AGES April 30th : Line bundles on toric varieties I

I - Recap & examples

$T = (\mathbb{C}^*)^n \subseteq X_\Sigma$ ↺ $(\mathbb{C}^*)^n$

Σ = rational polyhedral fan $\subseteq N_\mathbb{R}$

N free ab gp of rk n

$N_\mathbb{R} = N \otimes_\mathbb{Z} \mathbb{R}$

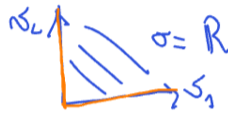
$\sigma \in \Sigma$ gen by integer vectors in N

$\tau, \sigma \in \Sigma \Rightarrow \sigma \cap \tau \in \Sigma$

$\tau < \sigma \in \Sigma \Rightarrow \tau \in \Sigma$

X_Σ is obtained by glueing U_σ, $\sigma \in \Sigma$ together s.t. $U_\sigma \cap U_\tau = U_{\sigma \cap \tau}$

Ex: $N = \mathbb{Z}^2$, $v_1 = \binom{1}{0}$, $v_2 = \binom{0}{1}$



$\sigma = \mathbb{R}_+ v_1 + \mathbb{R} v_2 \supseteq \mathbb{R}_+ v_2$, $\mathbb{R}_+ v_1$, $\{0\}$

$N^* = \mathrm{Hom}(N, \mathbb{Z})$

$U_\sigma \supseteq U_{\mathbb{R}_+ v_1}, U_{\mathbb{R}_+ v_2} \supseteq U_{\{0\}}$

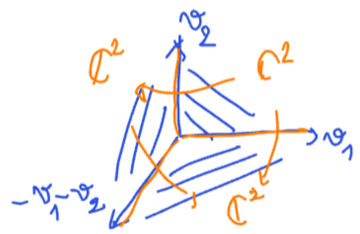
$U_\sigma = \mathrm{Spec}(\mathbb{C}[\sigma^\vee \cap N^*])$, $\{u \in N^* \mid u(\sigma) \subseteq \mathbb{R}_+\}$ monoid

Here: $U_\sigma = \mathrm{Spec}(\mathbb{C}[X, Y]) = \mathbb{C}^2$ ($X = v_1^*$, $Y = v_2^*$)

$U_{\mathbb{R}_+ v_2} = \mathrm{Spec}(\mathbb{C}[X^{\pm 1}, Y]) = \mathbb{C}^* \times \mathbb{C}$

$U_{\mathbb{R}_+ v_1} = \mathrm{Spec}(\mathbb{C}[X, Y^{\pm 1}]) = \mathbb{C} \times \mathbb{C}^*$

Ex: $\mathbb{CP}^2 = \{[X:Y:Z], (X,Y,Z) \in \mathbb{C}^3 \setminus \{0\}\}$



Covered by 3 copies of $\mathbb{C}^2$

II - Dictionary

X_Σ	Σ
U_σ affine open $\cup$ principal open U_τ	$\sigma \in \Sigma$, $\tau < \sigma$
$V_\sigma(\tau) \subseteq U_\sigma$ closed, dim $n-m$	$\tau < \sigma \in \Sigma$ (τ: m, σ: n); $Y_{V(\tau)} = \{X = 0\} \subseteq U_\sigma$
$\dim X_\Sigma = \dim U_\sigma$; $U_\sigma \supseteq V_\sigma(\tau)$; $V_\sigma(\tau)$ hypersurfaces; $V(\tau)$ ↺ $T = N \otimes_\mathbb{Z} \mathbb{C}^*$	$\tau < \sigma \in \Sigma$ (τ: 1, σ: n)
T-divisors χ^u $= X_1^{u_1} \cdots X_n^{u_n}$ fctn on T_N	$u \in N^* = \mathrm{Hom}(N, \mathbb{Z})$
U_σ contractible; Ex: $\mathbb{C}^2$, $\mathbb{C} \times \mathbb{C}^*$	σ max dim'l cone
$\chi(X)$	$\#\{\text{max dim'l cone}\}$
$H^2(X)$	$\mathrm{Ker}\left(\bigoplus_\sigma N^*/N^* \cap \sigma^\perp \to \bigoplus N^*/N^*(\sigma \cap \tau)^\perp\right)$
X_Σ complete	$N_\mathbb{R} = \lvert\Sigma\rvert$, Σ fan is complete
X_Σ quotient sing	# gen $\geq$ dim $\sigma = n$; Σ simplicial iff equality $\forall \sigma \in \Sigma$ maximal

