $H'' \to U$ has empty special fiber. We argue that H'' is empty. If H'' were nonempty, the order of $G[n]$ over the generic and special points of U would differ, which means that the maximal subtori and quotient abelian varieties over the generic and special points of U differ in rank and dimension. But these invariants are determined by the dual graph of both curves (see [BLR90, §9.2, Example 8]), which coincide assumption of locally constant topological type. □

Step 2. Now that $G[n] \to U$ is a trivial cover, we argue that $\mathcal{C}_U \to U$ satisfies the assumptions of Theorem 4. First, complete local rings are excellent, so all assumptions on S are satisfied. Now, any discrete valuation ring of $k(\eta)$ containing R is the local ring of an irreducible component of D; let $\mathcal{O}$ be one such local ring. As inertia acts trivially on $G_\eta[n]$, it acts unipotently on $G_\eta[n^m]$ for any $m \geq 1$. If we assume $\mathcal{C}_U$ is smooth, so that $G \to U$ is an abelian scheme, it follows from [GRR72, Proposition 3.5] that G_η extends to a semi-abelian scheme over $\mathcal{O}$.

Assume first that $\mathcal{C}_\eta$ is smooth. Then it follows from [DM69, Proposition 2.4] that it extends to a stable curve over $\mathcal{O}$. If $\mathcal{C}_\eta$ is not longer smooth, observe that the $\operatorname{Gal}(\overline{\eta}/\eta)$ acts trivially on its topological type[2], so the singular points of $\mathcal{C}_\eta$ define sections over η. Taking the normalization of $\mathcal{C}_\eta$ yields smooth pointed curves $(\mathcal{D}_{j,\eta}, \tau_{i,j,\eta})$, where $\tau_{j,i,\eta}$ are sections over η along which to glue to recover $\mathcal{C}_\eta$. Applying [DM69, Proposition 2.4] (rather its extension to smooth pointed curves) we find smooth pointed curves $(\mathcal{D}_j, \tau_{i,j})$ over $\mathcal{O}$. Gluing them along the sections $\tau_{i,j}$ defines a stable curve $\mathcal{C}_{\mathcal{O}}$ over $\mathcal{O}$.

[2] I do not understand why.

Step 3. Applying Theorem 4 we find a normalized blowing up $\tilde{S} \to S$ with center the closed point of S, such that $\mathcal{C}_U$ extends to a stable curve $\tilde{\mathcal{C}} \to \tilde{S}$. We may assume that the normalized blowing up is a sequence of simple blowing ups using:

Fact 7 ([SR66], Corollary p.34)**:** *Let S be an irreducible, noetherian, two-dimensional, regular scheme. Given a reduced scheme $\tilde{S}$ and a proper, birational morphism $f : \tilde{S} \to S$, there exists a reduced scheme S' and a proper, birational morphism $g : S' \to \tilde{S}$ such that $g \circ f : S' \to S$ is a composite of blowing ups along smooth points.*

We will now show that $\tilde{\mathcal{C}} \to \tilde{S}$ descends along simple blowing ups. Thus we may assume that $\tilde{S} \to S$ is one simple blowing up of the closed point $\{s\}$.

Step 4. Let $\tilde{E}$ be the exceptional fiber, isomorphic to the projective line over the residue field k of R. Now $\mathcal{C}_E := \tilde{\mathcal{C}}_{|E}$ is a stable curve over E. Let τ' denote the topological type of its generic fiber, let W denote the subset of closed point where the topological type jumps; W is locally closed in E, hence closed, hence finite. We claim that W is contained in the strict transform of D, which then implies that W consists of at most 2 closed points because S is regular of dimension 2. To show the claim, we recall the following fact on the structure of the moduli stack of genus $g \geq 2$ stable curves $\mathscr{M}$:

Fact 8 ([dJO97], §3, iii)**:** *Locally in the étale topology on $\mathscr{M}$ there exist Cartier divisors D_i, $i \in I$, which are normal crossing relative to* $\operatorname{Spec} \mathbb{Z}$*, such that the topological type of the universal curve is constant on each stratum $D_J = (\bigcap_{j \in J} D_j) \setminus (\bigcup_{i \in I \setminus J} D_i)$, where $J \subseteq I$.*

Let $p \in W$. We can find, étale locally at p, Cartier divisors $\tilde{D}_i$, $i \in I$ such that $\tilde{C}$ has constant topological type on each stratum $\tilde{D}_J$. We know that $\tilde{\mathcal{C}}$ has constant topological type outside $\pi^{-1}(D) \subseteq \tilde{S}$, so if p is not in the strict transform of D this means that some D_i is contained in an étale neighborhood of p in E which is thus of constant topological type. But this implies that some Zariski open neighborhood of p in E has constant topological type, which contradicts the assumption that $\mathcal{C}_p$ is not of type τ'. Hence p lies in the strict transform of D in $\tilde{S}$.

Step 5. Applying Lemma 5 we find that $\mathcal{C}_E \to E$ is constant. Denote $\tilde{f} : \tilde{\mathcal{C}} \to \tilde{S}$. As in Section 2.2 we define the coherent sheaves on S:

$$\tilde{\mathcal{F}}_n = \tilde{f}_*(\omega_{\tilde{\mathcal{C}}/\tilde{S}}^{\otimes 3n})$$

and let $\tilde{\mathcal{F}} := \bigoplus_{n \geq 0} \tilde{\mathcal{F}}_n$ so that $\tilde{\mathcal{C}} = \operatorname{Proj}(\tilde{F})$. We show below that the natural morphism $\pi^*\pi_*\tilde{\mathcal{F}}_n \to \tilde{\mathcal{F}}_n$ is an isomorphism; we can thus define $\mathcal{C} := \operatorname{Proj}(\pi_*\tilde{\mathcal{F}})$ which is a stable curve over S extending $\mathcal{C}_U$. To show the isomorphism, we begin by arguing that the $\tilde{\mathcal{F}}_n$ are locally free. For this, we apply Fact 9 for $i = 0$ to the line bundle $\omega^{\otimes 3n}$ over $\tilde{\mathcal{C}}$.

Fact 9 ([Har77], III, Corollary 12.9)**:** *Let $X \to Y$ be a projective morphism between noetherian schemes with Y integral. Let $\mathcal{G}$ be a coherent sheaf on X, flat over Y. Let $i \geq 0$. If the function*

$$y \in Y \longmapsto \dim_{k(y)} H^i(X_y, \mathcal{G} \otimes k(y))$$

is constant, then $R^i f_* \mathcal{G}$ is locally free over Y.

The isomorphism now follows from the fact that $\pi_*\mathcal{O}_{\tilde{S}} = \mathcal{O}_S$ using [Sta18, Tag 0AY8].

2.4. **General case.** We now assume $\dim R = d \geq 3$ and that Theorem 1 holds for dimension at most $d-1$. It follows from Lemma 2 that $\mathcal{C}_U$ extends to a stable curve $\mathcal{C}_V$ over $V := S \setminus \{s\}$. There are two steps to the argument.

- Step 1: Let D_1 be a component of D. We argue that the restriction of $\mathcal{C}_V$ over $V \cap D_1$ extends to a family $\mathcal{C}_{D_1} \to D_1$, and that IF there is a formal extension $\widehat{\mathcal{C}} \to \widehat{Z}$ (here $\widehat{Z}$ denotes the formal completion of S along D_1) then it can be algebraized into a stable curve $\mathcal{C} \to S$.
- Step 2: We show that such a formal extension indeed exists, by lifting $\mathcal{C}_{D_1} \to D_1$ to infinitesimal neighborhoods of D_1 in S.
- Step 3: We show that the restriction $(\mathcal{C})_V$ and $\mathcal{C}_V$ are isomorphic. Thus $\mathcal{C} \to S$ extends $\mathcal{C}_V$, which concludes the proof.

