DISTINCT G-SETS WITH ISOMORPHIC G-MODULES

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Contents

1. Introduction

Fix $K = \mathbb{F}_q$ a finite field and $n \geq 3$. Let $V = K^n$, $W = V^\vee$ its dual, $G = \mathrm{GL}(V)$ its linear automorphism group. Set $X = \mathbb{P}(V)$ and $Y = \mathbb{P}(W)$, which are left-G-sets. Salim Alloun noticed that although X and Y are not equivalent as G-sets, the induced G-modules $\mathbb{Q}[X]$ and $\mathbb{Q}[Y]$ are isomorphic. In this note we investigate why.

Remark 1: A triple (G, X, Y) where X and Y are G-sets such that $\mathbb{Q}[X]$ and $\mathbb{Q}[Y]$ are isomorphic G-modules is called a *Gassmann triple.*

2. X and Y are not isomorphic

We show the following.

Proposition 2: *The G-sets X and Y are not isomorphic.*

Remark 3: If $n = 1$ or $n = 2$, then X and Y are isomorphic.

Let $v = (1, 0, \ldots, 0) \in V$. The stabilizer of $K \cdot v \in X$ in G is the parabolic subgroup

$$P_1 = \left\{ \begin{pmatrix} x & * \\ 0 & A \end{pmatrix} ; x \in K^\times,\ A \in \mathrm{GL}_{n-1}(K) \right\}.$$

Let $\ell : K^n \to K$ denote the scalar product with v, so that $K \cdot \ell \in Y$. The stabilizer of $K \cdot \ell$ is the parabolic subgroup

$$P_2 = \left\{ \begin{pmatrix} y & 0 \\ * & B \end{pmatrix} ; y \in K^\times,\ B \in \mathrm{GL}_{n-1}(K) \right\}.$$

This subgroup is conjugate to

$$P_2' = \left\{ \begin{pmatrix} B & * \\ 0 & y \end{pmatrix} ; y \in K^\times,\ B \in \mathrm{GL}_{n-1}(K) \right\}$$

via the action of the permutation matrix reversing the order of the canonical basis. Since the actions are transitive, if X and Y were isomorphic then P_1 and P_2 (hence also P_2') would be conjugate.

Definition 4: Let B denote the subgroup of upper-triangular matrices. The parabolic subgroups of $\mathrm{GL}_n(K)$ containing B are called *standard.*

The subgroups B, P_1 and P_2' are all standard parabolics. We use the following fact from the theory of reductive groups.

Fact 5: *Each $\mathrm{GL}_n(K)$-conjugacy class of parabolic subgroups contains a unique standard parabolic.*

Fact 5 shows that P_1 and P_2' cannot be conjugate, which concludes the proof of Proposition 2.

3. $\mathbb{Q}[X]$ and $\mathbb{Q}[Y]$ are isomorphic

For any G-set Z, the induced $\mathbb{Q}$-linear representation $\mathbb{Q}[Z]$ is equivalent to a $\mathbb{Q}[G]$-module structure on $\mathbb{Q}[Z]$. To argue that the $\mathbb{Q}[G]$-modules $\mathbb{Q}[X]$ and $\mathbb{Q}[Y]$ are isomorphic, we use the following.

Theorem 6 (Brauer-Nesbitt)**:** *Let k be a field of characteristic zero, A a (not necessarily commutative) k-algebra. Let M and N be semisimple A-modules that are finite dimensional over k. Then M and N are isomorphic over A if and only if the traces of the actions of A on M and N coincide.*

For $A = k[G]$, Theorem 6 says that M and N are isomorphic if and only if the characters agree.

Remark 7: Theorem 6 fails if k has characteristic $p > 0$. For instance if $k = \mathbb{F}_3$, and $G = \mathbb{Z}/2\mathbb{Z}$, G can act on k^3 either trivially, or via $-\mathrm{id}$. Both actions are semisimple and have equal characters.

