POINTWISE TO GLOBAL GOOD REDUCTION WITH LEVEL STRUCTURE

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Contents

1. Introduction

1.1. **Motivation and main result.** Let $\mathcal{O}_F$ denote a complete DVR with fraction field F. Consider $\mathcal{X}$ a smooth separated geometrically connected scheme of finite type over $\mathcal{O}_F$, let X/F denote its generic fiber. Fix $Y \to X$ a smooth and proper family of varieties. We are concerned with the following problem: assuming (sufficiently many of) the closed fibers Y_x have good reduction, does Y extend to a smooth and proper scheme over $\mathcal{X}$? Does pointwise good reduction imply global good reduction?

This question is asked in the recent work [CT26, §4.3], where it is answered positively in the case of pointed smooth proper curves of genus $g \geq 2$, stable proper curves of genus $g \geq 2$, and abelian schemes if F has absolute ramification index $e \leq p-1$. The two main inputs of their argument are [CT26, Theorem 1] stating that a local system on X extends to $\mathcal{X}$ if and only if it is pointwise unramified for a relevant collection of closed points of X, as well as Serre-Tate-type criteria. It is also discussed (see [CT26, §4.3, Example (2)]) how the *smooth integral canonical models* of Shimura varieties could be leveraged for a strategy when $\mathcal{O}_F$ is the ring of integers of a p-adic field. Indeed, by definition these models satisfy a property akin to the Néron extension property of Néron models, thus providing a Serre-Tate criterion over general bases.

In this document, we formalize the discussion of [CT26, §4.3, Example (2)]. We reduce the pointwise-to-global good reduction problem to the existence of a smooth, finite type $\mathcal{O}_F$-scheme $\mathscr{S}$ equipped with a local system $\mathbb{L}$ such that the pair $(\mathscr{S}, \mathbb{L})$ satisfies some extension property, and show that smooth integral canonical models of Shimura varieties can fill this role. Consequently, we establish:

Theorem 1.1: *Assume $p > 2$. Let $F/\mathbb{Q}_p$ be a finite unramified field extension. Assume that $Y \to X$ is either:*

(a) *a polarized abelian scheme with neat, prime-to-p level structure, or*
(b) *a primitively polarized K3 surface of degree $2d$ not divisible by p, with appropriate prime-to-p level structure.*

If $Y \to X$ has pointwise good reduction, then it has global good reduction as an abelian scheme or a K3 surface.

We give a precise definition of pointwise/global good reductions in Definition 2.3. Level structures are triviality conditions on the monodromy of the relative cohomology of $Y \to X$ which are made explicit in Section 6; they ensure the representability of relevant moduli stacks. Case (b) is new, and is enabled by the work of Madapusi Pera in [MP16] and [MP15]. Case (a) is weaker than the statement in [CT26, §4.3, Example (3)]; however building upon [VZ10, Theorem 28] we also show the optimality of their assumption $e \leq p-1$, and relate it to the absence of a smooth canonical integral model for

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Siegel modular varieties if $e \geq p$. We have not investigated the existence of a similar obstruction for orthogonal-type Shimura varieties involved in case (b).

1.2. **Outline.** We begin Section 2 by formulating the pointwise-to-global good reduction problem. We then argue that even in the usually well-behaved case of abelian schemes the answer is negative: if $F/\mathbb{Q}_p$ has ramification index $e \geq p$, Corollary 2.7 provides a counterexample with arbitrary level structure. To obtain it, we recall the work of Vasiu-Zink in [VZ10] and descend their construction to finite type bases over $\mathcal{O}_F$.

In Section 3 we introduce some formalism. If we assume that $\mathcal{U}/\mathcal{O}_F$ is a fine moduli space of smooth proper varieties, with generic fiber U/F, then $Y \to X$ having pointwise good reduction exactly corresponds to the map $X \to U$ being a PGR-morphism (Definition 3.2), while pointwise good reduction implying global good reduction exactly means that $\mathcal{U}$ has the PGR-extension property (Definition 3.4). We show in Proposition 3.7 that one can obtain schemes with the PGR-extension property as open subschemes of special schemes $\mathscr{S}/\mathcal{O}_F$ equipped with a local system $\mathbb{L}$ such that the pair $(\mathscr{S}, \mathbb{L})$ satisfies an extension property stated in Definition 3.1.

Prime candidates for pairs $(\mathscr{S}, \mathbb{L})$ are provided by smooth integral models of Shimura varieties and their canonical local systems. We take a detour in section Section 4 to properly define canonical local systems associated to projective systems of torsors. When the schemes in a projective system are also moduli spaces parametrizing level structures, we show in Proposition 4.14 that the canonical local system coincides with the universal local system from the moduli problem. This generalizes constructions and statements from [UY13] and [CM20].

We recall the definition of smooth integral canonical models of Shimura varieties in Section 5, and show that, together with their canonical local systems from Section 4, they form a pair $(\mathscr{S}, \mathbb{L})$ satisfying the desired extension property. This result and a sketch of proof had been previously discussed in [BST24, §1.3].

We prove Theorem 1.1 in Section 6. To do so, we recall the constructions of Shimura varieties and moduli spaces involved in cases (a) and (b), and we observe that the latter are realized as open subschemes of the smooth integral canonical models of the former. In particular, these moduli spaces have the PGR-extension property, which is the key to Theorem 1.1.

1.3. **Conventions and notations.** All schemes are locally Noetherian. A variety over a field is a reduced separated scheme of finite type over the field.

Let $\mathcal{O}_F$ denotes a complete DVR with fraction field F. The integral closure of $\mathcal{O}_F$ in some fixed algebraic closure of F is denoted $\overline{\mathcal{O}_F}$.

If X is a scheme, the set of its closed points is denoted by $|X|$. A *smooth model* of a variety X/F is a smooth $\mathcal{O}_F$-scheme $\mathcal{X}$ with generic fiber X/F.

We fix $p > 0$ a prime number. We denote $\widehat{\mathbb{Z}}^p = \prod_{\ell \neq p} \mathbb{Z}_\ell$ the prime-to-p profinite integers, and $\mathbb{A}_f^p$ the prime-to-p adèles.

2. Pointwise and global good reduction problems

2.1. **Formulation of the question.** We define the notions of pointwise and global good reduction.

Definition 2.1: Let Y/F be a smooth proper variety. We say that it has *good reduction* if it is the generic fiber of some smooth proper $\mathcal{O}_F$-scheme $\mathcal{Y}$.

We define the integral locus of a variety with smooth model.

Definition 2.2: Let X/F be a smooth variety, and $\mathcal{X}/\mathcal{O}_F$ a smooth model. We define the *integral locus* of X (relative to $\mathcal{X}$) as

$$|X|^{\text{int}} = \text{im}(\mathcal{X}(\overline{\mathcal{O}_F}) \to |X|).$$

If $\mathcal{X} = X$, the integral locus is empty and if $\mathcal{X}/\mathcal{O}_F$ is proper, then $|X|^{\text{int}} = |X|$.

Definition 2.3: Let X/F be a smooth variety, and $\mathcal{X}/\mathcal{O}_F$ a smooth model. Consider a smooth proper map $Y \to X$. We say that this family has *pointwise good reduction* (relative to $\mathcal{X}$) if for every $x \in |X|^{\text{int}}$, the fiber $Y_x/F(x)$ has good reduction. We say that this family has *global good reduction* (relative to $\mathcal{X}$) if there exists a smooth proper map $\mathcal{Y} \to \mathcal{X}$ over $\mathcal{O}_F$ with generic fiber $Y \to X$.

We are interested in whether pointwise good reduction implies global good reduction, and if not, what may be some obstructions.

2.2. **The ramification obstruction.** For abelian schemes over a sufficiently ramified p-adic field F, pointwise good reduction does not imply global good reduction. We build a counterexample upon the work of Vasiu-Zink in [VZ10][1], of which we recall some steps. Let R denote a regular local ring of dimension 2 and mixed characteristic $(0,p)$. Let $\mathfrak{m}$ denote its maximal ideal, and set $U = \operatorname{Spec} R\backslash\{\mathfrak{m}\}$.