Step 1. Let D' denote the reunion of the components of D distinct from D_1. Now $\mathcal{C}_V$ restricts to a family of stable curve over $V_1 := V \times_S D_1$, of locally constant topological type outside of the normal crossing divisor $D' \cap D_1$. The induction hypothesis implies that this family extends to $\mathcal{C}_{D_1} \to D_1$. We will show in Step 2 that $\mathcal{C}_{D_1}$ extends to a family $\widehat{\mathcal{C}} \to \widehat{Z}$ which coincides with the base change of $\mathcal{C}_V$ along $\widehat{V} := V \times_S \widehat{Z} \to V$. More precisely, let Z_n denote the n-th infinitesimal neighborhood of D_1 in S, and set $V_n := V \times_S Z_n$: we will argue that $\mathcal{C}_{D_1}$ extends to $\mathcal{C}_n \to Z_n$ for all $n \geq 1$ compatibly and so as to coincide with the base change of $\mathcal{C}_V$ along $\widehat{V}_n \to V$, then take the limit over n to define $\widehat{\mathcal{C}} \to \widehat{Z}$. According to Fact 10 below applied to the ideal of definition of D_1 (note that R is also complete for the adic topology with respect to this ideal) we can algebraize this morphism to find a projective family $\mathcal{C} \to S$, which is again a stable curve[3].

Fact 10 ([DG67], III, Théorème 5.4.5)**:** *Let A be a complete, Hausdorff topological ring, let I be an open ideal such that $(I^n)_{n\geq 1}$ defines a fundamental system of neighborhoods of 0*[4]*. Let $S = \operatorname{Spec} A$, $Z_k = \operatorname{Spec} A/I^k$, and $\widehat{Z} = \operatorname{Spf} A$ the formal scheme associated with the pair (A, I). Let $f : \mathfrak{X} \to \widehat{Z}$ be a proper morphism of formal schemes, set $X_1 = \mathfrak{X} \times_{\widehat{Z}} Z_1$. Let $\mathscr{L}$ be an invertible $\mathcal{O}_{\mathfrak{X}}$-module, and assume $\mathscr{L}_1 := \mathscr{L}/I\mathscr{L}$ is an ample $\mathcal{O}_{X_1}$-module.*

Then there exists a proper S-scheme $X \to S$ and an ample $\mathcal{O}_X$-module $\mathscr{M}$ such that
- *the formal completion of X along I recovers $\mathfrak{X}$, and*
- *the formal completion of $\mathscr{M}$ along I recovers $\mathscr{L}$.*

Step 2. Let $f_{n-1} : \mathcal{C}_{n-1} \to Z_{n-1}$ be a stable curve as desired. Following [Ill71, III, Proposition 2.1.2.3 (i)], the obstruction to the existence of a flat lift to Z_n is a cohomology class

$$\partial u \in \operatorname{Ext}^2_{\mathcal{C}_{n-1}}(L_{\mathcal{C}_{n-1}/Z_{n-1}}, f_{n-1}^* J)$$

where J denotes the ideal sheaf of $Z_{n-1} \hookrightarrow Z_n$[5], and $L_{\mathcal{C}_{n-1}/Z_{n-1}}$ is the cotangent complex of $\mathcal{C}_{n-1}$ over Z_{n-1}. If this class vanishes, the same reference shows that the set of lifts is a torsor under the group

$$\operatorname{Ext}^1_{\mathcal{C}_{n-1}}(L_{\mathcal{C}_{n-1}/Z_{n-1}}, f_{n-1}^* J).$$

The following statements makes both groups more tractable.

Lemma 11: *Let $f : \mathcal{C} \to S$ be a stable curve, consider its cotangent complex $L_{\mathcal{C}/S}$. Then there is a quasi-isomorphism*

$$Rf_* \circ R\mathcal{H}om_{\mathcal{C}}(L_{\mathcal{C}/S}, -)(\mathcal{O}_{\mathcal{C}}) \simeq \mathcal{E}[-1]$$

where $\mathcal{E}$ is a locally free sheaf of rank $3g-3$ on S.

We delay the proof and instead show the following:

Corollary 12: *With notations from Lemma 11,*

$$\operatorname{Ext}^i(L_{\mathcal{C}/S}, \mathcal{O}_{\mathcal{C}}) = H^{i-1}(S, \mathcal{E}).$$

Proof. Let $\Gamma_{\mathcal{C}}$ (resp. Γ_S) denote the global section functor on $\mathcal{C}$ (resp. S). Thus:

$$R\Gamma_S \circ Rf_* \circ R\mathcal{H}om_{\mathcal{C}}(L_{\mathcal{C}/S}, -)(\mathcal{O}_{\mathcal{C}}) \simeq R\Gamma_S(\mathcal{E}[-1]).$$

Observe that

$$R\Gamma_S \circ Rf_* = R(\Gamma_S \circ f_*) = R\Gamma_{\mathcal{C}}$$

Indeed, this follows from [Sta18, Tag 015M]: it suffices to show that for any injective sheaf $\mathcal{I}$ on $\mathcal{C}$, $f_*\mathcal{I}$ is Γ_S-acyclic. This is true because injective sheaves are flasque, pushforward of flasque sheaves are flasque, and flasque sheaves are Γ_S-acyclic. For similar reasons,

$$R\Gamma_{\mathcal{C}} \circ R\mathcal{H}om_{\mathcal{C}}(L_{\mathcal{C}/S}, -) = R(\Gamma_{\mathcal{C}} \circ \mathcal{H}om_{\mathcal{C}}(L_{\mathcal{C}/S}, -)) = R\operatorname{Hom}_{\mathcal{C}}(L_{\mathcal{C}/S}, -).$$

(this time the justification is [God73, Lemme 7.3.2]) Summarizing:

$$R\operatorname{Hom}_{\mathcal{C}}(L_{\mathcal{C}/S}, -)(\mathcal{O}_{\mathcal{C}}) = R\Gamma_S(\mathcal{E}[-1]).$$

Taking second cohomology on both sides yields the result. □

[3] I do not understand why; it would suffice to show that the family is flat.

[4] Such an A is an *adic ring*, and such an I is an *ideal of definition*. Note that I may not be maximal, even if A is local.

[5] J is a priori a sheaf on Z_n, but as $J^2 = 0$ it is supported over Z_{n-1}; it is in fact a free line bundle.

We may now complete our lifting procedure. From Corollary 12, we find

$$\mathrm{Ext}^i_{\mathcal{C}_{n-1}}(L_{\mathcal{C}_{n-1}/Z_{n-1}}, f_{n-1}^*J) = H^{i-1}(Z_{n-1}, \mathcal{E}\otimes J)$$

which vanishes for $i \geq 2$ since Z_{n-1} is affine and $\mathcal{E}\otimes J$ is coherent: there is no obstruction to lifting $\mathcal{C}_{n-1} \to Z_{n-1}$ over Z_n. Let $\mathcal{C}'_n \to Z_n$ be such a lift, its restriction over V_n and the restriction $(\mathcal{C}_V)_{V_n}$ differ by an element of $H^0(V_{n-1}, \mathcal{E}\otimes J)$. We wish to show that this element lifts under the restriction map

$$H^0(Z_{n-1}, \mathcal{E}\otimes J) \to H^0(V_{n-1}, \mathcal{E}\otimes J),$$

we do this by showing that this map is surjective. Since the underlying topological space of Z_{n-1} (resp. V_{n-1}) equals that of Z_1 (resp. V_1), it suffices to show the surjectivity of

$$H^0(Z_1, \mathcal{E}\otimes J) \to H^0(V_1, \mathcal{E}\otimes J).$$

This is a consequence of the algebraic Hartogs' lemma, since Z_1 is a regular scheme of dimension ≥ 2, and V_1 is the complement of its closed point. Thus we can replace $\mathcal{C}'_n \to Z_n$ with some stable curve $\mathcal{C}_n \to Z_n$ such that its base change to V_n matches $(\mathcal{C}_V)_{V_n}$, as desired.

It remains to prove Lemma 11 above.

