POINTWISE TO GLOBAL GOOD REDUCTION PROBLEMS

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Contents

1. Introduction

1.1. **Motivation and main results.** Let $\mathcal{O}_F$ denote a complete DVR with fraction field F. Consider $\mathcal{X}$ a smooth, separated, geometrically connected scheme of finite type over $\mathcal{O}_F$, let X/F denote its generic fiber. Fix $Y \to X$ a smooth and proper family of varieties. We are concerned with the following problem: assuming (sufficiently many of) the closed fibers Y_x have good reduction, does Y extend to a smooth and proper scheme over $\mathcal{X}$? Does pointwise good reduction imply global good reduction?

This question is asked in the recent work of Cadoret and Tamagawa [CT26, §4.3], where it is answered positively in the case of pointed, smooth, proper curves of genus $g \geq 2$, of stable, proper curves of genus $g \geq 2$, and of abelian schemes if F has absolute ramification index $e \leq p-1$. The two main inputs of their argument are [CT26, Theorem 1] stating that a local system on X extends to $\mathcal{X}$ if and only if it is pointwise unramified for a relevant collection of closed points of X, as well as Serre-Tate-type criteria. It is also discussed in [CT26, §4.3, Example (2)] how *smooth integral canonical models* of Shimura varieties could be leveraged for a strategy when $\mathcal{O}_F$ is the ring of integers of a p-adic field. Indeed, by definition these models satisfy a property akin to the Néron extension property of Néron models, thus providing a Serre-Tate criterion over general bases.

The main goal of this document is to formalize the discussion in [CT26, §4.3, Example (2)]. We reduce the pointwise-to-global good reduction problem to the existence of a Deligne-Mumford stack $\mathscr{S}/\mathcal{O}_F$ equipped with a local system $\mathbb{L}$ such that the pair $(\mathscr{S}, \mathbb{L})$ satisfies some "L-extension property", and show that moduli spaces of curves, as well as smooth integral canonical models of Shimura varieties, can fill this role. Consequently, we establish:

Theorem 1.1: *Assume $p > 2$. Let $F/\mathbb{Q}_p$ be a finite field extension. Assume that $Y \to X$ is either:*

(a) *a smooth, proper family of genus g, n-pointed, geometrically connected curves with $g \geq 2$, $2-2g < n$,*
(b) *($F/\mathbb{Q}_p$ unramified) a polarized abelian scheme,*
(c) *($F/\mathbb{Q}_p$ unramified) a primitively polarized K3 surface of degree $2d$ not divisible by p,*
(d) *($F/\mathbb{Q}_p$ unramified) a smooth, proper family of cubic fourfolds.*

If $Y \to X$ has pointwise good reduction, then it has global good reduction.

We give a precise definition of pointwise/global good reductions in Definition 3.3. Cases (c) and (d) are new, and are enabled by the work of Madapusi Pera in [MP16] and [MP15] on integral canonical models of orthogonal-type Shimura varieties. Case (a) was already known, and our proof is the same as in [CT26, §4.3, Example (1)], only reformulated to fit our framework. Case (b) is weaker than the statement in [CT26, §4.3, Example (3)]; however building upon [VZ10, Theorem 28] we also show the optimality of their assumption $e \leq p-1$, and relate it to the non-existence of a smooth integral

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Date: May 19, 2026.

canonical model for Siegel modular varieties if $e \geq p$. We have not investigated the possibility of a similar obstruction for orthogonal-type Shimura varieties involved in cases (c) and (d).

1.2. **Outline.** Section 2 introduces the formalism of local systems and torsors on stacks used in the rest of the document. Of particular importance to us are towers of stacks with finite étale transition maps inducing compatible systems of torsors: associated with this setup is a *canonical local system*. When the stacks carry specific moduli interpretations, we define a *universal local system* and argue that canonical and universal local systems agree, thus generalizing constructions and statements from [UY13] and [CM20].

We formulate in Section 3 the pointwise-to-global good reduction problem. We then show that even in the usually well-behaved case of abelian schemes the answer is negative: if $F/\mathbb{Q}_p$ has ramification index $e \geq p$, Corollary 3.8 provides a counterexample with arbitrary level structure. To obtain it, we recall the work of Vasiu-Zink in [VZ10] and descend their construction to finite type bases over $\mathcal{O}_F$.

Section 4 is the core of the proof toward Theorem 1.1. For the convenience of the reader, we motivate the technical results with case (c) in mind. With X/F and $\mathcal{X}/\mathcal{O}_F$ as above, let $Y \to X$ be a family of K3 surfaces (with extra structure) having pointwise good reduction, we wish to extend it to a family $\mathcal{Y} \to \mathcal{X}$ of K3 surfaces. Let $\mathscr{M}$ denote the moduli stack of K3 surfaces (with extra structure) over $\mathcal{O}_F$, the family defines a *PGR-morphism* $X \to \mathscr{M}$ (Definition 4.6) which we wish to extend to $\mathcal{X}$. If any PGR-morphism to $\mathscr{M}$ admits an extension over $\mathcal{O}_F$, we say that $\mathscr{M}$ *has PGR-extension* (Definition 4.8).

To argue that $\mathscr{M}$ has PGR-extension, we need to introduce two other extension properties of moduli stacks: *L-extension* and *smooth-extension* (Definition 4.5 and Definition 4.16). The majority of Section 4 is spent establishing the implications below:

(i) If $\mathscr{S}_\infty/\mathcal{O}_F$ is a stack with smooth-extension which can be expressed as a certain limit
$$\mathscr{S}_\infty = \varprojlim_K \mathscr{S}_K,$$
then each $\mathscr{S}_K$ has L-extension (Proposition 4.17). This is where the notion of canonical local system from Section 2 is used. The implication had already been observed in [BST24, §1.3] in the context of Shimura varieties.

(ii) If $\mathscr{S}/\mathcal{O}_F$ has L-extension, then it also has PGR-extension (Proposition 4.10).

(iii) If $\mathscr{S}/\mathcal{O}_F$ has PGR-extension and $\mathcal{U} \to \mathscr{S}$ is étale, then $\mathcal{U}$ has PGR-extension (Theorem 4.11). This fails for L-extension, as shown by the open inclusion of the generic fiber in $\mathscr{S}$.

(iv) If $\mathcal{U} \to \mathscr{M}$ is a finite torsor over $\mathcal{O}_F$ and if $\mathcal{U}$ has PGR-extension, then so does $\mathscr{M}$ (Proposition 4.15).

Summarizing, the extension properties imply each-other from top-right to bottom-left in the following diagram, using (i) to (iv) in order:

$$\begin{tikzcd}
& & \mathscr{S}_\infty \arrow[d, "\text{projection}"] & \boxed{\text{smooth-extension}} \\
\boxed{\text{PGR-extension}} & \mathcal{U} \arrow[r, "\text{étale}"] \arrow[d, "\text{finite torsor}"'] & \mathscr{S} = \mathscr{S}_K & \boxed{\text{L-extension}} \\
\boxed{\text{PGR-extension}} & \mathscr{M} & &
\end{tikzcd}$$

We end Section 4 by introducing smooth integral canonical model of Shimura stacks (Definition 4.18) which have smooth-extension by definition.

In Section 5 we argue that the relevant moduli spaces $\mathscr{M}$ in Theorem 1.1 have PGR-extension. For case (a), it is known from [Sti05] that $\mathscr{M}$ has L-extension, hence PGR-extension. For case (b), the moduli space is the smooth integral canonical model of a Shimura stack of finite level, which also has L-extension; we then relate the counterexample from Section 3 to the non-existence of such models if F is too ramified (Proposition 5.7). Cases (c) and (d) require extra steps using the implications of Section 4: the period morphism defines an étale map from a moduli space $\mathcal{U} = \mathscr{M}_K$ with K-level structure to a smooth integral canonical model $\mathscr{S}_K$ of a Shimura stack of level K.Theorem 1.1 summarizes the contents of Theorem 5.2, Theorem 5.5, Theorem 5.15 and Theorem 5.19.

1.3. **Conventions and notations.** A variety over a field is a reduced separated scheme of finite type over the field.

Let $\mathcal{O}_F$ denotes a complete DVR with fraction field F. The integral closure of $\mathcal{O}_F$ in some fixed algebraic closure of F is denoted $\overline{\mathcal{O}_F}$.

A *smooth model* of a variety X/F is a smooth algebraic space $\mathcal{X}/\mathcal{O}_F$ with generic fiber X/F. A *smooth model* of a stack X/F is a smooth stack $\mathcal{X}/\mathcal{O}_F$ with generic fiber X/F.

If X is a stack, the set of closed points of its underlying topological space is denoted by $|X|$. If X is a scheme, this coincides with the closed points of its underlying topological space in the usual sense.

We fix $p > 0$ a prime number. We denote $\widehat{\mathbb{Z}}^p = \prod_{\ell \neq p} \mathbb{Z}_\ell$ the prime-to-p profinite integers, and $\mathbb{A}_f^p$ the prime-to-p adèles.

2. Canonical local systems on schemes and stacks

This section mostly serves as a toolbox of definitions: we advise the reader to skip it at first read, and come back to it whenever necessary. The first two paragraphs define what is meant in the rest of the document by local systems and torsors on schemes and stacks. In particular, to a projective system of schemes/stacks is attached a *canonical* local system which plays a role in later applications ; this construction is inspired from [UY13, §2] and [CM20, §3.1]. When the stacks carry a moduli interpretation, one can also define a *universal* local system: this is formalized in the third paragraph, where it is shown that it is canonically isomorphic to the canonical local system. This identification recovers [UY13, paragraph following Remark 2.8] (see also [CM20, Proposition 5.2]) by applying it to points on Siegel modular varieties, as well as [CM20, end of the proof of Theorem 6.6] for points on moduli spaces of K3 surfaces. This result is not used in the rest of the document.

Let S be a base scheme, set $\mathscr{C} := (\mathrm{Sch}/S)_{\text{proét}}$ the big pro-étale site over S[1]. All fibered categories, stacks and morphisms between them are over $\mathscr{C}$. We fix R a topological ring and V a free R-module of finite rank, we let $\mathrm{GL}(V)$ denote the topological group of R-linear automorphisms of V. We also fix Q a topological group.

2.1. Local systems and torsors on schemes.

Definition 2.1: Let A be a topological space. Let $\mathcal{F}_A$ denote the presheaf on $\mathscr{C}$ which associates to any $T \in \mathscr{C}$ the set of continuous functions $T \to A$. This is a sheaf by [BS13, Lemma 4.2.12].

If A is discrete and S is qcqs then $\mathcal{F}_A$ is the usual constant sheaf $\underline{A}$. The sheaf $\mathcal{F}_R$ is a sheaf of rings, and $\mathcal{F}_V$ is an $\mathcal{F}_R$-module.

Definition 2.2: An $\mathcal{F}_R$-module locally isomorphic to the $\mathcal{F}_R$-module $\mathcal{F}_V$ is called an *R-local system with fiber V*. The fibered category of R-local systems with fiber V over $\mathscr{C}$ is a stack denoted $\mathbf{Loc}_V$.

The sheaf $\mathcal{F}_Q$ is a sheaf of groups.

Definition 2.3: A *Q-sheaf* $\mathcal{G}$ is a sheaf of sets together with a morphism $\mathcal{F}_Q \times \mathcal{G} \to \mathcal{G}$ inducing a group action for every object of $\mathscr{C}$. A *Q-torsor* is a Q-sheaf on $\mathscr{C}$ locally isomorphic to the Q-sheaf $\mathcal{F}_Q$[2]. The fibered category of Q-sheaves (resp. the stack of Q-torsors) over $\mathscr{C}$ is denoted $Q\mathbf{Sh}$ (resp. $\mathbf{B}Q$).

Fact 2.4 ([Gir20], Corollaire 2.2.6)**:** *If $Q = \mathrm{GL}(V)$ then there is an equivalence*

$$\mathrm{Triv} : \mathbf{Loc}_V \to \mathbf{B}\mathrm{GL}(V)$$

mapping a local system $\mathbb{L}$ to its sheaf of trivializations $\underline{\mathrm{Isom}}(\mathbb{L}, \mathcal{F}_V)$.

Definition 2.5: Let $\phi : Q \to Q'$ be a morphism of topological groups. The forgetful functor from $Q'\mathbf{Sh}$ to $Q\mathbf{Sh}$ admits a left-adjoint

$$\mathrm{Ind}_Q^{Q'} : Q\mathbf{Sh} \to Q'\mathbf{Sh}.$$

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[1] To avoid set theoretic issues, one should restrict the class of covers using strong limit cardinals, as in [BS13, Remark 4.1.2].

[2] In other words, it is an $\mathcal{F}_Q$-torsor in the usual sense.

mapping any Q-sheaf $\mathcal{G}$ to the Q'-sheaf $\mathrm{Ind}_Q^{Q'}(\mathcal{G})$ defined as the quotient[3] of $\mathcal{F}_{Q'} \times \mathcal{G}$ under the Q-action:

$$\begin{aligned} \mathcal{F}_Q \times (\mathcal{F}_{Q'} \times \mathcal{G}) &\to \mathcal{F}_{Q'} \times \mathcal{G} \\ (q, q', g) &\mapsto (q'\phi(q)^{-1}, q \cdot g). \end{aligned}$$

This functor restricts to a functor $\mathrm{Ind}_Q^{Q'} : \mathbf{B}Q \to \mathbf{B}Q'$.