Lemma 2.4 ([VZ10], Lemma 27)**:** *Let $D \to H$ be a homomorphism of finite flat group schemes over* $\operatorname{Spec} R$ *which is not a monomorphism, but whose restriction over U is a monomorphism. Let B be an abelian scheme over* $\operatorname{Spec} R$ *in which H embeds. Define the quotient abelian scheme $A = B_U/D_U$ over U.*

If R' is a faithfully flat local R-algebra of relative dimension 0, with maximal ideal $\mathfrak{m}'$, let $U' := U \times_R \operatorname{Spec} R' = \operatorname{Spec} R' \setminus \{\mathfrak{m}'\}$. Then $A \times_U \operatorname{Spec} R' \to U'$ is an abelian scheme which does not extend into an abelian scheme over $\operatorname{Spec} R'$.

Proof. The morphism $\operatorname{Spec} R' \to \operatorname{Spec} R$ is fpqc, so the assumptions on monomorphisms still hold after base-change to R'. Let D', H', A', B' denote the group schemes base-changed to R', then $A' = B'_U/D'_U$. Hence we reduce to the case $R' = R$.

Suppose that A extends to an abelian scheme $\tilde{A} \to \operatorname{Spec} R$. By [Ray06, Corollaire IX 1.4], the isogeny $B_U \to A$ extends to a homomorphism $B \to \tilde{A}$ which is automatically an isogeny. Its kernel K is a finite flat group scheme which coincides with D_U over U, it is thus isomorphic to D by purity for finite flat group schemes over a regular base (see [MB85, Lemme 2]). This defines a monomorphism

$$D \simeq K \to B$$

of group schemes over R, whose restriction over U is the original morphism $D_U \to H_U$. Thus this monomorphism factors through $H \to B$, contradicting the assumption that $D \to H$ is not a monomorphism. ☐

Proposition 2.5: *Suppose $F/\mathbb{Q}_p$ has ramification index at least p. Then for any positive prime-to-p integer N there exists a finite étale local $\mathcal{O}_F[[T]]$-algebra R with maximal ideal $\mathfrak{m}$ and an abelian scheme $A \to U := \operatorname{Spec} R\backslash\{\mathfrak{m}\}$ with level N structure which does not extend to an abelian scheme over* $\operatorname{Spec} R$.

Proof. Consider first $F_0/\mathbb{Q}_p$ a totally ramified subextension of $F/\mathbb{Q}_p$ of degree at least p. Then according to [VZ10, Theorem 28] and its proof we can take the power series ring $R_0 = \mathcal{O}_{F_0}[[T]]$ and find a homomorphism of group schemes $D \to H$ over R_0 satisfying the assumptions of Lemma 2.4, hence an abelian scheme $A \to U_0$ which does not extend to an abelian scheme over $\operatorname{Spec} R_0$.

Set $R = \mathcal{O}_F[[T]]$ which is a faithfully flat R_0-algebra of relative dimension 0. Then by Lemma 2.4 the abelian scheme

$$A \times_{U_0} \operatorname{Spec} R \to U := U_0 \times_{\operatorname{Spec} R_0} \operatorname{Spec} R$$

does not extend to $\operatorname{Spec} R$. Finally, replace R with an appropriate finite étale local ring extension to ensure that $A \to U$ has the appropriate level structure. ☐

Next we argue that we can replace R with a finite type $\mathcal{O}_F$-algebra.

Proposition 2.6: *Suppose $F/\mathbb{Q}_p$ has ramification index at least p. Then for any positive prime-to-p integer N there exists a smooth local $\mathcal{O}_F$-algebra R with maximal ideal $\mathfrak{m}$ and an abelian scheme $A \to U := \operatorname{Spec} R \setminus \{\mathfrak{m}\}$ with level N structure which does not extend to an abelian scheme over* $\operatorname{Spec} R$.

Proof. We denote by $\tilde{R}$ the finite étale local $\mathcal{O}_F[[T]]$-algebra with maximal ideal $\tilde{\mathfrak{m}}$, by $\tilde{U} := \operatorname{Spec} \tilde{R}\backslash\{\tilde{\mathfrak{m}}\}$ the open subset and by $f : \tilde{A} \to \tilde{U}$ the abelian scheme with level N structure which does not extend into an abelian scheme over $\operatorname{Spec} \tilde{R}$, provided by Proposition 2.5. We wish to descend all of this data over a smooth $\mathcal{O}_F$-algebra.

We first argue that $\mathcal{O}_F \to \tilde{R}$ is a regular ring homomorphism, to do so we may assume $\tilde{R} = \mathcal{O}_F[[T]]$. It is clear that $k[[T]]$ is geometrically regular, where k is the residue field of $\mathcal{O}_F$. It remains to show

[1]Vasiu-Zink generalize a counterexample to [FC13, Chapter V, Corollary 6.8] due to Raynaud-Ogus-Gabber (see [dJO97, §6]).

that $\mathcal{O}_F[[T]] \otimes_{\mathcal{O}_F} F = \mathcal{O}_F[[T]][1/p]$ is regular. By [Sta26, Tag 07EL] we reduce to showing that $F \to \mathcal{O}_F[[T]][1/p]$ is formally smooth for the (T)-adic topology, which is easily checked.

We may now apply Popescu's theorem, see [Sta26, Tag 07GC]: there are smooth $\mathcal{O}_F$-algebras R_i such that $\tilde{R} = \varinjlim R_i$. All maps between the R_i and $\tilde{R}$ factor through local and integral rings, so we may assume that the R_i are local integral with maximal ideal $\mathfrak{m}_i$, and set $U_i := \operatorname{Spec} R_i \setminus \{\mathfrak{m}_i\}$. We let $p_i : \tilde{U} \to U_i$ be the maps induced from the projective system. We wish to descend f to an abelian scheme over some U_i.

Since the U_i are Noetherian, by [Sta26, Tag 0CNL] for large enough i there is a smooth and proper morphism $f_i : A_i \to \operatorname{Spec} U_i$ descending f. Using [Sta26, Tag 0CNR] we can assume that the group scheme structure and the level N structure descends to f_i, thus f_i defines an abelian scheme. Let $A = A_i$ and $U = U_i$ for this appropriate choice of i.

If $A \to U$ were to extend to an abelian scheme over $\operatorname{Spec} R$, its pullback to $\operatorname{Spec} \tilde{R}$ would extend $\tilde{A} \to \tilde{U}$, contradicting the definition of $\tilde{A}$. □

Corollary 2.7: *With notations from Proposition 2.6, set $\mathcal{X} = \operatorname{Spec} R$ which is a smooth model over $\mathcal{O}_F$ of its generic fiber $\iota\colon X \hookrightarrow \mathcal{X}$. Then $\iota^*A \to X$ has level N structure, pointwise good reduction, but not global good reduction as an abelian scheme.*

Proof. We argue that the abelian scheme $\iota^*A \to X$ does not have global good reduction. Let η denote the generic point of X, hence also of U and $\mathcal{X}$, fix a symmetric ample line bundle $\mathcal{L}_\eta$ on A_η. Assume the family extends to an abelian scheme $A' \to \mathcal{X}$. According to Fact 2.8 there are positive integers n, m such that $\mathcal{L}_\eta^{\otimes n}$ and $\mathcal{L}_\eta^{\otimes m}$ extend to relatively ample line bundles on $A \to U$ and $A' \to \mathcal{X}$ respectively. Let $j\colon U \to \mathcal{X}$ be the inclusion of open set; replacing n and m with nm, we find that the polarized abelian schemes

$$A \to U, \qquad j^*A' \to U$$

have isomorphic generic fibers. Separatedness of the stack of polarized abelian schemes implies that they are isomorphic. In particular, $A' \to \mathcal{X}$ extends $A \to U$, which contradicts Proposition 2.6.