Proof of Lemma 11. It suffices to show that such a quasi-isomorphism holds for the universal family; the result follows by pullback. Since $\mathscr{M}$ is smooth over $\operatorname{Spec}\mathbb{Z}$, it admits a regular, hence Gorenstein, atlas, and so we may assume that S is Gorenstein. Since f is proper between noetherian schemes of finite dimension, there is a relative dualizing complex K_f, and coherent duality states that for all coherent sheaves $\mathcal{F}$ on $\mathcal{C}$:

$$R\mathcal{H}om(Rf_*\mathcal{F}, \mathcal{O}_S) \simeq Rf_* \circ R\mathcal{H}om(\mathcal{F}, K_f).$$

As the base is Gorenstein and the relative dimension is 1, the complex K_f consists in a single invertible sheaf $\omega_{\mathcal{C}/S}$ in degree -1: $K_f = \omega_{\mathcal{C}/S}[1]$. We let $K_f^\vee := R\mathcal{H}om(K_f, \mathcal{O}_\mathcal{C})$. Choosing $\mathcal{F} = R\mathcal{H}om(L_{\mathcal{C}/S}, \mathcal{O}_\mathcal{C})$, the left-hand side rearranges:

$$\begin{aligned} Rf_* \circ R\mathcal{H}om(\mathcal{F}, K_f) &\simeq Rf_* \circ R\mathcal{H}om(\mathcal{F}\otimes K_f^\vee, \mathcal{O}_\mathcal{C}) \\ &= Rf_* \circ R\mathcal{H}om(R\mathcal{H}om(L_{\mathcal{C}/S}, \mathcal{O}_\mathcal{C})\otimes K_f^\vee, \mathcal{O}_\mathcal{C}) \\ &\simeq Rf_* \circ R\mathcal{H}om(R\mathcal{H}om(L_{\mathcal{C}/S}, K_f^\vee), \mathcal{O}_\mathcal{C}) \end{aligned}$$

As $\mathcal{C}$ is also Gorenstein, the structure sheaf is a dualizing sheaf (see [Sta18, Tag 0BFQ]), and so the functor $D_\mathcal{C} : \mathcal{F} \mapsto R\mathcal{H}om(\mathcal{F}, \mathcal{O}_\mathcal{C})$ on coherent sheaves is equipped with a canonical equivalence $\mathrm{id} \to D_\mathcal{C}\circ D_\mathcal{C}$ (see [Sta18, Tag 0A89]). Consequently:

$$\begin{aligned} Rf_* \circ R\mathcal{H}om(R\mathcal{H}om(L_{\mathcal{C}/S}, K_f^\vee), \mathcal{O}_\mathcal{C}) &\simeq Rf_* \circ R\mathcal{H}om(R\mathcal{H}om(L_{\mathcal{C}/S}\otimes K_f, \mathcal{O}_\mathcal{C}), \mathcal{O}_\mathcal{C}) \\ &\simeq Rf_*(L_{\mathcal{C}/S}\otimes K_f). \end{aligned}$$

Summarizing:

$$R\mathcal{H}om(Rf_* \circ R\mathcal{H}om(L_{\mathcal{C}/S}, \mathcal{O}_\mathcal{C}), \mathcal{O}_S) \simeq Rf_*(L_{\mathcal{C}/S}\otimes \omega_{\mathcal{C}/S}[1]). \tag{1}$$

It follows from [DM69, paragraph before Lemma 1.3] that the natural map

$$L_{\mathcal{C}/S} \to \Omega^1_{\mathcal{C}/S}$$

is a quasi-isomorphism. We now claim that the complex $Rf_*(\Omega^1_{\mathcal{C}/S}\otimes\omega_{\mathcal{C}/S}[1])$ is quasi-isomorphic to a locally free sheaf $\mathcal{E}^*$ in degree -1. Indeed, its cohomology is concentrated in degrees -1 and 0. However on all the fibers C/k of $\mathcal{C}\to S$ the cohomology group $H^1(C, \Omega^1_{C/k}\otimes\omega_{C/k}) \simeq \mathrm{Hom}(\Omega^1_{C/k}, \mathcal{O}_C)$ vanishes, and since the sheaf $\Omega^1_{\mathcal{C}/S}\otimes\omega_{\mathcal{C}/S}$ is flat over S it follows from Fact 9 that $R^1f_*(\Omega^1_{\mathcal{C}/S}\otimes\omega_{\mathcal{C}/S}[1]) = 0$: the degree 0 cohomology vanishes. Therefore the complex is quasi-isomorphic to its cohomology in degree -1, i.e. $f_*(\Omega^1_{\mathcal{C}/S}\otimes\omega_{\mathcal{C}/S}[1])$. We see from Fact 9 again that this is a locally free sheaf of rank $3g-3$. Applying $R\mathcal{H}om(-, \mathcal{O}_\mathcal{C})$ on both sides of Equation (1), the right-hand side is a complex made of a single locally free sheaf $\mathcal{E}$ in degree 1, whereas the left-hand side recovers $Rf_*\circ R\mathcal{H}om(L_{\mathcal{C}/S}, \mathcal{O}_\mathcal{C})$, hence the result. □

Lemma 13: *Let k be a field, consider a stable curve C/k with sheaf of differentials $\Omega_{C/k}$ and dualizing sheaf $\omega_{C/k}$. Then*

$$H^1(C, \Omega^1_{C/k} \otimes \omega_{C/k}) \simeq H^0(C, (\Omega^1_{C/k})^\vee) = 0,$$
$$H^0(C, \Omega^1_{C/k} \otimes \omega_{C/k}) \simeq H^1(C, (\Omega^1_{C/k})^\vee) \simeq k^{3g-3}.$$

Proof. The two left-most isomorphisms follow from Serre duality. We show the right-most equalities. Let $\pi : \tilde{C} \to C$ denote the disingularization of the curve. Let $\mathcal{T}_{C/k}$ stand for the tangent sheaf of C, i.e. the dual sheaf $(\Omega^1_{C/k})^\vee$, and similarly for $\mathcal{T}_{\tilde{C}/k}$. Let D denote the effective divisor on $\mathcal{C}$ defined as the pre-image under π of the singular points of C. We claim that

$$\mathcal{T}_{C/k} \simeq \pi_* \mathcal{T}_{\tilde{C}/k}(-D).$$

Assuming this, we argue that the line bundle $\mathcal{T}_{\tilde{C}/k}(-D)$ has vanishing cohomology on each connected component of $\tilde{C}$, i.e. that it has negative degree. This is automatic on components of genus ≥ 2. By definition of stable curves, every component of genus 1 (resp. 0) has at least one point (resp. three points) in the support of D, which implies $\deg(\mathcal{T}_{\tilde{C}/k}(-D)) < 0$ as required. It follows that this line bundle has no global sections; Riemann-Roch gives the remaining isomorphism.