Using Fact 2.4 and Definition 2.5, any continuous representation of Q on V defines a functor

$$\mathbf{B}Q \to \mathbf{Loc}_V$$

sending P to the R-local system with fiber V associated with $\mathrm{Ind}_Q^{\mathrm{GL}(V)} P$.

We will be interested in torsors over $\mathscr{C}$ of geometric origin.

Definition 2.6: Let $S' \to S$ be an algebraic space over S. Fix a homomorphism $Q \to \mathrm{Aut}_S(S')$. We say that $S' \to S$ is a *Q-torsor* if the functor of points $h_{S'}$ defined by

$$(T \to S) \in \mathscr{C} \mapsto h_{S'}(T) = \mathrm{Hom}_S(T, S'),$$

is a Q-torsor as in Definition 2.3 for the induced action of $\mathcal{F}_Q$. If Q is finite then $S' \to S$ is étale; if moreover $S' \to S$ is finite then S' is a scheme.

If K_0 is a topological group, let $\mathcal{C}(K_0)$ denote the set of all open normal subgroups of K_0.

Definition 2.7: Let K_0 be a profinite group. Let $(S_K)_{K \in \mathcal{C}(K_0)}$ be a projective system of schemes with $S_{K_0} = S$. We say that it is *fit for* K_0 if it satisfies the following assumptions:

(i) for each $K \in \mathcal{C}(K_0)$, the morphism $S_K \to S_{K_0}$ is a finite K_0/K-torsor as in Definition 2.6, and
(ii) for every $K' \subseteq K$ in $\mathcal{C}(K_0)$, the morphism $S_{K'} \to S_K$ is K_0-equivariant for the K_0-action induced from (i).

Then for every $K \in \mathcal{C}(K_0)$ the map $S_K \to S_{K_0}$ is a finite étale cover, thus it makes sense to define the projective limit

$$S_\infty := \varprojlim_{K \in \mathcal{C}(K_0)} S_K$$

which is a pro-étale cover of S_{K_0}, and a K_0-torsor as in Definition 2.6 with $h_{S_\infty} = \varprojlim_K h_{S_K}$ in $K_0\mathbf{Sh}$.

Definition 2.8: Let K_0 be a profinite group with a fixed continuous embedding $K_0 \hookrightarrow \mathrm{GL}(V)$. Let $(S_K)_{K \in \mathcal{C}(K_0)}$ be schemes fit for K_0, let h_{S_∞} be the K_0-torsor from Definition 2.7. The corresponding *canonical local system* $\mathbb{L}^{\mathrm{can}} \in \mathbf{Loc}_V(S)$ is the R-local system associated with $\mathrm{Ind}_{K_0}^{\mathrm{GL}(V)}(h_{S_\infty})$.

Remark 2.9: Let us give a more down-to-earth description of the canonical local system in case S is a connected scheme with a fit system $(S_K)_K$ for K_0. Fix $(\overline{s}_K)_K$ a compatible system of geometric points of $(S_K)_K$. For each $K \in \mathcal{C}(K_0)$, $S_K \to S$ is a $\underline{K_0/K}$-torsor on S in the usual sense for the étale topology on S. Having fixed geometric points, this torsor is uniquely described by a continuous morphism

$$\pi_1(S, \overline{s}_{K_0}) \to K_0/K.$$

The compatibility conditions allow to take the projective limit over K so as to find a continuous morphism

$$\pi_1(S, \overline{s}_{K_0}) \to K_0.$$

The composition with the embeddings $K_0 \hookrightarrow \mathrm{GL}(V)$ yields the monodromy action on the fibers of $\mathbb{L}^{\mathrm{can}}$. If S is locally noetherian but no longer connected, we can argue component-wise.

Example 2.10: Let (G, Ω) be a Shimura datum with reflex field E. Let K_0 be a neat compact open subgroup of $G(\mathbb{A}_f)$. If $K \in \mathcal{C}(K_0)$, let $\mathrm{Sh}_K(G, \Omega)/E$ denote the associated Shimura variety of level K. Then $(\mathrm{Sh}_K(G, \Omega))_K$ is fit for K_0, so $\mathrm{Sh}_{K_0}(G, \Omega)$ carries a canonical local system. The neatness assumption is equivalent to condition (i) of Definition 2.7, we can remove it by replacing schemes with stacks.

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[3] The quotient sheaf is the sheafification of the quotient presheaf. This is part of the reason why $\mathscr{C}$ is the proétale site on S, as opposed to the fpqc site which does not have sheafification in general.

2.2. **Local systems and torsors on stacks.** Let $\mathscr{X}$ be a stack on $\mathscr{C}$. We extend the notions of the previous paragraph to $\mathscr{X}$.

Definition 2.11: We denote by $Q\mathbf{Sh}(\mathscr{X})$ (resp. $\mathbf{Loc}_V(\mathscr{X})$, $\mathbf{B}Q(\mathscr{X})$) the category of morphisms of fibered categories $\mathscr{X} \to Q\mathbf{Sh}$ (resp. $\mathscr{X} \to \mathbf{Loc}_V$, $\mathscr{X} \to \mathbf{B}Q$), the objects of which are called *Q-sheaves on $\mathscr{X}$* (resp. *R-local systems with fiber V on $\mathscr{X}$*, *Q-torsors on $\mathscr{X}$*).

Explicitely, an object of $Q\mathbf{Sh}(\mathscr{X})$ is the data, for each $\xi : T \to \mathscr{X}$, of a Q-sheaf $F_\xi \in Q\mathbf{Sh}(T)$, collectively satisfying compatibility conditions as ξ and T vary. Identical descriptions hold for $\mathbf{Loc}_V(\mathscr{X})$ and $\mathbf{B}Q(\mathscr{X})$.

The induction functor and the equivalence of Fact 2.4 automatically extend to Q-sheaves, local systems and torsors on $\mathscr{X}$. Consequently, any continuous representation of Q on V defines a functor

$$\mathbf{B}Q(\mathscr{X}) \to \mathbf{Loc}_V(\mathscr{X})$$

sending P to the R-local system with fiber V associated with $\mathrm{Ind}_Q^{\mathrm{GL}(V)} P$.

Definition 2.12: Let $f : \mathscr{X}' \to \mathscr{X}$ be a representable morphism of stacks over $\mathscr{C}$. Assume $\mathscr{X}'$ is equipped with a pseudo-Q-action[4] preserving f[5]. For any scheme $T \to X$, the fiber product $T' := T \times_{\mathscr{X}} \mathscr{X}'$ is an algebraic space which inherits a Q-action $Q \to \mathrm{Aut}_T(T')$. We say that f is a *Q-torsor* if for all such T the morphism $T' \to T$ is a Q-torsor as in Definition 2.6. This defines a Q-torsor $h_{\mathscr{X}'}$ on $\mathscr{X}$ as in Definition 2.11.

Definition 2.13: Let K_0 be a profinite group. Let $(\mathscr{X}_K)_{K \in \mathcal{C}(K_0)}$ be a projective system of stacks. We say that it is *fit for K_0* if it satisfies the following assumptions:

(i) for each $K \in \mathcal{C}(K_0)$, the morphism $\mathscr{X}_K \to \mathscr{X}_{K_0}$ is a finite K_0/K-torsor as in Definition 2.12, and

(ii) for every $K' \subseteq K$ in $\mathcal{C}(K_0)$, the morphism $\mathscr{X}_{K'} \to \mathscr{X}_K$ is pseudo-K_0-equivariant[6] for the K_0-action induced from (i).

Then for every $K \in \mathcal{C}(K_0)$ the map $\mathscr{X}_K \to \mathscr{X}_{K_0}$ is finite étale. One can define the projective limit

$$\mathscr{X}_\infty := \varprojlim_{K \in \mathcal{C}(K_0)} \mathscr{X}_K$$

which is pro-étale over $\mathscr{X}_{K_0}$, and is a K_0-torsor as in Definition 2.12 with $h_{\mathscr{X}_\infty} = \varprojlim_K h_{\mathscr{X}_K}$ in $K_0\mathbf{Sh}(\mathscr{X}_{K_0})$.

Definition 2.14: Let K_0 be a profinite group with a fixed continuous embedding $K_0 \hookrightarrow \mathrm{GL}(V)$. Let $(\mathscr{X}_K)_{K \in \mathcal{C}(K_0)}$ be stacks fit for K_0, let $h_{\mathscr{X}_\infty}$ be the K_0-torsor defined in Definition 2.13. The corresponding *canonical local system* $\mathbb{L}^{\mathrm{can}} \in \mathbf{Loc}_V(\mathscr{X}_{K_0})$ is the R-local system associated to $\mathrm{Ind}_{K_0}^{\mathrm{GL}(V)}(h_{\mathscr{X}_\infty})$.

If $\mathscr{X}_{K_0}$ is a scheme, then so are all the $\mathscr{X}_K$, and Definitions 2.8 and 2.14 coincide.

Example 2.15: Let (G, Ω) be a Shimura datum with reflex field E. Let K_0 be a neat compact open subgroup of $G(\mathbb{A}_f)$. Let $\mathrm{Sh}_\infty(G, \Omega) = \varprojlim_{K \in \mathcal{C}(K_0)} \mathrm{Sh}_K(G, \Omega)$ on which $G(\mathbb{A}_f)$ acts. For K' *any* compact open subgroup of $G(\mathbb{A}_f)$, define $\mathrm{Sh}_{K'}[G, \Omega]$ as the quotient stack $[\mathrm{Sh}_\infty(G, \Omega)/K']$ (if K' is neat this agrees with the earlier definition of $\mathrm{Sh}_{K'}(G, \Omega)$). If we fix K_0' *any* compact open subgroup of $G(\mathbb{A}_f)$ then $(\mathrm{Sh}_{K'}[G, \Omega])_{K' \in \mathcal{C}(K_0')}$ is a system of stacks fit for K_0'; in particular $\mathrm{Sh}_{K_0'}[G, \Omega]$ carries a canonical local system.

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[4] Formally, a pseudo-Q-action is a 2-morphism $\mathbf{B}Q \to \mathbf{Cat}$ with essential image $\mathscr{X}'$. Explicitely, this is the data for every $q \in Q$ of an equivalence $a_q : \mathscr{X}' \xrightarrow{\sim} \mathscr{X}'$ such that there are isomorphisms $a_q \circ a_{q'} \simeq a_{qq'}$ satisfying the associativity condition.

[5] Formally, for any $q \in Q$ inducing an equivalence $a_q : \mathscr{X}' \to \mathscr{X}'$, there is an isomorphism $f \circ a_q \simeq f$ satisfying omitted compatibility conditions.

[6] This is the categorical version of an equivariant morphism, sending the pseudo-action on $\mathscr{X}_{K'}$ to the pseudo-action on $\mathscr{X}_K$ up to isomorphisms satisfying omitted compatibility conditions.

Remark 2.16: Let K_0 be a profinite group. If $K \in \mathcal{C}(K_0)$, the induction functor defines a finite K_0/K-torsor $\mathbf{B}K \to \mathbf{B}K_0$ on $\mathbf{B}K_0$ as in Definition 2.12, so that $(\mathbf{B}K)_{K\in\mathcal{C}(K_0)}$ is fit for K_0. The associated K_0-torsor recovers the universal torsor on $\mathbf{B}K_0$ defined by the identity. We call $(\mathbf{B}K)_{K\in\mathcal{C}(K_0)}$ the *universal system fit for* K_0.

If $\mathscr{X}_{K_0}$ is a stack together with a K_0-torsor $\mathscr{X}_{K_0} \to \mathbf{B}K_0$, pulling back the universal system yields a system $(\mathscr{X}_K)_{K\in\mathcal{C}(K_0)}$ fit for K_0. Conversely, if $(\mathscr{X}_K)_{K\in\mathcal{C}(K_0)}$ is fit for K_0, the associated K_0-torsor defines a morphism $\mathscr{X}_{K_0} \to \mathbf{B}K_0$. Pulling back $(\mathbf{B}K)_{K\in\mathcal{C}(K_0)}$ along this morphism recovers $(\mathscr{X}_K)_{K\in\mathcal{C}(K_0)}$ up to isomorphism. Thus systems fit for K_0 are equivalent to K_0-torsors on the base stack.