Let us compute the traces of the action of G on $\mathbb{Q}[X]$ and $\mathbb{Q}[Y]$. Because G acts via permutation matrices, the trace of $g \in G$ matches the number of fixed points of G. Notice that:

$$\mathrm{Fix}_V(g) = \ker(g - \mathrm{id} : V \to V)$$
$$\mathrm{Fix}_W(g) = \mathrm{im}(g^{-1} - \mathrm{id} : V \to V)^{\perp}.$$

The first equality is clear, while the second follows from the standard fact that for any $\phi\mathrm{End}(V)$,

$$\ker(\phi^T : W \to W) = \mathrm{im}(\phi : V \to V)^{\perp}.$$

Moreover, $\mathrm{rk}(g^{-1} - \mathrm{id}) = \mathrm{rk}(g - \mathrm{id})$, hence from the rank nullity theorem we find

$$\dim_K \mathrm{Fix}_V(g) = \dim_K \mathrm{Fix}_W(g).$$

Passing to projective space, we find that $|\mathrm{Fix}_X(g)| = |\mathrm{Fix}_Y(g)|$.

4. What about $\mathbb{Z}[X]$ and $\mathbb{Z}[Y]$?

Any $\mathbb{Q}$-linear isomorphism $\mathbb{Q}[X] \xrightarrow{\sim} \mathbb{Q}[Y]$ of G-modules extends to a $\mathbb{Z}[1/N]$-linear isomorphism $\mathbb{Z}[1/N][X] \xrightarrow{\sim} \mathbb{Z}[1/N][Y]$ for some integer N; in particular, for infinitely many prime numbers ℓ, $\mathbb{Z}_\ell[X]$ and $\mathbb{Z}_\ell[Y]$ are isomorphic G-modules. Is it possible to find an isomorphism over $\mathbb{Z}$? The answer is no.

Proposition 8: *The $\mathbb{Z}[G]$-modules $\mathbb{Z}[X]$ and $\mathbb{Z}[Y]$ are not isomorphic.*

Notice first that the product $X \times Y$ splits into two G-orbits: those pairs (d, H) where $d \subseteq H$, and those where $d \cap H = \{0\}$. Consequently, $\mathrm{Hom}_{\mathbb{Q}}[G](\mathbb{Q}[X], \mathbb{Q}[Y])$ is a two-dimensional Q-vector space. Let $A = (a_{d,H})_{d,H}$ denote the square matrix of size $\frac{q^n-1}{q-1}$ where

$$a_{(d,H)} = \begin{cases} 1 & \text{if } d \subseteq H \\ 0 & \text{otherwise} \end{cases}$$

and $J = (1)_{d,H}$ the all-1 matrix, then (A, J) is a basis of $\mathrm{Hom}_{\mathbb{Q}}[G](\mathbb{Q}[X], \mathbb{Q}[Y])$. Any G-equivariant morphism $\mathbb{Z}[X] \to \mathbb{Z}[Y]$ is of the form $aA + bJ$ for some $a, b \in \mathbb{Z}$. It remains to show that $aA + bJ$ cannot have determinant ± 1. Fix $a, b \in \mathbb{Z}$, let us compute the eigenvalues of $aA + bJ$. We leave it as an exercise to show that

$$A^T A = q^{n-2} I + \frac{q^{n-2} - 1}{q - 1} J.$$

In particular, $A^T A$ commutes with J, hence stabilizes $\ker J$ and $\mathrm{im}\, J = \mathbb{Q} \cdot \mathbb{1}$ where $\mathbb{1} = (1, \cdots, 1)$. This implies that A stabilizes these subspaces as well. The reader can now check the following:

(i) over $\mathbb{Q} \cdot \mathbb{1}$, the eigenvalue of $aA + bJ$ is $a\frac{q^{n-1}-1}{q-1} + b\frac{q^n-1}{q-1}$, and
(ii) over $\ker J$, the eigenvalues of $aA + bJ$ all have complex absolue value $aq^{\frac{n-2}{2}}$.

In particular,

$$\begin{aligned}|\det(aA+bJ)| &= \left|a\frac{q^{n-1}-1}{q-1}+b\frac{q^n-1}{q-1}\right|\left(aq^{\frac{n-2}{2}}\right)^{\dim\ker J} \\ &= \left|a\frac{q^{n-1}-1}{q-1}+b\frac{q^n-1}{q-1}\right|\left(aq^{\frac{n-2}{2}}\right)^{q\frac{q^{n-1}-1}{q-1}}\end{aligned}$$

This expression evaluates to 1 if and only if $n=1, b=\pm 1$, or $n=2, a+2b=\pm 1$.