The claim is established as follows. There is a map $\pi^* \mathcal{T}_{C/k} \to \mathcal{T}_{\tilde{C}/k}(-D)$ by pullback of derivations; by adjunction we find $\mathcal{T}_{C/k} \to \pi_* \mathcal{T}_{\tilde{C}/k}(-D)$ which is trivially an isomorphism outside of the singular points of C. At singular points, we may take formal completion to assume $C = \operatorname{Spf} k[[x,y]]/(xy) = \operatorname{Spec} A$. One checks as an exercice that the formal tangent sheaf corresponds to the A-module

$$k[[x]] \cdot x\frac{\partial}{\partial x} \oplus k[[y]] \cdot y\frac{\partial}{\partial y}.$$

Writing down coordinates for the blowing up, we see that (the formal completion of) $\tilde{C}$ corresponds to the k-algebra $k[[u]] \times k[[v]]$ so that the blowup morphism corresponds to

$$\begin{aligned} A = k[[x,y]]/(xy) &\longrightarrow \tilde{A} := k[[u]] \times k[[v]] \\ x &\longmapsto u \\ y &\longmapsto v. \end{aligned}$$

The tangent sheaf of $\tilde{A}$ is given by the $\tilde{A}$-module $k[[u]] \cdot u\frac{\partial}{\partial u} \oplus k[[v]] \cdot u\frac{\partial}{\partial v}$. To find the formal completion of $\mathcal{T}_{\tilde{C}/k}(-D)$ at the pre-images of the singular point, we tensor it with the free module generated by u and v, yielding the $\tilde{A}$-module

$$k[[u]] \cdot u\frac{\partial}{\partial u} \oplus k[[v]] \cdot v\frac{\partial}{\partial v}$$

The morphism between tangent sheaves now corresponds to the A-modules isomorphism

$$\begin{aligned} k[[x]] \cdot x\frac{\partial}{\partial x} \oplus k[[y]] \cdot y\frac{\partial}{\partial y} &\longrightarrow k[[u]] \cdot u\frac{\partial}{\partial u} \oplus k[[v]] \cdot v\frac{\partial}{\partial v} \\ x\frac{\partial}{\partial x} &\longmapsto u\frac{\partial}{\partial u} \\ y\frac{\partial}{\partial y} &\longmapsto v\frac{\partial}{\partial v}. \end{aligned}$$

□

Step 3. We now have two stable curves: the original $\mathcal{C}_V \to V$, and the algebraized $\mathcal{C} \to S$. It suffices to show that the latter extends the former, i.e. that both are isomorphic over V. Define

$$I = \underline{\operatorname{Isom}}_V(\mathcal{C}|_V, \mathcal{C}_V) \to V$$

which is finite unramified over V, since it is obtained by base-change of the diagonal of a Deligne-Mumford stack. By construction there is a section over $\widehat{V} := V \times_S \widehat{Z}$ denoted

$$\widehat{f} : \widehat{V} \to \widehat{I}$$

(where $\widehat{I}$ is the formal completion of I along the inverse image of V_1). This section can be algebraized up to finite étale cover[6]: there is a finite étale cover $V' \to V$ and a V-morphism $V' \to I$. But since $\mathrm{codim}_S(V) \geq 2$, it follows from Zariski-Nagata purity theorem (see [Gro68, X, Théorème 3.4]) that V' comes from a finite étale cover of S, all of which are trivial, so $V' \to V$ is a trivial cover. Thus we find a section $V \to I$, so $\mathcal{C} \to S$ is indeed an extension of $\mathcal{C}_V \to V$. The proof is now complete.

3. Proofs of intermediate results

3.1. Extension of curves over a maximal base.

We recall the statement.

Lemma 14: *Let S ba a locally Noetherian scheme, U a dense open subscheme, and $f : \mathcal{C}_U \to U$ a stable curve over U.*

(i) *Suppose V is an open subscheme of S containing U, such that for any $s \in V \setminus U$, the local ring $\mathcal{O}_{S,s}$ either has depth ≥ 2, or has depth 1 and is geometrically unibranch. Then there is at most one extension of $\mathcal{C}_U$ to a stable curve over V.*

(ii) *Let $s \in S$, consider $\operatorname{Spec} \mathcal{O}_{S,s} \to S$. If $\mathcal{C}_U$ extends to a stable curve over $\operatorname{Spec} \mathcal{O}_{S,s}$, then it extends to a stable curve over an open subscheme V of S containing s and U.*

We first prove (ii) which is easier. If $\mathcal{C}_U$ extends over $\operatorname{Spec} \mathcal{O}_{S,s}$, there is an open neighborhood W of s and a stable curve $\mathcal{C}_W$ extending the curve over the generic point of $\mathcal{O}_{S,s}$. Over this generic point, $\mathcal{C}_U$ and $\mathcal{C}_W$ coincide, so we can pick a smaller W and assume that $(\mathcal{C}_U)_{U\cap W} \simeq (\mathcal{C}_W)_{U\cap W}$. Then we can glue both curves into a stable curve over $V := U \cup W$.

We now prove (i). Consider $\mathcal{C}_1, \mathcal{C}_2$ two extensions of $\mathcal{C}_U$ over V as in the statement. Since the moduli space of stable curves is Deligne-Mumford separated, its diagonal is finite unramified, and so $I := \underline{\mathrm{Isom}}(\mathcal{C}_1, \mathcal{C}_2)$ is finite unramified over V. By assumption, $I \to V$ admits a section over U, we let V' denote the scheme theoretic closure in I of the section. We argue that the induced morphism

$$V' \to V$$

is an isomorphism, whose inverse then provides a section of $I \to V$ and concludes the proof. It suffices to show the isomorphism holds above every local ring. For each $s \in V$ we let $O(s) := \mathcal{O}_{S,s}$ and $O(s)' := O(s) \times_V V'$, we show that $O(s)' \to O(s)$ is an isomorphism by induction on $\dim O(s)$. This already holds true whenever $s \in U$.

If $\dim O(s) = 0$, then s has depth 0 and so $s \in U$. Assume now that the isomorphism holds up to dimension $d-1$ and suppose $\dim O(s) = d$. By assumption $p : O(s)' \to O(s)$ restricts to an isomorphism at all local rings outside of the closed point, so p is an isomorphism outside of the closed point. Thus the bottom horizontal arrow in the following diagram is an isomorphism.

$$\begin{array}{ccc} \Gamma(O(s), \mathcal{O}_{O(s)}) & \longrightarrow & \Gamma(O(s)', \mathcal{O}_{O(s)'}) \\ \downarrow & & \downarrow \\ \Gamma(O(s) \setminus \{s\}, \mathcal{O}_{O(s)}) & \xrightarrow{\sim} & \Gamma(O(s)' \setminus p^{-1}(\{s\}), \mathcal{O}_{O(s)'}) \end{array}$$

We claim that if s has depth ≥ 2 then the left vertical arrow is an isomorphism: this follows from the exact sequence

$$H^0_s(O(s), \mathcal{O}_{O(s)}) \longrightarrow H^0(O(s), \mathcal{O}_{O(s)}) \longrightarrow H^0(O(s) \setminus \{s\}, \mathcal{O}_{O(s)}) \longrightarrow H^1_s(O(s), \mathcal{O}_{O(s)})$$

where H^*_s denotes local cohomology, which vanishes in degree k if s has depth $> k$. It follows that the top horizontal arrow is injective, and the right vertical arrow is surjective. But the latter is also injective, because $O(s)' \setminus p^{-1}(\{s\})$ is schematically dense in $O(s)'$, by definition of the Zariski closure. Now all arrows but the one on top are isomorphisms, hence the one on top is an isomorphism.

It remains to show that p is an isomorphism whenever $O(s)$ has depth 1 and is geometrically unibranch. Let $O(s)^{sh}$ denote the strict henselization of $O(s)$. As the morphism

$$p^{sh} : O(s)' \times_{O(s)} O(s)^{sh} \longrightarrow O(s)^{sh}$$

[6] I do not understand why.

is finite unramified with strictly henselian base, it follows from [Sta18, Tag 04GL] that we can decompose

$$O(s)' \times_{O(s)} O(s)^{sh} = \coprod_i T_i$$

where each $T_i \to O(s)^{sh}$ is a closed immersion. Moreover, by [DG67, IV, Proposition 18.8.15] (see also [Sta18, Tag 0BQ4]) the ring $O(s)^{sh}$ is irreducible. We also know that p^{sh} is an isomorphism outside of the closed point ξ of O^{sh}, thus there is exactly one connected component T among the T_i's such that

$$T \to O(s)^{sh}$$

is an isomorphism outside of ξ. Now we argue as above (denoting s^{sh} the closed point of $O(s)^{sh}$):

$$\begin{array}{ccc} \Gamma(O(s)^{sh}, \mathcal{O}_{O(s)^{sh}}) & \twoheadrightarrow & \Gamma(T, \mathcal{O}_T) \\ \downarrow & & \downarrow \\ \Gamma(O(s)^{sh} \setminus \{s^{sh}\}, \mathcal{O}_{O(s)^{sh}}) & \xrightarrow{\sim} & \Gamma(T \setminus \{s^{sh}\}, \mathcal{O}_T) \end{array}$$

Depth 1 implies that the left vertical arrow is injective (via an exact sequence as before), so the top horizontal arrow is injective too. This shows that $T \to O(s)^{sh}$ is an isomorphism.