It remains to check that $\iota^*A \to X$ has pointwise good reduction. Let $\ell \neq p$ be a prime number, the local system $\mathbb{L} = R^1 f_* \mathbb{Q}_\ell$ over U corresponds to an ℓ-adic representation of the étale fundamental group $\pi_1(U)$. As $\mathcal{X} \setminus U$ has codimension at least 2 in $\mathcal{X}$, Zariski-Nagata purity implies that $\mathbb{L}$ extends to a local system on $\mathcal{X}$. In particular, for any $x \in |X|^{\mathrm{int}}$ the local system $\mathbb{L}_x$ is unramified, which by [ST71] means that $(\iota^*A)_x$ has good reduction. □

Fact 2.8 ([Ray06], Remarque XI 1.3 (e))**:** *Let S be a normal, integral, locally Noetherian scheme with generic point η, consider an abelian scheme $G \to S$, and a symmetric ample line bundle $\mathcal{L}_\eta$ on the generic fiber G_η. Then there exists a positive integer n such that $\mathcal{L}_\eta^{\otimes n}$ extends to a symmetric relatively ample line bundle on $G \to S$.*

3. Extension properties for good reduction

Definition 3.1: Let S/F be a smooth variety, and $\mathscr{S}/\mathcal{O}_F$ a smooth model. Let $\mathbb{L}$ be an étale local system on S. We say that the pair $(\mathscr{S}, \mathbb{L})$ *has the smooth extension property* if for every smooth variety X/F with smooth model $\mathcal{X}/\mathcal{O}_F$, any map of F-varieties

$$f : X \to S$$

such that the local system $f^{-1}\mathbb{L}$ on X extends to $\mathcal{X}$ has a unique extension over $\mathcal{O}_F$

$$\tilde{f} : \mathcal{X} \to \mathscr{S}.$$

We will show that if $(\mathscr{S}, \mathbb{L})$ has the smooth extension property, then any open subscheme $\mathcal{U}$ also inherits some form of extension property, which we now define.

Definition 3.2: Let $X_1, X_2/F$ be varieties with smooth models $\mathcal{X}_1, \mathcal{X}_2/\mathcal{O}_F$. A *PGR-morphism* (as in Pointwise Good Reduction) is a map $f : X_1 \to X_2$ between the generic fibers such that $f(|X_1|^{\mathrm{int}}) \subseteq |X_2|^{\mathrm{int}}$.

Remark 3.3: The acronym PGR is motivated by the case, of interest to us, where X_2/F and $\mathcal{X}_2/\mathcal{O}_F$ are fine moduli spaces for some families of smooth and proper schemes over F and $\mathcal{O}_F$ respectively.

In this case, a PGR-morphism $f : X_1 \to X_2$ corresponds to a smooth proper family $Y \to X_1$ with pointwise good reduction as in Definition 2.3.

Definition 3.4: Let U/F be a smooth variety, and $\mathcal{U}/\mathcal{O}_F$ a smooth model. We say that it *has the PGR-extension property* if for every smooth, geometrically connected variety X/F with smooth model $\mathcal{X}/\mathcal{O}_F$, any PGR-morphism

$$f : X \to U$$

has a unique extension over $\mathcal{O}_F$

$$\tilde{f} : \mathcal{X} \to \mathcal{U}.$$

Remark 3.5: Assuming that U/F and $\mathcal{U}/\mathcal{O}_F$ are fine moduli spaces for some families of smooth and proper schemes over F and $\mathcal{O}_F$ respectively, the PGR-extension property exactly states that a smooth proper family with pointwise good reduction has global good reduction.

Lemma 3.6: *Let U/F be a smooth variety, and $\mathcal{U}/\mathcal{O}_F$ a smooth model. Let E/F be a finite extension. Then any PGR-morphism $g : \operatorname{Spec} E \to U$ extends to an $\mathcal{O}_F$-morphism $\operatorname{Spec} \mathcal{O}_E \to \mathcal{U}$.*

Proof. Let $y \in |U|^{\text{int}}$ denote the image of g. The scheme-theoretic closure of y in $\mathcal{U}$ is isomorphic to $\operatorname{Spec} A$ where $A/\mathcal{O}_F$ is a finite local ring extension. The result follow from the valuative criterion of properness in the diagram

$$\begin{array}{ccc} \operatorname{Spec} E & \longrightarrow & \operatorname{Spec} A \\ \downarrow & \nearrow & \downarrow \\ \operatorname{Spec} \mathcal{O}_E & \longrightarrow & \operatorname{Spec} \mathcal{O}_F. \end{array}$$

□

Proposition 3.7: *Consider a pair $(\mathscr{S}, \mathbb{L})$ with the smooth extension property as in Definition 3.1. Let $\mathcal{U}$ be an open subscheme of $\mathscr{S}$. Then $\mathcal{U}$ has the PGR-extension property.*

Proof. Let X/F be a smooth, geometrically connected variety with smooth model $\mathcal{X}/\mathcal{O}_F$, and let $f : X \to U$ be a PGR-morphism. If $x \in |X|^{\text{int}}$, then $\mathbb{L}_{f(x)}$ is unramified, and so is its pullback to x. By [CT26, Theorem 1], the local system $f^{-1}\mathbb{L}$ extends to $\mathcal{X}$. By the smooth extension property, f extends uniquely to a map

$$\tilde{f} : \mathcal{X} \to \mathscr{S}.$$

It remains to show that the image of this map factors through $\mathcal{U}$. It suffices to do so for closed points of the special fiber of $\mathcal{X}$. Let x_0 be such a closed point, with residue field $k(x_0)$. Let E/F be a finite extension with valuation ring $\mathcal{O}_E$ and maximal ideal $\mathfrak{m}_E$ such that $k(x_0) \simeq \mathcal{O}_E/\mathfrak{m}_E$. By smoothness we may lift x_0 to $\mathcal{O}_E/\mathfrak{m}_E^n$-points of $\mathcal{X}$, and eventually to an $\mathcal{O}_E$-point denoted $\tilde{x}$. The generic fiber x of $\tilde{x}$ lives in $|X|^{\text{int}}$ by construction, so the composition map

$$\operatorname{Spec} E \xrightarrow{x} X \to U$$

is a PGR-morphism. By Lemma 3.6 it extends to an $\mathcal{O}_F$-morphism $\tilde{g} : \operatorname{Spec}\mathcal{O}_E \to \mathcal{U}$. The situation is summarized in the following commutative diagram:

$$\begin{array}{ccccccc} & & \operatorname{Spec} \mathcal{O}_E & & \xrightarrow{\quad\tilde{g}\quad} & \mathcal{U} & \hookrightarrow & \mathscr{S} \\ & \nearrow & \uparrow & & & \uparrow & & \uparrow \\ \operatorname{Spec} k(x_0) & & \operatorname{Spec} E & \xrightarrow{x} & X & \xrightarrow{f} U & \hookrightarrow & S \\ & \searrow & \downarrow & & \downarrow & & & \downarrow \\ & & \operatorname{Spec} \mathcal{O}_E & \xrightarrow{\tilde{x}} & \mathcal{X} & \xrightarrow{\quad\tilde{f}\quad} & & \mathscr{S} \end{array}$$

We see that middle row map $\operatorname{Spec} E \to S$ extends in two ways (top and bottom rows) to $\operatorname{Spec} \mathcal{O}_E \to \mathscr{S}$, which must coincide by the uniqueness of the smooth extension property. Hence the image of the bottom row factors through $\mathcal{U}$. As x_0 was defined by the composition

$$\operatorname{Spec} k(x_0) \to \operatorname{Spec} \mathcal{O}_E \xrightarrow{\tilde{x}} \mathcal{X},$$

its image through $\tilde{f}$ lies in $\mathcal{U}$. □

4. Canonical and universal local systems

In order to apply Proposition 3.7 to certain families of varieties, we need to produce pairs with the smooth extension property as in Definition 3.1. Taking inspiration from [UY13, §2] and [CM20, §3.1], we define canonical local systems arising from towers of torsors. We show that when these towers are also moduli spaces for level structures on the cohomology of varieties, the canonical local system agrees with the universal local system. This identification recovers [UY13, paragraph following Remark 2.8] (see also [CM20, Proposition 5.2]) by applying it to points on Siegel modular varieties, as well as [CM20, end of the proof of Theorem 6.6] for points on moduli spaces of K3 surfaces.