Remark 2.17: Assume K_0 is a finite group. The induction functor from the trivial group $\mathcal{E}$ to K_0 yields a finite K_0-torsor $\mathbf{B}\mathcal{E} \to \mathbf{B}K_0$ which we call the *universal* K_0*-torsor cover*. If $\mathscr{X}$ is a stack together with a K_0-torsor $\mathscr{X} \to \mathbf{B}K_0$, the corresponding finite étale cover $\mathscr{X}' \to \mathscr{X}$ is exactly the pullback of the universal K_0-torsor cover:

$$\begin{tikzcd}
\mathscr{X}' \arrow[r] \arrow[d] & \mathbf{B}\mathcal{E} \arrow[d] \\
\mathscr{X} \arrow[r] & \mathbf{B}K_0.
\end{tikzcd}$$

2.3. **Canonical vs universal local systems.** Let us first motivate our goal and the formalism we introduce to achieve it. Fix some integer $g > 0$ and let $\mathscr{M}_{K_0}$ denote the moduli stack over $\mathbb{Q}$ of principally polarized abelian schemes of dimension g with K_0-level structure, where K_0 is a compact open subgroup of $\mathrm{GSp}_{2g}(\mathbb{A}_f)$. Replacing K_0 with finite index open subgroups K, we find a projective system $(M_K)_K$ of stacks that is fit for K_0 in the sense of Definition 2.13, so we can consider the corresponding *canonical* $\mathbb{A}_f$-local system $\mathbb{L}^{\mathrm{can}}$ on $\mathscr{M}_{K_0}$. There is a second local system on $\mathscr{M}_{K_0}$: the relative cohomology of the universal abelian scheme, which we call the *universal* local system $\mathbb{L}_{K_0}$. We aim to show that $\mathbb{L}^{\mathrm{can}}$ and $\mathbb{L}_{K_0}$ are canonically isomorphic, which is the content of Proposition 2.22 below.

To do so, we formalize the notion of a moduli stack with level structure in Definition 2.18 and Definition 2.20. We sketch how this is done in the explicit case of $\mathscr{M}_{K_0}$. For any scheme $T/\mathbb{Q}$, an object in $\mathscr{M}_{K_0}(T)$ is a triple $(f : A \to T, \lambda, [\eta]_{K_0})$ where

- $(f : A \to T, \lambda)$ is a principally polarized abelian scheme of dimension g, and
- $[\eta]_{K_0}$ is a section of the quotient by $\mathcal{F}_{K_0}$ of some subsheaf of $\underline{\mathrm{Isom}}(R^1 f_* \mathbb{A}_f, \mathcal{F}_V)$ where $V = \mathbb{A}_f^{2g}$ is acted-on by K_0; namely we restrict to those isomorphisms that are compatible with the usual symplectic structure on V, and the symplectic structure on $R^1 f_* \mathbb{A}_f$ induced from λ.

To define $\mathscr{M}_{K_0}$, we first consider the stack $\mathscr{X}$ of principally polarized abelian schemes of dimension g and view relative cohomology as a morphism from $\mathscr{X}$ to $\mathbf{Loc}_V(\mathscr{X})$. From there, we could define a stack of triples $(f : A \to T, \lambda, [\eta]_{K_0})$ with the slight caveat that $[\eta]_{K_0}$ is a section of the whole sheaf $\underline{\mathrm{Isom}}(R^1 f_* \mathbb{A}_f, \mathcal{F}_V)/\mathcal{F}_{K_0}$ (no compatibility condition with the symplectic structures). To formalize replacing every $\mathrm{GL}_d(\mathbb{A}_f)$-torsor $\underline{\mathrm{Isom}}(R^1 f_* \mathbb{A}_f, \mathcal{F}_V)$ with the appropriate $\mathrm{GSp}_{2g}(\mathbb{A}_f)$-subsheaf (which is a $\mathrm{GSp}_{2g}(\mathbb{A}_f)$-torsor), we use the language of natural transformations between fibered categories.

Fix an embedding $Q \hookrightarrow \mathrm{GL}(V)$. Recall the equivalence of stacks

$$\mathrm{Triv} : \mathbf{Loc}_V \to \mathbf{B}\mathrm{GL}(V)$$

assigning to any $T \in \mathscr{C}$ and any $\mathbb{L} \in \mathbf{Loc}_V(T)$ the $\mathrm{GL}(V)$-torsor $\underline{\mathrm{Isom}}(\mathbb{L}, \mathcal{F}_V)$ over T.

Definition 2.18: A *Q-moduli datum* is a quadruple $(\mathscr{X}, \mathbb{L}_{\mathscr{X}}, \mathrm{Lev}, \iota)$ consisting of

(i) a stack $\mathscr{X}$ over $\mathscr{C}$,
(ii) a local system $\mathbb{L}_{\mathscr{X}} : \mathscr{X} \to \mathbf{Loc}_V$ on $\mathscr{X}$; we let $\mathrm{Triv}_{\mathscr{X}} = \mathrm{Triv} \circ \mathbb{L}_{\mathscr{X}}$,
(iii) a Q-torsor $\mathrm{Lev} : \mathscr{X} \to \mathbf{B}Q$ on $\mathscr{X}$, and

(iv) a natural monomorphism ι between the morphisms of fibered categories:

$$\begin{tikzcd}
& \mathbf{B}\mathrm{GL}(V) \arrow[dr, bend left, "\text{forget}"] & \\
\mathscr{X} \arrow[ur, bend left, "\mathrm{Triv}_{\mathscr{X}}"] \arrow[dr, bend right, "\mathrm{Lev}"'] & \Uparrow\iota & Q\mathbf{Sh}. \\
& \mathbf{B}Q \arrow[ur, bend right, "\text{forget}"'] &
\end{tikzcd}$$

A Q-moduli datum is to be understood as follows. The functor $\mathrm{Triv}_{\mathscr{X}} : \mathscr{X} \to \mathbf{B}\mathrm{GL}(V)$ assigns to any $T \in \mathscr{C}$ and any $A \in \mathscr{X}(T)$ the $\mathrm{GL}(V)$-torsor $\underline{\mathrm{Isom}}(\mathbb{L}_{\mathscr{X}}(A), \mathcal{F}_V)$ over T. The functor Lev assigns to the same data a Q-subsheaf of $\mathrm{Triv}_{\mathscr{X}}(A)$ which is a Q-torsor for the induced action, considering only isomorphisms with additional constraints. The condition of being a Q-subsheaf for each $T \in \mathscr{C}$ and $A \in \mathscr{X}(T)$ is formalized by the natural monomorphism ι.

Lemma 2.19: *Let $(\mathscr{X}, \mathbb{L}_{\mathscr{X}}, \mathrm{Lev}, \iota)$ be a Q-moduli datum. Then $\mathbb{L}_{\mathscr{X}}$ is the R-local system with fiber V associated with the* $\mathrm{GL}(V)$*-torsor* $\mathrm{Ind}_Q^{\mathrm{GL}(V)}(\mathrm{Lev})$.

Proof. Condition (iv) is equivalent to a natural isomorphism

$$\begin{tikzcd}
\mathscr{X} \arrow[rr, "\mathrm{Triv}_L"] \arrow[dr, bend right, "\mathrm{Lev}"'] & \Uparrow & \mathbf{B}\mathrm{GL}(V) \\
& \mathbf{B}Q \arrow[ur, bend right, "\mathrm{Ind}_Q^{\mathrm{GL}(V)}."'] &
\end{tikzcd}$$

The equivalence follows formally from the fact that $\mathrm{Ind}_Q^{\mathrm{GL}(V)} : Q\mathbf{Sh} \to \mathrm{GL}(V)\mathbf{Sh}$ is left adjoint to the forgetful functor $\mathrm{GL}(V)\mathbf{Sh} \to Q\mathbf{Sh}$. □

It follows that a Q-moduli datum is determined up to isomorphism by the Q-torsor Lev on $\mathscr{X}$.

We are interested in the following moduli stacks:

Definition 2.20: Let K_0 be a compact subgroup of Q which is profinite, and K an open subgroup of K_0. Let $(\mathscr{X}, \mathbb{L}_{\mathscr{X}}, \mathrm{Lev}, \iota)$ be a Q-moduli datum. The associated *moduli stack of level K* is the stack $\mathscr{M}_K$ over $\mathscr{C}$ assigning to each $T \in \mathscr{C}$ a pair $(A, [\eta]_K)$, where

(i) A is an object in $\mathscr{X}(T)$, and
(ii) $[\eta]_K \in H^0(T, \mathrm{Lev}(A)/\mathcal{F}_K)$ is a global section of the quotient of $\mathrm{Lev}(A)$ by the action of $\mathcal{F}_K$.

If K is normal in K_0, the obvious morphism $\mathscr{M}_K \to \mathscr{M}_{K_0}$ is a K_0/K-torsor as in Definition 2.12.

Fix K_0 and $(\mathscr{X}, \mathbb{L}_{\mathscr{X}}, \mathrm{Lev}, \iota)$ as in Definition 2.20. The projective system $(\mathscr{M}_K)_{K \in \mathcal{C}(K_0)}$ is fit for K_0. By Definition 2.14 we can consider the canonical local system $\mathbb{L}^{\mathrm{can}}$ on $\mathscr{M}_{K_0}$. We now define the universal local system on $\mathscr{M}_{K_0}$ and show that it is canonically isomorphic to $\mathbb{L}^{\mathrm{can}}$. Observe that the projection onto the first component defines a morphism of stacks $p : \mathscr{M}_{K_0} \to \mathscr{X}$.

Definition 2.21: The *universal local system* $\mathbb{L}_{K_0} \in \mathbf{Loc}_V(\mathscr{M}_{K_0})$ on $\mathscr{M}_{K_0}$ is the pullback of $\mathbb{L}_{\mathscr{X}}$ by p, that is:

$$\mathscr{M}_{K_0} \xrightarrow{p} \mathscr{X} \xrightarrow{\mathbb{L}_{\mathscr{X}}} \mathbf{Loc}_V.$$

The *universal torsor* $\mathrm{Lev}_{K_0} \in \mathbf{B}Q(X_{K_0})$ on $\mathscr{M}_{K_0}$ is the pullback of Lev by p, that is:

$$\mathscr{M}_{K_0} \xrightarrow{p} \mathscr{X} \xrightarrow{\mathrm{Lev}} \mathbf{B}Q.$$

By Lemma 2.19, $\mathbb{L}_{K_0}$ is the R-local system with fiber V associated with $\mathrm{Ind}_Q^{\mathrm{GL}(V)}(\mathrm{Lev}_{K_0})$.

Proposition 2.22: *There is a canonical isomorphism $\mathbb{L}^{\mathrm{can}} \simeq \mathbb{L}_{K_0}$.*

Proof. Fix $T \to \mathscr{M}_{K_0}$, let $(A, [\eta]_{K_0}) \in \mathscr{M}_{K_0}(T)$ be the corresponding object. Recall that $\mathbb{L}^{\mathrm{can}}$ is defined from a $\mathrm{GL}(V)$-torsor, itself induced from the K_0-torsor $h_{\mathscr{M}_\infty}$ as in Definition 2.13. A section of $h_{\mathscr{M}_\infty}(A, [\eta]_{K_0})$ is by definition a compatible collection of sections $T \to \mathscr{M}_K$ over $\mathscr{M}_{K_0}$ as K varies, which amounts to a projective system

$$([\eta]_K)_K \in \varprojlim_K H^0(T, \mathrm{Lev}(A)/\mathcal{F}_K) = H^0(T, \varprojlim_K \mathrm{Lev}(A)/\mathcal{F}_K) \simeq H^0(T, \mathrm{Lev}(A)) = H^0(T, \mathrm{Lev}_{K_0}(A, [\eta]_{K_0})).$$

Here we used that K_0 is profinite. Thus we find a natural monomorphism from the underlying K_0-sheaf of $h_{\mathscr{M}_\infty}$ to the underlying K_0-sheaf of Lev_{K_0}. As in Lemma 2.19 this defines a canonical isomorphism between the Q-torsors $\mathrm{Ind}_{K_0}^{Q}(h_{\mathscr{M}_\infty})$ and Lev_{K_0}. Applying $\mathrm{Ind}_{Q}^{\mathrm{GL}(V)}$, we find the desired isomorphism. □

Example 2.23: Let (G, Ω) be a Shimura datum with reflex field E. Let K_0 be a compact open subgroup of $G(\mathbb{A}_f)$. If $\mathrm{Sh}_{K_0}[G, \Omega]/E$ carries a modular interpretation (e.g. is Hodge-type), then the local system on $\mathrm{Sh}_{K_0}[G, \Omega]$ coming from the relative cohomology of the universal family coincides with the canonical local system.

3. Pointwise and global good reduction problems

3.1. Formulation of the question.

We define the notions of pointwise and global good reduction.

Definition 3.1: Let Y/F be a smooth proper variety. We say that it has *good reduction* if it is the generic fiber of some smooth proper model $\mathcal{Y}/\mathcal{O}_F$.

We define the integral locus of a variety with smooth model.

Definition 3.2: Let $\mathcal{X}/\mathcal{O}_F$ be a smooth stack with generic fiber X/F. We define the *integral locus* of X (relative to $\mathcal{X}$) as

$$|X|^{\mathrm{int}} = \mathrm{im}(\mathcal{X}(\overline{\mathcal{O}_F}) \to |X|).$$

If $\mathcal{X} = X$, the integral locus is empty and if $\mathcal{X}/\mathcal{O}_F$ is a proper scheme, then $|X|^{\mathrm{int}} = |X|$.

Definition 3.3: Let X/F be a smooth stack, and $\mathcal{X}/\mathcal{O}_F$ a smooth model. Consider a smooth proper map $Y \to X$ where Y is a scheme.