III - Divisors

$N^* \ni u \mapsto \chi^u \in \mathcal{O}(T_N) \subset \mathcal{M}(X_\Sigma)$ $\quad T_N = U_{\{0\}}$, $\{0\}^\vee = M_\mathbb{R}^*$

$\mathrm{div} : N^* \longrightarrow \mathbb{Z}\{\text{irr hypersurfaces}\}$

$\searrow \mathrm{Div}_T(X_\Sigma) = \bigoplus_{\dim \tau = 1} \mathbb{Z} V(\tau)$

$\{\text{irr T-div}\} = \{\text{hyp.} \circlearrowleft T_N\} = \{V(\tau)\}_{\dim \tau = 1}$

$\{\chi^u(t_1, \ldots, t_n) = 0\} = \{\chi^u = 0\}$

↳ $\mathrm{div}(\chi^u)$ is T-div

SES $0 \to N^* \to \mathrm{Div}_T(X_\Sigma) \to A_{n-1}(X_\Sigma) \to 0$

Idea: D Weil div is principal on T_N ↳ eq to div $\subseteq X_\Sigma \setminus T_N$

D Weil divisor is Cartier if locally

$D_{|U_i} = \mathrm{div}(\chi^{u_i})$ $\quad X = \bigcup_i U_i$, $u_i \in N^*$

$u(\sigma_i)$ satisfy cocycle condition.

$\mathrm{div}(\chi^{u(\sigma_i)}) = \sum_{\tau < \sigma_i} \langle u(\sigma_i), v_\tau \rangle V(\tau)$ $\quad$ PW lin fctn on $\lvert\Sigma\rvert$

$\tau < \sigma_i$, $v_\tau \in \tau$, lin 1

Cartier div D $\leftrightarrow$ $\psi : \lvert\Sigma\rvert \longrightarrow \mathbb{R}$ continuous, lin on $\sigma \in \Sigma$

$\mathcal{O}(D) \subseteq \mathcal{M}(X)$

$H^0(\mathcal{O}(D)_{|U_\sigma}) = \mathcal{O}(U_\sigma)\chi^{u(\sigma)}$

↳ $H^0(\mathcal{O}(D)) = \bigoplus_{u \in P_\psi \cap N^*} \mathbb{C}$

$P_\psi = \{u \in N_\mathbb{R}^* \mid \langle u, v \rangle \geq \psi(v)\ \forall v \in N_\mathbb{R}\}$

IV - Cohomology of line bdles

D a T-Cartier div / X_Σ

$\updownarrow$

$\psi : \lvert\Sigma\rvert \to \mathbb{R}$

$u \in N^*$

$Z(u) = \{v \in \lvert\Sigma\rvert \mid \langle u, v \rangle \geq \psi(v)\}$

$\subseteq N_\mathbb{R}$

$H^p_{Z(u)}(\lvert\Sigma\rvert) = H^p_c(\lvert\Sigma\rvert, \lvert\Sigma\rvert \setminus Z(u))$

$$\boxed{H^p(X_\Sigma, \mathcal{O}(D)) = \bigoplus_{u \in N^*} H^p_{Z(u)}(\lvert\Sigma\rvert)}$$

Ex: $\mathbb{Z}^2$, τ_2, σ, τ_1, $\langle v_1, -v_2 \rangle$ $\quad U_\sigma \supseteq V(\tau_1), V(\tau_2)$

$\mathbb{Z} V(\tau_1) \oplus \mathbb{Z} V(\tau_2) = \mathrm{Div}_T(U_\sigma)$

$\sigma^\vee =$ $\quad u_1 + 2u_2$, u_1, X

$\mathbb{C}[\sigma^\vee \cap N^*] = \mathbb{C}[X, Y, Z]/(XY - Z^2)$

$u \in N^* \to \mathrm{div}(\chi^u) = (2p - q)V(\tau_1) + q\, V(\tau_2)$

$u = (p\ q)$ $\quad \langle u, 2v_1 - v_2 \rangle$ $\quad \langle u, v_2 \rangle$

$\neq V(\tau_1), V(\tau_2)$

$\psi(v_2) = q$

$\psi(v_1) = 2p - q$