Let d be an integer. Fix R a topological ring and V a free R-module of rank d. Let $\mathrm{GL}(V) = \mathrm{Aut}_R(V)$ considered as a topological group. If K_0 denotes a compact subgroup of $\mathrm{GL}(V)$, we define $\mathcal{C}(K_0)$ to be the set of all open normal subgroups of K_0. We will systematically assume such K_0 to be profinite, so that

$$K_0 = \varprojlim_{K \in \mathcal{C}(K_0)} K_0/K$$

and make use of this fact without mention. When K_0 is clear from context, we will often omit writing "$\in \mathcal{C}(K_0)$" for collections indexed by K over this set.

4.1. **Canonical local systems.** Fix K_0 a compact subgroup of $\mathrm{GL}(V)$.

Definition 4.1: We say that a projective system of schemes $(X_K)_{K\in\mathcal{C}(K_0)}$ is *fit for* K_0 if it satisfies the following assumptions:

(i) for each $K \in \mathcal{C}(K_0)$, the projection map $X_K \to X_{K_0}$ is a K_0/K-torsor over X_{K_0}, and
(ii) for each $K' \subseteq K$ in $\mathcal{C}(K_0)$, the map $X_{K'} \to X_K$ is K_0-equivariant for the K_0 action induced from (i).

Fix $(X_K)_K$ that is fit for K_0. Since transition maps are finite étale (possibly disconnected) covers, we may define the projective limit

$$X := \varprojlim_K X_K.$$

A geometric point $\overline{x}$ on X is equivalent to a compatible system of geometric points $(\overline{x}_K)_K$ for $(X_K)_K$. For simplicity, let us now assume from now-on that X_{K_0} is connected. There is a function

$$\{\text{Geometric points on } X\}^2 \to \{R\text{-local systems with fiber } V \text{ on } X_{K_0}\} \tag{1}$$

defined as follows. Fix a geometric point $\overline{x}$ of X. For each $K \in \mathcal{C}(K_0)$ the K_0/K-torsor $X_K \to X_{K_0}$ is equivalent to the datum of a morphism

$$\pi_1(X_{K_0}, \overline{x}_{K_0}) \to K_0/K.$$

The compatibility in (ii) above and of the geometric points, together with the assumption that K_0 is profinite, yield a morphism

$$\pi_1(X_{K_0}, \overline{x}_{K_0}) \to \varprojlim_K K_0/K = K_0. \tag{2}$$

Composing with $K \hookrightarrow \mathrm{GL}(V)(R)$, this defines an R-local system of rank d on X_{K_0} with fiber at $\overline{x}_{K_0}$ naturally identified with V. We let $\mathbb{L}_{\mathrm{can}}^{\overline{x}}$ denote this local system. The isomorphism class of $\mathbb{L}_{\mathrm{can}}^{\overline{x}}$ is independent of $\overline{x}$ and is denoted $\mathbb{L}_{\mathrm{can}}$. We call this the *canonical local system* on X_{K_0} (with respect to the projective system $(X_K)_K$).

Remark 4.2: If we no longer assume X_{K_0} to be connected, the function in eq. (1) takes as an input a collection of geometric points of X, one mapping to each connected component of X_{K_0}. To avoid cumbersome statements X_{K_0} will be assumed to be connected. The general case is easily recovered by working componentwise.

Example 4.3: If each $X_K \to X_{K_0}$ is a trivial K_0/K-torsor, then $\mathbb{L}_{\mathrm{can}}^{\overline{x}} = \underline{V}$ on the nose, regardless of the choice of $\overline{x}$.

[2] We implicitly identify two geometric points $\mathrm{Spec}\, k_i \to X$ for $i \in \{1, 2\}$ if there is an algebraically closed overfield Ω containing k_1 and k_2 such that the composite maps $\mathrm{Spec}\,\Omega \to \mathrm{Spec}\, k_i \to X$ coincide.

Example 4.4: Let (G, X) be a Shimura datum with reflex field E. Let K_0 be a neat[3] compact open subgroup of $G(\mathbb{A}_f)$. Then $(\mathrm{Sh}_K(G,X))_{K \in \mathcal{C}(K_0)}$ is fit for K_0, so Shimura varieties have canonical local systems. Note that it may happen that $\mathrm{Sh}_{K_0}(G,X)$ is not connected.

Remark 4.5: Let Y_{K_0} be a scheme and $f : Y_{K_0} \to X_{K_0}$ be a morphism. Define Y_K as the pullback of X_K along f, then $(Y_K)_K$ is fit for K. Let $\overline{y}$ be a geometric point of its limit, set $\overline{x} = f(\overline{y})$. Then

$$f^{-1}\mathbb{L}_{\mathrm{can}}^{\overline{x}} = \mathbb{L}_{\mathrm{can}}^{\overline{y}}.$$

4.2. **Universal local systems.** Let S be a base scheme, set $\mathscr{C} := (\mathrm{Sch}/S)_{\text{Ét}}$ the big étale site over S. All fibered categories considered in this paragraph, and all morphisms between them, are implicitly defined over $\mathscr{C}$.

Definition 4.6: Denote by Loc_S^R (resp. $\mathrm{Loc}_S^{d,R}$) the fibered category (resp. stack) assigning to any $T \to S$ the category (resp. groupoid) of R-local systems (resp. of R-local systems of rank d) over T. Let $\underline{V}$ denote the constant local system with fiber V. Let

$$\mathrm{Triv} : \mathrm{Loc}_S^{d,R} \to B\underline{\mathrm{GL}(V)}$$

be the functor assigning to any $T \to S$ and any $\mathbb{L} \in \mathrm{Loc}_S^{d,R}(T)$ the $\underline{\mathrm{GL}(V)}$-torsor $\underline{\mathrm{Isom}}(\underline{V}, \mathbb{L})$ over T.

We spare a paragraph to motivate our goal and the formalism we introduce to achieve it. Fix some integer $g > 0$ and let $\mathscr{M}_{K_0}$ denote the moduli stack over S of polarized abelian schemes of dimension g with K_0-level structure, where K_0 is a neat compact open subgroup of $\mathrm{GSp}_{2g}(\mathbb{A}_f)$. This stack is representable by a scheme M_{K_0}. Replacing K_0 with finite index open subgroups K, we find a projective system $(M_K)_K$ that is fit for K_0 as in Definition 4.1, so we can consider the corresponding *canonical* $\mathbb{A}_f$-local system $\mathbb{L}_{\mathrm{can}}$ on $\mathscr{M}_{K_0}$. There is a second local system on $\mathscr{M}_{K_0}$: the relative cohomology of the universal abelian scheme arising from the representability of $\mathscr{M}_{K_0}$, which we call the *universal* local system $\mathbb{L}_{\mathrm{univ}}$. We aim to show that $\mathbb{L}_{\mathrm{can}}^{\overline{x}}$ and $\mathbb{L}_{\mathrm{univ}}$ are canonically identified once a geometric point $\overline{x}$ of $(M_K)_K$ is selected.

To do so, we formalize the notion of a moduli stack with level structure in Definition 4.7 and Definition 4.8. We sketch how this is done in the explicit case of $\mathscr{M}_{K_0}$. For any $T \to S$, an object in $\mathscr{M}_{K_0}(T)$ is a triple $(f : A \to T, \lambda, [\eta]_{K_0})$ where

- $(f : A \to T, \lambda)$ is a polarized abelian scheme of dimension g, and
- $[\eta]_{K_0}$ is a section of the quotient by K_0 of some subsheaf of $\mathrm{Triv}(R^1 f_* \mathbb{A}_f)$; namely we restrict to those isomorphisms that are compatible with the usual symplectic structure on $\mathbb{A}_f^{2g}$, and the symplectic structure on $R^1 f_* \mathbb{A}_f$ induced from λ.