We say that this family has *pointwise good reduction* (relative to $\mathcal{X}$) if for every $x \in |X|^{\mathrm{int}}$, the fiber $Y_x/F(x)$ has good reduction.

We say that this family has *global good reduction* (relative to $\mathcal{X}$) if there exists a smooth proper map $\mathcal{Y} \to \mathcal{X}$ of algebraic spaces over $\mathcal{O}_F$ with generic fiber $Y \to X$.

We are interested in whether pointwise good reduction implies global good reduction, and if not, what may some obstructions be.

Remark 3.4: By default, we require that the model $\mathcal{Y} \to \mathcal{X}$ extends as much of the structure of $Y \to X$ as possible. Typically if $Y \to X$ is a polarized abelian scheme or K3 surface with some level structure, we require $\mathcal{Y} \to \mathcal{X}$ to be so too.

3.2. The ramification obstruction.

For abelian schemes over a sufficiently ramified p-adic field F, pointwise good reduction does not imply global good reduction. We build a counterexample upon the work of Vasiu-Zink in [VZ10][7], of which we recall some steps.

Lemma 3.5 ([VZ10], Lemma 27)**:** *Let R denote a regular local ring of dimension 2 and mixed characteristic $(0, p)$. Let $\mathfrak{m}$ denote its maximal ideal, and set $U = \operatorname{Spec} R \setminus \{\mathfrak{m}\}$.*

Let $D \to H$ be a homomorphism of finite flat group schemes over $\operatorname{Spec} R$ which is not a monomorphism, but whose restriction over U is a monomorphism. Let B be an abelian scheme over $\operatorname{Spec} R$ in which H embeds. Define the quotient abelian scheme $A = B_U/D_U$ over U.

If R' is a faithfully flat local R-algebra of relative dimension 0, with maximal ideal $\mathfrak{m}'$, let $U' := U \times_R \operatorname{Spec} R' = \operatorname{Spec} R' \setminus \{\mathfrak{m}'\}$. Then $A \times_U \operatorname{Spec} R' \to U'$ is an abelian scheme which does not extend into an abelian scheme over $\operatorname{Spec} R'$.

---

[7] Vasiu-Zink generalize a counterexample to [FC13, Chapter V, Corollary 6.8] due to Raynaud-Ogus-Gabber (see [dJO97, §6]).

Proof. The morphism $\operatorname{Spec} R' \to \operatorname{Spec} R$ is fpqc, so the assumptions on monomorphisms still hold after base-change to R'. Let D', H', A', B' denote the group schemes base-changed to R', then $A' = B'_U/D'_U$. Hence we reduce to the case $R' = R$.

Suppose that A extends to an abelian scheme $\tilde{A} \to \operatorname{Spec} R$. By [Ray06, Corollaire IX 1.4], the isogeny $B_U \to A$ extends to a homomorphism $B \to \tilde{A}$ which is automatically an isogeny. Its kernel K is a finite flat group scheme which coincides with D_U over U, it is thus isomorphic to D by purity for finite flat group schemes over a regular base (see [MB85, Lemme 2]). This defines a monomorphism

$$D \simeq K \to B$$

of group schemes over R, whose restriction over U is the original morphism $D_U \to H_U$. Thus this monomorphism factors through $H \to B$, contradicting the assumption that $D \to H$ is not a monomorphism. □

Proposition 3.6: *Suppose $F/\mathbb{Q}_p$ has ramification index at least p. Then for any positive prime-to-p integer N there exists a finite étale local $\mathcal{O}_F[[T]]$-algebra R with maximal ideal $\mathfrak{m}$ and an abelian scheme $A \to U := \operatorname{Spec} R \backslash \{\mathfrak{m}\}$ with level N structure which does not extend to an abelian scheme over* $\operatorname{Spec} R$.

Proof. According to [VZ10, Theorem 28] and its proof we can take the power series ring $R = \mathcal{O}_F[[T]]$ and find a homomorphism of group schemes $D \to H$ over R_0 satisfying the assumptions of Lemma 3.5, hence an abelian scheme $A \to U_0$ which does not extend to an abelian scheme over $\operatorname{Spec} R_0$. By Lemma 3.5 we can replace R with an appropriate finite étale local ring extension to ensure that $A \to U$ has the desired level structure. □

Next we argue that we can replace R with a finite type $\mathcal{O}_F$-algebra.

Proposition 3.7: *Suppose $F/\mathbb{Q}_p$ has ramification index at least p. Then for any positive prime-to-p integer N there exists a smooth local $\mathcal{O}_F$-algebra R with maximal ideal $\mathfrak{m}$ and an abelian scheme $A \to U := \operatorname{Spec} R \setminus \{\mathfrak{m}\}$ with level N structure which does not extend to an abelian scheme over* $\operatorname{Spec} R$.

Proof. We denote by $\tilde{R}$ the finite étale local $\mathcal{O}_F[[T]]$-algebra with maximal ideal $\tilde{\mathfrak{m}}$, by $\tilde{U} := \operatorname{Spec} \tilde{R} \backslash \{\tilde{\mathfrak{m}}\}$ the open subset and by $f : \tilde{A} \to \tilde{U}$ the abelian scheme with level N structure which does not extend into an abelian scheme over $\operatorname{Spec} \tilde{R}$, provided by Proposition 3.6. We wish to descend all of this data over a smooth $\mathcal{O}_F$-algebra.

We first argue that $\mathcal{O}_F \to \tilde{R}$ is a regular ring homomorphism, to do so we may assume $\tilde{R} = \mathcal{O}_F[[T]]$. It is clear that $k[[T]]$ is geometrically regular, where k is the residue field of $\mathcal{O}_F$. It remains to show that $\mathcal{O}_F[[T]] \otimes_{\mathcal{O}_F} F = \mathcal{O}_F[[T]][1/p]$ is regular. By [Sta26, Tag 07EL] we reduce to showing that $F \to \mathcal{O}_F[[T]][1/p]$ is formally smooth for the (T)-adic topology, which is easily checked.

We may now apply Popescu's theorem, see [Sta26, Tag 07GC]: there are smooth $\mathcal{O}_F$-algebras R_i such that $\tilde{R} = \varinjlim R_i$. All maps between the R_i and $\tilde{R}$ factor through local and integral rings, so we may assume that the R_i are local integral with maximal ideal $\mathfrak{m}_i$, and set $U_i := \operatorname{Spec} R_i \setminus \{\mathfrak{m}_i\}$. We let $p_i : \tilde{U} \to U_i$ be the maps induced from the projective system. We wish to descend f to an abelian scheme over some U_i.

Since the U_i are Noetherian, by [Sta26, Tag 0CNL] for large enough i there is a smooth and proper morphism $f_i : A_i \to \operatorname{Spec} U_i$ descending f. Using [Sta26, Tag 0CNR] we can assume that the group scheme structure and the level N structure descends to f_i, thus f_i defines an abelian scheme. Let $A = A_i$ and $U = U_i$ for this appropriate choice of i.

If $A \to U$ were to extend to an abelian scheme over $\operatorname{Spec} R$, its pullback to $\operatorname{Spec} \tilde{R}$ would extend $\tilde{A} \to \tilde{U}$, contradicting the definition of $\tilde{A}$. □

Corollary 3.8: *With notations from Proposition 3.7, set $\mathcal{X} = \operatorname{Spec} R$ which is a smooth model over $\mathcal{O}_F$ of its generic fiber $\iota \colon X \hookrightarrow \mathcal{X}$. Then $\iota^* A \to X$ has level N structure, pointwise good reduction, but not global good reduction as an abelian scheme.*

Proof. We argue that the abelian scheme $\iota^* A \to X$ does not have global good reduction. Let η denote the generic point of X, hence also of U and $\mathcal{X}$, fix a symmetric ample line bundle $\mathscr{L}_\eta$ on A_η. Assume the family extends to an abelian scheme $A' \to \mathcal{X}$. According to Fact 3.9 there are positive integers n, m

such that $\mathscr{L}_\eta^{\otimes n}$ and $\mathscr{L}_\eta^{\otimes m}$ extend to relatively ample line bundles on $A \to U$ and $A' \to \mathcal{X}$ respectively. Let $j\colon U \to \mathcal{X}$ be the inclusion of open set; replacing n and m with nm, we find that the polarized abelian schemes

$$A \to U, \qquad j^*A' \to U$$

have isomorphic generic fibers. Separatedness of the stack of polarized abelian schemes implies that they are isomorphic (Proposition 3.10). In particular, $A' \to \mathcal{X}$ extends $A \to U$, which contradicts Proposition 3.7.

It remains to check that $\iota^*A \to X$ has pointwise good reduction. Let $\ell \neq p$ be a prime number, the local system $\mathbb{L} = R^1f_*\mathbb{Q}_\ell$ over U corresponds to an ℓ-adic representation of the étale fundamental group $\pi_1(U)$. As $\mathcal{X} \setminus U$ has codimension at least 2 in $\mathcal{X}$, Zariski-Nagata purity implies that $\mathbb{L}$ extends to a local system on $\mathcal{X}$. In particular, for any $x \in |X|^{\mathrm{int}}$ the local system $\mathbb{L}_x$ is unramified, which by [ST71] means that $(\iota^*A)_x$ has good reduction. □

Fact 3.9 ([Ray06], Remarque XI 1.3 (e))**:** *Let S be a normal, integral, locally Noetherian scheme with generic point η, consider an abelian scheme $G \to S$, and a symmetric ample line bundle $\mathscr{L}_\eta$ on the generic fiber G_η. Then there exists a positive integer n such that $\mathscr{L}_\eta^{\otimes n}$ extends to a symmetric relatively ample line bundle on $G \to S$.*

For this next proposition, we recall that a stack is *algebraic* if it admits a surjective, smooth, representable morphism from a scheme, and it is *Deligne-Mumford* if it admits a surjective, étale, representable morphism from a scheme. A Deligne-Mumford stack is separated if its diagonal is proper (or equivalently, finite).

Proposition 3.10: *Let $\mathscr{X}$ be a separated Deligne-Mumford stack over some base scheme S. Let Y be a normal reduced locally Noetherian S-scheme and U a dense open subscheme of Y. Let $f, g : Y \to \mathscr{X}$ be two morphisms such that $f_{|U} \simeq g_{|U}$. Then $f \simeq g$.*

Proof. Let $I = \underline{\mathrm{Isom}}(f, g)$ the algebraic stack obtained by the pullback of the diagonal Δ

$$\begin{tikzcd} I \arrow[r, "\theta"] \arrow[d] & Y \arrow[d, "f\times g"] \\ \mathscr{X} \arrow[r, "\Delta"] & \mathscr{X} \times_S \mathscr{X}. \end{tikzcd}$$

By assumption, Δ, and hence also θ, is finite, so by [LMB18, Théorème A.2] I is a scheme. The assumption $f_{|U} \simeq g_{|U}$ means that θ has a section over U $s : U \to I_{|U}$. Let Z denote the scheme-theoretic closure of $s(U)$ in I. We want to show that $\theta_{|Z} : Z \to Y$ is an isomorphism. Working componentwise, we can assume that Y, and so U and Z, are irreducible. Since $\theta(Z)$ contains U, $\theta_{|Z}$ is dominant. By [Sta26, Tag 0356] Z is reduced. Thus $\theta_{|Z} : Z \to Y$ is a finite birational morphism between integral schemes. By [Sta26, Tag 0AB1] it is an isomorphism. □

Remark 3.11: The normality assumption on Y in Proposition 3.10 cannot be dropped. Over some base field k, consider Y to be $\mathbb{P}^1$ with 0 and ∞ glued together, let U be Y minus the singular point, and set $\mathscr{X} = \mathbf{B}\mathbb{Z}/2\mathbb{Z}$. Let Y_1 be two disjoint copies of Y and Y_2 be two copies of $\mathbb{P}^1$ glued at zero and glued at infinity. Then there are obvious projection maps $Y_i \to Y$, $i \in \{1, 2\}$, which are double étale covers, and which both restrict to the trivial double cover over U. In this way, we find two non-isomorphic maps $Y \to \mathscr{X}$ which are isomorphic over the dense open subset U.

The following corollary will be used repeatedly.

Corollary 3.12: *Let $\mathcal{X}/\mathcal{O}_F$ be a regular scheme, $\mathscr{S}/\mathcal{O}_F$ a smooth, separated Deligne-Mumford stack. Let X/F and $\mathcal{S}/F$ be their generic fibers. Let $f : X \to \mathcal{S}$ be an F-morphism. Let $g, h : \mathcal{X} \to \mathscr{S}$ be two $\mathcal{O}_F$-morphisms isomorphic over X. Then $g \simeq h$.*

4. Extension properties for good reduction

4.1. **L-extension and PRG-extension.**

Definition 4.1: A *requirement* is a pair $\tau = (C_1, C_2)$ consisting of

- a collection C_1 of $\mathcal{O}_F$-schemes stable under étale covers, and
- a collection C_2 of morphisms $f : X \to \mathcal{S}$ where X/F is the generic fiber of an $\mathcal{O}_F$ scheme $\mathcal{X}$ in C_1, and $\mathcal{S}$ is the generic fiber of an algebraic stack $\mathscr{S}/\mathcal{O}_F$, satisfying the following condition:

  if $\mathscr{S} \to \mathscr{S}_0$ is a surjective étale morphism of $\mathcal{O}_F$-algebraic stacks, then the composition $f : X \to \mathcal{S} \to \mathcal{S}_0$ is again in C_2.