Let s' denote the closed point of $O(s)'$ in the image of $T \to O(s)'$, let T_0 denote the spectrum of the local ring $\mathcal{O}_{O(s)',s'}$. We claim that $T_0 \to O(s)$ is an isomorphism. If this is true, then p admits a section which is an open and closed immersion as p is separated unramified. In particular, T_0 is the scheme theoretic closure in $O(s)'$ of $p^{-1}(O(s) \setminus \{s\})$; but this closure is all of $O(s)'$ by construction, so $O(s)' = T_0 \simeq O(s)$ as desired.

We show the claim as follows. We have that $O(s)^{sh} \to O(s)$ is faithfully flat, hence so is $O(s)' \times_{O(s)} O(s)^{sh} \to O(s)'$, and $T \to T_0$ as well. The composition $T \to T_0 \to O(s)$ recovers $O(s)^{sh} \to O(s)$ and so is faithfully flat: it follows that $T_0 \to O(s)$ is faithfully flat by commutative algebra. Since this map is already unramified, it is surjective étale. The claim is a consequence of Lemma Lemma 15 below.

Lemma 15: *Let T_0, S be spectra of local rings, and $f : T_0 \to S$ a surjective étale morphism. Assume f is an isomorphism outside of the closed point of S. Then f is an isomorphism.*

Proof. It suffices to show that f is an isomorphism étale locally, so it suffices to show that the base-change

$$f' : T_0 \times_S T_0 \to T_0$$

is an isomorphism. In this setting, the diagonal defines a section $\Delta : T_0 \to T_0 \times_S T_0$ which is an open and closed embedding. Let U, V denote the complements of the closed point in S and T respectively, by assumption the restriction

$$\Delta_V : V \to V \times_U V$$

is an isomorphism. Now $V \times_U V$ is dense in $T_0 \times_S T_0$, so Δ is a surjective open embedding, i.e. an isomorphism. ☐

3.2. Extending curves after blowing up.

We recall the statement.

Theorem 16: *Let S be a quasi-compact, excellent, reduced, normal scheme of dimension 2, with function field $K(S)$. Let U be a dense open subscheme, and $f : \mathcal{C}_U \to U$ a stable curve over U. Assume the following holds:*

For all valuation ring R in $K(S)$ and $\phi : \operatorname{Spec} R \to S$ with $\phi(\eta) \in U$ (here $\eta \in \operatorname{Spec} R$ denotes the generic point), the pullback curve $\mathcal{C}_\eta$ under ϕ extends to a stable curve over $\operatorname{Spec} R$.

Then there exists a normalized blowing up $\tilde{S} \to S$ with center in $S \setminus U$ such that $\mathcal{C}_U$ extends to a stable curve over $\tilde{S}$.

Let us prove Theorem 16. Let s be a codimension 1 point of S. Since S is normal, it is regular in codimension 1, thus $R = \mathcal{O}_{S,s}$ is a DVR in $K(S)$. According to Corollary 3 along with our assumption, $\mathcal{C}_U$ extends to a stable curve over an open subscheme V such that $S \setminus V$ has codimension ≥ 2, i.e. consists of closed points. It suffices to show that the result holds when S is the spectrum of the local ring at such a closed point, with U the complement of this point.

We use the following result, which we state under general assumptions.

Lemma 17 ([Del85], Lemme 1.6)**:** *Let $\mathscr{M}$ be a separated Deligne-Mumford stack proper over* $\operatorname{Spec}\mathbb{Z}$*, assume there exists a proper scheme S_0 together with a surjective morphism $S_0 \to \mathscr{M}$.*

Let U be a dense quasi-compact open in an algebraic space (resp. a scheme) S, and $u : U \to \mathscr{M}$ a morphism. Then there exists an algebraic space (resp. a scheme) S' together with a proper surjective morphism $f : S' \to S$ such that $U \times_S S' \to \mathscr{M}$ extends to $S' \to \mathscr{M}$.

Proof. Consider the two morphisms $f, g : U \times S_0 \rightrightarrows \mathscr{M}$, let $I := \underline{\operatorname{Isom}}(f, g) = U \times_{\mathscr{M}} S_0$, which fits inside the 2-cartesian diagram

$$\begin{array}{ccc} I & \xrightarrow{\theta} & U \times S_0 \\ \downarrow & & \downarrow{\scriptstyle f\times g} \\ \mathscr{M} & \xrightarrow{\Delta} & \mathscr{M} \times_B \mathscr{M}. \end{array}$$

It follows that θ is finite. I is an algebraic space; if U is a scheme then it is a scheme according to [LM00, Théorème A.2]. Let Γ denote the image of θ, and $\overline{\Gamma}$ its closure inside $S \times S_0$. Then $h : I \to \Gamma$ extends into a finite morphism $\overline{I} \to \overline{\Gamma}$; indeed h is affine, hence corresponds to a finite $\mathcal{O}_\Gamma$-algebra $\mathcal{A}$, so we define $\overline{I}$ to correspond to the finite $\mathcal{O}_{\overline{\Gamma}}$-algebra $j_*\mathcal{A}$, where $j : \Gamma \hookrightarrow \overline{\Gamma}$. ☐

Thus we can find a proper surjective $f : T \to S$ such that the pullback $\mathcal{C}_U \times_S T$ over $f^{-1}(U)$ extends to a stable curve $\mathcal{C}_T$ over T. Let t be a closed point in the generic fiber of f, let T_0 denote the scheme theoretic closure of $\{t\}$ in T. Then $f_{|T_0}$ is generically finite, proper, and dominant hence surjective. Replacing T with T_0 we may assume that f is also generically finite and that T is reduced. Replacing T with its normalization, which is finite over T since R is excellent, we may also assume that T is normal.

We need the following theorem

Fact 18 ([RG71], Théorème 5.2.2)**:** *Let*

- *S be a quasi-compact, quasi-separated scheme,*
- *$V \subseteq S$ an quasi-compact open subscheme,*
- *$f : X \to S$ an S-scheme of finite presentation,*
- *$\mathcal{F}$ a quasi-coherent $\mathcal{O}_X$-module of finite type such that $\mathcal{F}_V$ is flat over V.*

Then there exists a blowup $b : \tilde{S} \to S$ with center in $S \setminus V$ such that the strict transform $\tilde{\mathcal{F}}$ of $\mathcal{F}$ is flat over $\tilde{S}$.

We apply Fact 18 to $\mathcal{F} = \mathcal{O}_T$ to find a blowup $\tilde{S} \to S$ such that the strict transform T' of T is flat over $\tilde{S}$. Since $T' \to \tilde{S}$ is flat and generically finite, it is quasi-finite; it is also proper, so $T' \to \tilde{S}$ is finite. Applying the following result to $\tilde{S}$, we may also assume that $\tilde{S}$ is regular.

Fact 19 ([Lip78], Introduction, Theorem and remark C)**:** *Let S be an excellent normal scheme of dimension 2. Then there exists a blowup $S' \to S$ with zero-dimensional center such that S' is regular.*

Let $\tilde{T} \to T'$ be the normalization, which is finite as R is excellent. Then $\pi : \tilde{T} \to \tilde{S}$ is finite, so $\dim \tilde{T} = 2$; in this case the normality of $\tilde{T}$ implies that it is Cohen-Macaulay. By miracle flatness, π is flat. We let $\mathcal{C}_{\tilde{T}}$ denote the pullback stable curve over $\tilde{T}$.