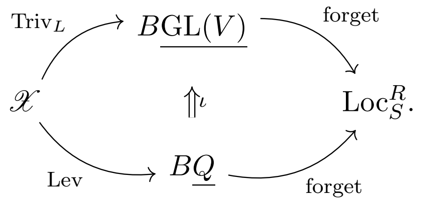
To define $\mathscr{M}_{K_0}$, we first consider the stack $\mathscr{X}$ of polarized abelian schemes of dimension g and view relative cohomology as a morphism from $\mathscr{X}$ to $\mathrm{Loc}_S^{d,R}$. From there, we can define a stack of triples $(f : A \to T, \lambda, [\eta]_{K_0})$ with the slight caveat that $[\eta]_{K_0}$ is a section of the whole sheaf $\mathrm{Triv}(R^1 f_* \mathbb{A}_f)/K_0$ (no compatibility condition with the symplectic structures). To formalize replacing every $\underline{\mathrm{GL}_d(\mathbb{A}_f)}$-torsor $\mathrm{Triv}(R^1 f_* \mathbb{A}_f)$ with an appropriate subsheaf (which will be a torsor for a fixed subgroup of $\mathrm{GL}_d(\mathbb{A}_f)$), we will use the language of natural transformations between fibered categories (rather than just stacks).

Definition 4.7: Let Q be a topological subgroup of $\mathrm{GL}(V)$. A *level moduli datum* is a quadruple $(\mathscr{X}, L, \mathrm{Lev}, \iota)$ consisting of

(i) a stack $\mathscr{X}$ over $\mathscr{C}$,
(ii) a morphism of stacks $L : \mathscr{X} \to \mathrm{Loc}_S^{d,R}$; we let $\mathrm{Triv}_L = \mathrm{Triv} \circ L$,
(iii) a morphism $\mathrm{Lev} : \mathscr{X} \to B\underline{Q}$, and

[3]The neat assumption exactly translates to $\mathrm{Sh}_K(G,X) \to \mathrm{Sh}_{K_0}(G,X)$ being a K_0/K-torsor for all K.

(iv) a $\underline{Q}$-equivariant natural monomorphism ι between the morphisms of fibered categories:



A *morphism of level moduli data* $(\mathscr{X}, L, \mathrm{Lev}, \iota) \to (\mathscr{X}', L', \mathrm{Lev}', \iota')$ is a morphism of stacks $\mathcal{X} \to \mathcal{X}'$ satisfying the expected compatibility conditions.

A level moduli datum is to be understood as follows. The functor $\mathrm{Triv}_L \colon \mathscr{X} \to \mathrm{B}\underline{\mathrm{GL}(V)}$ assigns to any $T \to S$ and any $A \in \mathscr{X}(T)$ the $\mathcal{G}$-torsor $\underline{\mathrm{Isom}}(\underline{V}, L(A))$ over T. The functor Lev assigns to the same data a subsheaf of $\mathrm{Triv}_L(A)$ which is a $\underline{Q}$-torsor for the induced action, considering only isomorphisms with additional constraints. The condition of being a subsheaf for each $T \to S$ and $A \in \mathscr{X}(T)$ is formalized by the natural monomorphism ι.

We are interested in the following moduli stacks:

Definition 4.8: Let Q be a topological subgroup of $\mathrm{GL}(V)$, K_0 a compact open subgroup of Q which is profinite, and K an open subgroup of K_0. Let $(\mathscr{X}, L, \mathrm{Lev}, \iota)$ be a level moduli datum. The associated *moduli stack of level* K is the stack $\mathscr{M}_K$ over $\mathscr{C}$ assigning to each $T \to S$ a pair $(A, [\eta]_K)$, where

(i) A is an object in $\mathscr{X}(T)$, and
(ii) the *level structure* $[\eta]_K \in H^0(T, \mathrm{Lev}(A)/K)$ is a global section of the quotient sheaf $\mathrm{Lev}(A)/K$.

If moreover $K \in \mathcal{C}(K_0)$, the induced morphism $\mathscr{M}_K \to \mathscr{M}_{K_0}$ is a K_0/K-torsor.

Remark 4.9: With notations as in Definition 4.8, let $T \to S$, $A \in \mathscr{X}(T)$ and assume $L(A) = \underline{W}$ is a constant local system on T for some rank d free R-module W. Then

$$\mathrm{Triv}_L(A) = \underline{\mathrm{Isom}_R(V, W)}$$

is a trivial $\underline{\mathrm{GL}(V)}$-torsor. For any $K \in \mathcal{C}(K_0)$ the sheaf of sets $\mathrm{Triv}_L(A)/K$ is non-canonically isomorphic to $\underline{\mathrm{GL}(V)/K}$. The same holds true for $\mathrm{Lev}(A)$ replacing $\mathrm{GL}(V)$ with Q, and these identifications are compatible under the monomorphism $\iota(A) : \mathrm{Lev}(A) \hookrightarrow \mathrm{Triv}_L(A)$. In particular, if $(A, [\eta]_K) \in \mathscr{M}_K(T)$, then

$$[\eta]_K \in H^0(T, \mathrm{Lev}(A)/K) \hookrightarrow H^0(T, \mathrm{Triv}(A)/K) = \mathrm{Isom}_R(V, W)/K$$

is represented by some isomorphism $\eta : V \to W$.

From now-on we fix Q, K_0 and $(\mathscr{X}, L, \mathrm{Lev}, \iota)$ as in Definition 4.8, and we assume that $\mathscr{M}_{K_0}$ is represented by an S-scheme M_{K_0}. We denote by $(\mathcal{A}_{K_0}, [\alpha]_{K_0}) \in \mathscr{M}_{K_0}(M_{K_0})$ the corresponding universal object. Then for every $K \in \mathcal{C}(K_0)$, $\mathscr{M}_K$ is represented by an S-scheme M_K and the projective system $(M_K)_{K \in \mathcal{C}(K_0)}$ is fit for K_0.

Definition 4.10: We define the following R-local systems of rank d on M_{K_0}:

- $\mathbb{L}_{\mathrm{can}}(M_{K_0})$ the *canonical local system* attached to $(M_K)_K$, and
- $\mathbb{L}_{\mathrm{univ}}(M_{K_0}) = L(\mathcal{A}_{K_0})$ the *universal local system* attached to the universal object.

If no confusion can arise, we simplify the notation to $\mathbb{L}_{\mathrm{can}}$ and $\mathbb{L}_{\mathrm{univ}}$. The universal local system has the expected functoriality.

Lemma 4.11: *Let $\mathscr{M}_1$, $\mathscr{M}_2$ be stacks over $\mathscr{C}$ which are representable by S-schemes M_1 and M_2. Let $A_1 \in \mathscr{M}_1(M_1)$, $A_2 \in \mathscr{M}_2(M_2)$ be their respective universal objects. Let $f : \mathscr{M}_2 \to \mathscr{M}_1$ be a morphism of stacks. Consider the induced functors*

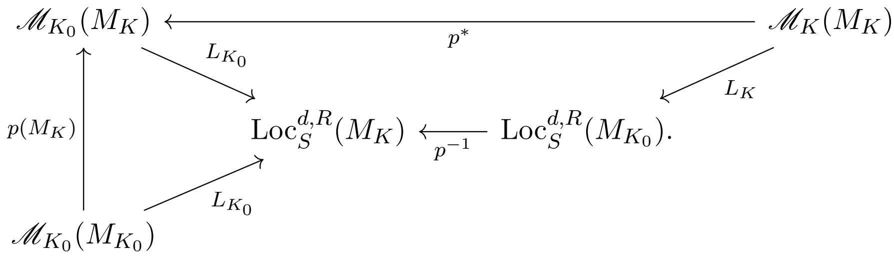
$$f^* : \mathscr{M}_1(M_1) \to \mathscr{M}_1(M_2), \qquad f(M_2) : \mathscr{M}_2(M_2) \to \mathscr{M}_1(M_2).$$

Then $f^*(A_1) = f(M_2)(A_2)$.