Definition 4.2: Let $\tau = (C_1, C_2)$ be a requirement, let $\mathscr{S}/\mathcal{O}_F$ be a separated Deligne-Mumford stack with generic fiber $\mathcal{S}/F$. We say that $\mathscr{S}$ *has τ-extension* if for any scheme $\mathcal{X}/\mathcal{O}_F$ in C_1 with generic fiber X/F, any morphism

$$f : X \to \mathcal{S}$$

in C_2 admits a unique (up to isomorphism) extension

$$\tilde{f} : \mathcal{X} \to \mathscr{S}.$$

The uniqueness in Definition 4.2 will be ensured by Corollary 3.12 in all cases below, so we omit it. The technical conditions in Definition 4.1 ensure the following:

Lemma 4.3: *Let τ be a requirement, let $\mathscr{S}/\mathcal{O}_F$ be a stack with τ-extension. Consider $\mathscr{S}'/\mathcal{O}_F$ an étale atlas over $\mathscr{S}/\mathcal{O}_F$ which is separated over $\mathcal{O}_F$. Then $\mathscr{S}'$ has τ-extension.*

Proof. Write $\tau = (C_1, C_2)$. Let $\mathcal{S}, \mathcal{S}'/F$ denote the generic fibers of $\mathscr{S}, \mathscr{S}'/\mathcal{O}_F$. Let $\mathcal{X}/\mathcal{O}_F$ be in C_1, let $f : X \to S'$ be in C_2. We want to show that f extends to a morphism $\mathcal{X} \to \mathscr{S}'$ over $\mathcal{O}_F$. Using τ-extension for $\mathscr{S}$, the composition $X \to S' \to S$ extends uniquely to $h : \mathcal{X} \to \mathscr{S}$. The situation is summarized in the following commutative diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, hook] & \mathcal{S}' \arrow[r] \arrow[d, hook] & \mathcal{S} \arrow[d, hook] \\
\mathcal{X} \arrow[r, dashed, "?"] \arrow[rr, bend right, "h"'] & \mathscr{S}' \arrow[r] & \mathscr{S}.
\end{tikzcd}$$

Let $\mathcal{X}' := \mathcal{X} \times_{\mathscr{S}} \mathscr{S}'$ which is an étale cover of $\mathcal{X}$, let X'/F denote its generic fiber. We wish to show that $h' : \mathcal{X}' \to \mathscr{S}'$ descends to a morphism $\mathcal{X} \to \mathscr{S}'$, in other words we need to show that both composition arrows

$$\mathcal{X}' \times_{\mathcal{X}} \mathcal{X}' \rightrightarrows \mathcal{X}' \to \mathscr{S}'$$

coincide. This follows from Corollary 3.12. ☐

Remark 4.4: In Definition 4.1, Definition 4.2 and Lemma 4.3 one can replace stacks with pairs $(\mathscr{S}, \mathbb{L})$ comprised of a stack $\mathscr{S}$ with a local system $\mathbb{L}$, and morphisms of stacks with arrows $(\mathscr{S}', \mathbb{L}') \to (\mathscr{S}, \mathbb{L})$ in the 2-category of stacks with local systems[8]. This is needed to make Definition 4.5 below rigorous.

Definition 4.5: Let $\mathscr{S}/\mathcal{O}_F$ be a smooth separated Deligne-Mumford stack with generic fiber $\mathcal{S}/F$. Let $\mathbb{L}$ be an étale local system on $\mathscr{S}$. We say that the pair $(\mathscr{S}, \mathbb{L})$ *has L-extension* if it has τ-extension for the requirement $\tau = (C_1, C_2)$ where

- C_1 is the collection of smooth schemes $\mathcal{X}/\mathcal{O}_F$, and
- C_2 is the collection of morphisms $f : X \to \mathscr{S}$ such that the pullback $f^{-1}\mathbb{L}$ extends to a local system on $\mathcal{X}$.

If $(\mathscr{S}, \mathbb{L})$ has L-extension, then any étale atlas $\mathscr{S}' \to \mathscr{S}$ also has it with respect to the pullback of $\mathbb{L}$ by Lemma 4.3 and Remark 4.4. In our applications however, we will be considering non-surjective étale morphisms $\mathcal{U} \to \mathscr{S}$: it is no longer true that $\mathcal{U}$ inherits L-extension. Fortunately, L-extension implies a weaker property called the PGR-extension, which we define below. In the next paragraph we show that PGR-extension pulls back under étale maps.

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[8] That is, a morphism $\mathscr{S}' \to \mathscr{S}$ together with and isomorphism between $\mathbb{L}'$ and the pullback of $\mathbb{L}$.

Definition 4.6: Let $\mathcal{X}, \mathscr{M}/\mathcal{O}_F$ be smooth algebraic stacks with generic fibers $X, M/F$. A *PGR-morphism* (as in Pointwise Good Reduction) is a morphism $f : X \to M$ between the generic fibers such that $f(|X|^{\text{int}}) \subseteq |M|^{\text{int}}$.

The restriction of an $\mathcal{O}_F$-morphism $\mathcal{X} \to \mathscr{M}$ to the special fibers is a PGR-morphism.

Remark 4.7: The acronym PGR is motivated by the case, of interest to us, where $\mathscr{M}/\mathcal{O}_F$ is a moduli stack for some families of smooth and proper algebraic spaces over $\mathcal{O}_F$. In this case, a PGR-morphism $f : X \to M$ corresponds to a smooth proper family $Y \to X$ with pointwise good reduction as in Definition 3.3.

Definition 4.8: Let $\mathscr{M}/\mathcal{O}_F$ be a smooth separated Deligne-Mumford stack with generic fiber M/F. We say that it *has PGR-extension* if it has τ-extension for the requirement $\tau = (C_1, C_2)$ where

- C_1 is the collection of smooth, geometrically connected[9] schemes $\mathcal{X}/\mathcal{O}_F$, and
- C_2 is the collection of PGR-morphisms $f : X \to M$.

Remark 4.9: Assuming $\mathscr{M}/\mathcal{O}_F$ is a moduli stack for some smooth and proper families of schemes over $\mathcal{O}_F$, having PGR-extension exactly states that such a family with pointwise good reduction has global good reduction.

Proposition 4.10: *Let $\mathscr{S}/\mathcal{O}_F$ be a smooth, separated Deligne-Mumford stack with a local system $\mathbb{L}$. Assume the pair $(\mathscr{S}, \mathbb{L})$ has L-extension as in Definition 4.5. Then $\mathscr{S}$ has PGR-extension.*

Proof. Let X/F be a smooth, geometrically connected variety with smooth model $\mathcal{X}/\mathcal{O}_F$, together with a PGR-morphism $f : X \to \mathcal{S}$, where $\mathcal{S}$ is the generic fiber of $\mathscr{S}$. We want to show that f extends to a morphism $\mathcal{X} \to \mathscr{S}$ over $\mathcal{O}_F$. It suffices to show that $f^{-1}\mathbb{L}$ extends to a local system on $\mathcal{X}$ and apply L-extension.

Let $x \in |X|^{\text{int}}$, then $f(x) \in |\mathcal{S}|^{\text{int}}$ is in the image of some $\overline{\mathcal{O}_F}$-point of $\mathscr{S}$. In particular, the Galois representation defined by the pullback of $\mathbb{L}$ to $f(x)$ is unramified, hence so is its pullback to x. Thus $f^{-1}\mathbb{L}$ is unramified at every point of $|X|^{\text{int}}$; by [CT26, Theorem 1] it extends to a local system over $\mathcal{X}$. ☐

4.2. **Pulling back PGR-extension along étale maps.**

We aim to show the following statement.

Theorem 4.11: *Let $\mathscr{S}/\mathcal{O}_F$ be a smooth, separated Deligne-Mumford stack with PGR-extension. Let $\mathcal{U}/\mathcal{O}_F$ be an algebraic stack étale over $\mathscr{S}$. Then $\mathcal{U}$ has PGR-extension.*

We first consider the case when $\mathscr{S}$ is a scheme.

Proposition 4.12: *Let $\mathscr{S}/\mathcal{O}_F$ be a smooth, separated scheme with PGR-extension. Let $\mathcal{U}/\mathcal{O}_F$ be a scheme étale over $\mathscr{S}$. Then $\mathcal{U}$ has PGR-extension.*

Proof. Let X/F be a smooth, geometrically connected variety with smooth model $\mathcal{X}/\mathcal{O}_F$, together with a PGR-morphism $f : X \to U$, where U is the generic fiber of $\mathcal{U}$. We want to show that f extends to a morphism $\mathcal{X} \to \mathcal{U}$ over $\mathcal{O}_F$. Let $\mathcal{S}$ denote the generic fiber of $\mathscr{S}/\mathcal{O}_F$; the composition $X \to U \to \mathcal{S}$ is again PGR, hence extends to $h : \mathcal{X} \to \mathscr{S}$. The situation is summarized in the following commutative diagram:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, hook] & U \arrow[r, "g"] \arrow[d, hook] & \mathcal{S} \arrow[d, hook] \\
\mathcal{X} \arrow[r, dashed, "?"] \arrow[rr, bend right, "h"'] & \mathcal{U} \arrow[r, "\tilde{g}"] & \mathscr{S}.
\end{tikzcd}$$

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[9] This condition should be replaced with "geometrically connected irreducible components" so as to ensure stability under étale covers. We omit it for readability, and because there is no harm in working componentwise.

Let $\Gamma_f \hookrightarrow X \times_F U$ be the graph of f, let Z denote its closure in $\mathcal{X} \times_{\mathcal{O}_F} \mathcal{U}$. It suffices to show that the projection $Z \to \mathcal{X}$ is an isomorphism, so that the extension of f is provided by

$$\tilde{f} : \mathcal{X} \to Z \hookrightarrow \mathcal{X} \times_{\mathcal{O}_F} \mathcal{U} \to \mathcal{U}.$$

Let $\Gamma_h \hookrightarrow \mathcal{X} \times_{\mathcal{O}_F} \mathcal{U}$ denote the graph of h. Let $\tilde{Z}$ denote the pullback of Γ_h along $\tilde{g}$:

$$\begin{tikzcd}
\tilde{Z} \arrow[r, hook] \arrow[d] & \mathcal{X} \times_{\mathcal{O}_F} \mathcal{U} \arrow[r, "p_2"] \arrow[d, "\mathrm{id}\times\tilde{g}"] & \mathcal{U} \arrow[d, "\tilde{g}"] \\
\Gamma_h \arrow[r, hook] & \mathcal{X} \times_{\mathcal{O}_F} \mathscr{S} \arrow[r, "p_1"] & \mathcal{X}.
\end{tikzcd}$$

By definition $\tilde{Z}$ is a closed subscheme of $\mathcal{X} \times_{\mathcal{O}_F} \mathcal{U}$ containing Γ_f, hence Z is a closed subscheme of $\tilde{Z}$. By Lemma 4.13 below, the projection $Z \to \mathcal{X}$ is surjective. The commutative diagram

$$\begin{tikzcd}
Z \arrow[r, hook] \arrow[d] & \mathcal{X} \times_{\mathcal{O}_F} \mathcal{U} \arrow[r, "p_1"] \arrow[d, "\mathrm{id}\times\tilde{g}"] & \mathcal{X} \arrow[d, "\mathrm{id}"] \\
\Gamma_h \arrow[r, hook] \arrow[rr, bend right, "\sim"'] & \mathcal{X} \times_{\mathcal{O}_F} \mathscr{S} \arrow[r, "p_1"] & \mathcal{X}
\end{tikzcd}$$

shows that this projection factors through $Z \to \Gamma_h \xrightarrow{\sim} \mathcal{X}$, thus $\theta : Z \to \Gamma_h$ is surjective. By Lemma 4.14 below, $Z \to \Gamma_h$ is étale. By [Gro66, Proposition 18.2.8] the number of points in the geometric fibers of θ is lower semicontinuous, bounded below by 1 by surjectivity, and equals 1 on the dense open subscheme Γ_f (as $\Gamma_f \to \Gamma_{g \circ f}$ is an isomorphism); it is thus constant equal to 1. By *loc.cit.*, θ is finite, and so by [Sta26, Tag 0AB1] it is an isomorphism. ☐

Lemma 4.13: *Let X/F be a smooth variety with smooth model $\mathcal{X}/\mathcal{O}_F$. Let U/F be a variety with model $\mathcal{U}/\mathcal{O}_F$. Let $f : X \to U$ be a PGR-morphism, denote $\Gamma \hookrightarrow X \times_F U$ its graph. Let Z be the closure of Γ in $\mathcal{X} \times_{\mathcal{O}_F} \mathcal{U}$. Then the projection $Z \to \mathcal{X}$ is surjective.*

Proof. We show that the set-theoretic image of $p : Z \to \mathcal{X}$ contains all closed points. Then by Chevalley's theorem the image is a locally constructible subset of $\mathcal{X}$ containing all closed points, thus it is the whole of $\mathcal{X}$. Since $\Gamma \subseteq Z$, X is contained in $p(Z)$, so it remains to show that the closed points of the special fiber $\mathcal{X}_s$ of $\mathcal{X}$ are attained.