Invoking Corollary 3 again, $\mathcal{C}_U \to U$ extends to $\mathcal{C}_{V'} \to V'$ where V' is an open subscheme of $\tilde{S}$ containing the generic point of the closed fiber of $\tilde{S} \to S$. We claim that the stable curves $\pi^*(\mathcal{C}_{V'})$ and $\mathcal{C}_{\tilde{T}}$ over $\pi^{-1}(V)$ are isomorphic. Indeed, they are isomorphic over $\pi^{-1}(U)$, and $\pi^{-1}(V')$ is normal, so we deduce the claim from the following statement.

Proposition 20: *Let $\mathscr{M}$ be a separated Deligne-Mumford stack over some base scheme B. Let Y be a normal reduced locally Noetherian B-scheme and U a dense open subscheme of Y. Let $f, g : Y \to \mathscr{M}$ be two morphisms such that $f_{|U} \simeq g_{|U}$. Then $f \simeq g$.*

Proof. Let $I = \underline{\operatorname{Isom}}(f, g)$ the algebraic stack obtained by the pullback of the diagonal Δ

$$\begin{array}{ccc} I & \xrightarrow{\theta} & Y \\ \downarrow & & \downarrow{\scriptstyle f\times g} \\ \mathscr{M} & \xrightarrow{\Delta} & \mathscr{M} \times_B \mathscr{M}. \end{array}$$

By assumption, Δ, and hence also θ, is finite, so by [LM00, Théorème A.2] I is a scheme. The assumption $f_{|U} \simeq g_{|U}$ means that θ has a section over U $s : U \to I_{|U}$ which is an open immersion by [Sta18, Tag 024T]. Let Z denote the scheme-theoretic closure of $s(U)$ in I. We want to show that $\theta_{|Z} : Z \to Y$ is an isomorphism. Working componentwise, we can assume that Y, and so U and Z, are irreducible. Since $\theta(Z)$ contains U, $\theta_{|Z}$ is dominant. By [Sta18, Tag 0356] Z is reduced. Thus $\theta_{|Z} : Z \to Y$ is a finite birational morphism between integral schemes. By [Sta18, Tag 0AB1] it is an isomorphism. □

Let $\tilde{T}(2) := \tilde{T} \times_{\tilde{S}} \tilde{T}$, and $\mathcal{C}_1, \mathcal{C}_2$ the pullbacks of $\mathcal{C}_{\tilde{T}}$ along both projections. Since $(\mathcal{C}_{\tilde{T}})_{|V'} \simeq \pi^*(\mathcal{C}_{V'})$, $\mathcal{C}_1$ and $\mathcal{C}_2$ are isomorphic over $\pi^{-1}(V') \times_{V'} \pi^{-1}(V')$. We claim that this open subscheme of $\tilde{T}(2)$ contains all depth ≤ 1 points. Assuming this, it follows from Lemma 14 that there is an isomorphism

$$\epsilon : \mathcal{C}_1 \longrightarrow \mathcal{C}_2.$$

Then, ϵ satisfies the cocycle relations over $\tilde{T} \times_{\tilde{S}} \tilde{T} \times_{\tilde{S}} \tilde{T}$, because it does so over the dense open subscheme $\pi^{-1}(V') \times_{V'} \pi^{-1}(V') \times_{V'} \pi^{-1}(V')$. Since descent of stable curves is effective, we find that $\mathcal{C}_{\tilde{T}}$ descends to a stable curve over $\tilde{S}$.

The claim above is proved as follows. As $g : \tilde{T}(2) \to \tilde{T}$ is flat, with finite fibers, and $\tilde{T}$ is Cohen-Macaulay, we deduce from [DG67, IV, Corollaire 6.3.3] that $\tilde{T}(2)$ is Cohen-Macaulay too. Any $x \in \tilde{T}(2)$ with ≤ 1 depth is a codimension ≤ 1 point, which is mapped to a codimension ≤ 1 point, all of which lie in V'. This concludes the proof of Theorem 16.

3.3. **Triviality of stable curves of locally constant topological type over $\mathbb{P}^1$.** We recall the statement.

Lemma 21: *Let k be a field, let $(\mathcal{C}, \Sigma) \to \mathbb{P}^1_k$ be a stable r-pointed curve of genus g. If the topological type of $(\mathcal{C}, \Sigma)$ is locally constant over $\mathbb{P}^1_k \setminus \{0, \infty\}$, then $(\mathcal{C}, \Sigma)$ is constant, i.e. there exists a stable pair (C_0, Σ_0) over k such that*

$$(C, \Sigma) \simeq (C_0 \times_k \mathbb{P}^1_k, \Sigma_0 \times_k \mathbb{P}^1_k).$$

We denote $\Sigma = (\sigma_1, \ldots, \sigma_r)$ the ordered sections of the curve $\mathcal{C}$. We begin by proving the result in a simpler setting.

Lemma 22: *Let k is an algebraically closed field, let $(\mathcal{C}, \Sigma) \to \mathbb{P}^1_k$ be a stable r-pointed curve of genus g, with locally constant topological type over $\mathbb{P}^1_k \setminus \{0, \infty\}$. Assume that $\mathcal{C}_\eta$ is smooth, η being the generic point of $\mathbb{P}^1$. Then (C, Σ) is constant.*

Proof. We let $\mathscr{M}_{g,r}$ denote the moduli space of stable r-pointed curves of genus g; it is separated, Deligne-Mumford stack. Now $(\mathcal{C}, \Sigma)$ corresponds to a morphism $\mathbb{P}^1 \to \mathscr{M}_{g,r}$ which we argue is constant.

By assumption the restriction of $\mathcal{C}$ over $\mathbb{P}^1_k \setminus \{0, \infty\}$ is smooth. We consider cases $g = 0$, $g = 1$ and $g = 2$ separately.

For $g = 0$, Σ consists of at least 3 sections, otherwise (C, Σ) would not be stable. There is only one smooth, 3-pointed surve of genus 0, hence

$$(\mathcal{C}, (\sigma_1, \sigma_2, \sigma_3))_{|\mathbb{P}^1_k \setminus \{0,\infty\}} \simeq (\mathbb{P}^1_k, (0, 1, \infty)) \times (\mathbb{P}^1_k \setminus \{0, \infty\}).$$

Consider a section σ_i for $i > 3$, the isomorphism above induces a map

$$\sigma_i : \mathbb{P}^1_k \setminus \{0, \infty\} \to \mathbb{P}^1_k \setminus \{0, 1, \infty\}.$$

Such a map is necessarily constant: otherwise we could complete it to a nonconstant $\mathbb{P}^1 \to \mathbb{P}^1$ which is an isomorphism, and restricting back to $\mathbb{P}^1_k \setminus \{0, \infty\}$ cannot map to $\mathbb{P}^1_k \setminus \{0, 1, \infty\}$. Letting p_i denote the image of σ_i, and $\Sigma_0 := (p_0, \ldots, p_r)$, we have shown:

$$(\mathcal{C}, \Sigma)_{|\mathbb{P}^1_k \setminus \{0,\infty\}} \simeq (\mathbb{P}^1_k, \Sigma_0) \times (\mathbb{P}^1_k \setminus \{0, \infty\}).$$

Now $\mathbb{P}^1 \to \mathscr{M}_{0,r}$ restricts to a constant morphism over the dense open $\mathbb{P}^1_k \setminus \{0, \infty\}$, thus it is constant by Proposition 20.