Proof. This follows from the diagram $\begin{CD} M_2 @>{\mathrm{id}}>> M_2 \\ @VfVV @VVfV \\ M_1 @>{\mathrm{id}}>> M_1. \end{CD}$ ☐

Corollary 4.12: *Let $K \in \mathcal{C}(K_0)$ and denote $p\colon \mathscr{M}_K \to \mathscr{M}_{K_0}$. Then $p^{-1}\mathbb{L}_{\mathrm{univ}}(M_{K_0}) = \mathbb{L}_{\mathrm{univ}}(M_K)$.*

Proof. Let us write L_{K_0} for the composition of $\mathscr{M}_{K_0} \to \mathscr{X}$ with L, and similarly for L_K. Then the result follows from Lemma 4.11 and the diagram



☐

Corollary 4.13: *Let $(\mathscr{X}, L, \mathrm{Lev}, \iota) \to (\mathscr{X}', L', \mathrm{Lev}', \iota')$ be a morphism of moduli data, $f : \mathscr{M}'_{K_0} \to \mathscr{M}_{K_0}$ the induced morphism between the level moduli stacks. If both stacks are representable by M_{K_0} and M'_{K_0}, then $f^{-1}\mathbb{L}_{\mathrm{univ}}(M_{K_0}) = \mathbb{L}_{\mathrm{univ}}(M'_{K_0})$.*

Proof. The proof is nearly identical to that of Corollary 4.12. ☐

The local systems $\mathbb{L}_{\mathrm{can}}(M_{K_0})$ and $\mathbb{L}_{\mathrm{univ}}(M_{K_0})$ are isomorphic. Namely, letting $M = \varprojlim_{K\in\mathcal{C}(K_0)} M_K$:

Proposition 4.14: *Fix $\overline{x} = (\overline{x}_K)_K$ a geometric point of M. Then there is a canonical identification*

$$\mathbb{L}_{\mathrm{can}}^{\overline{x}} \simeq \mathbb{L}_{\mathrm{univ}}.$$

We first establish the identification in the case of constant local systems.

Lemma 4.15: *If $\mathbb{L}_{\mathrm{univ}}$ is constant over M_{K_0}, then there is a canonical identification $\mathbb{L}_{\mathrm{can}} \simeq \mathbb{L}_{\mathrm{univ}}$.*

Remark 4.16: By Example 4.3, there is no need to specify a geometric point in the statement of Lemma 4.15.

Proof of Lemma 4.15. Let $K \in \mathcal{C}(K_0)$, we first argue that the universal level structure $[\alpha]_{K_0}$ admits a canonical lift $[\alpha]_K \in H^0(M_{K_0}, \mathrm{Lev}(\mathcal{A}_{K_0})/K)$. Let $(\mathcal{A}_K, [\beta]_K) \in \mathscr{M}_K(M_K)$ be the universal object over M_K. Then by Corollary 4.12, both $\mathbb{L}_{\mathrm{univ}}(M_{K_0})$ and $\mathbb{L}_{\mathrm{univ}}(M_K)$ are constant, and they share a common unverlying free R-module of rank d denoted W. By Remark 4.9, choosing representatives α and β for the universal level structures gives rise to two isomorphisms

$$(\alpha\colon V \to W) \in H^0(M_{K_0}, \mathrm{Lev}(\mathcal{A}_{K_0}))$$
$$(\beta\colon V \to W) \in H^0(M_K, \mathrm{Lev}(\mathcal{A}_K)).$$

which, by Lemma 4.11, agree modulo the action of K_0 on V. We can thus change the representative of $[\alpha]_{K_0}$ to assume $\alpha = \beta$. Define $[\alpha]_K \in H^0(M_{K_0}, \mathrm{Lev}(\mathcal{A}_{K_0})/K)$ as the image of α in $\mathrm{Isom}(V, W)/K$, which is independent of the choice of representative β. The object $(\mathcal{A}_{K_0}, [\alpha]_K) \in \mathscr{M}_K(M_{K_0})$ defines a section $M_{K_0} \to M_K$ of the K_0/K-torsor $M_K \to M_{K_0}$, which is trivial. By Example 4.3 we find $\mathbb{L}_{\mathrm{can}}(M_{K_0}) = \underline{V}$.

Moreover, the projective system of lifts $([\alpha]_K)_{K\in\mathcal{C}(K_0)}$ defines a canonical global section of $\mathrm{Lev}(\mathcal{A}_{K_0})$, hence a canonical isomorphism $\mathbb{L}_{\mathrm{can}} = \underline{V} \to \mathbb{L}_{\mathrm{univ}}$. ☐

Remark 4.17: With notations from Corollary 4.13, if $(\mathscr{M}_K)_{K\in\mathcal{C}(K_0)}$ satisfies the assumptions of Lemma 4.15 then so does $(\mathscr{M}'_K)_{K\in\mathcal{C}(K_0)}$, and it follows from the proof of Lemma 4.15 that the canonical identification $\mathbb{L}_{\mathrm{can}}(M_{K_0}) \simeq \mathbb{L}_{\mathrm{univ}}(M_{K_0})$ pulls back to the canonical identification over M'_{K_0}.

Proof of Proposition 4.14. Let $\{U_i \to M_{K_0}\}_i$ be an étale cover such that each pullback $\mathbb{L}_{\mathrm{univ}}(M_{K_0})_{|U_i}$ is constant. By Remark 4.5, Corollary 4.13 and Lemma 4.15, there are canonical identifications

$$\mathbb{L}_{\mathrm{univ}}(M_{K_0})_{|U_i} = \mathbb{L}_{\mathrm{univ}}(U_i) \simeq \mathbb{L}_{\mathrm{can}}(U_i) = (\mathbb{L}_{\mathrm{can}}(M_{K_0})^{\overline{x}})_{|U_i}.$$

By Remark 4.17 these isomorphisms coincide on $U_i \times_{M_{K_0}} U_j$. Hence they glue to a global canonical isomorphism between $\mathbb{L}_{\mathrm{can}}(M_{K_0})^{\overline{x}}$ and $\mathbb{L}_{\mathrm{univ}}(M_{K_0})$. □

5. Extension property for models of Shimura varieties

We recall from [Vas99], [Moo98] and [Kis09] the notion of smooth integral canonical models of Shimura varieties. We equip these objects with canonical local systems as in paragraph 4.1, and show that they form pairs with the smooth extension property as in Definition 3.1.

5.1. **Integral canonical models of Shimura varieties.**

Let (G, X) be a Shimura datum with reflex field $E/\mathbb{Q}$, and $K_p \subseteq G(\mathbb{Q}_p)$ a hyperspecial subgroup. Let v be a place of E lying above p, $\mathcal{O}_{(v)}$ the local ring of $\mathcal{O}_E$ at v and $\mathcal{O}_v$ its v-adic completion. Consider a level $K = K^pK_p$ where K^p is a compact open subgroup of $G(\mathbb{A}_f^p)$, write $\mathrm{Sh}_K(G, X)/E$ for the corresponding variety over E. Set

$$\mathrm{Sh}_{K_p}(G, X) = \varprojlim_{K^p} \mathrm{Sh}_{K^pK_p}(G, X).$$

In addition, let $\mathcal{O}$ be a discrete valuation ring faithfully flat over $\mathcal{O}_{(v)}$, write $\tilde{E}$ for its quotient field and $\mathfrak{m}_{\mathcal{O}}$ for its maximal ideal. We will be considering suitable models of $\mathrm{Sh}_{K_p}(G, X)$ over $\mathcal{O}$.

Definition 5.1: A *smooth integral model* of $\mathrm{Sh}_{K_p}(G, X)$ over $\mathcal{O}$ is a faithfully flat $\mathcal{O}$-scheme $\mathscr{S}_{K_p}(G, X)$ with a continuous $G(\mathbb{A}_f^p)$-action, such that

(i) there is a $G(\mathbb{A}_f^p)$-equivariant identification $\mathscr{S}_{K_p}(G, X) \times_{\mathcal{O}} \tilde{E} \simeq \mathrm{Sh}_{K_p}(G, X) \times_E \tilde{E}$, and
(ii) there exists a compact open subgroup $K_0^p \subseteq G(\mathbb{A}_f^p)$ such that for any $K_2^p \subseteq K_1^p \subseteq K_0^p$ open subgroups, the canonical quotient map

$$\mathscr{S}_{K_p}(G, X)/K_1^p \to \mathscr{S}_{K_p}(G, X)/K_2^p$$

is a finite étale morphism between smooth schemes of finite type over $\mathcal{O}$.

If $K = K^pK_p \subseteq K_0^pK_p$ is an open subgroup, we let $\mathscr{S}_K(G, X) := \mathscr{S}_{K_p}(G, X)/K^p$.