Let $x_s \in \mathcal{X}_s$ be a closed point. By the infinitesimal lifting criterion for smoothness, x_s lifts to an $\mathcal{O}_E$-point $x : \operatorname{Spec} \mathcal{O}_E \to \mathcal{X}$ for E/F a finite extension. Let $x_0 : \operatorname{Spec} E \to X$ denote the special fiber of x. Let $y_0 = f(x_0)$, $y_0 \in |U|^{\mathrm{int}}$ by the PGR assumption, hence y_0 is the special fiber of some $\mathcal{O}_{E'}$-point $y : \operatorname{Spec} \mathcal{O}_{E'} \to \mathcal{U}$ for E'/F a finite extension. We may assume $E = E'$. The restriction $x_0 \mapsto y_0$ defines a one-point graph Γ_0:

$$\Gamma_0 \hookrightarrow \operatorname{Spec} E \times_F \operatorname{Spec} E$$

which coincides with the diagonal morphism. After compositition with the open embedding $\operatorname{Spec} E \times_F \operatorname{Spec} E \hookrightarrow \operatorname{Spec} \mathcal{O}_E \times_{\mathcal{O}_F} \operatorname{Spec} \mathcal{O}_E$, the closure of Γ_0 coincides with the diagonal

$$\Delta_{\mathcal{O}_E} : \operatorname{Spec} \mathcal{O}_E \hookrightarrow \operatorname{Spec} \mathcal{O}_E \times_{\mathcal{O}_F} \operatorname{Spec} \mathcal{O}_E.$$

The situation is summarized in the following commutative diagram:

$$\begin{tikzcd}
\Gamma_0 \arrow[r, hook, "x_0 \mapsto y_0"] \arrow[d, hook] \arrow[rrd, hook] & \Gamma_f \arrow[d, hook] \arrow[rrd] & & \\
\operatorname{Spec} E \times_F \operatorname{Spec} E \arrow[r, "{(x_0,y_0)}"] \arrow[rrd, hook] & X \times_F U \arrow[rrd, hook] & \operatorname{Spec} \mathcal{O}_E \arrow[r, dashed, "?"'] \arrow[d, hook, "\Delta_{\mathcal{O}_E}"'] & Z \arrow[d, hook] \\
& & \operatorname{Spec} \mathcal{O}_E \times_{\mathcal{O}_F} \operatorname{Spec} \mathcal{O}_E \arrow[r, "{(x,y)}"'] & \mathcal{X} \times_{\mathcal{O}_F} \mathcal{U}.
\end{tikzcd}$$

The composition $\operatorname{Spec} \mathcal{O}_E \xrightarrow{\Delta_{\mathcal{O}_E}} \mathcal{O}_E \times_{\mathcal{O}_F} \operatorname{Spec} \mathcal{O}_E \xrightarrow{(x,y)} \mathcal{X} \times_{\mathcal{O}_F} \mathcal{U}$ factors through Z. Indeed, the pullback of Z along (x,y) is a closed subscheme of $\mathcal{O}_E \times_{\mathcal{O}_F} \operatorname{Spec} \mathcal{O}_E$ containing Γ_0, thus it contains the image of the diagonal $\Delta_{\mathcal{O}_E}$. We conclude by observing that the morphism $\operatorname{Spec}\mathcal{O}_E \to Z \to \mathcal{X}$ is the $\mathcal{O}_E$-point x, so x_s lies in its image. ☐

Lemma 4.14: *Let $\phi : \tilde{Z} \to \Gamma$ be an étale morphism of schemes with Γ regular irreducible. Let $Z \hookrightarrow \tilde{Z}$ be an irreducible closed subscheme such that $\theta : Z \to \Gamma_2$ is again surjective. Then θ is étale.*

Proof. The surjectivity of θ ensures $\dim \tilde{Z} \geq \dim Z \geq \dim \Gamma_2 = \dim \tilde{Z}$, so Z is an irreducible component of $\tilde{Z}$. As ϕ is étale, $\tilde{Z}$ is also regular, hence so are its irreducible components. By miracle flatness θ is flat. It is also unramified, hence étale. ☐

Proof of Theorem 4.11. Let X/F be a smooth, geometrically connected variety with smooth model $\mathcal{X}/\mathcal{O}_F$, together with a PGR-morphism $f : X \to U$, where U is the generic fiber of $\mathcal{U}$. We want to show that f extends to a morphism $\mathcal{X} \to \mathcal{U}$ over $\mathcal{O}_F$.

Consider $\mathscr{S}'/\mathcal{O}_F$ an étale atlas over $\mathscr{S}/\mathcal{O}_F$. Up to replacing $\mathscr{S}'$ with a disjoint union of open affines, we can assume that $\mathscr{S}'/\mathcal{O}_F$ is separated. By Lemma 4.3, $\mathscr{S}'$ has PGR-extension. Let $\mathcal{X}'/\mathcal{O}_F$ and $\mathcal{U}'/\mathcal{O}_F$ be algebraic spaces obtained by base change by $\mathscr{S}' \to \mathscr{S}$, write X'/F and U'/F for their generic fibers. Since $\mathscr{S}' \to \mathscr{S}$ is an étale presentation of a Deligne-Mumford stack with separated and quasi-compact diagonal, $\mathcal{X}'$ and $\mathcal{U}'$ are schemes. By Proposition 4.12, $\mathcal{U}'$ has PGR-extension, and since $f' : X' \to U'$ is again PGR, thus we find a unique morphism

$$\tilde{f}' : \mathcal{X}' \to \mathcal{U}'$$

extending f'. Now it suffices to show that the composition $\mathcal{X}' \to \mathcal{U}' \to \mathcal{U}$ descends to a morphism $\mathcal{X} \to \mathcal{U}$, in other words we need to show that both composition arrows

$$\mathcal{X}' \times_{\mathcal{X}} \mathcal{X}' \rightrightarrows \mathcal{X}' \to \mathcal{U}' \to \mathcal{U}$$

are isomorphic. This follows from Corollary 3.12. ☐

We end this paragraph by showing that PGR-extension descends along finite torsors.

Proposition 4.15: *Let H be a finite group. Let $\mathscr{M}/\mathcal{O}_F$ be a smooth, separated Deligne-Mumford stack. Let $P : \mathscr{M} \to \mathbf{B}H$ denote an H-torsor on $\mathscr{M}$, consider $\mathcal{U} \to \mathscr{M}$ the corresponding finite étale cover. If $\mathcal{U}$ has PGR-extension, then so does $\mathscr{M}$.*

Proof. Let X/F be a smooth, geometrically connected variety with smooth model $\mathcal{X}/\mathcal{O}_F$, together with a PGR-morphism $f : X \to M$, where M is the generic fiber of $\mathscr{M}$. We want to show that f extends to a morphism $\mathcal{X} \to \mathscr{M}$ over $\mathcal{O}_F$.

Let $X' := X \times_M U$, with U the generic fiber of $\mathcal{U}$. We will argue that X'/F has a smooth model $\mathcal{X}'/\mathcal{O}_F$ along with a finite étale cover $\mathcal{X}' \to \mathcal{X}$ extending $X' \to X$. It then follows that $X' \to U$ is a PGR-morphism, hence extends to $\mathcal{X}' \to \mathcal{U}$. It remains to show that the composition $\mathcal{X}' \to \mathcal{U} \to \mathscr{M}$ descends to a morphism $\mathcal{X} \to \mathscr{M}$, this is done exactly as in the proof of Theorem 4.11 using Corollary 3.12.

We construct $\mathcal{X}'$ as follows. From Remark 2.17 we have the 2-cartesian diagram

$$\begin{tikzcd}
\mathcal{U} \arrow[r] \arrow[d] & \mathbf{B}\mathcal{E} \arrow[d] \\
\mathscr{M} \arrow[r] & \mathbf{B}H.
\end{tikzcd}$$

Arguing as in Proposition 4.10, it follows from [CT26, Theorem 1] that the H-torsor $f^{-1}H$ on X extends to $\mathcal{X}$, hence a morphism $\mathcal{X} \to \mathbf{B}H$. We define the finite étale cover $\mathcal{X}' \to \mathcal{X}$ via the 2-cartesian diagram

$$\begin{tikzcd}
\mathcal{X}' \arrow[r] \arrow[d] & \mathbf{B}\mathcal{E} \arrow[d] \\
\mathcal{X} \arrow[r] & \mathbf{B}H.
\end{tikzcd}$$

It follows that the generic fiber of $\mathcal{X}'$ recovers X'. ☐

4.3. Smooth extension.

Definition 4.16: Let $\mathscr{S}/\mathcal{O}_F$ be a smooth separated Deligne-Mumford stack with generic fiber $\mathcal{S}/F$. We say that it *has smooth-extension* if it has τ-extension for the requirement $\tau = (C_1, C_2)$ where

- C_1 is the collection of regular, formally smooth schemes $\mathcal{X}/\mathcal{O}_F$, and
- C_2 is the collection of F-morphisms $f : X \to \mathcal{S}$.

Restricting C_1 to smooth separated $\mathcal{O}_F$ schemes in Definition 4.16 would recover the definition of Néron models. As we will be dealing with pro-schemes, we need to remove the finite type assumption. The following statement is a reformulation of [BST24, §1.3].

Proposition 4.17: *Let R be a topological ring, let V be a free R-module of finite rank, let K_0 be a profinite subgroup of* $\mathrm{GL}(V)$. *Recall that $\mathcal{C}(K_0)$ denotes the set of all open normal subgroups of K_0.*

Let $(\mathscr{S}_K)_{K\in\mathcal{C}(K_0)}$ be a projective system of $\mathcal{O}_F$-stacks fit for K_0 as in Definition 2.13. Let

$$\mathscr{S}_\infty := \varprojlim_{K\in\mathcal{C}(K_0)} \mathscr{S}_K.$$

Let $\mathbb{L}^{\mathrm{can}}$ be the canonical local system on $\mathscr{S}_{K_0}$ as in Definition 2.14. If $\mathscr{S}_\infty$ has smooth-extension, then $(\mathscr{S}_{K_0}, \mathbb{L}^{\mathrm{can}})$ has L-extension.

Proof. For $K \in \mathcal{C}(K_0)$, write $\mathcal{S}_K/F$ for the generic fiber of $\mathscr{S}_K$, similarly for $\mathcal{S}_\infty/F$. Let $\mathcal{X}/\mathcal{O}_F$ be a smooth scheme with generic fiber X/F, let $f : X \to \mathcal{S}_{K_0}$ such that $f^{-1}\mathbb{L}^{\mathrm{can}}$ extends to a local system $\mathbb{L}$ on $\mathcal{X}$. We want to show that f extends to $\tilde{f} : \mathcal{X} \to \mathscr{S}_{K_0}$.

By definition, $\mathbb{L}^{\mathrm{can}}$ is induced from a K_0-torsor on $\mathscr{S}_{K_0}$, thus $\mathbb{L}$ is induced from a K_0-torsor on $\mathcal{X}$. From Remark 2.16, this defines a projective system of schemes $(\mathcal{X}_K)_{K\in\mathcal{C}(K_0)}$ fit for K_0, with $\mathcal{X}_{K_0} = \mathcal{X}$. Let

$$\mathcal{X}_\infty := \varprojlim_K \mathcal{X}_K$$

which is a regular formally smooth $\mathcal{O}_F$-scheme, with generic fiber X_∞/F. If X_K/F denotes the generic fiber of $\mathcal{X}_K$, the system $(X_K)_K$ coincides with the pullback along $f : X \to \mathcal{S}_{K_0}$ of the system $(\mathcal{S}_K)_K$. In particular, there is an induced F-morphism

$$X_\infty \to \mathcal{S}_\infty$$

with $\mathcal{S}_\infty/F$ the generic fiber of $\mathscr{S}_\infty$. By smooth extension, there is an induced morphism

$$\mathcal{X}_\infty \to \mathscr{S}_\infty.$$

Now it suffices to show that the composition $\mathcal{X}_\infty \to \mathscr{S}_\infty \to \mathscr{S}_{K_0}$ descends to a morphism $\mathcal{X}_{K_0} \to \mathscr{S}_{K_0}$, in other words we need to show by fpqc descent that both composition arrows

$$\mathcal{X}_\infty \times_{\mathcal{X}_{K_0}} \mathcal{X}_\infty \rightrightarrows \mathcal{X}_\infty \to \mathscr{S}_\infty \to \mathscr{S}_{K_0}$$

are isomorphic. This follows from Corollary 3.12. □

4.4. **Applications to Shimura varieties.** Let (G, Ω) be a Shimura datum with reflex field $E/\mathbb{Q}$, and $K_p \subseteq G(\mathbb{Q}_p)$ a hyperspecial subgroup. Let v be a place of E lying above p, $\mathcal{O}_{(v)}$ the local ring of $\mathcal{O}_E$ at v and $\mathcal{O}_v$ its v-adic completion. Consider a neat level $K = K^pK_p$ where K^p is a compact open subgroup of $G(\mathbb{A}_f^p)$, write $\mathrm{Sh}_K(G,\Omega)/E$ for the corresponding variety over E. Set

$$\mathrm{Sh}_{K_p}(G,\Omega) = \varprojlim_{K^p} \mathrm{Sh}_{K^pK_p}(G,\Omega)$$

which is an E-scheme with $G(\mathbb{A}_f^p)$-action. If $K = K^pK_p$ is no longer neat, we let $\mathrm{Sh}_K[G,\Omega] = [\mathrm{Sh}_{K_p}(G,\Omega)/K^p]$ denote the quotient stack.