For $g = 1$, Σ consists of at least one section σ_1, and so we view $(\mathcal{C}, \sigma_1)_{|\mathbb{P}^1_k \setminus \{0,\infty\}}$ as a family of elliptic curves. The fiber $\mathcal{C}_0$ is a stable curve of genus 1, so $\mathcal{C}$ has semistable reduction at 0. It follows that the local monodromy is unipotent, i.e. the inertia subgroup of $\pi_1(\mathbb{P}^1_k \setminus \{0,\infty\},1)$ acts unipotently on the ℓ-adic Tate module $T_\ell(\mathcal{C}_1)$, $\ell \neq p := \mathrm{char}(k)$. Fixing a basis for $T_\ell(\mathcal{C}_1)$, we find a morphism

$$\rho : \pi_1(\mathbb{P}^1_k \setminus \{0,\infty\},1) \to \mathrm{GL}_2(\mathbb{Z}_\ell).$$

Unipotent subgroups of $\mathrm{GL}_2(\mathbb{Z}_\ell)$ are pro-ℓ-groups, thus the wild inertia subgroup, which is pro-p, has trivial image: ρ factors through the tame fundamental group $\pi_1^t(\mathbb{P}^1_k \setminus \{0,\infty\},1)$. We know from [Gro71, XIII, Corollaire 2.12] that this group is abelian. (in fact, $\pi_1^t(\mathbb{P}^1_k \setminus \{0,\infty\},1) \simeq \widehat{\mathbb{Z}}/\mathbb{Z}_p$) Thus there exists some $v \in T_\ell(\mathcal{C}_1)$ fixed under monodromy. For any $n > 0$, the reduction modulo ℓ^n of v defines some ℓ^n-torsion element $v_n \in \mathcal{C}_1[\ell^n]$ which is therefore fixed by the monodromy action: this defines a section of the étale group scheme

$$\mathcal{C}[\ell^n] \to \mathbb{P}^1_k \setminus \{0,\infty\}$$

and thus a map $\mathbb{P}^1_k \setminus \{0,\infty\} \to X_1(\ell^n)$, where $X_1(\ell^n)$ denotes the appropriate modular curve. This map extends to $\mathbb{P}^1_k \to X_1(\ell^n)$, but the genus of $X_1(\ell^n)$ tends to infinity as n increases ([LMF26, Modular curve family $X_1(N)$]). Thus for large enough n this map is constant, and so is $(\mathcal{C},\sigma_1)_{|\mathbb{P}^1_k\setminus\{0,\infty\}}$. A similar argument as in $g = 0$ show that sections $\sigma_i : \mathbb{P}^1_k \setminus \{0,\infty\} \to \mathcal{C}$ are constant as well, so $(\mathcal{C},\Sigma)_{|\mathbb{P}^1_k\setminus\{0,\infty\}}$ is constant. Conclusion as above invoking Proposition 20.

We are left with the $g \geq 2$ case. Let $p : \mathcal{C} \to \mathcal{C}'$ be the contraction of $\mathcal{C}$ to a minimal model above $\mathbb{P}^1_k$[7]. We claim that this map only contracts curves within fibers above 0 and ∞. Indeed, p is proper, and so induces a morphism $p_* : \mathrm{CH}_1(\mathcal{C}) \to \mathrm{CH}_1(\mathcal{C}')$ between Chow groups; if $Z \subseteq \mathcal{C}$ is a curve, it is contracted if and only if $p_*[Z] = 0$, but this value only depends on the rational equivalence class of Z. Now all fibers of $\mathcal{C}$ above $\mathbb{P}^1_k \setminus \{0,\infty\}$ are rationally equivalent, contracting one would contract all of them and so $\mathcal{C}'$ would not be a surface.

According to [Szp81, Théorème 4 b)], $\mathcal{C}' \to \mathbb{P}^1_k$ is isotrivial in the sense of [Szp81, §0]. Szpiro writes that a semistable isotrivial family is smooth, and there exists some finite étale cover $\pi : B \to \mathbb{P}^1_k$ such that $\mathcal{C}'_B \to B$ is constant. Here B is a disjoint union of copies of $\mathbb{P}^1_k$, so $\mathcal{C}' \to \mathbb{P}^1_k$ is constant. In particular, so is $\mathcal{C}_{|\mathbb{P}^1_k\setminus\{0,\infty\}} \to P^1_k \setminus \{0,\infty\}$: for some smooth, proper, genus 2 curve C_0,

$$\mathcal{C}_{|\mathbb{P}^1_k\setminus\{0,\infty\}} \simeq C_0 \times_k (\mathbb{P}^1_k \setminus \{0,\infty\}).$$

Arguing as before, the sections σ_i define morphisms $\mathbb{P}^1_k \setminus \{0,\infty\} \to C_0$ which are necessarily constant. We now invoke Proposition 20 to argue that $(\mathcal{C},\Sigma)$ is constant over $\mathbb{P}^1_k$ as a whole. □

We prove Lemma 21 by reducing to Lemma 22. First we reduce to the case of k algebraically closed. Let (C_0,Σ_0) be the fiber above a k-rational point x of $\mathbb{P}^1_k$; if $(\mathcal{C},\Sigma) \times_k \overline{k}$, then the $\mathbb{P}^1_k$-scheme

$$I = \underline{\mathrm{Isom}}_{\mathbb{P}^1_k}((C_0 \times_k \mathbb{P}^1_k, \Sigma_0 \times_k \mathbb{P}^1_k),(\mathcal{C},\Sigma))$$

is such that $I_{\overline{k}}$ is isomorphic to a disjoint union of copies of $\mathbb{P}^1_k$. Moreover, it admits a k-rational point lying over x, hence some connected component is isomorphic to $\mathbb{P}^1_k$. Thus $I \to \mathbb{P}^1_k$ admits a section, and so $(\mathcal{C},\Sigma)$ is isomorphic to $(C_0 \times_k \mathbb{P}^1_k, \Sigma_0 \times_k \mathbb{P}^1_k)$.

From now on we assume k algebraically closed. We need the following result.

Proposition 23: *Let $X \to T$ be a proper morphism of schemes, with all fibers equidimensional of dimension d. Assume T is locally Noetherian, irreducible with generic point η. Suppose the locus where π is smooth is dense in any fiber of $X \to T$. There exists a finite étale morphism $W \to T$ such that*

$$W_{\overline{\eta}} \simeq \{\textit{irreducible components of } X_{\overline{\eta}}\}$$

as $\mathrm{Gal}(\overline{\eta}/\eta)$*-sets.*

Before proving Proposition 23, let us finish the proof of Lemma 21. Applying Proposition 23 to $\mathcal{C} \to \mathbb{P}^1_k$ yields a finite étale cover $W \to \mathbb{P}^1_k$ as in the statement. But $\mathbb{P}^1_k$ is simply connected, so W is trivial, and hence the Galois action on the irreducible components of $\mathcal{C}_{\overline{\eta}}$, $\eta \in \mathbb{P}^1_k$ the generic point,

[7] I do not understand why the minimal model is needed. It seems to me that [Szp81, Théorème 4 b)] can be applied to the original family $\mathcal{C}$ directly.

is trivial. Consequently, all irreducible components of $\mathcal{C}_\eta$ are geometrically irreducible. Taking the closure in $\mathcal{C}$ of each of these components, we find

$$\mathcal{C} = \mathcal{C}_1 \cup \dots \mathcal{C}_m$$

where each $\mathcal{C}_{i,\eta}$ is geometrically irreducible. Let $\mathcal{S}$ denote the closure in $\mathcal{C}$ of the singular locus of $\mathcal{C}_\eta$, that is, the points of intersection of the $\mathcal{C}_{i,\eta}$. It follows from Fact 24 below that $\mathcal{S} \to \mathbb{P}^1$ if finite unramified, hence étale as the base scheme is a nonsingular curve.

Fact 24 ([MB85], Lemme 3)**:** *Let $f : Y \to T$ be a stable curve over a scheme T, denote* $\operatorname{Sing}(g)$ *the relative singular locus of f defined by the appropriate Fitting ideal of $\Omega_{Y/T}$. Then* $\operatorname{Sing}(f)$ *is finite unramified over T, and is locally defined by two equations in Y. In particular,* $\operatorname{Codim}_Y(\operatorname{Sing}(f)) \geq 2$.