A smooth integral model is canonical if it satisfies the following property.

Definition 5.2: A smooth integral model $\mathscr{S}_{K_p}(G, X)$ over $\mathcal{O}$ is canonical if, for any regular formally smooth $\mathcal{O}$-scheme $\mathcal{T}$, any $\tilde{E}$-morphism $\mathcal{T} \times_{\mathcal{O}} \tilde{E} \to \mathrm{Sh}_{K_p}(G, X)$ extends uniquely to an $\mathcal{O}$-morphism $\mathcal{T} \to \mathscr{S}_{K_p}(G, X)$.

Remark 5.3: The extension property of Definition 5.2 is refered to in the literature as the *smooth extension property* of the pro-scheme $\mathscr{S}_{K_p}(G, X)$. The terminology overlaps with Definition 3.1, but this should cause no issue: the latter applies to a pair $(\mathscr{S}, \mathbb{L})$ with $\mathscr{S}/\mathcal{O}_F$ of finite type, and Proposition 5.7 shows that it is implied by the former for Shimura varieties at finite level.

It follows from Definition 5.2 that a smooth integral canonical model, if it exists, is unique. Moreover, being canonical is preserved under certain ring extensions:

Fact 5.4 ([Vas99], Proposition 3.2.3.3)**:** *Let $\mathscr{S}_{K_p}(G, X)$ be a smooth integral canonical model of $\mathrm{Sh}_{K_p}(G, X)$. Let $\mathcal{O} \hookrightarrow \mathcal{O}'$ be a faithfully flat extension of discrete valuation rings with separable residue field extension, and such that $\mathfrak{m}_{\mathcal{O}}\mathcal{O}'$ is maximal in $\mathcal{O}'$. Let $\tilde{E}'$ be the fraction field of $\mathcal{O}'$. Then the base-change $\mathscr{S}_{K_p}(G, X)_{\mathcal{O}'}$ is a smooth integral canonical model of $\mathrm{Sh}_{K_p}(G, X)_{\tilde{E}'}$.*

Remark 5.5: The condition that $\mathfrak{m}_{\mathcal{O}}\mathcal{O}'$ is maximal in $\mathcal{O}'$ is also a requirement in the theory of Néron models (see [BLR12, Theorem 7.2.1 (ii)]).

5.2. **Canonical local systems on integral models.** With notations as in paragraph 5.1, we fix $\mathscr{S}_{K_p}(G,X)$ a smooth integral canonical model of $\mathrm{Sh}_{K_p}(G,X)$ over $\mathcal{O}$. Let $K_0^p \subseteq G(\mathbb{A}_f^p)$ be a compact open subgroup as in Definition 5.1, and set $K_0 = K_0^p K_p \subseteq G(\mathbb{A}_f)$. Then if $K \subseteq K_0$ is an open normal subgroup, the finite étale map

$$\mathscr{S}_K(G,X) \to \mathscr{S}_{K_0}(G,X)$$

is a K_0^p/K^p-torsor, and the projective system $(\mathscr{S}_K(G,X))_{K^p \in \mathcal{C}(K_0^p)}$ is fit for K_0^p as in Definition 4.1.

Definition 5.6: We denote $\mathbb{L}_{\mathrm{can}}$ the (isomorphism class of) canonical local system on $\mathscr{S}_{K_0}(G,X)$.

By construction, any choice of geometric point induces a homomorphism $\pi_1(\mathscr{S}_{K_0}(G,X)) \to K_0^p$ characterizing $\mathbb{L}_{\mathrm{can}}$ on a connected component.

Proposition 5.7 ([BST24], §1.3)**:** *The pair $(\mathscr{S}_{K_0}(G,X), \mathbb{L}_{\mathrm{can}})$ has the smooth extension property as in Definition 3.1.*

Proof. Let $X/\tilde{E}$ be a smooth variety with smooth model $\mathcal{X}/\mathcal{O}$. Let $f : X \to \mathrm{Sh}_{K_p}(G,X)_{\tilde{E}}$ be a map of $\tilde{E}$-varieties such that the local system $f^{-1}\mathbb{L}_{\mathrm{can}}$ over X extends to $\mathcal{X}$. We may assume that X is connected. By choosing geometric points appropriately, we find a homomorphism

$$\pi_1(\mathcal{X}) \to K_0^p.$$

For any $K^p \subseteq K_0^p$ open subgroup, let $K = K^p K_p$ and define $\mathcal{X}_K \to \mathcal{X}$ as the finite étale cover defined by the kernel of the map

$$\pi_1(\mathcal{X}) \to K_0^p \to K_0^p/K^p.$$

The cover between the generic fibers $X_K \to X$ coincides with the pullback of $\mathrm{Sh}_K(G,X)_{\tilde{E}} \to \mathrm{Sh}_{K_0}(G,X)_{\tilde{E}}$ to X. Then the regular formally smooth $\mathcal{O}$-scheme $\mathcal{X}_{K_p} := \varprojlim_{K^p} \mathcal{X}_K$ has generic fiber $X_{K_p} := \varprojlim_{K^p} X_K$. The map f induces

$$X_{K_p} \to \mathrm{Sh}_{K_p}(G,X)_{\tilde{E}}$$

which extends to $\mathcal{X}_{K_p} \to \mathscr{S}_{K_p}(G,X)$. To show that this map descends to $\mathcal{X} \to \mathscr{S}_{K_0}(G,X)$ it suffices to show that the two morphisms

$$\mathcal{X}_{K_p} \times_{\mathcal{X}} \mathcal{X}_{K_p} \rightrightarrows \mathcal{X}_{K_p} \to \mathscr{S}_{K_p}(G,X) \to \mathscr{S}_{K_0}(G,X)$$

coincide and invoke fpqc descent. By assumption they agree between the generic fibers, hence they coincide everywhere as $\mathscr{S}_{K_0}(G,X)$ is separated. □

6. Applications

Assume $p \neq 2$. Unless specified otherwise, F denotes a finite unramified extension of $\mathbb{Q}_p$.

6.1. **Abelian varieties.** Fix $g \geq 1$ and integer, (V,ψ) a symplectic space over $\mathbb{Q}$ of dimension $2g$. Let $G = \mathrm{GSp}(\psi)$ and X the space of complex structures on $V_{\mathbb{R}}$ for which $(V, \pm\psi)$ is a polarized Hodge structure. Fix $K_p \subseteq G(\mathbb{Q}_p)$ a hyperspecial level at p, $K_0 = K_0^p K_p \subseteq G(A_f)$ a neat compact open subgroup, and $K \subseteq K_0$ an open subgroup. Consider the level K Siegel modular variety $\mathrm{Sh}_K(G,X)$ over its reflex field $\mathbb{Q}$.

Fact 6.1 ([Moo98], Corollary 3.8)**:** *The Shimura variety $\mathrm{Sh}_{K_p}(G,X)$ admits a smooth integral canonical model $\mathscr{S}_{K_p}(G,X)$ over $\mathbb{Z}_{(p)}$.*

Corollary 6.2: *The $\mathcal{O}_F$-scheme $\mathscr{S}_K(G,X)_{\mathcal{O}_F}$ has the PGR-extension property.*

Proof. Since $F/\mathbb{Q}_p$ is unramified, by Fact 5.4 the base change $\mathscr{S}_{K_p}(G,X)_{\mathcal{O}_F}$ is a smooth integral canonical model of $\mathrm{Sh}_{K_p}(G,X)_F$. By Proposition 5.7, $(\mathscr{S}_K(G,X)_{\mathcal{O}_F}, \mathbb{L}_{\mathrm{can}})$ has the smooth, extension properties, and the conclusion follows from Proposition 3.7. □

Theorem 6.3: *Let X/F be a smooth, geometrically connected variety, and $\mathcal{X}/\mathcal{O}_F$ a smooth model. Consider $A \to X$ a polarized abelian scheme with K^p-level structure. If it has pointwise good reduction, then it has unique global good reduction as an abelian scheme.*

Proof. By Remark 3.3 the abelian scheme is defined by a PGR-morphism $X \to \mathrm{Sh}_K(G,X)_F$. Global good reduction follows from Corollary 6.2 and Remark 3.5, and uniqueness follows from the separatedness of $\mathscr{S}_K(G,X)_{\mathcal{O}_F}$. □

Theorem 6.3 is weaker than the following result of Cadoret-Tamagawa, which relies on [VZ10, Corollary 5] without involving Shimura varieties.