Let $\mathcal{O}$ be a discrete valuation ring, faithfully flat over $\mathcal{O}_{(v)}$, write $\tilde{E}$ for its fraction field and $\mathfrak{m}_{\mathcal{O}}$ for its maximal ideal. We recall from [Vas99], [Moo98] and [Kis09] the notion of smooth integral canonical models of Shimura varieties.

Definition 4.18: A *smooth integral model* of $\mathrm{Sh}_{K_p}(G,\Omega)$ over $\mathcal{O}$ is a faithfully flat $\mathcal{O}$-scheme $\mathscr{S}_{K_p}(G,\Omega)$ with a $G(\mathbb{A}_f^p)$-action, such that

(i) there is a $G(\mathbb{A}_f^p)$-equivariant identification $\mathscr{S}_{K_p}(G,\Omega) \times_{\mathcal{O}} \tilde{E} \simeq \mathrm{Sh}_{K_p}(G,\Omega) \times_E \tilde{E}$, and

(ii) there exists a compact open subgroup $K_0^p \subseteq G(\mathbb{A}_f^p)$ such that for any $K_2^p \subseteq K_1^p \subseteq K_0^p$ open subgroups, the canonical quotient map

$$\mathscr{S}_{K_p}(G,\Omega)/K_1^p \to \mathscr{S}_{K_p}(G,\Omega)/K_2^p$$

is a finite étale morphism between smooth separated schemes of finite type over $\mathcal{O}$.

Moreover, $\mathscr{S}_{K_p}(G,\Omega)$ is a *smooth integral canonical model* if it also has smooth-extension as in Definition 4.16.

Fix $\mathscr{S}_{K_p}(G,\Omega)$ a smooth integral canonical model of $\mathrm{Sh}_{K_p}(G,\Omega)$ over $\mathcal{O}$. For any level $K = K^pK_p$, we let $\mathscr{S}_K[G,\Omega] := [\mathscr{S}_{K_p}(G,\Omega)/K^p]$ denote the quotient stack. Fix K_0^p any compact open subgroup of $G(\mathbb{A}_f^p)$, then the projective system $(\mathscr{S}_{K^pK_p}[G,\Omega])_{K^p\in\mathcal{C}(K_0^p)}$ is fit for K_0^p. From Definition 2.14 there is a canonical local system $\mathbb{L}^{\mathrm{can}}$ on $\mathscr{S}_{K_0^pK_p}[G,\Omega]$.

Proposition 4.19: *The pair* $(\mathscr{S}_{K_0^pK_p}[G,\Omega],\mathbb{L}^{\mathrm{can}})$ *has L-extension, hence also PGR-extension.*

Proof. This follows from Proposition 4.17 and Proposition 4.10. □

Let $\mathcal{O} \to \mathcal{O}'$ be a faithfully flat extension of discrete valuation rings. It is not a priori clear whether the base change $\mathscr{S}_{K_p}(G,\Omega)_{\mathcal{O}'}/\mathcal{O}'$ also has smooth-extension, as schemes $\mathcal{X}/\mathcal{O}'$ need not be formally smooth over $\mathcal{O}$. This holds under additional assumptions.

Fact 4.20 ([Vas99], Proposition 3.2.3.3)**:** *Let* $\mathcal{O} \hookrightarrow \mathcal{O}'$ *be a faithfully flat extension of discrete valuation rings with separable residue field extension, and such that* $\mathfrak{m}_{\mathcal{O}}\mathcal{O}'$ *is maximal in* $\mathcal{O}'$*. Let* $\tilde{E}'$ *be the fraction field of* $\mathcal{O}'$*. Then the base-change* $\mathscr{S}_{K_p}(G,\Omega)_{\mathcal{O}'}$ *is a smooth integral canonical model of* $\mathrm{Sh}_{K_p}(G,\Omega)_{\tilde{E}'}$.

Remark 4.21: The assumptions of Fact 4.20 are also required for the analogous statement in the theory of Néron models (see [BLR12, Theorem 7.2.1 (ii)]).

5. Applications

Assume $p \neq 2$. Let $F/\mathbb{Q}_p$ be a finite field extension.

5.1. Smooth curves.

Let $g \geq 2$, $n \geq 0$ be integers satisfying $2-2g < n$. Let $\mathscr{M}_{g,n}$ denote the smooth, separated Deligne-Mumford stack of genus-g, n-pointed, geometrically connected, smooth, proper curves. Let ℓ_1, ℓ_2 be prime numbers different from p, set $\mathscr{L} = \{\ell_1,\ell_2\}$, and $\mathbb{Z}_{\mathscr{L}} = \mathbb{Z}_{\ell_1}\times\mathbb{Z}_{\ell_2}$. In [Sti05, Remark 2.8 (a)] Stix defines a $\mathbb{Z}_{\mathscr{L}}$-local system $\mathbb{L}$[10] over $\mathscr{M}_{g,n}\times_{\mathbb{Z}}\mathbb{Z}\left[\frac{1}{\mathscr{L}}\right]$.

Let $\mathscr{S} := \mathscr{M}_{g,n}\times_{\mathbb{Z}}\mathcal{O}_F$, the pullback of $\mathbb{L}$ to $\mathscr{S}$ is again denoted $\mathbb{L}$.

Proposition 5.1: *The pair* $(\mathscr{S},\mathbb{L})$ *has L-extension, hence* $\mathscr{S}$ *has PGR-extension.*

Proof. This is a corollary of [Sti05, Theorem 2.10 (B)] and Proposition 4.10. □

Consequently, we recover [CT26, §4.3, Example (1)].

Theorem 5.2: *Let* $\mathscr{X}/\mathcal{O}_F$ *be a smooth, geometrically connected scheme with generic fiber* X/F*. Let* $f: Y \to X$ *be a smooth proper family of genus-g, n-pointed, geometrically connected, smooth, proper curves. If it has pointwise good reduction, then it has global good reduction.*

5.2. Abelian varieties.

Let $g \geq 1$ be an integer, let V be a free abelian group of rank $2g$, let ψ be a nondegenerate symplectic form on V, set $G = \mathrm{GSp}(V_{\mathbb{Q}},\psi_{\mathbb{Q}})$. Let Ω be the space of complex structures on $V_{\mathbb{R}}$ for which (V,ψ) is a polarized Hodge structure. Fix $K_p \subseteq G(\mathbb{Q}_p)$ a hyperspecial level at p, $K = K^pK_p \subseteq G(\mathbb{A}_f)$ a compact open subgroup. The Siegel modular stack $\mathrm{Sh}_K[G,\Omega]$ over $\mathbb{Q}$ has been defined in Section 4.4.

Remark 5.3: It follows from Example 2.23 that the canonical local system on $\mathscr{S}_K[G,\Omega]$ coincides with the universal Tate module.

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[10] We warn the reader that in [Sti05] the notation $\mathbb{L}$ refers to the set denoted $\mathscr{L}$ here.

Fact 5.4: *Assume $F/\mathbb{Q}_p$ is unramified. The Shimura variety $\mathrm{Sh}_{K_p}(G,\Omega)$ admits a smooth integral canonical model $\mathscr{S}_{K_p}(G,\Omega)$ over $\mathcal{O}_F$ as in Definition 4.18.*

Proof. The existence of $\mathscr{S}_{K_p}(G,\Omega)$ over $\mathbb{Z}_{(p)}$ is shown in [Mil92, Theorem 2.10], except the last part of the proof has to be modified following [Moo98, Corollary 3.8]. By Fact 4.20, its base-change to $\mathcal{O}_F$ is again a smooth integral canonical model. ☐

From Proposition 4.19 we find that the quotient stack $\mathscr{S}_K[G,\Omega]$ has PGR-extension. Thus, for $F/\mathbb{Q}_p$ unramified, polarized abelian schemes with K-level structure having pointwise good reduction also have global good reduction. By choosing $K^p = G(\widehat{\mathbb{Z}}^p)$, we find:

Theorem 5.5: *Assume $F/\mathbb{Q}_p$ is unramified. Let X/F be a smooth, geometrically connected variety, and $\mathcal{X}/\mathcal{O}_F$ a smooth model. Consider $A \to X$ a polarized abelian scheme. If it has pointwise good reduction, then it has global good reduction.*

Proof. We may choose the symplectic module (V,ψ) so that the abelian scheme $A \to X$ defines a PGR-morphism $X \to \mathrm{Sh}_K[G,\Omega]$ with $K^p = G(\widehat{\mathbb{Z}}^p)$, which extends to $\mathcal{X} \to \mathscr{S}_K[G,\Omega]$ by PGR-extension. ☐

Theorem 5.5 is a weaker version of the following result of Cadoret-Tamagawa, which relies on [VZ10, Corollary 5] with no need for Shimura varieties.

Fact 5.6 ([CT26], §4.3.2, Example (3))**:** *Assume $F/\mathbb{Q}_p$ has ramification index $e \leq p-1$. Let X/F be a smooth, geometrically connected variety, and $\mathcal{X}/\mathcal{O}_F$ a smooth model. Let $A \to X$ be an abelian scheme. If it has pointwise good reduction, then it has global good reduction.*

The bound on e is optimal, as shown by Corollary 3.8. The theory of Shimura varieties provides a second interpretation for this obstruction. It was asked by Vasiu in [Vas99, Remark 3.2.12.1] if Fact 4.20 still holds for ramified extensions. For the Siegel modular variety, the answer is negative in general.

Proposition 5.7: *Assume $F/\mathbb{Q}_p$ has ramification index $e \geq p$. There are symplectic modules (V,ψ) such that smooth integral canonical models for $\mathrm{Sh}_{K_p}(G,\Omega)_F$ do not exist.*

Proof. If such models were to exist, Fact 5.4 and Theorem 5.5 would hold for F, but this is not the case as shown by Corollary 3.8. ☐

5.3. **K3 surfaces.** Fix $d \geq 1$ an integer such that $p \nmid 2d$. The K3 lattice is the rank 22 quadratic lattice $\Lambda = E_8^{\oplus 2} \oplus U^{\oplus 3}$, where E_8 is the E8 lattice of rank 8 and U is the hyperbolic lattice of rank 2. Let $e, f \in U \subseteq \Lambda$ be isotropic elements satisfying $[e,f]=1$ in some copy of the hyperbolic lattice, where $[\cdot,\cdot]$ is the bilinear symmetric form on U. Set $L_d = \langle e - df\rangle^{\perp} \subseteq \Lambda$ a sublattice of discriminant $2d$ and rank 21 and write $W \coloneqq L_{d,\mathbb{Q}}$. We let $L_d^{\vee} \subseteq W$ denote the dual lattice. To this data is associated a Shimura stack of orthogonal type, following [MP15, §3.1].

Definition 5.8: Let $G = \mathrm{SO}(W)$ and Ω the space of oriented negative planes in $L_{d,\mathbb{R}}$. Let $K_0 = K_0^p K_p \subseteq G(\mathbb{A}_f)$ denote the largest subgroup of $\mathrm{SO}(L_d)(\widehat{\mathbb{Z}})$ acting trivially on $L_d^{\vee}/L_d$. For any level $K = K^p K_p \subseteq K_0$, let $\mathrm{Sh}_K[G,\Omega]$ be the Shimura stack of level K associated to the Shimura datum (G,Ω), over its reflex field $\mathbb{Q}$.

Fact 5.9: *Assume $F/\mathbb{Q}_p$ is unramified. The Shimura variety $\mathrm{Sh}_{K_p}(G,\Omega)$ admits a smooth integral canonical model $\mathscr{S}_{K_p}(G,\Omega)$ over $\mathcal{O}_F$.*

Proof. The existence of $\mathscr{S}_{K_p}(G,\Omega)$ over $\mathbb{Z}_{(p)}$ is established in [MP16, Proposition 7.9]. By Fact 4.20, its base-change to $\mathcal{O}_F$ is again a smooth integral canonical model. ☐

We assume from now-on that $F/\mathbb{Q}_p$ is unramified and that $\mathscr{S}_{K_p}(G,\Omega)$ is as in Fact 5.9. From Proposition 4.19 we find that the quotient stack $\mathscr{S}_K[G,\Omega]$ has PGR-extension.

Next we introduce the moduli space of primitively polarized K3 surfaces with level structure, following [MP15, §2.1, §2.10].