As $\mathbb{P}^1_k$ is simply connected, $\mathcal{S}$ is a disjoint union of sections of $\mathcal{C} \to \mathbb{P}^1_k$, and so the normalization $\mathcal{D}$ of $\mathcal{C}$ is a disjoint union

$$\mathcal{D} = \mathcal{D}_1 \sqcup \dots \sqcup \mathcal{D}_m.$$

The inverse image of $\mathcal{S}$ in $\mathcal{D}$, mapping two to one étale to $\mathcal{S}$, is a union of sections τ_{ji} with $j = 1, \dots, m$ and $i = 1, \dots n_j$. Each section of Σ mapping to $\mathcal{C}_j$ lifts uniquely to a section of $\mathcal{D}_j$ denoted σ_{ji}, $i = 1, \dots, t_j$. By assumption of stability, each of the pairs

$$(\mathcal{D}_j, (\sigma_{j1}, \dots, \sigma_{j,t_j}, \tau_{j1}, \dots, \tau_{jn_j}))$$

is a stable pointed curve over $\mathbb{P}^1_k$ of locally constant topological type over $\mathbb{P}^1_k \setminus \{0, \infty\}$. By Lemma 22 they are constant. As $\mathcal{C}$ is obtained by gluing the $\mathcal{D}_j$ along the images of the τ_{ij}, it is also constant. This concludes the proof of Lemma 21.

To prove Proposition 23, we begin with $W_\eta \to \eta$ the finite étale cover prescribed by the $\operatorname{Gal}(\overline{\eta}/\eta)$-sets in the statement, and argue that it extends to a finite étale cover $W \to T$. To do so, we proceed in several steps:

- Step 1: for all t, let $T_{t^{sh}}$ denote $\operatorname{Spec} \mathcal{O}^{sh}_{T,t}$, let η^{sh} be a generic point of $T_{t^{sh}}$ and $W_{\eta^{sh}} \to \eta^{sh}$ the pullback of $W_\eta \to \eta$. We show that $W_{\eta^{sh}} \to \eta^{sh}$ is trivial, hence extends to a finite étale (trivial) cover $W_{t^{sh}} \to T_{t^{sh}}$.
- Step 2: we show that $W_{t^{sh}} \to T_{t^{sh}}$ descends to a finite étale cover $W_t \to T_t := \operatorname{Spec} \mathcal{O}_{T,t}$.
- Step 3: We show that all $W_t \to T_t := \operatorname{Spec} \mathcal{O}_{T,t}$ glue together into one finite étale cover $W \to T$.

Proof of Step 1. Fix t, η^{sh} as in the statement. Now $W_{\eta^{sh}} \to \eta^{sh}$ corresponds to the $\operatorname{Gal}(\overline{\eta^{sh}}/\eta^{sh})$-set of irreducible components of $X_{\overline{\eta^{sh}}}$; we argue that the action is trivial. So we may assume that T is strictly Henselian local, and show that the irreducible components of $X_{\overline{\eta}}$ are all defined over η. Let $Z_\eta \subseteq X_\eta$ correspond to a Galois orbit of irreducible components of $X_{\overline{\eta}}$, we argue that Z_η is geometrically irreducible.

Let $Z \subseteq X$ be the closure of Z_η. Since $Z \to T$ is proper and dominant, we get that Z_t is nonempty. The dimension of this fiber is at least the dimension of Z_η ([DG67, IV, 13.1.1]), so

$$\dim Z_t \geq \dim Z_\eta = \dim X_\eta = \dim X_t \geq \dim Z_t.$$

It follows that Z_t contains an irreducible component of X_t. Thus π is generically smooth on Z_t, in particular it is smooth at some closed point of Z_t. As T is strictly Henselian local, any smooth map admits a section, hence a morphism $T \to T$. The image of η defines a rational point of Z_η. The conclusion follows from Lemma 25. □

Lemma 25: *Let K be a field Z be an irreducible K-scheme, x a rational point of Z. Assume the local ring at x is normal. Then Z is geometrically irreducible.*

Proof. Let $K(Z)$ be the function field of Z. We need to show that $K(Z)/K$ is separably closed. Let $f \in K(Z)$ be separable over K, it is a root of an irreducible monic polynomial P with coefficients in K. For any height 1 prime $\mathfrak{p}$ of $\mathcal{O}_{Z,x}$, the valuation $v_{\mathfrak{p}}(f) = 0$: this follows from $P(f) = 0$. By the algebraic Hartogs' Lemma, $f \in \mathcal{O}_{Z,x} \setminus \mathfrak{m}_x$. Reducing $P(f) = 0$ modulo $\mathfrak{m}_x$ we find that $f \mod \mathfrak{m}_x$ is a root of P in K. But P is irreducible, so it has degree 1, and so $f \in K$, as desired. □

Remark 26: Lemma 25 is false without normality assumption (consider $\mathbb{R}[x, y]/(x^2 + y^2)$).

We extend the trivial cover $W_{\eta^{sh}} \to \eta^{sh}$ to a trivial cover $W_{t^{sh}} \to T_{t^{sh}}$. Step 2 follows from

Lemma 27: *Let T be an irredicible local scheme with generic point η, $S \to T$ an irreducible faithfully flat local extension with generic point $\tilde{\eta}$. Let $W_S \to S$ be a finite étale trivial cover, and assume its restriction $W_{\tilde{\eta}} \to \tilde{\eta}$ descends to a finite étale cover $W_\eta \to \eta$. Then $W_S \to S$ descends to a finite étale cover $W_T \to T$.*

Proof. Once we have shown that $W_S \to S$ descends to some $W_T \to T$, the latter is finite étale since it is so fpqc locally.

We define $W_T \to T$ by associating to $W_S \to S$ a descent datum (see [Sta18, Tag 023V]), which is effective for affine morphisms (see [Sta18, Tag 0245]). We begin with the descent datum associated to $W_{\tilde{\eta}} \simeq W_\eta \times_\eta \tilde{\eta}$ which is an isomorphism over $\tilde{\eta} \times_\eta \tilde{\eta}$

$$\varphi : W_{\tilde{\eta}} \times_\eta \tilde{\eta} \xrightarrow{\sim} \tilde{\eta} \times_\eta W_{\tilde{\eta}}$$

satisfying appropriate cocycle relations. Since $W_{\tilde{\eta}} \to \tilde{\eta}$ is trivial, say of degree r, it amounts to an isomorphism over $\tilde{\eta} \times_\eta \tilde{\eta}$

$$\coprod_{1 \le i \le r} \tilde{\eta} \times_\eta \tilde{\eta} \xrightarrow{\simeq} \coprod_{1 \le i \le r} \tilde{\eta} \times_\eta \tilde{\eta}.$$

In other words, φ is equivalent to a permutation σ of $\{1, \ldots, r\}$. Since $W_T \to T$ is also trivial of degree r, any descent datum is also some permutation of $\{1, \ldots, r\}$, provided the cocycle relations are then satisfied. We claim that the same σ does the trick: the cocycle relations hold because they do over the dense subscheme

$$\tilde{\eta} \times_\eta \tilde{\eta} \times_\eta \tilde{\eta} \hookrightarrow S \times_T S \times_T S.$$

□

Only Step 3 remains. Let $\mathcal{U}$ be the set of open subschemes U of T over which there exists a finite étale cover $W_U \to U$ extending $W_\eta \to \eta$. It follows from Step 2 and spreading out that any $t \in T$ admits an open neighborhood in $\mathcal{U}$. Moreover, $\mathcal{U}$ is stable under arbitrary union, thanks to Lemma 28 below. Thus $T \in \mathcal{U}$, which concludes the proof of Proposition 23.

Lemma 28: *Let T be an irreducible scheme with generic point η, $W_1, W_2 \to T$ finite étale covers. Any isomorphism $W_{1,\eta} \simeq W_{2,\eta}$ over η extends uniquely to an isomorphism $W_1 \simeq W_2$ over T.*

Proof. This follows from $\mathrm{Gal}(\overline{\eta}/\eta) \to \pi_1(T, \eta)$ being an epimorphism. □

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