Fact 6.4 ([CT26], §4.3.2, Example (3))**:** *Assume $F/\mathbb{Q}_p$ has ramification index $e \leq p-1$. Let X/F be a smooth, geometrically connected variety, and $\mathcal{X}/\mathcal{O}_F$ a smooth model. Let $A \to X$ be an abelian scheme. If it has pointwise good reduction, then it has global good reduction as an abelian scheme.*

The bound on e is optimal, as shown by Corollary 2.7. The theory of Shimura varieties provides a second interpretation for this obstruction. It was asked by Vasiu in [Vas99, Remark 3.2.12.1] if Fact 5.4 still holds for ramified extensions. For the Siegel modular variety, the answer is negative in general.

Proposition 6.5: *Assume $F/\mathbb{Q}_p$ has ramification index $e \geq p$. There is no smooth integral canonical model for $\mathrm{Sh}_{K_p}(G,X)_F$.*

Proof. If such a model were to exist, Corollary 6.2 and Theorem 6.3 would hold for F, but this is not the case as shown by Corollary 2.7. □

6.2. **K3 surfaces.** Fix $d \geq 1$ an integer such that $p \nmid 2d$. The K3 lattice is the rank 22 quadratic lattice $\Lambda = E_8^{\oplus 2} \oplus U^{\oplus 3}$, where E_8 is the E8 lattice of rank 8 and U is the hyperbolic lattice of rank 2. Let $e, f \in U \subseteq \Lambda$ be isotropic elements satisfying $[e,f]=1$ in some copy of the hyperbolic lattice, where $[\cdot,\cdot]$ is the bilinear symmetric form on U. Set $L_d = \langle e - df \rangle^{\perp} \subseteq \Lambda$ a sublattice of discriminant $2d$ and rank 21 and write $V \coloneqq L_{d,\mathbb{Q}}$. We let $L_d^{\vee} \subseteq V$ denote the dual lattice. To this data is associated a Shimura variety of orthogonal type, following [MP15, §3.1].

Definition 6.6: Let $G = \mathrm{SO}(V)$ and X the space of oriented negative planes in $L_{d,\mathbb{R}}$. Let $K_0 = K_0^p K_p \subseteq G(\mathbb{A}_f)$ denote the largest subgroup of $\mathrm{SO}(L_d)(\widehat{\mathbb{Z}})$ acting trivially on $L_d^{\vee}/L_d$. For any level $K = K^p K_p \subseteq K_0$, let $\mathrm{Sh}_K(G,X)$ be the Shimura variety of level K associated to the Shimura datum (G,X), over its reflex field $\mathbb{Q}$.

Fact 6.7 ([MP16], Proposition 7.9)**:** *The Shimura variety $\mathrm{Sh}_{K_p}(G,X)$ admits a smooth integral canonical model $\mathscr{S}_{K_p}(G,X)$ over $\mathbb{Z}_{(p)}$.*

Next we introduce the moduli space of primitively polarized K3 surfaces with level structure, following [MP15, §2.1, §2.10].

Definition 6.8: Let $\mathscr{M}_{2d}^{\circ}$ be the moduli problem that assigns to every $\mathbb{Z}_{(p)}$-scheme T the groupoid of tuples $(f : Y \to T, \xi)$ where
- $f : Y \to T$ is a K3 surface, and
- $\xi \in \underline{\mathrm{Pic}}(X/T)$ is a primitive polarization of degree $2d$.

It is a separated Deligne-Mumford stack of finite type over $\mathbb{Z}_{(p)}$ ([Riz05] Theorem 4.3.3).

If $(f : Y \to T, \xi) \in \mathscr{M}_{2d}^{\circ}(T)$, let $\mathbf{H}^2_{\widehat{\mathbb{Z}}^p}(f) = \prod_{\ell \neq p} \mathbf{H}^2_{\ell}(f)$ be the $\widehat{\mathbb{Z}}^p$-local system on T defined as the product of the relative ℓ-adic cohomology sheaves. The polarization defines a relative Chern class $\mathrm{ch}_{\widehat{\mathbb{Z}}^p}(\xi) \in \mathbf{H}^2_{\widehat{\mathbb{Z}}^p}(f)(1)$, the sub-local-system orthogonal to this class is the relative primitive cohomology sheaf $\mathbf{P}^2_{\widehat{\mathbb{Z}}^p}(f)(1)$. With notations from Definition 4.6, define the functor

$$\begin{aligned} L : \mathscr{M}_{2d}^{\circ} &\longrightarrow \mathrm{Loc}_{\mathbb{Z}_{(p)}}^{21,\widehat{\mathbb{Z}}^p} \\ (f : Y \to T, \xi) &\longmapsto \mathbf{P}^2_{\widehat{\mathbb{Z}}^p}(f)(1). \end{aligned}$$

The cup-product induces a quadratic form on (primitive) cohomology. We define

$$\begin{aligned} \mathrm{Lev} : \mathscr{M}_{2d}^{\circ} &\longrightarrow B\underline{\mathrm{SO}(L_d)(\widehat{\mathbb{Z}}^p)} \\ (f : Y \to T, \xi) &\longmapsto [(U \to T) \mapsto \{\text{Isometries } L_d \otimes_{\mathbb{Z}} \underline{\widehat{\mathbb{Z}}^p}_U \xrightarrow{\sim} \mathbf{P}^2_{\widehat{\mathbb{Z}}^p}(f)(1)_U\}]. \end{aligned}$$

Now $(\mathscr{M}^\circ_{2d}, L, \mathrm{Lev}, \iota)$ is a level moduli datum as in Definition 4.7, where ι is the obvious sheaf inclusion of $\mathrm{Lev}(f : Y \to T, \xi)$ inside $\underline{\mathrm{Isom}}(L_d \otimes_{\mathbb{Z}} \underline{\widehat{\mathbb{Z}}^p}, \mathbf{P}^2_{\widehat{\mathbb{Z}}^p}(f)(1))$. For any open subgroup $K = K^pK_p \subseteq K_0$ we may consider the associated moduli stack of level K denoted $\mathscr{M}^\circ_{2d,K}$. If K is small enough it is represented by an algebraic space (see [Riz05, Theorem 6.1.2]).

Assume from now-on that the compact open subgroup $K = K^pK_p \subseteq K_0$ is neat.

Fact 6.9 ([MP15], Corollary 4.15)**:** *The stack $\mathscr{M}^\circ_{2d,K}$ is represented by a scheme $M^\circ_{2d,K}$, and there is a (noncanonical) open immersion $M^\circ_{2d,K} \hookrightarrow \mathscr{S}_K(G, X)$.*

Remark 6.10: It follows from Proposition 4.14 that the pullback of $\mathbb{L}_{\mathrm{can}}$ to $M^\circ_{2d,K}$ coincides with the universal relative primitive cohomology $\mathbf{P}^2_{\widehat{\mathbb{Z}}^p}(1)$.

As a consequence of Proposition 5.7 and Proposition 3.7, we find:

Corollary 6.11: *The $\mathcal{O}_F$-scheme $M^\circ_{2d,K,\mathcal{O}_F}$ has the PGR-extension property.*

Theorem 6.12: *Let X/F be a smooth, geometrically connected variety, and $\mathcal{X}/\mathcal{O}_F$ a smooth model. Consider $Y \to X$ a primitively polarized K3-surface of degree $2d$ with K^p-level structure. If it has pointwise good reduction, then it has unique global good reduction as a K3 surface.*

Proof. By Remark 3.3 f is defined by a PGR-morphism $X \to M^\circ_{2d,K,\mathcal{O}_F}$. Global good reduction follows from Corollary 6.11 and Remark 3.5, and the uniqueness follows from the separatedness of $M^\circ_{2d,K,\mathcal{O}_F}$. □

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