Definition 5.10: Let $\mathscr{M}^\circ_{2d}$ be the moduli problem that assigns to every $\mathbb{Z}_{(p)}$-scheme T the groupoid of tuples $(f : Y \to T, \xi)$ where

- $f : Y \to T$ is a K3 surface[11], and
- $\xi \in \underline{\mathrm{Pic}}(X/T)(T)$ is a primitive polarization of degree $2d$.

It is a separated Deligne-Mumford stack of finite type over $\mathbb{Z}_{(p)}$ ([Riz05] Theorem 4.3.3).

If $(f : Y \to T, \xi) \in \mathscr{M}^\circ_{2d}(T)$, let $\mathbf{H}^2_{\widehat{\mathbb{Z}}^p}(f) = \prod_{\ell \neq p} \mathbf{H}^2_\ell(f)$ be the $\widehat{\mathbb{Z}}^p$-local system on T defined as the product of the relative ℓ-adic cohomology sheaves. The polarization defines a relative Chern class $\mathrm{ch}_{\widehat{\mathbb{Z}}^p}(\xi) \in \mathbf{H}^2_{\widehat{\mathbb{Z}}^p}(f)(1)$, the sub-local-system orthogonal to this class is the relative primitive cohomology sheaf $\mathbf{P}^2_{\widehat{\mathbb{Z}}^p}(f)(1)$. Let $V := L_d \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}^p$. With notations from Section 2 there is a local system

$$\begin{aligned} \mathbb{L} : \mathscr{M}^\circ_{2d} &\longrightarrow \mathbf{Loc}_V \\ (f : Y \to T, \xi) &\longmapsto \mathbf{P}^2_{\widehat{\mathbb{Z}}^p}(f)(1). \end{aligned}$$

The cup-product induces a quadratic form on (primitive) cohomology. We define

$$\begin{aligned} \mathrm{Lev} : \mathscr{M}^\circ_{2d} &\longrightarrow \mathbf{B}\mathrm{SO}(L_d)(\widehat{\mathbb{Z}}^p) \\ (f : Y \to T, \xi) &\longmapsto [(U \to T) \mapsto \{\text{Isometries } \mathbf{P}^2_{\widehat{\mathbb{Z}}^p}(f)(1)_U \xrightarrow{\sim} \mathcal{F}_V\}]. \end{aligned}$$

Let $R = \widehat{\mathbb{Z}}^p$ and $Q = \mathrm{SO}(L_d)(R)$, now $(\mathscr{M}^\circ_{2d}, \mathbb{L}, \mathrm{Lev}, \iota)$ is a Q-moduli datum as in Definition 2.18, where ι is the obvious sheaf inclusion of $\mathrm{Lev}(f : Y \to T, \xi)$ inside $\underline{\mathrm{Isom}}(\mathbf{P}^2_{\widehat{\mathbb{Z}}^p}(f)(1), \mathcal{F}_V)$. For any open subgroup $K = K^p K_p \subseteq K_0$ we may consider the associated moduli stack of level K denoted $\mathscr{M}^\circ_{2d,K}$. As K_0 is a finite index open normal subgroup of Q, we find that the projection $\mathscr{M}^\circ_{2d,K_0} \to \mathscr{M}^\circ_{2d}$ is a Q/K_0-torsor.

Ideally, $\mathrm{Sh}_K[G, \Omega]$ would serve as a period domain for the cohomology of K3 surfaces, so as to induce a morphism $\mathscr{M}^\circ_{2d,K,\mathbb{C}} \to \mathrm{Sh}_K[G, \Omega]_{\mathbb{C}}$ which would descend over $\mathbb{Q}$, and finally extend to $\mathscr{M}^\circ_{2d,K,\mathcal{O}_F} \to \mathscr{S}_K[G, \Omega]$ by invoking some extension property. To make this argument work however, $\mathscr{M}^\circ_{2d}$ needs to be replaced with a double cover.

Definition 5.11: Let $\tilde{\mathscr{M}}^\circ_{2d} \to \mathscr{M}^\circ_{2d}$ be the double finite étale cover parametrizing isometric trivializations $\det(\mathbf{P}^2_2) \xrightarrow{\sim} \mathcal{F}_{\det(L_d) \otimes_{\mathbb{Z}} \mathbb{Z}_2}$. Its objects are called *oriented, primitively polarized K3-surfaces of degree* $2d$.

From [MP15, Proposition 4.2, Corollary 4.4 and Proposition 4.7] there is a natural period map $\tilde{\mathscr{M}}^\circ_{2d,\mathbb{C}} \to \mathrm{Sh}_{K_0}(G, \Omega)_{\mathbb{C}}$ which extends to an $\mathcal{O}_F$-morphism $\tilde{\mathscr{M}}^\circ_{2d,\mathcal{O}_F} \to \mathscr{S}_{K_0}(G, \Omega)$.

Fact 5.12 ([MP15], Proposition 4.8)**:** *The map* $\tilde{\mathscr{M}}^\circ_{2d,\mathcal{O}_F} \to \mathscr{S}_{K_0}[G, \Omega]$ *is étale.*

Let $\tilde{\mathscr{M}}^\circ_{2d,K}$ denote the pullback of $\mathscr{M}^\circ_{2d,K} \to \mathscr{M}^\circ_{2d}$ along $\tilde{\mathscr{M}}^\circ_{2d} \to \mathscr{M}^\circ_{2d}$. The morphism in Fact 5.12 lifts to an étale morphism at level $K = K^p K_p \subseteq K_0$:

$$\tilde{\mathscr{M}}^\circ_{2d,K,\mathcal{O}_F} \to \mathscr{S}_K[G, \Omega].$$

Remark 5.13: It follows from Proposition 2.22 that the pullback of $\mathbb{L}_{\mathrm{can}}$ on $\mathscr{S}_K[G, \Omega]$ to $\tilde{\mathscr{M}}^\circ_{2d,K}$ coincides with the universal relative primitive cohomology $\mathbf{P}^2_{\widehat{\mathbb{Z}}^p}(1)$.

Corollary 5.14: *The stacks* $\tilde{\mathscr{M}}^\circ_{2d,\mathcal{O}_F}$, $\mathscr{M}^\circ_{2d,\mathcal{O}_F}$, $\tilde{\mathscr{M}}^\circ_{2d,K,\mathcal{O}_F}$, $\mathscr{M}^\circ_{2d,K,\mathcal{O}_F}$ *and all have PGR-extension.*

Proof. For $\tilde{\mathscr{M}}^\circ_{2d,K,\mathcal{O}_F}$ and $\tilde{\mathscr{M}}^\circ_{2d,K,\mathcal{O}_F}$ this follows from $\mathscr{S}_K[G, \Omega]$ having PGR-extension and Theorem 4.11. As $\tilde{\mathscr{M}}^\circ_{2d} \to \mathscr{M}^\circ_{2d}$ is a $\mathbb{Z}/2\mathbb{Z}$-torsor, PGR-extension for $\mathscr{M}^\circ_{2d,\mathcal{O}_F}$ and $\mathscr{M}^\circ_{2d,K,\mathcal{O}_F}$ is then deduced from Proposition 4.15. □

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[11] That is, Y is a proper and smooth algebraic space over T, with geometric fibers that are K3 surfaces in the usual sense.

Thus, for $F/\mathbb{Q}_p$ unramified, (oriented) primitively polarized K3-surfaces with K-level structure having pointwise good reduction also have global good reduction. In the particular case of $\mathscr{M}^{\circ}_{2d,\mathcal{O}_F}$, we find:

Theorem 5.15: *Assume $F/\mathbb{Q}_p$ is unramified. Let X/F be a smooth, geometrically connected variety, and $\mathcal{X}/\mathcal{O}_F$ a smooth model. Consider $Y \to X$ a primitively polarized K3-surface of degree $2d$.If it has pointwise good reduction, then it has global good reduction.*

Remark 5.16: All statements of this paragraph still hold true if $\mathscr{M}^{\circ}_{2d}$ is replaced with $\mathscr{M}_{2d}$ the moduli stack defined as in Definition 5.10, except ξ is now a primitive quasi-polarization (as in [MP15, §2.1]).

Remark 5.17: In trying to generalize the argument from K3 surfaces to general hyperkähler varieties, only the analogue Fact 5.12 is still missing. One can attach a Shimura datum $(G^{\mathrm{HK}}, \Omega^{\mathrm{HK}})$ to the quadratic lattice defined by the second cohomology group of a fixed hyperkähler variety as in [Bin21, §4.5], as well as introduce the moduli stack $\mathscr{M}_{2d}$ (resp. $\tilde{\mathscr{M}}_{2d}$) of degree-$2d$ polarized, (resp. oriented) hyperkähler varieties over $\mathbb{Z}_{(p)}$, $p \neq d$, as in [Bin21, Definition 3.3.1, Definition 4.3.3][12]. By [MP16, Proposition 7.9] the variety $\mathrm{Sh}_{K_p}(G^{\mathrm{HK}}, \Omega^{\mathrm{HK}})/\mathbb{Q}$ has a smooth integral canonical model $\mathscr{S}_{K_p}(G^{\mathrm{HK}}, \Omega^{\mathrm{HK}})/\mathbb{Z}_{(p)}$, where $K_p \subseteq G^{\mathrm{HK}}(\mathbb{A}_p)$ suitably chosen. By [Bin21, Theorem 4.5.2] there is an étale period map $\tilde{\mathscr{M}}_{2d,K,\mathbb{Q}} \to \mathrm{Sh}_K(G^{\mathrm{HK}}, \Omega^{\mathrm{HK}})$ over $\mathbb{Q}$, and the extension property yields

$$\tilde{\mathscr{M}}_{2d,K} \to \mathscr{S}_K(G^{\mathrm{HK}}, \Omega^{\mathrm{HK}})$$

over $\mathbb{Z}_{(p)}$. If this map were shown to be étale, it would follow as in Corollary 5.14 that $\tilde{\mathscr{M}}_{2d,K}$, $\mathscr{M}_{2d,K}$ and $\mathscr{M}_{2d}$ have PGR-extension, and so Theorem 5.15 would extend to hyperkähler varieties.

5.4. **Cubic fourfolds.** The argument for K3 surfaces applies *mutatis mutandis* to cubic fourfolds; we give references to the main ingredients following [MP15, §5.13].

Let M_0 be the even rank 2 quadratic lattice with bilinear form represented by $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ and set

$$M = E_8^{\oplus 2} \oplus U^{\oplus 2} \oplus M_0$$

which is has signature $(20, 2)$ and is self dual away from 2 and 3.

Definition 5.18: Let $G = \mathrm{SO}(V)$ and Ω the space of oriented negative planes in $M_{\mathbb{R}}$. Let $K_0 = K_0^p K_p \subseteq G(\mathbb{A}_f)$ denote the largest subgroup of $\mathrm{SO}(M)(\widehat{\mathbb{Z}})$ acting trivially on $M^{\vee}/M$. For any level $K = K^p K_p \subseteq K_0$, let $\mathrm{Sh}_K[G, \Omega]$ be the Shimura stack of level K associated to the Shimura datum (G, Ω), over its reflex field $\mathbb{Q}$.

Consider $\mathscr{M}$ the moduli stack of cubic fourfolds over $\mathbb{Z}_{(p)}$, and $\tilde{\mathscr{M}}$ its double cover trivializing the determinant of primitive cohomology. For K as in Definition 5.18, we let $\mathscr{M}_K$ (resp. $\tilde{\mathscr{M}}_K$) be the moduli stack with K-level structure on degree-4 primitive cohomology which is a finite étale cover of $\mathscr{M}$ (resp. $\tilde{\mathscr{M}}$). In particular, $\mathscr{M}_{K_0} \to \mathscr{M}$ is a finite torsor.

From the modular description of $\mathrm{Sh}_{K_0}(G, \Omega)_{\mathbb{C}}$ (see [MP15, Proposition 3.3]) we find a period map:

$$\tilde{\mathscr{M}}_{\mathbb{C}} \to \mathrm{Sh}_{K_0}(G, \Omega)_{\mathbb{C}}$$

which descends over $\mathbb{Q}$ as in [MP15, Corollary 4.4]. By smooth-extension we find a period map $\tilde{\mathscr{M}} \to \mathscr{S}_{K_0}[G, \Omega]$ which is étale by [MP15, Theorem 5.14]. Thus for K as in Definition 5.18 we find étale maps $\tilde{\mathscr{M}}_K \to \mathscr{S}_K(G, \Omega)$. It follows that $\tilde{\mathscr{M}}_K$, hence also $\mathscr{M}_K$ and $\mathscr{M}$, have PGR. This proves the pointwise to global good reduction for (oriented) cubic fourfolds with K-level structure, and in the particular case of $\mathscr{M}$:

Theorem 5.19: *Assume $F/\mathbb{Q}_p$ is unramified. Let X/F be a smooth, geometrically connected variety, and $\mathcal{X}/\mathcal{O}_F$ a smooth model. Consider $Y \to X$ a smooth, proper family of cubic fourfolds. If it has pointwise good reduction, then it has global good reduction.*

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[12] The stack introduced in [Bin21] is the disjoint union over d of the $\tilde{\mathscr{M}}_{2d,\mathbb{Q}}$.

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